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The simplest parametrized normal forms of Hopf and generalized Hopf bifurcations

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Abstract This paper considers the computation of the simplest parameterized normal forms (SPNF) of Hopf and generalized Hopf bifurcations. Although the notion of the simplest normal form has been studied for more than two decades, most of the efforts have been spent on the systems that do not involve perturbation parameters due to the restriction of the computational complexity. Very recently, two singularities – single zero and Hopf bifurcation – have been investigated, and the SPNFs for these two cases have been obtained. This paper extends a recently developed method for Hopf bifurcation to compute the SPNF of generalized Hopf bifurcations. The attention is focused on a codimension-2 generalized Hopf bifurcation. It is shown that the SPNF cannot be obtained by using only a near-identity transformation. Additional transformations such as time and parameter rescaling are further introduced. Moreover, an efficient recursive formula is presented for computing the SPNF. Examples are given to demonstrate the applicability of the new method.

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1 Introduction

Bifurcation and stability are two main concerns in the study of nonlinear dynamical systems. There exist many methodologies in analyzing dynamical behaviors; one of the frequently used approaches is the normal form theory. The basic idea of the normal form theory is to transform the original system to a simpler one that retains the essential dynamical behavior of the original system, which greatly simplifies the dynamical analysis of the original system. Normal forms are not unique and can be given in equivalent forms, depending upon what approaches are used. Normal form usually refers to Poincaré normal form, Birkhoff normal form, or Takens normal form. However, it is not clear which "form" would be a "simpler" or even the simplest for a particular singularity. In fact, it has been found that the conventional (or classical) normal form (CNF) [1–5] can be further reduced to the so-called simplest normal form (SNF) or unique normal form. Takens is probably the first one to notice that the CNF can be further simplified, therefore he [6] considered a further reduction of the normal form associated with degenerate simple zero singularity. Ten years later, Ushiki [7] introduced the method of infinitesimal deformation based on Lie algebra to study the SNF for a number of singularities,

but the SNF computation was limited to lower-order terms. Following Ushiki's work, many researchers have applied the homological operator theory to study the SNF and made significant contributions to this area in both theoretical and computational developments (e.g., see [8–16]).

It has been noticed that the computation of the SNF is much more complicated than that of the CNF. This is why only very few singularities have been investigated by using the SNF in the past two decades, where the main attention has been focused on the SNF of the systems without perturbation parameters. However, a physical or an engineering problem always involves some system parameters, which play an important role in the study of bifurcation and stability. Therefore, finding the SNFs with perturbation (bifurcation) parameters is more important in practice. The terms in a normal form involving perturbation parameters are usually called unfolding. Unfortunately, it has been found that the computation of the SNF with unfolding is more complicated than that of the SNF without unfolding. Even with the aid of computer algebra systems such as Maple and Mathematica, it is not easy to compute such SNFs. Only very recently has attention been turned to the computation of the SNFs with unfolding. Two kinds of singularities have been investigated: the simple zero [17] and the Hopf bifurcation [18]. Efficient computational methods have also been developed for these two singularities, with programs coded using Maple.

Normal forms of nonlinear systems are usually obtained via near-identity transformations, particularly for systems without perturbation parameters. However, it has been shown that near-identity transformations are not enough for computing the SNFs with unfolding. Additional transformations like time rescaling and reparmetrization need to be used [17, 18]. This paper will extend a recently developed method for computing the SNF of Hopf bifurcation [19] to generalized Hopf bifurcations. One key issue in the SNF computation is the computational efficiency, particularly for the SNFs with unfolding. In fact, it has been noted that even with powerful computers, one cannot go very far if the method used is not efficient. Recently, some efficient computational approaches have been developed [19–21], which considerably save the computational time and computer memory, making it possible to obtain higher-order SNFs. One novel approach, called the matching pursuit technique, has been developed to completely solve the Takens–Bogdanov singularity (a double-zero eigenvalue) [21]. This technique can automatically "match" a correct form of the SNF to a particular system, without knowing any conditions or restrictions that are generally needed by other methods (e.g., see [7, 10–12, 14, 15, 20, 22]). This technique will be employed in this paper.

The rest of the paper is organized as follows. In the next section, an efficient computational formula is presented. Section 3 is devoted to the computation of the SPNF of generalized Hopf bifurcations. Two examples are given in Section 4 to illustrate the application of the new method. Finally, conclusion is drawn in Section 5.

2 Efficient computational method

In this section, we present an efficient recursive formula for computing the SPNF, which is applicable for general differential equations and for any type of singularities (not restricted to the Hopf bifurcation).

Consider the following general parameterized nonlinear differential equation:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \ \ \boldsymbol{\mu} \in \boldsymbol{R}^m, \tag{1}$$

where $\boldsymbol{x}$ and $\boldsymbol{\mu}$ are the n-dimensional state vector and m-dimensional parameter vector, respectively. It is assumed that $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium of the system for all real values of $\boldsymbol{\mu}$, i.e., $\boldsymbol{f}(\boldsymbol{0}, \boldsymbol{\mu}) \equiv \boldsymbol{0}$. Further, it is assumed that the nonlinear function $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu})$ is analytic with respect to both $\boldsymbol{x}$ and $\boldsymbol{\mu}$, so that one may expand Equation (1) as

$$\begin{aligned}\frac{d\boldsymbol{x}}{dt} &= L\,\boldsymbol{x} + \boldsymbol{f}_2(\boldsymbol{x}, \boldsymbol{\mu}) + \boldsymbol{f}_3(\boldsymbol{x}, \boldsymbol{\mu}) \\ &\quad + \cdots + \boldsymbol{f}_k(\boldsymbol{x}, \boldsymbol{\mu}) + \cdots\end{aligned} \tag{2}$$

where $L\,\boldsymbol{x} \triangleq \boldsymbol{v}_1(\boldsymbol{x})$ represents the linear part and L is the Jacobian matrix evaluated at the equilibrium $\boldsymbol{x} = \boldsymbol{0}$ and the critical point $\boldsymbol{\mu} = \boldsymbol{0}$. It is assumed that all eigenvalues of L have zero real parts and, without loss of generality, L is given in Jordan canonical form. $\boldsymbol{f}_k(\boldsymbol{x}, \boldsymbol{\mu})$ denotes the kth-degree vector homogeneous polynomials of $\boldsymbol{x}$ and $\boldsymbol{\mu}$.

The basic idea of the normal form theory is to find a near-identity nonlinear transformation

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{y} + \boldsymbol{h}_2(\boldsymbol{y}, \boldsymbol{\nu}) + \boldsymbol{h}_3(\boldsymbol{y}, \boldsymbol{\nu}) \\ &\quad + \cdots + \boldsymbol{h}_k(\boldsymbol{y}, \boldsymbol{\nu}) + \cdots,\end{aligned} \tag{3}$$

where $\boldsymbol{\nu}$ is a new scaled parameter, via the relation

$$\boldsymbol{\mu} = \boldsymbol{\nu} + \boldsymbol{p}_2(\boldsymbol{\nu}) + \boldsymbol{p}_3(\boldsymbol{\nu}) + \cdots + \boldsymbol{p}_k(\boldsymbol{\nu}) + \cdots \tag{4}$$

so that a simpler form (normal form) can be obtained as

$$\frac{d\boldsymbol{y}}{d\tau} = (L + L_1(\boldsymbol{\nu}))\,\boldsymbol{y} + \boldsymbol{g}_2(\boldsymbol{y}) + \boldsymbol{g}_3(\boldsymbol{y}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y}) + \cdots \tag{5}$$

where the terms $\boldsymbol{g}_k(\boldsymbol{y})$ are the same as that of the CNF [1], while $L_1(\boldsymbol{\nu})$ is an $n \times n$ matrix linear function of $\boldsymbol{\nu}$, to be determined in the process of computation, representing the unfolding. The new time variable τ is defined through the time rescaling [17]

$$t = [\,T_0 + T_1(\boldsymbol{y}, \boldsymbol{\nu}) + T_2(\boldsymbol{y}, \boldsymbol{\nu}) + \cdots + T_k(\boldsymbol{y}, \boldsymbol{\nu}) + \cdots]\,\tau. \tag{6}$$

According to the normal form theory (e.g., see [1, 2]), we further define a so-called Lie bracket operator as follows:

$$\begin{aligned} &L_k : \mathcal{H}_k \mapsto \mathcal{H}_k, \\ &U_k \in \mathcal{H}_k \mapsto L_k(U_k) = [\,U_k,\ \boldsymbol{v}_1\,] \in \mathcal{H}_k, \\ &[\,U_k,\ \boldsymbol{v}_1] \ =\ D\boldsymbol{v}_1 \cdot U_k - DU_k \cdot \boldsymbol{v}_1, \end{aligned} \tag{7}$$

where $\mathcal{H}_k$ denotes a linear vector space consisting of the kth-degree homogeneous vector polynomials. The operator $[\,\bullet, \bullet\,]$ is called the Lie bracket.

Next, define the space $\mathcal{R}_k$ as the range of L_k, and the complementary space of $\mathcal{R}_k$ as $\mathcal{K}_k$. Thus,

$$\mathcal{H}_k = \mathcal{R}_k \oplus \mathcal{K}_k, \tag{8}$$

and we can choose desirable vector bases for spaces $\mathcal{R}_k$ and $\mathcal{K}_k$. Consequently, a vector homogeneous polynomial $\boldsymbol{f}_k \in \mathcal{H}_k$ can be split into two parts: one is spanned by the basis of $\mathcal{R}_k$ and the other by that of $\mathcal{K}_k$. Normal form theory shows that the part belonging to $\mathcal{R}_k$ can be eliminated while the part belonging to $\mathcal{K}_k$ must be retained in the normal form. The idea of finding the SPNF is similar to the CNF: Finding an appropriate nonlinear transformation which further simplifies the normal form. The "simplest" here means that the number of the terms in each order retained in the SPNF reaches the minimum. More precisely, differentiating Equation (3) with respect to t and then applying Equations (2) and (3)–(5) yields a set of algebraic equations in each order for solving the coefficients of the SPNF and the nonlinear transformation. The main idea in further reduction from the CNF to the SPNF is to find appropriate $\boldsymbol{h}_k(\boldsymbol{y}, \boldsymbol{\nu})$'s such that some terms of $\boldsymbol{g}_k(\boldsymbol{y})$'s can be removed.

The key step in the computation of the kth-order normal form is to find the kth-order algebraic equations, which take most of the computational time and computer memory. The solution procedures given in most of the existing normal form computation methods contain all lower-order and many higher-order terms in the kth-order equations, which tremendously increase the memory requirement and the computational time. Therefore, from the computational point of view, a crucial step in normal form computation is to find the kth-order algebraic equations that contain only the kth-order nonlinear terms.

An efficient recursive formula for computing the kth-order algebraic equations has been developed in [18], which is given by

$$\begin{aligned} \boldsymbol{g}_k = {} & y\boldsymbol{f}_k + [\,\boldsymbol{h}_k,\ \boldsymbol{v}_1\,] \\ & + \sum_{i=2}^{k-1} \{D\boldsymbol{f}_i\,\tilde{\boldsymbol{h}}_{k-i+1} - D\boldsymbol{h}_{k-i+1}\,\boldsymbol{g}_i\,\} \\ & + \sum_{m=0}^{k-2} T_m \sum_{i=2}^{[\frac{k}{2}]-m} \frac{1}{i} \sum_{j=i}^{k-m-i} D^i \boldsymbol{f}_j \\ & \times \sum_{\substack{l_1 + l_2 + \cdots + l_i = k-m-(j-i) \\ 2 \le l_1, l_2, \ldots, l_i \le k-m+2-(i+j)}} \tilde{\boldsymbol{h}}_{l_1} \ \ldots\ \tilde{\boldsymbol{h}}_{l_i} \\ & + \sum_{i=1}^{k-3} \sum_{j=2}^{k-1-i} T_i\, D\boldsymbol{f}_j\, \tilde{\boldsymbol{h}}_{k+1-i-j} \\ & + \sum_{i=1}^{k-2} T_i\,(\boldsymbol{f}_{k-i} + L\boldsymbol{h}_{k-i}) \\ & + (T_{k-1}\boldsymbol{v}_1 - D\boldsymbol{h}_{k-1}L_2), \end{aligned} \tag{9}$$

for $k = 2, 3, \ldots$, where $\boldsymbol{v}_1(\boldsymbol{y}) = L\,\boldsymbol{y}$, representing the linear part of the system, and $L_2 = L_1(\boldsymbol{\nu})\,\boldsymbol{y}$ is the unfolding.

Note that in Equation (9), the variables $\boldsymbol{y}$ and $\boldsymbol{\nu}$ in $\boldsymbol{g}_i, \boldsymbol{f}_i, \boldsymbol{h}_i, \boldsymbol{v}_1$ and T_i have been dropped for simplicity, and $\tilde{\boldsymbol{h}}_1 \equiv (\boldsymbol{y},\ \boldsymbol{\nu})^{\mathrm{T}}$. And $\boldsymbol{f}_j(\boldsymbol{x}, \boldsymbol{\mu})$, $\boldsymbol{g}_j(\boldsymbol{y})$ and $\boldsymbol{h}_j(\boldsymbol{y}, \boldsymbol{\nu})$ are all jth-degree vector homogeneous

polynomials in their arguments, where $\boldsymbol{x}$ and $\boldsymbol{\mu}$ are given in Equations (3) and (4), respectively. Moreover, $T_j(\boldsymbol{y}, \boldsymbol{\nu})$ is a scalar function of $\boldsymbol{y}$ and $\boldsymbol{\nu}$, while $\boldsymbol{p}(\boldsymbol{\nu})$ is a vector polynomial of $\boldsymbol{\nu}$. The notation $D^i \boldsymbol{f}_j \tilde{\boldsymbol{h}}_{l_1} \tilde{\boldsymbol{h}}_{l_2} \ldots \tilde{\boldsymbol{h}}_{l_i}$ denotes the ith-order terms of the Taylor expansion of the jth-degree vector homogeneous polynomial $\boldsymbol{f}_j(\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{\nu}), \boldsymbol{\nu} + \boldsymbol{p}(\boldsymbol{\nu})) \equiv \boldsymbol{f}_j(\tilde{\boldsymbol{y}} + \tilde{\boldsymbol{h}}(\tilde{\boldsymbol{y}}))$ about $\tilde{\boldsymbol{y}}$. Here, $\tilde{\boldsymbol{y}} = (\boldsymbol{y}, \boldsymbol{\nu})^{\mathrm{T}}$ and $\tilde{\boldsymbol{h}} = (\boldsymbol{h}, \boldsymbol{p})^{\mathrm{T}}$. More precisely,

$$D^i \boldsymbol{f}_j(\tilde{\boldsymbol{y}} + \tilde{\boldsymbol{h}}(\tilde{\boldsymbol{y}})) = D(D(\cdots D((D\boldsymbol{f}_j)\tilde{\boldsymbol{h}}_{l_1})\tilde{\boldsymbol{h}}_{l_2}) \ldots \tilde{\boldsymbol{h}}_{l_i-1})\tilde{\boldsymbol{h}}_{l_i},$$

where each differential operator D affects only function $\boldsymbol{f}_j$, not $\tilde{\boldsymbol{h}}_{l_m}$ (i.e., $\tilde{\boldsymbol{h}}_{l_m}$ is treated as a constant vector in the process of differentiation), and thus $i \leq j$. Note that at each level of the differentiation, the D operator is actually a Frechét differentiation with respect to $\tilde{\boldsymbol{y}}$, which yields a matrix. This matrix is multiplied by a vector to generate another vector, and then to another level of the Frechét differentiation, and so on.

If system (1) does not contain parameter $\boldsymbol{\mu}$, then $T_i = 0$, $L_1 = 0$, $\boldsymbol{p} = \boldsymbol{0}$, $\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{\nu}) = \boldsymbol{f}(\boldsymbol{y})$, $\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{\nu}) = \boldsymbol{h}(\boldsymbol{y})$, and thus Equation (9) is reduced to the formula that has been obtained for the "reduced" system [20]. Also, note that the only operation involved in the formula is the Frechét differentiation in $D\boldsymbol{h}_i$, $D^i \boldsymbol{f}_j$ and the Lie bracket $[\bullet, \bullet]$. This operation can be easily implemented by using a computer algebra system like Maple.

3 The SPNFs of Hopf and generalized Hopf bifurcations

We are now ready to discuss the SPNFs of Hopf and generalized Hopf bifurcations. Consider the following system:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \quad \boldsymbol{x} \in \boldsymbol{R}^2, \quad \boldsymbol{\mu} \in \boldsymbol{R}^q, \quad \boldsymbol{f}: \boldsymbol{R}^{2+q} \to \boldsymbol{R}^2, \tag{10}$$

where $q \geq 1$. It is assumed that $\boldsymbol{f}(\boldsymbol{0}, \boldsymbol{\mu}) = \boldsymbol{0}$ for all real values of $\boldsymbol{\mu} \in \boldsymbol{R}^q$, indicating that $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium of system (10). Suppose that at $\boldsymbol{\mu} = \boldsymbol{\mu}^*$, the Jacobian of system (10) has one pair of purely imaginary eigenvalues. Then Hopf and generalized Hopf bifurcations may bifurcate from the equilibrium $\boldsymbol{x} = \boldsymbol{0}$. Further, without loss of generality, it is assumed that the critical point is $\boldsymbol{\mu}^* = \boldsymbol{0}$, and that the Jacobian of system (10) evaluated at the critical point and at the equilibrium $\boldsymbol{x} = \boldsymbol{0}$ is given in the following form:

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{11}$$

implying that system (10) is described on a two-dimensional center manifold. The rest of the paper concentrates on the SPNF computation based on this center manifold.

If system (10) does not contain parameter $\boldsymbol{\mu}$, the SNFs of Hopf and generalized Hopf bifurcations have been obtained using only near-identity transformations [13], which are summarized later. For simplicity, suppose that the CNF (without parameters) has been found from system (10) (e.g., by using the Maple program developed in [5]), given in polar coordinates

$$\begin{aligned} \frac{dr}{dt} &= a_{13} r^3 + a_{15} r^5 + a_{17} r^7 + \cdots \\ \frac{d\theta}{dt} &= 1 + a_{23} r^2 + a_{25} r^4 + a_{27} r^6 + \cdots, \end{aligned} \tag{12}$$

where r and θ denote the amplitude and phase of the motion, respectively. Then, the SNFs without perturbation parameters for Hopf and generalized Hopf bifurcations are given in the following theorem.

Theorem 1 ([13]). *When the CNF of a system is given by Equation (12), we have*

(a) For Hopf bifurcation: if $a_{13} \neq 0$, then the SNF is

$$\begin{aligned} \frac{dR}{dt} &= a_{13} R^3 + a_{15} R^5, \\ \frac{d\Theta}{dt} &= 1 + a_{23} R^2, \end{aligned} \tag{13}$$

up to an arbitrary order, where R and Θ represent the amplitude and phase of the motion.

(b) For generalized Hopf bifurcations:

$$\text{(i) if } \begin{cases} a_{13} = a_{15} = \cdots = a_{1(2k-1)} = 0, \ a_{1(2k+1)} \neq 0, \\ a_{23} = a_{25} = \cdots = a_{2(2k-1)} = 0, \end{cases}$$

then the SNF is

$$\frac{dR}{dt} = a_{1\,(2k+1)}\, R^{2k+1} + b_{1\,(4k+1)}\, R^{4k+1},$$
$$\frac{d\Theta}{dt} = 1 + a_{2\,(2k+1)}\, R^{2k}; \tag{14}$$

(ii) *if* $\begin{cases} a_{13} = a_{15} = \cdots = a_{1\,(2k-1)} = 0, \;\; a_{1\,(2k+1)} \neq 0, \\ a_{23} = a_{25} = \cdots = a_{2\,(2j-3)} = 0, \;\; a_{2\,(2j-1)} \neq 0 \\ \quad (\text{for } \; 2 \leq j \leq k), \end{cases}$

then the SNF is given by

$$\frac{dR}{dt} = a_{1\,(2k+1)}\, R^{2k+1} + b_{1\,(4k+1)}\, R^{4k+1},$$
$$\frac{d\Theta}{dt} = 1 + a_{2\,(2j-1)}\, R^{2(j-1)} + \cdots + b_{2\,(2k+1)}\, R^{2k}. \tag{15}$$

Here, b_{ij}'s are explicitly expressed in terms of a_{ij}'s.

Later, the SPNF, namely the SNF with perturbation parameters for Hopf bifurcation was obtained, as summarized in the following theorem.

Theorem 2 ([18]). *The SNF with perturbation parameters (unfolding) for system (10) is given in polar coordinates as follows:*

$$\frac{dR}{d\tau} = R(\alpha\, \nu + b_{13}\, R^2),$$
$$\frac{d\Theta}{d\tau} = 1 + b_{23}\, R^2 + b_{25}\, R^4 + b_{27}\, R^6 + \cdots \tag{16}$$

where b_{ij}'s are explicitly given in terms of the original system parameters, and $\alpha\nu$ denotes the unfolding; in addition, $b_{13} \neq 0$.

It is observed, by comparing Equation (16) with Equation (13), that the amplitude equation of the SPNF is simpler than that of the SNF. However, the phase equation of the SPNF has *infinite* terms, while that of the SNF has only *one* term besides the frequency term, 1. Nevertheless, this does not increase difficulty in analysis since only the first equation of Equation (16) will be used for bifurcation and stability studies.

In the remainder of the section, we use the method developed in [18] to derive the SPNF of generalized Hopf bifurcations for system (10). In order to clearly present the idea of the SPNF, we need to define the codimension of a generalized Hopf bifurcation. It is well known that the codimension of Hopf bifurcation (with condition $a_{13} \neq 0$) is 1. When $a_{13} = 0$, but $a_{15} \neq 0$, the codimension of the generalized Hopf bifurcation is 2, which requires two bifurcation parameters in the unfolding. So, the dimension of $\boldsymbol{\mu}$ in Equation (10) should be 2, i.e., $q = 2$ in this case. In general, if $a_{13} = 0 = a_{15} = \cdots = a_{1(2k-1)} = 0$, but $a_{1(2k+1)} \neq 0$, the codimension of the generalized Hopf bifurcation is $q = k$. Other codimension cases can be similarly discussed.

In this paper, we restrict ourselves to a codimension-2 generalized Hopf bifurcation, i.e., it is assumed that

$$a_{13} = 0, \quad \text{but} \quad a_{15} \neq 0. \tag{17}$$

In this case, we have the following theorem.

Theorem 3. *Under the transformations (3)–(6) and condition (17), the SPNF of system (10) for codimension-2 generalized Hopf bifurcation is given in polar coordinates as follows:*

$$\frac{dR}{d\tau} = R(\boldsymbol{\alpha}_1\, \boldsymbol{\nu} + \boldsymbol{\alpha}_2\, \boldsymbol{\nu}\, R^2 + b_{15}\, R^4),$$
$$\frac{d\Theta}{d\tau} = 1 + b_{23}\, R^2 + b_{25}\, R^4 + \cdots, \tag{18}$$

up to any order, where $\boldsymbol{\nu} = (\nu_1,\, \nu_2)^{\mathrm{T}}$, and $\boldsymbol{\alpha}_1 = (\alpha_{11}, \alpha_{12})$ and $\boldsymbol{\alpha}_2 = (\alpha_{21},\, \alpha_{22})$ represent the coefficients of unfolding terms.

Note that the coefficients b_{1j} in Equation (18) are not given in terms of the coefficients a_{1j} of Equation (12), but the polynomial functions of the coefficients of system (10), since system (10) is not given in a CNF form.

Proof: Since the proof is similar to that of the Hopf bifurcation case [18], we only outline the main steps. To do so, use the transformations $x_1 = \frac{1}{2}(z + \bar{z})$ and $x_2 = \frac{i}{2}(z - \bar{z})$ to obtain

$$\frac{dz}{dt} = i\, z + f(z, \bar{z}, \boldsymbol{\mu}),$$
$$\frac{d\bar{z}}{dt} = -i\, \bar{z} + \bar{f}(z, \bar{z}, \boldsymbol{\mu}), \tag{19}$$

where f is a polynomial in z, $\bar{z}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)^{\mathrm{T}}$, starting from the 2nd-order terms, and $\bar{z}$ and $\bar{f}$ are the complex conjugates of z and f, respectively. The form of the CNF term $\boldsymbol{g}_k$ can be easily determined by using the Poincaré normal form theory as

$$\begin{aligned}\boldsymbol{g}_k(u, \bar{u}) &= \begin{pmatrix} g_k(u, \bar{u}) \\ \bar{g}_k(u, \bar{u}) \end{pmatrix} \\ &= \begin{pmatrix} (b_{1k} + i\, b_{2k})\, u^{(k+1)/2}\, \bar{u}^{(k-1)/2} \\ (b_{1k} - i\, b_{2k})\, \bar{u}^{(k+1)/2}\, u^{(k-1)/2} \end{pmatrix},\end{aligned} \tag{20}$$

for odd integers $k \geq 3$. In the SPNF computation, one wants to eliminate one or both of the b coefficients, in each order, by nonlinear transformations.

Without loss of generality, assume that the homogeneous polynomial $\boldsymbol{f}_k(z, \bar{z}, \boldsymbol{\mu})$ and the nonlinear transformation $\boldsymbol{h}_k(u, \bar{u}, \boldsymbol{\nu})$ are given in the general forms of $\boldsymbol{f}_k(z, \bar{z}, \boldsymbol{\mu}) = (f_k(z, \bar{z}, \boldsymbol{\mu}),\ \bar{f}_k(z, \bar{z}, \boldsymbol{\mu}))^{\mathrm{T}}$ and $\boldsymbol{h}_k(u, \bar{u}, \boldsymbol{\nu}) = (h_k(u, \bar{u}, \boldsymbol{\nu}),\ \bar{h}_k(u, \bar{u}, \boldsymbol{\nu}))^{\mathrm{T}}$, with

$$\begin{aligned} f_k &= \sum_{j+l+m=k} (\boldsymbol{a}_{1\,j\,l\,m} + i\, \boldsymbol{a}_{2\,j\,l\,m})\, z^j\, \bar{z}^l\, \boldsymbol{\mu}^m, \\ h_k &= \sum_{j+l+m=k} (\boldsymbol{c}_{1\,j\,l\,m} + i\, \boldsymbol{c}_{2\,j\,l\,m})\, u^j\, \bar{u}^l\, \boldsymbol{\nu}^m, \end{aligned} \tag{21}$$

and then $z = u + \sum_{k=2} h_k(u, \bar{u}, \boldsymbol{\nu})$. Further, assume that the time and parameter rescaling are given by

$$\begin{aligned} t &= \left\{ 1 + \sum_{k=1} \sum_{j+m=k} \boldsymbol{t}_{jm} \left[\frac{1}{2}(u + \bar{u}) \right]^j \boldsymbol{\nu}^m \right\} \tau, \\ \mu &= \boldsymbol{\nu} + \sum_{k=2} \boldsymbol{p}_k\, \boldsymbol{\nu}^k. \end{aligned} \tag{22}$$

Here, $\boldsymbol{\mu}^m = (\mu_1^{m_1},\ \mu_2^{m_2})^{\mathrm{T}}$ for all nonnegative integers m_1 and m_2 satisfying $m_1 + m_2 = m$, and similar definitions are applied to $\boldsymbol{\nu}^m$ and $\boldsymbol{\nu}^j$. Moreover, $\boldsymbol{a}_{1\,j\,l\,m}$ is a 2-dimensional low vector, satisfying $m_1 + m_2 = m$. Similar meanings apply to the notations $\boldsymbol{a}_{2\,j\,l\,m}$, $\boldsymbol{c}_{1\,j\,l\,m}$, $\boldsymbol{c}_{2\,j\,l\,m}$, $\boldsymbol{t}_{jm}$, and $\boldsymbol{p}_j$.

The universal unfolding can be assumed in the form of

$$\begin{aligned} L_2(u, \bar{u}, \boldsymbol{\nu}) &= \boldsymbol{\alpha}_1 \boldsymbol{\nu}\, (u,\ \bar{u})^{\mathrm{T}} = (\alpha_{11}\, \nu_1 + \alpha_{12}\, \nu_2)\, (u,\ \bar{u})^{\mathrm{T}}, \\ L_4(u, \bar{u}, \boldsymbol{\nu}) &= \boldsymbol{\alpha}_2 \boldsymbol{\nu}\, (u^2 \bar{u},\ u\, \bar{u}^2)^{\mathrm{T}} \\ &= (\alpha_{21}\, \nu_1 + \alpha_{22}\, \nu_2)\, (u^2 \bar{u},\ u\, \bar{u}^2)^{\mathrm{T}}. \end{aligned} \tag{23}$$

The reason for choosing this form can be explained as follows: since $b_{13} = a_{13} = 0$, the first nonzero coefficient in the normal form is b_{15}, and thus, the amplitude equation should be in the form of

$$\frac{dR}{d\tau} = b_{15}\, R^5 + \cdots$$

Therefore, it requires two terms in the unfolding so that the normal form can exhibit generic dynamical behavior of the original system (e.g., see [4]). As a matter of fact, if we process the SPNF computation without assuming the unfolding (23), we would find from the 2nd- and 4th-order computations that these two terms must be retained in the normal form since there do not exist any $\boldsymbol{c}$, $\boldsymbol{p}$, and $\boldsymbol{t}$ coefficients that can be used for solving the algebraic equations associated with the unfolding terms.

Next, we use the coefficients $\boldsymbol{c}_{1\,jlm}$, $\boldsymbol{c}_{2\,jlm}$, $\boldsymbol{t}_{jm}$, and $\boldsymbol{p}_j$ to eliminate $b_{1k}, b_{2k}, \boldsymbol{\alpha}_1$, and $\boldsymbol{\alpha}_2$, as many as possible. Based on the notation f_k given in Equation (22), the degenerate condition for the codimension-2 generalized Hopf bifurcation can be obtained as

$$a_{12100} - a_{11100}\, a_{22000} - a_{12000}\, a_{21100} = 0. \tag{24}$$

We begin with the 2nd-order equations ($k = 2$). Applying the formula $L_2 + \boldsymbol{g}_2 = \boldsymbol{f}_2 + [\boldsymbol{h}_2, \boldsymbol{v}_1] + T_1 \boldsymbol{v}_1$ yields the following seven $(3 + 2 \times 2)$ 2nd-order complex algebraic equations:

$$\begin{aligned} -(c_{22000} + a_{12000}) + i\, (c_{12000} - a_{22000}) &= 0, \\ 3\, c_{20200} - a_{10200} - i\, (3\, c_{10200} + a_{20200}) &= 0, \\ c_{21100} - a_{11100} - i\, (c_{11100} + a_{21100}) &= 0, \\ 2\, c_{20110} - a_{10110} - i\, (2\, c_{10110} + a_{20110}) &= 0, \\ 2\, c_{20101} - a_{10101} - i\, (2\, c_{10101} + a_{20101}) &= 0, \\ \alpha_{11} - a_{11010} - i\, (t_{010} + a_{21010}) &= 0, \\ \alpha_{12} - a_{11001} - i\, (t_{001} + a_{21001}) &= 0, \end{aligned} \tag{25}$$

which do not contain the b coefficients, as expected. Solving the first five equations of Equation (25) uniquely determines the $\boldsymbol{c}$ coefficients

$$c_{12000} = a_{22000}, \quad c_{22000} = -a_{12000},$$

$$c_{10200} = -\frac{1}{3}\, a_{20200},$$

$$c_{20200} = \frac{1}{3}\, a_{10200}, \quad c_{11100} = -a_{21100},$$

$$c_{21100} = a_{11100}$$

$$c_{10110} = -\frac{1}{2}\, a_{20110}, \quad c_{20110} = \frac{1}{2}\, a_{10110},$$

$$c_{10101} = -\frac{1}{2}\, a_{20101}, \quad c_{20101} = \frac{1}{2}\, a_{10101}.$$

The last two equations of Equation (25) result in

$$\alpha_{11} = a_{11010}, \quad \alpha_{12} = a_{11001},$$

$$t_{010} = -\, a_{21010}, \quad t_{001} = -\, a_{21001}.$$

The above calculations show that except for one unfolding term, $\boldsymbol{\alpha}_1 = (\alpha_{11},\ \alpha_{12}) \neq \mathbf{0}$, all the 2nd-order terms can be removed by using the $\boldsymbol{c}$ and $\boldsymbol{t}$ coefficients. The $\boldsymbol{p}$ coefficients, however, do not appear in the 2nd-order algebraic equations.

Next, consider $k = 3$. Similarly, we apply the formula $\boldsymbol{g}_3 = \boldsymbol{f}_3 + [\boldsymbol{h}_3,\ \boldsymbol{v}_1] + (D\boldsymbol{f}_2\, \tilde{\boldsymbol{h}}_2 - D\boldsymbol{h}_2\, \boldsymbol{g}_2) + T_1(\boldsymbol{f}_2 + L\, \boldsymbol{h}_2) + T_2\, \boldsymbol{v}_1 - D\boldsymbol{h}_2\, L_2$ to find 16 algebraic equations, which are obtained from balancing the coefficients of the terms u^3, $\bar{u}^3$, $u^2\,\bar{u}$, $u\,\bar{u}^2$, $u^2\,v_1$, $u^2\,v_2$, $\bar{u}^2\,v_1$, $\bar{u}^2\,v_2$, $u\,\bar{u}\,v_1$, $u\,\bar{u}\,v_2$, $u\,v_1^2$, $u\,v_2^2$, $u\,v_1\,v_2$, $\bar{u}\,v_1^2$, $\bar{u}\,v_2^2$, and $\bar{u}\,v_1\,v_2$, respectively. These equations are listed in the Appendix. It is seen from these equations that, except for the 3rd, 11th, 12th, and 13th equations, all the equations can be solved by uniquely choosing the 3rd-order $\boldsymbol{c}$ coefficients (those on the left-hand side of the equations). The 3rd, 11th, 12th, and 13th equations do not contain any 2nd- or 3rd-order $\boldsymbol{c}$ coefficients, and thus, other coefficients must be used. It is seen from the 3rd equation that $b_{13} = 0$, due to the degenerate condition (24), and that either b_{23} or t_{200} may be used to balance the imaginary part. We use t_{200} and leave b_{23} undetermined. The reason for choosing t_{200} is similar to that in the study of Hopf bifurcation [18]. The 11th, 12th, and 13th equations have nine coefficients, t_{020}, t_{011}, t_{002}, p_{120}, p_{111}, p_{102} and p_{220}, p_{211}, p_{202}. It can be shown that the coefficients p_{2ij}'s must be used in the 5th-order computation. Therefore, all t_{0ij}'s and p_{1ij}'s should be used to solve the three equations. This clearly indicates that it is not enough to use only near-identity transformations to compute the SPNF. It is also noted that some 2nd-order $\boldsymbol{c}$ and $\boldsymbol{t}$ coefficients have not been used in terms of this order. However, it can be shown from higher-order equations that these coefficients can be set to zero. Therefore, from the 3rd-order equations, we obtain

$$\begin{aligned}
b_{13} &= 0,\\
t_{200} &= 2\,(A_{2210} - a_{2210} + b_{23}\,),\\
p_{120} &= -\,\frac{1}{a_{11010}}\,(A_{11020} + a_{11020} + p_{220}\, a_{11001}),\\
p_{111} &= -\,\frac{1}{a_{11010}}\,(A_{11011} + a_{11011} + p_{211}\, a_{11001}),\\
p_{102} &= -\,\frac{1}{a_{11010}}\,(A_{11002} + a_{11002} + p_{202}\, a_{11001}),\\
t_{020} &= -\,(A_{21020} + a_{21020} + p_{120}\, a_{21010}\\
&\quad + p_{220}\, a_{21001}),\\
t_{011} &= -\,(A_{21011} + a_{21011} + p_{111}\, a_{21010}\\
&\quad + p_{211}\, a_{21001}),\\
t_{002} &= -\,(A_{21002} + a_{21002} + p_{102}\, a_{21010}\\
&\quad + p_{202}\, a_{21001}),\\
c_{11010} &= c_{21010} = c_{11001} = c_{21001} = t_{110} = t_{101} = 0,
\end{aligned} \tag{26}$$

and all other $\boldsymbol{c}$ coefficients are uniquely determined.

It should be noted in the above calculations that the coefficient of $z\mu_1$ in the original Equation (19) has been assumed nonzero, i.e.,

$$a_{11010} \neq 0, \tag{27}$$

under which the coefficients p_{1ij}'s are uniquely determined and the first term of the unfolding exists. In case $a_{11010} = 0$, but $a_{11001} \neq 0$, we can still find the unfolding but the roles of p_{1ij} and p_{2ij} need to be exchanged. Alternatively, we may just simply exchange v_1 and v_2. So, without loss of generality, we assume $a_{11010} \neq 0$. However, when both a_{11010} and a_{11001} equal zero, it gives another kind of generalized Hopf bifurcation, which is out of the scope of this paper and thus not further discussed here.

The situation for $k = 4$ is similar to that of $k = 2$, for which we obtain

$$\alpha_{21} = a_{12110} + (\cdots) \quad \text{and} \quad \alpha_{22} = a_{12101} + (\cdots), \tag{28}$$

Table 1 Solving the irregular equations ($k \geq 2$)

$k = 2$	α_{11}	(R)	$\Longrightarrow$	$u\,v_1$
	t_{010}	(I)		
	α_{12}	(R)	$\Longrightarrow$	$u\,v_2$
	t_{001}	(I)		
$k = 4$	α_{21}	(R)	$\Longrightarrow$	$u^2\,\bar{u}\,v_1$
	t_{210}	(I)		
	α_{22}	(R)	$\Longrightarrow$	$u^2\,\bar{u}\,v_2$
	t_{201}	(I)		
	$p_{2m_1m_2}$	(R)	$\Longrightarrow$	$u^2\,\bar{u}\,v_1^{m_1}v_2^{m_2}$
	$t_{2m_1m_2}$	(I)		$(m_1 + m_2 = k - 3)$
	$(i = 2)$			
$k \geq 2$, $k \neq 2, 4$, $i - j = 1$, $k - i - j \neq 0$	$c_{1(i-2)(j-2)m_1m_2}$	(R)	$\Longrightarrow$	$u^i\,\bar{u}^j\,v_1^{m_1}v_2^{m_2}$
	$t_{(2i-2)m_2m_2}$	(I)		$(m_1 + m_2 = k - i - j)$
	$(i \neq 2)$			

where $(\cdots)$ represents a lengthy known expression in terms of a_{ijklm}'s. Thus, it is unlikely to have $\alpha_{21} = 0$ or $\alpha_{22} = 0$. However, similar to Equation (26), it needs at least one of a_{12110} and a_{12110} to be nonzero in order to determine the coefficients p_{ijk} $(j + k = 4)$. Therefore, once again without loss of generality, we assume that

$$a_{12110} \neq 0. \tag{29}$$

This procedure used in the 2nd- and 3rd-order equations can be carried forward to higher-order ($k \geq 4$) equations.

The method of mathematical induction has been applied to find the following general computational rules:

(1) The b_{1j} coefficients, except for b_{15}, can be set zero, while $b_{2(2l-1)}$ $(l \geq 2)$ must be used to solve the real part of the equation corresponding to the term $u^{l+2}\,\bar{u}^{l+1}$. The $\boldsymbol{t}$ coefficient $t_{2l\,00}$ can still be used to eliminate the imaginary part of the equation associated with the term $u^{l+1}\,\bar{u}^l$.

(2) For the kth-order ($k \geq 3$) case, the equation corresponding to the term $u\,v_1^{m_1}v_2^{m_2}$ $(m_1 + m_2 = k - 1)$ must be solved by using $p_{1\,m_1m_2}$ and $t_{0\,m_1m_2}$, respectively, for its real and imaginary parts.

(3) For each order k, there exist some "irregular" equations that must be solved by using lower order $\boldsymbol{c}$ and $\boldsymbol{t}$ coefficients. The rules for solving the "irregular" equations are given in Table 1, where R and I means solving the real and imaginary parts, respectively.

(4) All other "regular" equations can be solved by using the kth-order $\boldsymbol{c}$ coefficients. In fact, $c_{2\,jlm_1m_2}$ and $c_{1\,jlm_1m_2}$ are used, respectively, to solve the real and imaginary parts of the equations corresponding to the term $u^j\,\bar{u}^l\,v_1^{m_1}v_2^{m_2}$, where j, l, m_1, and m_2 are nonnegative integers satisfying $j + l + m_1 + m_2 = k$, and j and l are not simultaneously equal to zero. The rules for solving the "regular" equations are given in Table 2.

Having found the coefficients b_{ij} and also having determined the coefficients of the unfolding, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_1$, one can write the SPNF in the form of $\frac{du}{d\tau} = i\,u + (\alpha_{11}\,v_1 + \alpha_{12}\,v_2)\,u + (\alpha_{21}\,v_1 + \alpha_{22}\,v_2)\,u^2\,\bar{u} + \sum g_k(u, \bar{u})$, i.e.,

$$\frac{du}{d\tau} = i\,u + (\alpha_{11}v_1 + \alpha_{12}v_2)u + (\alpha_{21}v_1 + \alpha_{22}v_2)u^2\bar{u} + b_{15}\,u^3\bar{u}^2 + i\sum_{m=1}^{\infty} b_{2(2m+1)}\,u^{m+1}u^m, \tag{30}$$

where b_{15} and b_{2j} are explicitly obtained, given in terms of the original system coefficients $a_{ijlm_1m_2}$'s.

Letting $u = R\,e^{i\Theta}$, where R and Θ are, respectively, the amplitude and phase of motion, and splitting the real and imaginary parts in Equation (30) results in

Table 2 Solving regular equations ($k \geq 3$)

N.T.		R/I	Corresponding term
$p_{1(k-1)m_1m_2}$	$\Longrightarrow$	R	$u\,\nu_1^{m_1}\nu_2^{m_2}$
$t_{0(k-1)m_1m_2}$	$\Longrightarrow$	I	$u\,\nu_1^{m_1}\nu_2^{m_2}$
$b_{1k}=0$			$(k=3,\ k\geq 7)$
b_{15}	$\Longrightarrow$	R	$u^3\bar{u}^2 (k=5)$
$b_{2(k-4)}$	$\Longrightarrow$	R	$u^{(k+1)/2}\bar{u}^{(k-1)/2}$ $(k\geq 7,\ \text{odd})$
$t_{(k-1)00}$	$\Longrightarrow$	I	$u^{(k+1)/2}\bar{u}^{(k-1)/2}$ $(k\geq 3,\ \text{odd})$
$c_{2jlm_1m_2}$	$\Longrightarrow$	R	$u^j\bar{u}^l\nu_1^{m_1}\nu_2^{m_2}$
			$(j+l+m_1+m_2\geq 7,\ j\neq l+1)$
$c_{1jlm_1m_2}$	$\Longrightarrow$	I	$u^j\bar{u}^l\nu_1^{m_1}\nu_2^{m_2}$

the real SPNF, given in polar coordinates

$$\begin{aligned}\frac{dR}{d\tau} &= R\,[\alpha_{11}\,\nu_1+\alpha_{12}\,\nu_2+(\alpha_{21}\,\nu_1+\alpha_{22}\,\nu_2)R^2\\ &\quad +b_{15}R^4],\\ \frac{d\Theta}{d\tau} &= 1+b_{23}\,R^2+b_{25}\,R^4+\cdots,\end{aligned}\tag{31}$$

which is Equation (18). This completes the proof of Theorem 3. □

Once the SPNF of a parameterized nonlinear system is found, it is easy to derive bifurcation solutions and their stability conditions. For convenience, let

$$\bar{\boldsymbol{\nu}}=\begin{pmatrix}\bar{\nu}_1\\ \bar{\nu}_2\end{pmatrix}=\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}\equiv\boldsymbol{\alpha}\,\boldsymbol{\nu},\tag{32}$$

so that $\boldsymbol{\nu}=\boldsymbol{\alpha}^{-1}\bar{\boldsymbol{\nu}}$ if $\boldsymbol{\alpha}$ is nonsingular, and the first equation of Equation (31) can be rewritten as

$$\frac{dR}{d\tau}=R(\bar{\nu}_1+\bar{\nu}_2\,R^2+b_{15}R^4).\tag{33}$$

On the basis of Equation (33), the steady-state solutions are readily obtained as

(I) $R_0=0$, and
(II) $R_\pm^2=\dfrac{1}{2\,b_{15}}\big(-\,\bar{\nu}_2\pm\sqrt{\bar{\nu}_2^2-4\,b_{15}\,\bar{\nu}_1}\big)$.

Solution (I) actually represents the original equilibrium, while solution (II) denotes the families of limit cycles. It is seen from Solution (II) that, unlike Hopf bifurcation which has maximal one family of limit cycles, generalized Hopf bifurcations may have multiple families of limit cycles. For example, a codimension-2 generalized Hopf bifurcation may have maximal two families of limit cycles by appropriately choosing the parameters $\bar{\nu}_1$ and $\bar{\nu}_2$ (or ν_1 and ν_2) such that both the two solutions R_+^2 and R_-^2 are positive. In general, for a codimension-q generalized Hopf bifurcation, there may exist maximal q families of limit cycles.

The stability of the steady-state solutions can be easily determined from the Jacobian of Equation (33), which is given by

$$J=\frac{d}{dR}\left(\frac{dR}{d\tau}\right)=\bar{\nu}_1+3\,\bar{\nu}_2\,R^2+5\,b_{15}R^4.\tag{34}$$

Substituting solution (I) into Equation (34) yields $J_0=\bar{\nu}_1$, implying that Solution (I) is stable (unstable) if $\bar{\nu}_1<0$ (>0). The critical value at which limit cycles bifurcate from the original equilibrium is $\bar{\nu}_1=0$, represented by a straight line on the $\nu_1-\nu_2$ parameter space.

Evaluating Jacobian (34) on Solution (II) gives

$$\begin{aligned}J_\pm &= -4\,\bar{\nu}_1-2\,\bar{\nu}_2\,R_\pm^2\\ &= -4\,\bar{\nu}_1-\frac{\bar{\nu}_2}{b_{15}}\big(-\,\bar{\nu}_2\pm\sqrt{\bar{\nu}_2^2-4\,b_{15}\,\bar{\nu}_1}\big)\\ &= \frac{1}{b_{15}}\sqrt{\bar{\nu}_2^2-4\,b_{15}\,\bar{\nu}_1}\,\big[\sqrt{\bar{\nu}_2^2-4\,b_{15}\,\bar{\nu}_1}\mp\bar{\nu}_2\,\big],\end{aligned}\tag{35}$$

where it is assumed that $\bar{\nu}_2^2-4\,b_{15}\,\bar{\nu}_1>0$, implying that $\bar{\nu}_1=O(\bar{\nu}_2^2)$, and thus $\sqrt{\bar{\nu}_2^2-4\,b_{15}\,\bar{\nu}_1}=O(\bar{\nu}_2)$.

Next, we want to find the conditions under which there exist two families of limit cycles. In order to have both $R_+>0$ and $R_-<0$, in addition to the condition $\bar{\nu}_2^2-4\,b_{15}\,\bar{\nu}_1>0$, it requires either

(a) if $b_{15}>0$, then $\bar{\nu}_2<0$ and $\bar{\nu}_1>0$; or
(b) if $b_{15}<0$, then $\bar{\nu}_2>0$ and $\bar{\nu}_1<0$.

Therefore, by using the stability condition (35) one easily concludes that when $b_{15}>0$, solution R_+ is unstable (since the term in the square bracket of

Table 3 Stability of limit cycles of system (31)

b_{15}		+	−
$\bar{\nu}_2$		−	+
$\bar{\nu}_1$		+	−
$\bar{\nu}_2^2 - 4\,b_{15}\,\bar{\nu}_1$		+	+
Stability	R_+	Unstable	Unstable
	R_-	Stable	Stable
	R_0	Unstable	Stable

Equation (35) is positive), while R_- is stable (since the term in the square bracket of Equation (35) is negative). In this case, $R_+ > R_-$ (see Solution (II)), indicating that the limit cycle with amplitude R_- is inside the limit cycle having amplitude R_+. When $b_{15} < 0$, similarly, we found that R_- is stable but R_+ is unstable. However, for this case, $R_+ < R_-$, implying that the two limit cycles have exchanged their positions. The above discussion shows that the limit cycle with amplitude R_- is always stable while the limit cycle having amplitude R_+ is always unstable. Further, the stability of R_0 will be consistent with the stability changes of the limit cycles. For example, if $b_{15} < 0$, then the inner limit cycle (R_+) is unstable, while the original equilibrium (R_0) and the outer limit cycle (R_-) must be stable. This is shown in Fig. 1. The stability conditions are summarized in Table 3.

It should be pointed out that generalized (or degenerate) Hopf bifurcations have been studied by many researchers. However, many results only showed one family of limit cycles like Hopf bifurcation, since the term $\bar{\nu}_2\,R^3$ in Equation (33) is not presented. For example, the solution of the so-called "flat Hopf bifurcation" [23] is actually given in the form of $\bar{\nu}_1 = -\,b_{15}\,R^4$.

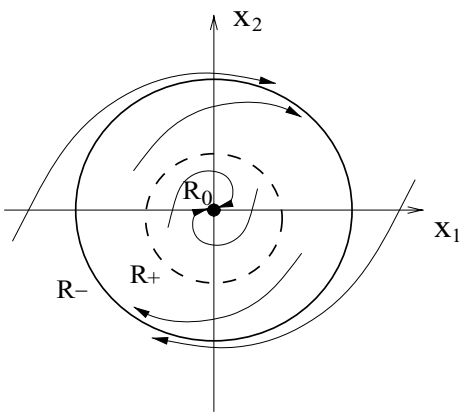

Fig. 1 Existence of multiple limit cycles when $b_{15} < 0$

It should also be noted that the computation of the SPNF is much more involved than the CNF computation, and thus it is almost impossible to find the SPNF by hand calculation even for very lower-order SPNF. Fortunately, there are now many computer algebra systems that can be used to implement the solution procedure described in this paper. In fact, the procedure presented in the proof earlier has been used as a guideline for developing symbolic programs using Maple for the SPNFs of generalized Hopf bifurcations. The computation involves finding the coefficients of the SPNF, the unfolding, the near-identity nonlinear transformation, and the time and parameter rescaling.

4 Applications

In this section, we present two examples to illustrate the application of the theoretical results obtained in the previous sections. The first one is a chemical reaction process, showing a Hopf bifurcation, while the second one is a nonlinear electrical circuit, demonstrating a codimension-2 generalized Hopf bifurcation.

4.1 A chemical system

Since Bray [24] reported that hydrogen peroxide decomposes in the presence of acidic iodate with an oscillatory rate, periodic behavior of reactions in homogeneous solution has received considerable attention. The mechanisms of many chemical oscillations can be very complex.

Consider a simplified model, the chlorine dioxide–iodine–malonic acid (ClO_2–I_2–MA) reaction, described by the following differential equations:

$$\frac{dw_1}{dt} = a - w_1 - \frac{4\,w_1\,w_2}{1+w_1^2} \equiv f_1(w_1, w_2; a, b),$$
$$\frac{dw_2}{dt} = b w_1 \left(1 - \frac{w_2}{1+w_1^2}\right) \equiv f_2(w_1, w_2; a, b), \quad (36)$$

where w_1 and w_2 denote the dimensionless concentrations of I^- and ClO_2^-, respectively. The parameters $a > 0$ and $b > 0$ depend on the empirical rate constants and the concentrations assumed for the slow reactants. It's not difficult to find that the system has a unique

equilibrium point given by

$$P_0 = (w_{10},\ w_{20}) = \left(\frac{a}{5},\ 1+\left(\frac{a}{5}\right)^2\right) \equiv \left(w_{10},\ 1+w_{10}^2\right). \tag{37}$$

The Jacobian matrix evaluated at the point P_0 is

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \frac{\partial f_1}{\partial w_2} \\ \frac{\partial f_2}{\partial w_1} & \frac{\partial f_2}{\partial w_2} \end{bmatrix} = \frac{1}{1+w_{10}} \begin{bmatrix} 3\,w_{10}^2-5 & -4\,w_{10} \\ 2\,b\,w_{10}^2 & -b\,w_{10} \end{bmatrix}, \tag{38}$$

which, in turn, gives the characteristic polynomial

$$\lambda^2 - \tau\lambda + \Delta = 0, \tag{39}$$

where

$$\tau = \frac{3\,w_{10}^2 - 5 - b\,w_{10}}{1+w_{10}^2} \quad \text{and} \quad \Delta = \frac{5\,b\,w_{10}}{1+w_{10}^2}. \tag{40}$$

Since $\Delta > 0$, we know that when $\tau = 0$, that is, when

$$b_c = \frac{3a}{5} - \frac{25}{a}, \tag{41}$$

a Hopf bifurcation may occur at the equilibrium point P_0 when $b = b_c$ (where $a > 5\sqrt{\frac{5}{3}}$ since $b > 0$).

Introducing the following transformation:

$$w_1 = w_{10} + y_1, \quad w_2 = w_{20} + y_2, \tag{42}$$

we can shift the equilibrium point P_0 to the origin and then expand the right-hand side of the resulting equation in the form of Taylor series

$$\begin{aligned} \frac{dy_1}{dt} = -\frac{1}{25+a^2}\Bigg\{ & 20a\,y_2 + (125-3a^2)\,y_1 \\ & + 100\left(1-\frac{2a^2}{25+a^2}\right) y_1\,y_2 \\ & - 10a\left(1+\frac{3125+50a^2-3a^4}{(25+a^2)^2}\right) y_1^2 \\ & - \frac{500a}{25+a^2}\left(1+\frac{50-2a^2}{25+a^2}\right) y_1^2 y_2 \Bigg\} + \cdots, \end{aligned}$$

$$\begin{aligned} \frac{dy_2}{dt} = \frac{1}{25+a^2}\Bigg\{ & -5ab\,y_2 + 2a^2 b\,y_1 \\ & + 25b\left(\frac{2a^2}{25+a^2}-1\right) y_1 y_2 \\ & + 5ab\left(3-\frac{4a^2}{25+a^2}\right) y_1^2 \\ & + \frac{125ab}{25+a^2}\left(1-\frac{2(a^2-25)}{25+a^2}\right)\Bigg\} y_1^2 y_2 + \cdots. \end{aligned} \tag{43}$$

When a Hopf bifurcation appears, the eigenvalues of the system evaluated at the equilibrium become

$$\lambda = \pm i\,\omega, \quad \text{where} \quad \omega = \sqrt{\frac{15a^2-625}{a^2+25}} \quad \text{and} \quad i^2 = -1, \tag{44}$$

and the associated complex eigenvector is $V = (\frac{i\,\omega - J_{22}}{J_{21}},\ 1)^{\mathrm{T}}$. Therefore, under the linear transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -\frac{J_{22}}{J_{21}} & \frac{\omega}{J_{21}} \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{45}$$

setting

$$b = b_c - \mu \tag{46}$$

yields the center manifold as follows (here, a has been chosen as $a = 10$ for definiteness):

$$\begin{aligned} \frac{dx_1}{dt} = {} & \sqrt{7}\,x_2 - \frac{2\sqrt{7}}{7}\mu\,x_2 + \frac{\sqrt{7}}{20}x_1x_2 + \frac{7}{80}x_1^2 \\ & - \frac{1}{16}x_2^2 - \frac{29}{640}x_1x_2^2 + \frac{1}{56}\mu x_2^2 \\ & - \frac{37\sqrt{7}}{3200}x_1^2x_2 - \frac{\sqrt{7}}{70}\mu x_1x_2 - \frac{21}{3200}x_1^3 \\ & - \frac{1}{40}\mu x_1^2 - \frac{\sqrt{7}}{128}x_2^3 + \cdots, \end{aligned}$$

$$\begin{aligned} \frac{dx_2}{dt} = {} & -\sqrt{7}x_1 + \frac{2}{5}\mu x_2 + \frac{1}{4}x_1x_2 + \frac{\sqrt{7}}{16}x_1^2 - \frac{5\sqrt{7}}{112}x_2^2 \\ & - \frac{29\sqrt{7}}{896}x_1x_2^2 - \frac{\sqrt{7}}{280}\mu x_2^2 \end{aligned}$$

$$-\frac{37}{640}x_1^2x_2+\frac{1}{50}\mu x_1x_2-\frac{3\sqrt{7}}{640}x_1^3+\frac{\sqrt{7}}{200}\mu x_1^2 -\frac{5}{128}x_2^3+\cdots. \quad (47)$$

By using the Maple programs developed in [18], we obtain the SPNF up to the fifth order, given in complex form as follows:

$$\frac{du}{d\tau}=i\sqrt{7}\,u+\frac{1}{5}\,\nu\,u-\frac{81}{2800}\,u^2\,\bar{u}-i\,\frac{180683\sqrt{7}}{1587600}\,u^2\,\bar{u} +i\,\frac{33276534620493449\sqrt{7}}{14839702619696640000}\,u^3\,\bar{u}^2, \quad (48)$$

and the relative parameter scaling is

$$\mu=\nu-\frac{16}{1225}\,\nu^3-\frac{8}{8575}\,\nu^4+\frac{284}{1500625}\,\nu^5+\cdots \quad (49)$$

The time scaling and the nonlinear transformation are also obtained, but not presented here for brevity.

Let $u=R\,e^{i\Theta}$, where R and Θ are the amplitude and phase of motion, respectively. Then, the SPNF given in polar coordinates is

$$\frac{dR}{d\tau}=\frac{1}{5}\,\nu\,R-\frac{81}{2800}\,R^3,$$
$$\frac{d\Theta}{d\tau}=\sqrt{7}\left(1-\frac{180683}{1587600}R^2 +\frac{33276534620493449}{14839702619696640000}R^4+\cdots\right). \quad (50)$$

It follows from Equation (50) that the system has a unique equilibrium solution $R=0$ (representing the P_0 for the original system), which is stable (unstable) for $\nu<0$ ($\nu>0$). At $\nu=0$, a family of limit cycles bifurcates from the equilibrium point P_0. The solutions for the family of limit cycles are given by

$$R^2=\frac{560}{81}\,\nu,$$
$$\Theta=\sqrt{7}\left(1-\frac{180683}{229635}\,\nu +\frac{33276534620493449}{310469671198436400}\,\nu^2\right). \quad (51)$$

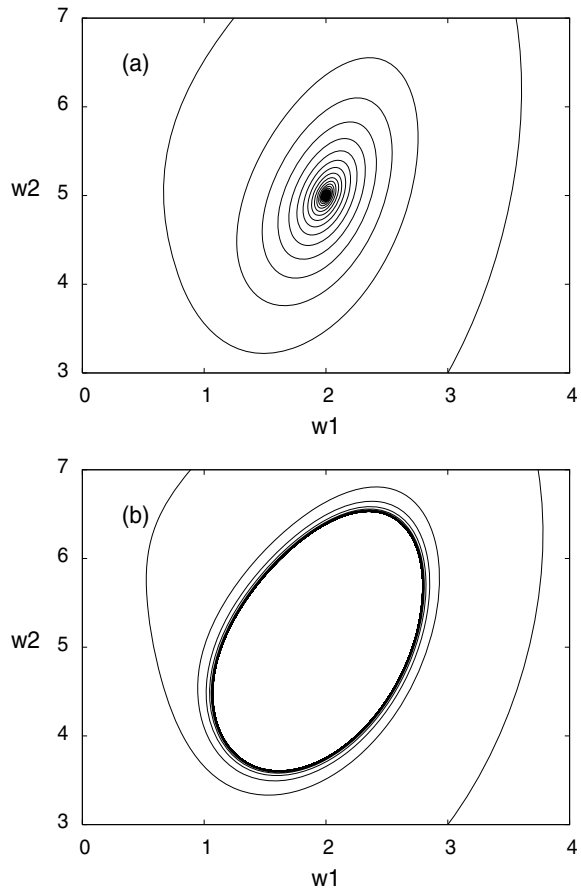

Fig. 2 Simulated trajectories of system (36) for $a=10$ with the initial point $w_{10}=w_{20}=3$: (a) convergent to the equilibrium point $P_0=(w_{10},\ w_{20})=(2,\ 5)$ when $b=4$; and (b) convergent to a stable limit cycle when $b=3$

Since the coefficient of R^3 is $-\frac{81}{2800}<0$ (see the first equation of Equation (50)), we know that the Hopf bifurcation is supercritical, that is, the bifurcating limit cycles are stable for $\nu>0$.

Numerical simulation results based on the original system (36) are shown in Fig. 2, where the parameter a is taken as $a=10$ (so $b_c=\frac{7}{2}$), and b are chosen as $b=4>b_c$ and $b=3<b_c$, respectively, for Fig. 2(a) and 2(b). It is shown that when $b=4$ (i.e., $\mu=-\frac{1}{2}$), the trajectory converges to the equilibrium point $P_0=(3,\ 3)$, as shown in Fig. 2(a), while for $b=3$ (i.e., $\mu=\frac{1}{2}$), the trajectory converges to a stable limit cycle (see Fig. 2(a)). These results completely agree with the analytical predictions, as expected.

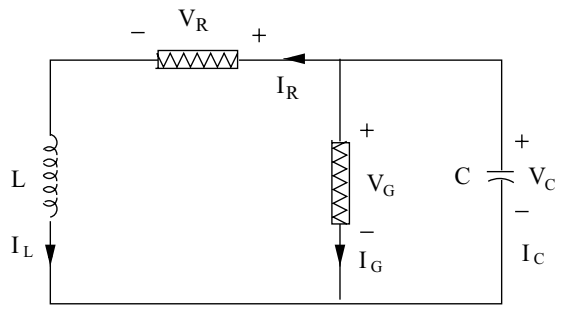

Fig. 3 A nonlinear electrical circuit

4.2 An electrical circuit

Now we present a simple nonlinear electrical circuit, shown in Fig. 3, to demonstrate the application of the SPNF of generalized Hopf bifurcation. The circuit consists of one inductor, L, one capacitor, C, and two nonlinear conductances characterized, respectively, by

$$V_R = \eta_1 I_R + \beta_1 I_R^2 - \beta_2 I_R^3 \qquad \text{and}$$
$$I_G = -\eta_2 V_G + \beta_3 V_G^3, \tag{52}$$

where I_G, I_R, and V_G, V_R, represent the currents and voltages of the conductances, respectively, and η_1 and η_2 are two bifurcation parameters, while the parameters β_1, β_2 and β_3 are positive and can be varied, considered as control parameters.

Choosing the current in the inductor and the voltage across the capacitor as the state variables (as shown in Fig. 3) yields the equations of the circuit, given by

$$L\,\frac{dI_L}{dt} = V_C - V_R,$$
$$C\,\frac{dV_C}{dt} = -I_L - I_G. \tag{53}$$

Denoting the state variables I_L and V_C by w_1 and w_2, respectively, we may rewrite Equation (53) as

$$\frac{dw_1}{dt} = \frac{1}{L}\left(-\eta_1 w_1 + w_2 - \beta_1 w_1^2 + \beta_2 w_1^3\right),$$
$$\frac{dw_2}{dt} = \frac{1}{C}\left(-w_1 + \eta_2 w_2 - \beta_3 w_2^3\right). \tag{54}$$

It is clear that $(w_1,\ w_2) = (0,\ 0)$ is the unique equilibrium of system (54). Let C and L take its unit values, and further let

$$\eta_1 = \frac{4}{5} + \mu_1 \quad \text{and} \quad \eta_2 = \frac{4}{5} + \mu_2. \tag{55}$$

Then, the Jacobian of system (54) evaluated at the critical point, $\mu_1 = \mu_2 = 0$, and at the equilibrium, $w_1 = w_2 = 0$, has one pair of purely imaginary eigenvalues: $\lambda_\pm = \pm\frac{3}{5}\,i \equiv \pm\omega i$.

Next, introducing the following transformation:

$$w_1 = 3\,x_1 + 4\,x_2, \quad w_2 = 5\,x_2, \tag{56}$$

into system (54) yields

$$\dot{x}_1 = \frac{3}{5}\,x_2 - \mu_1\,x_1 - \frac{4}{3}\,(\mu_1 + \mu_2)\,x_2 - \frac{1}{3}\,\beta_1\,(3x_1 + 4x_2)^2 + \frac{1}{3}\,\beta_2\,(3x_1 + 4x_2)^3 + \frac{100}{3}\,\beta_3\,x_2^3,$$
$$\dot{x}_2 = -\frac{3}{5}\,x_1 + \mu_1\,x_2 - 25\,\beta_3\,x_2^3. \tag{57}$$

Transforming Equation (57) into complex form and then applying the degenerate condition (24) yields the following result:

$$\beta_2 = \beta_3 + \frac{40}{27}\,\beta_1^2, \tag{58}$$

under which the system undergoes a codimension-2 generalized Hopf bifurcation. For determination, let

$$\beta_1 = \frac{1}{4}, \quad \beta_3 = \frac{2}{27}, \quad \text{and} \quad \beta_2 = \frac{1}{6}. \tag{59}$$

Finally, executing the Maple program gives the following complex SPNF:

$$\frac{du}{d\tau} = i\,\frac{3}{5}\,u - \frac{1}{2}\,(\nu_1 - \nu_2)\,u - \left(\frac{4335304717}{527212800}\,\nu_1 - \frac{1729242217}{527212800}\,\nu_2\right)u^2\bar{u} + \frac{8828125}{839808}\,u^3\,\bar{u}^2 + i\,\frac{2811217217}{439344000}\,u^2\,\bar{u}, \tag{60}$$

from which the real form of the SNF is obtained, as

$$\frac{dR}{d\tau} = \frac{3}{5} R\left[\bar{\nu}_1 + \bar{\nu}_2 R^2 + \frac{44140625}{2519424} R^4 \right],$$
$$\frac{d\Theta}{d\tau} = \frac{3}{5} + \frac{2811217217}{439344000} R^2 + \cdots, \tag{61}$$

where

$$\bar{\nu}_1 = \frac{5}{6}(\nu_2 - \nu_1),$$
$$\bar{\nu}_2 = \frac{1729242217}{316327680}\left(\nu_2 - \frac{4335304717}{1729242217}\nu_1\right). \tag{62}$$

Since $b_{15} = \frac{1729242217}{527212800} > 0$, in order to have two families of limit cycles, the following conditions must be satisfied (see Table 3):

$$\nu_2 - \nu_1 > 0,$$
$$\nu_2 - \frac{4335304717}{1729242217}\nu_1 < 0,$$
$$\left(\frac{1729242217}{316327680}\nu_2 - \frac{4335304717}{316327680}\nu_1\right)^2 - \frac{220703125}{104976}(\nu_2 - \nu_1) > 0, \tag{63}$$

which defines a region in the parameter space, as shown in Fig. 4(b), where the initial equilibrium solution, R_0, is unstable, the family of limit cycles defined by R_- is stable, while the family of limit cycles represented by R_+ is unstable. The solutions $R_\pm$ are given by

$$R_\pm = \frac{1259712}{44140525}\left[\frac{4335304717}{316327680}\nu_1 - \frac{1729242217}{316327680}\nu_2 \pm\sqrt{\left(\frac{1729242217}{316327680}\nu_2 - \frac{4335304717}{316327680}\nu_1\right)^2 - \frac{220703125}{104976}(\nu_2 - \nu_1)}\right]. \tag{64}$$

Since it requires that $\bar{\nu}_1 = O(\bar{\nu}_2^2)$, it is seen from Equation (62) that for this example ν_1 and ν_2 should be chosen close to the line $\nu_2 = \nu_1$. The third condition given in Equation (63) represents a hyperbola in the $\nu_1 - \nu_2$ parameter space (see Fig. 4(a)), which approximates the straight line, $\nu_2 = \nu_1$, near the origin. The first and second conditions given in Equation (63) represent two

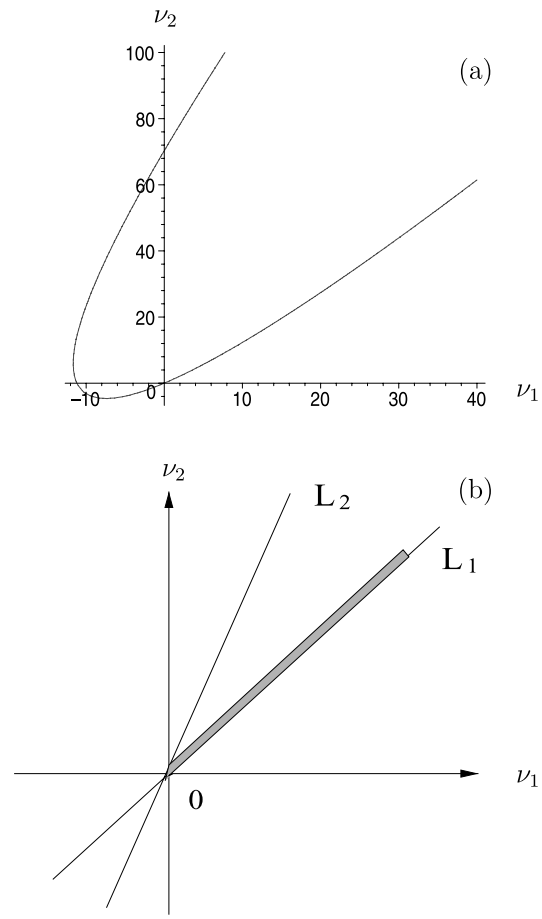

Fig. 4 Existence of multiple limit cycles: (a) the third condition of Equation (63); and (b) the region of existence

straight lines, L_1 and L_2, with slopes 1 and 2.5071, respectively (see Fig. 4(b)), which define a region in the first quadrant between the two boundaries. However, the third condition requires $\nu_2 \approx \nu_1$, and therefore, the region of existence for the multiple families of limit cycles should be located above but near the line $\nu_2 = \nu_1$, denoted by the shadowed area in Fig. 4(b).

Numerical simulation results are shown in Fig. 5. The parameter values for the computer simulation are chosen as $\nu_1 = 0.1$ and $\nu_2 = 0.1003$, which guarantee that the three conditions given in Equation (63) are satisfied. Applying formula (64) yields the amplitude $R_- = 0.13$ for the inner (stable) limit cycle, and $R_+ = 0.17$ for the outer (unstable) limit cycle. To find the values of the parameters μ_1 and μ_2 for the numerical simulation, we need the parameter rescaling, which has been obtained from the computer program as follows (other lengthy nonlinear transformations are omitted

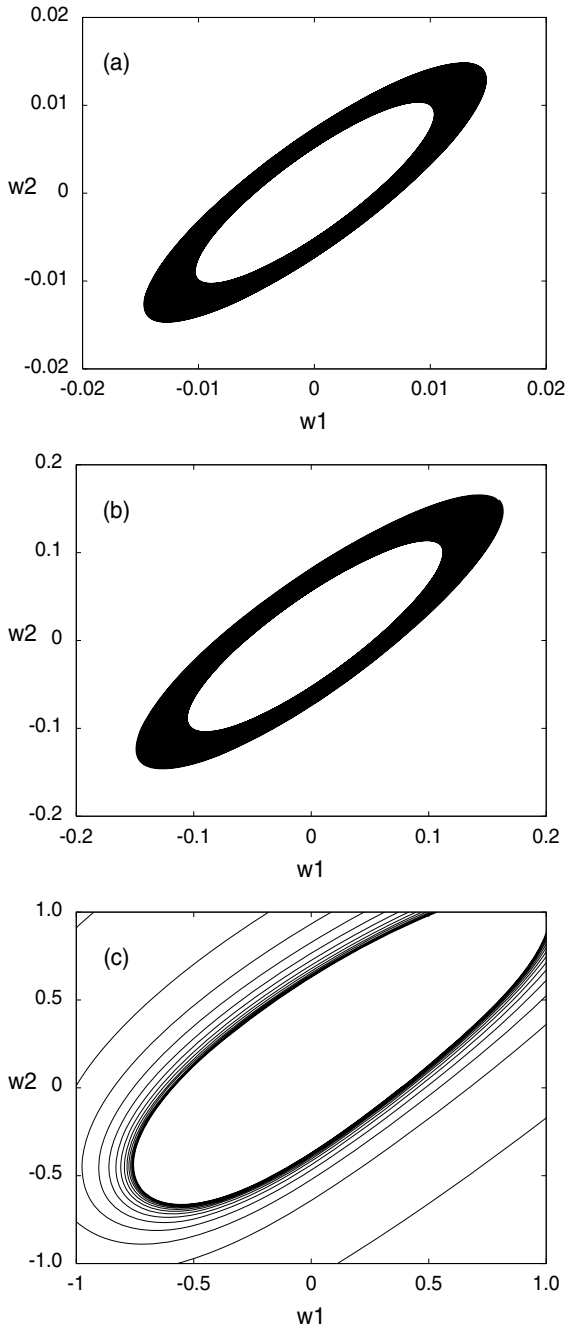

Fig. 5 Simulated trajectories of system (54) for $L = C = 1$, $\eta_1 = 0.865964$, $\eta_2 = 0.866110$, $\beta_1 = 0.25$, $\beta_2 = 0.166667$, $\beta_3 = 0.074074$: (a) convergent to R_- from $(w_{10}, w_{20}) = (0.01, 0.01)$; (b) convergent to R_- from $(w_{10}, w_{20}) = (0.16, 0.16)$; and (c) divergent to ∞ from $(w_{10}, w_{20}) = (0.39, 0.01)$

here):

$$\begin{aligned}
\mu_1 = {} & \nu_1 - \frac{7959420665328859706969}{21563374216500000000000}\nu_1^2 \\
& - \frac{4160639042078991483685 7}{21563374216500000000000}\nu_1\nu_2 \\
& - \frac{14767192893813975853087}{10781687108250000000000}\nu_2^2 \\
& - \frac{400840255476067331343537717977775248018067214153}{58380710176547021936182500000000000000000000000000}\nu_1^3 \\
& + \frac{140172064015569291051836860944299327578629994601}{14595177544136755484045625000000000000000000000000}\nu_1\nu_2^2 \\
& + \frac{115115664391906775453893067836829574780666335 41}{11676142035309404387236500000000000000000000000000}\nu_1^2\nu_2 \\
& - \frac{52941109986427146978214567768498650474416401 63}{48650591813789184946818750000000000000000000000000}\nu_2^3,
\end{aligned}$$

$$\begin{aligned}
\mu_2 = {} & \nu_2 + \frac{1599988401967114029303 1}{21563374216500000000000}\nu_1^2 \\
& - \frac{4160639042078991483685 7}{21563374216500000000000}\nu_1\nu_2 \\
& - \frac{26746845236313975853087}{10781687108250000000000}\nu_2^2 \\
& - \frac{320344186826450428611758416677827348018067214153}{58380710176547021936182500000000000000000000000000}\nu_1^3 \\
& + \frac{126233833360420304674358970604818002578629994601}{14595177544136755484045625000000000000000000000000}\nu_1\nu_2^2 \\
& - \frac{321287588869987980552978103330088825219333664 59}{11676142035309404387236500000000000000000000000000}\nu_1^2\nu_2 \\
& + \frac{108274290515178203673091970266031849525583598 37}{48650591813789184946818750000000000000000000000000}\nu_2^3,
\end{aligned}$$

which result in $\mu_1 = 0.065864$ and $\mu_2 = 0.066110$ when $\nu_1 = 0.1$ and $\nu_2 = 0.1003$.

The numerical results given in Fig. 5 show the trajectories of system (54), which agree with the analytical predictions: the original equilibrium R_0 (i.e., $w_1 = w_2 = 0$) is unstable; the inner limit cycle with amplitude R_- is stable, while the outer limit cycle with amplitude R_+ is unstable. Figure 5(a) depicts a trajectory starting from the point (0.01, 0.01) that converges to a stable limit cycle (R_-), and Fig. 5(b) shows a trajectory starting from the point (0.16, 0.16) that converges to the same limit cycle (R_-). However, when the initial point, (0.39, 0.01), is chosen far outside of the stable limit cycle (R_-), the trajectory diverges to ∞. This implies that there exists an unstable limit cycle (R_+) enclosing the stable limit cycle (R_-). Note that the convergent speed shown in Figs. 5(a) and 5(b) is very slow because the parameter values chosen in the simulation are very close to the critical stability boundary $\nu_2 = \nu_1$.

5 Conclusion

The simplest parameterized normal forms of Hopf and generalized Hopf bifurcations have been studied. It has

been shown that similar to Hopf bifurcation, the SPNF of a generalized Hopf bifurcation cannot be found by using only near-identity transformations. Time and parameter rescaling must be used. It has been shown that the computation of the SPNF of generalized Hopf bifurcations is more involved than the SPNF of Hopf bifurcation. An efficient method for computing the algebraic equations has been developed, which reduces the computation effort to minimum. This greatly saves computational time and computer memory. Symbolic computation programs have been coded using Maple and then applied to two examples. The numerical simulation results of the two examples agree well with the analytical predictions. This paper only deals with a codimension-2 generalized Hopf bifurcation, however. Nevertheless, the method presented in the paper is being generalized to higher-codimension generalized Hopf bifurcations and other singularities, targeting more engineering applications.

Appendix

In the proof of the main theorem, the sixteen 3rd-order algebraic equations used are

$$
\begin{aligned}
&-2c_{23000} - a_{13000} + i(2c_{13000} - a_{23000}) \\
&\quad = A_{13000} + i\left(A_{23000} + \frac{1}{4}t_{200}\right), \\
&-4\,c_{20300} + a_{10300} + i\,(4\,c_{10300} + a_{20300}) \\
&\quad = A_{10300} + i\,A_{20300}, \\
&-\ b_{13} - i\left(b_{23} - a_{22100} - \frac{1}{2}t_{200}\right) = i\,A_{22100}, \\
&-\ 2\,c_{21200} + a_{11200} + i\,(2\,c_{11200} + a_{21200}) \\
&\quad = A_{11200} + i\left(A_{21200} - \frac{1}{4}t_{200}\right), \\
&-\ c_{22010} - a_{12010} \quad + i\,(c_{12010} - a_{22010}) \\
&\quad = A_{12010} + 2\,c_{11010}\,a_{12000} - 2\,c_{21010}\,a_{22000} \\
&\quad + i\left(A_{22010} + 2c_{11010}\,a_{22000} + 2\,c_{21010}\,a_{12000} + \frac{1}{2}t_{110}\right), \\
&-c_{22001} - a_{12001} + i\,(c_{12001} - a_{22001}) \\
&\quad = A_{12001} + 2\,c_{11001}\,a_{12000} - 2\,c_{21001}\,a_{22000} \\
&\quad + i\left(A_{22001} + 2c_{11001}\,a_{22000} + 2\,c_{21001}\,a_{12000} + \frac{1}{2}t_{101}\right), \\
&-3\,c_{20210} + a_{10210} + i\,(3\,c_{10210} + a_{20210}) \\
&\quad = A_{10210} - 2\,c_{11010}\,a_{10200} - 2\,c_{21010}\,a_{20200} \\
&\quad + i\,(A_{20210} - 2\,c_{11010}\,a_{20200} + 2\,c_{21010}\,a_{10200}), \\
&-3\,c_{20201} + a_{10201} + i\,(3\,c_{10201} + a_{20201}) \\
&\quad = A_{10201} - 2\,c_{11001}\,a_{10200} - 2\,c_{21001}\,a_{20200} \\
&\quad + i\,(A_{20201} - 2\,c_{11001}\,a_{20200} + 2\,c_{21001}\,a_{10200}), \\
&-c_{21110} + a_{11110} + i\,(c_{11110} + a_{21110}) \\
&\quad = A_{11110} - 2\,c_{11010}\,a_{11100} \\
&\quad + i\left(A_{21110} - 2\,c_{11010}\,a_{21100} - \frac{1}{2}t_{110}\right), \\
&-c_{21101} + a_{11101} + i\,(c_{11101} + a_{21101}) \\
&\quad = A_{11101} - 2\,c_{11001}\,a_{11100} \\
&\quad + i\left(A_{21101} - 2\,c_{11001}\,a_{21100} - \frac{1}{2}t_{110}\right), \\
&-p_{120}\,a_{11010} - p_{220}\,a_{11001} - a_{11020} - i\,(t_{020} \\
&\quad + p_{120}\,a_{21010} + p_{220}\,a_{21001} + a_{21020}) \\
&\quad = A_{11020} + i\,A_{21020}, \\
&-p_{111}\,a_{11010} - p_{211}\,a_{21001} - a_{11011} - i\,(t_{011} \\
&\quad + p_{111}\,a_{21010} + p_{211}\,a_{21001} + a_{21011}) \\
&\quad = A_{11011} + i\,A_{21011}, \\
&-p_{102}\,a_{11010} - p_{202}\,a_{21001} - a_{11002} - i\,(t_{002} \\
&\quad + p_{102}\,a_{21010} + p_{202}\,a_{21001} + a_{21002}) \\
&\quad = A_{11002} + i\,A_{21002}, \\
&-2\,c_{20120} + a_{10120} + i\,(2\,c_{10120} + a_{20120}) \\
&\quad = A_{10120} - c_{11010}\,a_{10110} - c_{21010}\,a_{20110} \\
&\quad - p_{120}\,a_{10110} - p_{220}\,a_{10101} \\
&\quad - i\,(A_{20120} + c_{11010}\,a_{20110} + c_{21010}\,a_{10110} \\
&\quad + p_{120}\,a_{20110} + p_{220}\,a_{20101}), \\
&-2\,c_{20111} + a_{10111} + i\,(2\,c_{10111} + a_{20111}) \\
&\quad = A_{10111} - c_{11010}\,a_{10101} - c_{11001}\,a_{10110} \\
&\quad - c_{21010}\,a_{20101} - c_{21001}\,a_{20110} - p_{111}\,a_{10110} \\
&\quad - p_{211}\,a_{10101} - i\,(A_{20111} + c_{11010}\,a_{20101} \\
&\quad + c_{11001}\,a_{20110} - c_{21010}\,a_{10101} - c_{21001}\,a_{10110} \\
&\quad + p_{111}\,a_{20110} + p_{211}a_{20101}),
\end{aligned}
$$

$$
\begin{aligned}
&-2\,c_{20102} + a_{10102} + i\,(2\,c_{10102} + a_{20102}) \\
&\quad = A_{10102} - c_{11001}\,a_{10101} - c_{21001}\,a_{20101} \\
&\quad - p_{102}\,a_{10110} - p_{202}\,a_{10101} \\
&\quad - i\,(A_{20102} + c_{11001}\,a_{20101} + c_{21001}\,a_{10101} \\
&\quad + p_{102}\,a_{20110} + p_{202}\,a_{20101}),
\end{aligned}
$$

where A_{jkl}'s are explicitly given in terms of the original system's coefficients a_{ijlm}'s in Equation (22).

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