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Equivalence of the MTS Method and CMR Method for Differential Equations Associated with Semisimple Singularity*

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In this paper, the equivalence of the multiple time scales (MTS) method and the center manifold reduction (CMR) method is proved for computing the normal forms of ordinary differential equations and delay differential equations. The delay equations considered include general delay differential equations (DDE), neutral functional differential equations (NFDE) (or neutral delay differential equations (NDDE)), and partial functional differential equations (PFDE). The delays involved in these equations can be discrete or distributed. Particular attention is focused on dynamics associated with the semisimple singularity, and both the MTS and CMR methods are applied to compute the normal forms near the semisimple singular point. For the ordinary differential equations (ODE), we show that the two methods are equivalent up to any order in computing the normal forms; while for the differential equations with delays, we obtain the conditions under which the normal forms, derived by using the MTS and CMR methods, are identical up to third order. Different types of practical examples with delays are presented to demonstrate the application of the theoretical results, associated with Hopf, Hopf-zero and double-Hopf singularities.

Keywords: Ordinary differential equation (ODE); delay differential equation (DDE); neutral functional differential equation (NFDE); partial functional differential equation (PFDE); discrete delay; distributed delay; semisimple singularity; normal form; multiple time scales (MTS); center manifold reduction (CMR); equivalence of MTS and CMR.

1. Introduction

As we all know, it is important to compute normal forms of differential equations in the study of nonlinear dynamical systems, particularly for stability and bifurcation properties. The center manifold reduction (CMR) (e.g. see [Carr, 1981; Wiggins, 1990; Guckenheimer & Holmes, 1990; Kuznetsov, 2004]) and multiple time scales (MTS, or simply multiple scales (MS)) (e.g. see [Nayfeh, 1973, 1981; Yu, 1998]) are two useful techniques for computing

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the normal forms of differential equations. The CMR method is widely used by researchers from mathematical society, while the MTS method is mainly used by applied scientists and researchers from engineering society. Van Dyke [1975] perhaps is the first to discuss the problem of multiple time scales, referred to as the method of strained coordinates. The MTS method is sometimes attributed to Poincaré, though Poincaré credits the basic idea to the astronomer Lindstedt [Kevorkian & Cole, 1996], leading to one of the standard perturbation approaches, nowadays called the Lindstedt–Poincaré technique. Lighthill [1949] introduced a more general version of the MTS method in 1949. Later, Krylov and Bogoliubov (a development of the method of Krylov and Bogoliubov may be found in [Minorsky, 1947]), and Kevorkian and Cole [1996] introduced the two-scale expansion, which is now the more standard method. On the other hand, in order to study complex behavior of dynamical systems, the two-scale approach had been extended to multiple (more than two) time scales in the study of second-order scalar differential equations (e.g. see [Nayfeh, 1973]). Further, this technique was generalized to consider the stability and bifurcations of general n-dimensional, first-order differential systems [Yu, 1998]. For a dynamical system described by ordinary differential equations (ODEs), the MTS method is systematic and can be directly applied to the original nonlinear system [Yu, 1998, 2002; Zheng & Wang, 2010]. In fact, this approach combines the two steps involved in using center manifold theory and normal form theory into one unified step to obtain the normal form and nonlinear transformation simultaneously. Based on the MTS method, Yu [1998, 2001, 2002] developed Maple programs for computing the normal forms associated with Hopf bifurcation and other singularities. These programs can be "automatically" executed with a computer algebra system for a given ODE system. The basic idea of center manifold theory is applying successive coordinate transformations to systematically construct a simpler system which has less dimension compared to the original system, and thus greatly simplifies the dynamical analysis of the system.

The MTS method can also be directly applied to delay differential equations (DDEs) (for fundamental theory of functional differential equations, see [Hale, 1977; Das & Chatterjee, 2002; Nayfeh, 2008]). Compared to the MTS method, the CMR method is more complex in computing the normal forms of DDEs, since one needs to first change a DDE to an operator differential equation, and then decompose the solution space of their linearized form into stable and center manifolds; next, with adjoint operator equations, one computes the center manifold by projecting the whole space to the center manifold, and finally calculate the normal form restricted to the center manifold (e.g. see [Hassard *et al.*, 1981; Faria & Magalhães, 1995a, 1995b]).

To be more specific in defining the singularity of a given system, consider the m-dimensional autonomous differential equation,

$$\dot{x} = g(x, \alpha), \quad x \in \mathrm{R}^m, \;\; \alpha \in \mathrm{R}^n, \\ g : \mathrm{R}^m \times \mathrm{R}^n \to \mathrm{R}^m, \tag{1}$$

where x is a state vector, α is a parameter vector, and g is a general nonlinear function, assumed to be analytic. Further, assume $g(0, \alpha) = 0$, implying that $x = 0$ is an equilibrium solution for any real value of α. When the characteristic equation of the linearized system of (1) at $x = 0$, evaluated at a critical point, $\alpha = \alpha_c$, has n_1 pairs of purely imaginary roots $\pm \mathrm{i}\omega_j$ ($j = 1, 2, \ldots, n_1$), n_2 zero roots, and $m - 2n_1 - n_2$ roots with nonzero real part, we say that system (1) undergoes an n_1-Hopf–n_2-zero bifurcation, where $n_1 \geq 1$ and $n_2 \geq 0$. Suppose under a linear transformation, the Jacobian matrix of the linearized system of (1) can be put in a diagonal Jordan canonical form, namely, $J = \mathrm{diag}(J_1, J_2)$, where

$$J_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n_1+n_2}),$$

$$J_2 = \mathrm{diag}(\lambda_{2n_1+n_2+1}, \ldots, \lambda_m),$$

in which $\lambda_{2k-1} = \mathrm{i}\omega_k$, $\lambda_{2k} = -\mathrm{i}\omega_k$, $k = 1, 2, \ldots, n_1$, $\lambda_l = 0$, $l = 2n_1 + 1, \ldots, 2n_1 + n_2$, $\lambda_j, j = 2n_1 + n_2 + 1, \ldots, m$ satisfying $\mathrm{Re}(\lambda_j) \neq 0$, then system (1) is said to undergo a semisimple n_1-Hopf–n_2-zero bifurcation.

Many authors have considered some types of bifurcations in DDEs, by using the CMR method (e.g. see [Yu *et al.*, 2002; Yuan *et al.*, 2004; Chen & Yu, 2005; Wei & Jiang, 2005; Yuan & Wei, 2006; Jiang & Yuan, 2007; Ma *et al.*, 2008; Wang & Jiang, 2010]). Nayfeh [2008] used both the MTS and CMR methods to derive equivalent normal forms of Hopf bifurcation for some simple delayed nonlinear dynamical systems. Ding *et al.* [2012, 2013a, 2013b] applied the two methods to obtain the normal forms near Hopf-zero and double-Hopf critical points in DDEs and NFDEs, and showed their equivalence. Due to complexity in computing the center manifold and normal forms of DDEs, in recent years, researchers have paid attention to developing algorithms using numerical algorithm such as Fortran package [Aboud *et al.*, 1988] or using computer algebra systems such as Maple [Campbell, 2009]. However, it has been found that even with the help of computer systems, the computation using the CMR method is still not an easy job, in particular for those who are not familiar with the CMR method. On the other hand, many researchers from engineering or physical society prefer to apply a simple method, such as the MTS approach, to calculate the center manifold and normal forms for ODEs and DDEs. But since no rigorous proof has been given to show the equivalence of the MTS and CMR methods in general, people often have reservations or even suspicions on the results obtained by using the MTS method. That's why, as mentioned above, some researchers applied both the MTS and CMR methods to derive the normal form for a given dynamical system in order to show the correctness of their results. This certainly wastes researchers' time and thus a general proof is needed for the equivalence of the two methods.

The aim of this paper is to provide a rigorous proof for the equivalence of the MTS and CMR methods for general delay differential equations, associated with the semisimple n_1-Hopf–n_2-zero singularity. The differential equations considered in this paper include ordinary differential equations, general delay differential equations (DDE), neutral functional differential equations (NFDE) (or neutral delay differential equations (NDDE)), and partial functional differential equations (PFDE). The delays involved in these equations can be discrete or distributed. The NFDEs have been proposed in the study of population dynamics, neural network, engineering problems, etc. (e.g. see [Brayton, 1967; Kuang, 1999; El-Morshedy & Gopalsamy, 2000]). Several articles have been published [Guo & Lamb, 2008; Wang & Wei, 2008, 2010; Weedermann, 2001, 2006; Wu, 1993], focusing on bifurcation theory of NFDEs, such as normal form of Hopf bifurcation, global existence of periodic solutions, and equivariant Hopf bifurcation theory. Based on the work of Kazarinoff *et al.* [1978] who introduced Hopf bifurcation theory to differential-difference and integro-differential equations, Wang and Wei [2010] applied normal form theory and center manifold theory to study Hopf bifurcation properties (such as the direction of bifurcation and the stability of bifurcating periodic solutions) of NFDEs. The CMR method developed by Faria and Magalhães [1995a, 1995b] for DDEs was used by Weedermann [2001] to compute the normal forms of NFDEs. Later, Weedermann [2006], Wang and Wei [2008] extended the idea of [Faria & Magalhães, 1995b] to investigate NFDEs with parameters.

Compared to the DDE and NFDE systems, the PFDE systems have even wider applications, though they have more difficulty in analysis, since many physical systems are not only evolved temporally, but also varied spatially. For example, when an HIV model is focused on in-house dynamics, an ODE or a DDE model is good enough for studying the dynamical behavior of the system such as instability and bifurcations (e.g. see [Perelson *et al.*, 1993; Culshaw & Ruan, 2000]). However, when species in different patches are involved in such a model, then a PFDE model is necessary to be developed (e.g. see [Arino & van den Driessche, 2003]). Fundamental theory for general PFDEs has been established and applied to solve many physical, engineering and biological problems (e.g. see [Wu, 1996]). Other studies have mainly focused on dynamics of the systems like the existence of solutions [Travis & Webb, 1974; Hernández & Henŕiquez, 1998], stability and Hopf bifurcation [Busenberg & Huang, 1996; Azevedo & Ladeira, 2004], boundedness and almost periodicity of solutions [Furumochi *et al.*, 2002], and state-dependent delay involved in the systems [Hernández *et al.*, 2006], etc.

Another direction in the study of FDEs is to consider distributed delays rather than discrete delays, since using discrete delays is no longer appropriate in modeling such FDE systems. For example, when a more realistic age structure is introduced into an HIV model, distributed delays must be introduced in order to obtain more realistic dynamical solution of the system (e.g. see [Nelson *et al.*, 2004]). Recently, for the standard SIRS model, reinvestigation of this model by introducing distributed delays reveals that the shape of the distributions can destabilize oscillations, while fixed delays may yield stable oscillations for certain parameter values [Goncalves *et al.*, 2011]. The references mentioned above (e.g. see [Wu, 1996] and references therein) also consider FDEs with distributed delays.

Although the semisimple case considered in this paper is simpler than nonsemisimple case, most real applications actually fall in this category, rather than the nonsemisimple case. We will show in the proofs of theorems and the examples in the applications that the MTS method is simpler than the CMR method, which is particularly useful in applications. Another advantage of the MTS method over the CMR method is that the MTS method can easily treat multiple time delays with variations (perturbation) while the CMR method is restricted to single fixed constant delays or to the delays with their ratios to the maximum delay being constants (e.g. see [Faria, 2001]). From the viewpoint of applications, normal forms up to third order are usually enough for real practical systems. Thus, in this paper, we will show under certain conditions that the normal forms derived by using the MTS and the CMR methods are identical up to third order. Actually, the specific examples in the literature we refer to showing equivalence of the two methods all satisfy the conditions obtained in this paper. In order to show the basic idea in proving the equivalence of the two methods for DDEs, we will start our analysis from ODEs. In fact, the proof for the ODEs provides an independent rigorous proof for the equivalence of the MTS and CMR methods, which does not exist in the literature.

The rest of the paper is organized as follows. In the next section, the MTS method is proved to be equivalent to the CMR method up to any order for the ODE systems. In Sec. 3, particular attention is focused on the DDE systems, and a proof is given to show the equivalence of the two normal forms up to third order by using the MTS and CMR methods, associated with the semisimple n_1-Hopf–n_2-zero bifurcation. The proofs on the equivalence of the MTS and CMR methods for the NFDE and PFDE systems are given in Secs. 4 and 5, respectively. The DDEs, NFDEs and PFDEs with distributed delays are considered in Sec. 6 to show the equivalence of the MTS and CMR methods. Several different types of practical examples with discrete or distributed delays are present in Sec. 7 to demonstrate the application of the theoretical results. Finally, conclusion and discussion are given in Sec. 8.

2. Equivalence of the MTS and CMR Methods for ODEs

First, in this section we prove that the normal forms associated with the semisimple n_1-Hopf–n_2-zero bifurcation, derived by using the MTS and CMR methods are identical provided that the corresponding nonlinear transformations for the two methods are properly chosen for the normal forms. In other words, the MTS and CMR methods are equivalent in deriving normal forms.

Assume system (1) undergoes a semisimple n_1-Hopf–n_2-zero bifurcation at a critical point, $\alpha = \alpha_c$, with all eigenvalues of the linearized system of (1) having nonpositive real part. Without loss of generality, we may rewrite system (1) at the critical point $\alpha = \alpha_c$ as

$$\begin{aligned} \dot{x}_1 &= J_1 x_1 + g_1(x_1, x_2), \quad x_1 \in \mathrm{C}^{2n_1+n_2}, \\ \dot{x}_2 &= J_2 x_2 + g_2(x_1, x_2), \quad x_2 \in \mathrm{C}^{m-2n_1-n_2}, \end{aligned} \tag{2}$$

where $J_1 = J_1(\alpha_c) = \mathrm{diag}\{\mathrm{i}\omega_1, -\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_{n_1}, -\mathrm{i}\omega_{n_1}, \overbrace{0, \ldots, 0}^{n_2}\}$ and $J_2 = J_2(\alpha_c) = \mathrm{diag}\{\lambda_{2n_1+n_2+1}, \ldots, \lambda_m\}$ with $\mathrm{Re}(\lambda_k) < 0$, $k = 2n_1 + n_2 + 1, \ldots, m$, and $g_j(0,0) = \mathrm{D}g_j(0,0) = 0$ $(j = 1, 2)$, namely, system (2) has a trivial equilibrium solution $(x_1, x_2) = (0, 0)$. Note that one half of the equations in the first $2n_1$ equations of $\dot{x}_1$ are actually complex conjugates of the other half. Also, note that system (2) is assumed to not contain unstable manifold, which is usually the case in practical applications.

For system (2), we have the following theorem.

Theorem 1. *Assume that system (2) undergoes a semisimple n_1-Hopf–n_2-zero $(n_1 \geq 1, n_2 \geq 0, n = n_1 + n_2 \geq 1)$ bifurcation from the trivial equilibrium at the critical point $\alpha = \alpha_c$. Then, the normal forms associated with the semisimple n_1-Hopf–n_2-zero bifurcation, derived by using the multiple time scales and center manifold reduction methods, are identical up to any order provided that the corresponding nonlinear transformations associated with the two methods are properly chosen for their normal forms.*

Remark 1. Here, for simplicity, we ignore the derivation of unfolding terms, but focus on the normal forms, since the unfolding is obviously the same for the normal forms derived by using the two methods.

Proof. We apply the method of mathematical induction to prove the equivalence of the MTS and CMR methods order by order. We first describe the procedures of the CMR and MTS methods, respectively, and show that the conclusion is true up to third order. Then, under the assumption that the conclusion is true up to the $(k-1)$th order, prove that the conclusion is also true for the kth order.

First, consider the CMR method. By center manifold theorem [Wiggins, 1990; Guckenheimer & Holmes, 1990], system (2) is locally topologically equivalent (near the origin) to the following system:

$$\dot{x}_1 = J_1 x_1 + g_1(x_1, l(x_1)), \quad x_1 \in \mathrm{C}^{2n_1+n_2}, \tag{3}$$

where $l(x_1)$ satisfies

$$\begin{aligned} &\mathrm{D}_{x_1} l(x_1)[J_1 x_1 + g_1(x_1, l(x_1))] \\ &\quad - J_2 l(x_1) - g_2(x_1, l(x_1)) = 0, \end{aligned} \tag{4}$$

and the center manifold is defined by $M_c \triangleq \{(x_1, x_2) \,|\, x_2 = l(x_1)\}$. It can be seen from the definition of the center manifold that the "noncritical" state variable x_2 (associated with the eigenvalues having negative real part) is expressed in terms of the "critical" state variable x_1 (associated with the eigenvalues having zero real part), starting from second-order terms. This is the basic idea of center manifold theory, implying that the influence of the eigenvalues with negative real part on the "noncritical" state variable x_2 has been neglected from the asymptotic property, and only the influence from the "critical" state variable x_1 is considered. Equation (3) describes the dynamics of system (2) restricted to its center manifold, M_c. To find the normal form of (3), we apply a general nonlinear transformation to system (3) and choose appropriate terms in the transformation to simplify the system.

Suppose the nonlinear transformation is

$$x_1 = z + h_2(z) + h_3(z) + \cdots, \tag{5}$$

which is differentiated with respect to time t to yield

$$\dot{x}_1 = (\mathrm{I} + \mathrm{D}_z h_2 + \mathrm{D}_z h_3 + \cdots)\dot{z}.$$

Then, the equation for deriving the normal form is obtained as

$$\begin{aligned} \dot{z} &= (\mathrm{I} + \mathrm{D}_z h_2 + \mathrm{D}_z h_3 + \cdots)^{-1}\dot{x}_1 \\ &= (\mathrm{I} + \mathrm{D}_z h_2 + \mathrm{D}_z h_3 + \cdots)^{-1} \\ &\quad \times [J_1(z + h_2 + h_3 + \cdots) + g_1(z + h_2 \\ &\quad + h_3 + \cdots, l(z + h_2 + h_3 + \cdots))]. \end{aligned} \tag{6}$$

For convenience, we denote

$$\begin{aligned} l(x_1) &= l(z + h_2(z) + h_3(z) + \cdots) \\ &:= \sum_{k \geq 2} l_k(z), \\ g_j(z + h_2 &+ h_3 + \cdots, l(z + h_2 + h_3 + \cdots)) \\ &:= \sum_{k \geq 2} g_{jk}(z), \quad j = 1, 2. \end{aligned} \tag{7}$$

Here, h_k, l_k and g_{jk} represent kth degree homogeneous polynomials with respect to z. We introduce a linear operator $L_{J_1}^k : H_{2n_1+n_2}^k \to H_{2n_1+n_2}^k$ $(k \geq 2)$, defined by

$$L_{J_1}^k \tilde{h}_k(z) = \mathrm{D}_z \tilde{h}_k J_1 z - J_1 \tilde{h}_k, \quad \forall\, \tilde{h}_k \in H_{2n_1+n_2}^k, \tag{8}$$

which is usually called Homological operator or Lie bracket operator [Wiggins, 1990; Guckenheimer & Holmes, 1990]. Here, $H_{2n_1+n_2}^k$ denotes a linear space, spanned by the kth degree homogeneous polynomials in $(z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_{n_1}, \bar{z}_{n_1}, z_{n_1+1}, \ldots, z_n)$. Moreover, we decompose $H_{2n_1+n_2}^k$ as $H_{2n_1+n_2}^k = \mathrm{I}_k \oplus \mathrm{C}_k$, where I_k represents the image of $L_{J_1}^k$, and C_k is the complementary space to I_k.

Now, suppose the normal form of (3) has been obtained up to $(k-1)$th order, given by $\dot{z} = J_1 z + q_{22}(z) + \cdots + q_{(k-1)2}(z) + \cdots$, where $q_{j2} \in \mathrm{C}_j$, $j = 2, 3, \ldots, k-1$. Then, (6) becomes

$$\begin{aligned} \dot{z} &= J_1 z + q_{22}(z) + \cdots + q_{(k-1)2}(z) \\ &\quad + [g_{1k}(z) - L_{J_1}^k h_k(z)] + \cdots, \quad k \geq 3. \end{aligned} \tag{9}$$

Further, we split $g_{1k}(z)$ into two parts as $g_{1k}(z) = q_{k1}(z) + q_{k2}(z)$, where $q_{k1}(z)$ satisfies

$$q_{k1}(z) - L_{J_1}^k h_k(z) = 0,$$

and $q_{k2}(z)$ is the kth order normal form, and thus the normal form up to kth order becomes

$$\dot{z} = J_1 z + q_{22}(z) + \cdots + q_{(k-1)2}(z) + q_{k2}(z) + \cdots. \tag{10}$$

Next, for the MTS method, we do not directly apply the center manifold theory, but instead assume that the solution of (2) is given in the form of

$$
\begin{aligned}
\tilde{x}_1(t) &= \epsilon x_{11}(T_0,T_1,T_2,\ldots) + \epsilon^2 x_{12}(T_0,T_1,T_2,\ldots) \\
&\quad + \epsilon^3 x_{13}(T_0,T_1,T_2,\ldots) + \cdots, \\
\tilde{x}_2(t) &= \epsilon x_{21}(T_0,T_1,T_2,\ldots) + \epsilon^2 x_{22}(T_0,T_1,T_2,\ldots) \\
&\quad + \epsilon^3 x_{23}(T_0,T_1,T_2,\ldots) + \cdots,
\end{aligned} \tag{11}
$$

where $T_k = \epsilon^k t$, $k = 0,1,2,\ldots$, are called multiple time scales, and $\tilde{x}_j(t)$ ($j = 1,2$) is used to distinguish from the variable used in the CMR method, $x_j(t)$. The derivative with respect to t now becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} &= \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots \\
&= \mathrm{D}_0 + \epsilon \mathrm{D}_1 + \epsilon^2 \mathrm{D}_2 + \cdots,
\end{aligned} \tag{12}
$$

where the differential operator $\mathrm{D}_k = \frac{\partial}{\partial T_k}$, $k = 0,1,2,\ldots$. Substituting (11) into $g_j(x_1,x_2)$ ($j = 1,2$) yields

$$
\begin{aligned}
g_j(x_1,x_2) = \sum_{k\geq 2} \epsilon^k g_{jk}(&x_{11},x_{21},x_{12},x_{22},\ldots, \\
&x_{1(k-1)},x_{2(k-1)}), \quad j = 1,2.
\end{aligned}
$$

Further, substituting (11) with the multiple scales (12) and the above expressions into (2) and balancing the coefficients of ϵ^k ($k = 1,2,\ldots$) yields a set of ordered linear differential equations (LDEs).

First, consider the ϵ^1-order LDEs, given by

$$
\begin{aligned}
\mathrm{D}_0 x_{11} - J_1 x_{11} &= 0, \\
\mathrm{D}_0 x_{21} - J_2 x_{21} &= 0.
\end{aligned} \tag{13}
$$

Since $\mathrm{Re}(\lambda_j) < 0$, $j = 2n_1 + n_2 + 1,\ldots,m$, the solution of the second equation of (13) $x_{21} \to 0$ as $t \to +\infty$. Therefore, in the sense of asymptotic behavior with respect to x_{21}, we write the solution x_{21} as $x_{21} = 0$.

For the ϵ^2-order LDEs:

$$
\begin{aligned}
\mathrm{D}_0 x_{12} - J_1 x_{12} &= -\mathrm{D}_1 x_{11} + g_{12}(x_{11},0), \\
\mathrm{D}_0 x_{22} - J_2 x_{22} &= g_{22}(x_{11},0).
\end{aligned} \tag{14}
$$

Letting the secular terms in the first equation of (14) be zero, we can solve $\mathrm{D}_1 x_{11}$ in terms of x_{11}, and then obtain x_{12} expressed in x_{11}. By using the second equation of (14), we obtain x_{22} expressed in x_{11}, denoted by $x_{22}(x_{11})$.

The above procedure can in principle continue indefinitely (to any high order). For general ϵ^k-order LDEs ($k \geq 3$), we have

$$
\begin{aligned}
\mathrm{D}_0 x_{1k} - J_1 x_{1k} &= -\sum_{j=1}^{k-1} \mathrm{D}_j x_{1(k-j)} + g_{1k}(x_{11},x_{21}, \\
&\qquad x_{12},x_{22},\ldots,x_{1(k-1)},x_{2(k-1)}), \\
\mathrm{D}_0 x_{2k} - J_2 x_{2k} &= -\sum_{j=1}^{k-2} \mathrm{D}_j x_{2(k-j)} + g_{2k}(x_{11},x_{21}, \\
&\qquad x_{12},x_{22},\ldots,x_{1(k-1)},x_{2(k-1)}).
\end{aligned} \tag{15}
$$

Substituting $\mathrm{D}_j x_{1(k-j)}$ ($j = 1,2,\ldots,k-2$) into the first equation of (15) and letting the secular terms equal zero, we can solve $\mathrm{D}_{k-1} x_{11}$ in terms of x_{11}, and then obtain x_{1k} expressed in x_{11}. By using the second equation of (15), we obtain x_{2k} expressed in x_{11}.

The normal form derived using the MTS method can now be written as

$$
\begin{aligned}
\dot{x}_{11} &= \frac{\mathrm{d}x_{11}}{\mathrm{d}t} \\
&= \frac{\partial x_{11}}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial x_{11}}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial x_{11}}{\partial T_2}\frac{\partial T_2}{\partial t} \\
&\quad + \cdots + \frac{\partial x_{11}}{\partial T_{k-1}}\frac{\partial T_{k-1}}{\partial t} + \cdots \\
&= J_1 x_{11} + \epsilon \mathrm{D}_1 x_{11} + \epsilon^2 \mathrm{D}_2 x_{11} \\
&\quad + \cdots + \epsilon^{k-1} \mathrm{D}_{k-1} x_{11} + \cdots.
\end{aligned}
$$

Note that $\mathrm{D}_j x_{11}$ ($j = 1,2,\ldots$) is $(j+1)$-order linear homogeneous polynomial involving x_{11}. With the use of backwards scaling $x_{11} \mapsto x_{11}/\epsilon$, the above equation becomes

$$
\begin{aligned}
\dot{x}_{11} &= J_1 x_{11} + \mathrm{D}_1 x_{11} + \mathrm{D}_2 x_{11} \\
&\quad + \cdots + \mathrm{D}_{k-1} x_{11} + \cdots,
\end{aligned} \tag{16}
$$

which is the normal form derived using the MTS method.

Having described the procedures of the CMR and MTS methods, we are now ready to prove the equivalence of the two normal forms (10) and (16), derived by using the CMR and MTS methods respectively, order by order. The proof is divided into four steps.

Step 1. We show that the solutions of the linearized system in the center subspace for the two methods are identical.

Actually, it is seen from (10) that the linear solution z in the center subspace by using the CMR method satisfies

$$\dot{z} = J_1 z.$$

Similarly, it follows from (13) that the linear solution x_{11} on the center manifold by using the MTS method is described by

$$\mathrm{D}_0 x_{11} = \frac{\partial x_{11}}{\partial T_0} = \frac{\partial x_{11}}{\partial t} = J_1 x_{11}.$$

Thus, the linear solution z in the CMR method corresponds to the linear solution x_{11} in the MTS method, since these two equations have the exact same form. This is obvious because linear normal forms must be identical.

Step 2. We show that the second-order normal forms obtained by using the CMR and MTS methods are identical.

First note that $g_{j2}(z)$ $(j = 1, 2)$ is exactly in the same form as that of $g_{j2}(x_{11}, 0)$ with z corresponding to x_{11}. Directly using (6) for the CMR method and (13)–(14) for the MTS method, we obtain

$$\begin{aligned} q_{22}(z) &= g_{12}(z) + J_1 h_2(z) - \mathrm{D}_z h_2(z) J_1 z, \\ \mathrm{D}_1 x_{11} &= g_{12}(x_{11}, 0) + J_1 x_{12}(x_{11}) - \mathrm{D}_0 x_{12}(x_{11}) \\ &= g_{12}(x_{11}, 0) + J_1 x_{12}(x_{11}) \\ &\quad - \mathrm{D}_{x_{11}} x_{12}(x_{11}) J_1 x_{11}. \end{aligned}$$

Thus, as long as $h_2(z)$ takes the same form of $x_{12}(x_{11})$, $q_{22}(z)$ and $\mathrm{D}_1 x_{11}$ are identical, with z corresponding to x_{11}. Then, the second-order normal forms derived using the CMR method

$$\dot{z} = J_1 z + q_{22}(z)$$

and the normal form up to second order derived by using the MTS method

$$\dot{x}_{11} = J_1 x_{11} + \mathrm{D}_1 x_{11}$$

are identical.

Remark 2. Due to the choice of the basis for the complementary space C_k in the CMR method being not unique, the choice of the nonlinear transformation h_k is not unique and hence q_{k2} is not unique; while in the MTS method, the solution of x_{12} by solving the particular solution of the differential equation is unique. Thus, in order for the two second-order normal forms to be identical, $h_2(z)$ must be chosen as the same form as that of the $x_{12}(x_{11})$.

Further, for the CMR method, it is easy to see from (4) and (5) that the second-order terms in the center manifold, denoted by $l_2(z)$, satisfy

$$\mathrm{D}_z l_2(z) J_1 z - J_2 l_2(z) = g_{22}(z). \tag{17}$$

On the other hand, for the MTS method, with the use of (13), the second equation of (14) can be rewritten as

$$\begin{aligned} \mathrm{D}_{x_{11}} x_{22}(x_{11}) J_1 x_{11} - J_2 x_{22}(x_{11}) \\ = g_{22}(x_{11}, 0). \end{aligned} \tag{18}$$

Obviously, $g_{j2}(z)$ $(j = 1, 2)$ is exactly in the same form as that of $g_{j2}(x_{11}, 0)$, with z corresponding to x_{11}, and so is Eq. (17) as that of Eq. (18), and thus $l_2(z)$ and $x_{22}(x_{11})$ have the exact same solution, with z corresponding to x_{11}.

Step 3. We show that the third-order normal forms obtained using the CMR and MTS methods are identical.

Note that $g_{j3}(z)$ $(j = 1, 2)$ is exactly in the same form as that of $g_{j3}(x_{11}, 0, x_{12}, x_{22})$ with z corresponding to x_{11}, $h_2(z)$ to $x_{12}(x_{11})$ and $l_2(z)$ to $x_{22}(x_{11})$. For the CMR method, it follows from (6) that

$$\begin{aligned} q_{32}(z) = g_{13}(z) &+ J_1 h_3(z) \\ &- \mathrm{D}_z h_2(z)[J_1 h_2(z) + g_{12}(z)] \\ &- \mathrm{D}_z h_3(z) J_1 z + \mathrm{D}_z h_2(z) \mathrm{D}_z h_2(z) J_1 z, \end{aligned}$$

which, due to $q_{22}(z) = g_{12}(z) + J_1 h_2(z) - \mathrm{D}_z h_2(z) \times J_1 z$, is reduced to

$$\begin{aligned} q_{32}(z) = g_{13}(z) + J_1 h_3(z) - \mathrm{D}_z h_3(z) J_1 z \\ - \mathrm{D}_z h_2(z) q_{22}(z). \end{aligned}$$

For the MTS method, by the first equation of (15) with $k = 3$, we have

$$\begin{aligned} &\mathrm{D}_0 x_{13} - J_1 x_{13} \\ &\quad = -\mathrm{D}_1 x_{12} - \mathrm{D}_2 x_{11} + g_{13}(x_{11}, 0, x_{12}, x_{22}), \end{aligned}$$

which can be rewritten as

$$\begin{aligned}\mathrm{D}_2 x_{11} = g_{13}(x_{11}, 0, x_{12}, x_{22}) + J_1 x_{13} \\ - \mathrm{D}_{x_{11}} x_{13} J_1 x_{11} - \mathrm{D}_{x_{11}} x_{12} \mathrm{D}_1 x_{11}.\end{aligned}$$

Thus, similarly as long as $h_3(z)$ takes the same form of $x_{13}(x_{11})$, $q_{32}(z)$ and $\mathrm{D}_2 x_{11}$ are identical, with z corresponding to x_{11}. Hence, the third-order normal form derived using the CMR method,

$$\dot{z} = J_1 z + q_{22}(z) + q_{32}(z),$$

and the normal form up to the third order derived using the MTS method,

$$\dot{x}_{11} = J_1 x_{11} + \mathrm{D}_1 x_{11} + \mathrm{D}_2 x_{11},$$

are identical.

Moreover, note that for the CMR method, using (4) and (5) yields the third-order terms on the center manifold, denoted by $l_3(z)$, satisfying

$$\begin{aligned}\mathrm{D}_z l_3(z) J_1 z + \mathrm{D}_z l_2(z)[J_1 h_2(z) + g_{12}(z) \\ - \mathrm{D}_z h_2(z) J_1 z] - J_2 l_3(z) - g_{23}(z) = 0. \quad (19)\end{aligned}$$

The second equation of (15) with $k = 3$ can be rewritten as

$$\begin{aligned}\mathrm{D}_{x_{11}} x_{23}(x_{11}) J_1 x_{11} + \mathrm{D}_{x_{11}} x_{22}[J_1 x_{12}(x_{11}) \\ + g_{12}(x_{11}, 0) - \mathrm{D}_{x_{11}} x_{12}(x_{11}) J_1 x_{11}] \\ - J_2 x_{23}(x_{11}) - g_{23}(x_{11}, 0, x_{12}, x_{22}) = 0. \quad (20)\end{aligned}$$

Obviously, $g_{j3}(z)$ $(j = 1, 2)$ is exactly in the same form as that of $g_{j3}(x_{11}, 0, x_{12}, x_{22})$, with z corresponding to x_{11}, and so is Eq. (19) as that of Eq. (20), and thus $l_3(z)$ and $x_{23}(x_{11})$ have the exact same solution with z corresponding to the x_{11}.

Step 4. Finally we prove that the normal forms obtained using the CMR and MTS methods are identical up to any order.

Having proved that the conclusion of Theorem 1 is true for second order and third order ($k = 2$ and $k = 3$). According to the method of mathematical induction, we assume that the conclusion of Theorem 1 is true up to $(k-1)$th order ($k \geq 4$). That is, q_{j2} $(j = 2, 3, \ldots, k-1)$ and $\mathrm{D}_{j-1} x_{11}$ are identical with $h_j(z)$ corresponding to x_{1j}, and both x_{1j} $(j = 2, 3, \ldots, k-1)$ and x_{2j} are expressed in terms of x_{11}, and $l_j(z)$ and $x_{2j}(x_{11})$ $(j = 2, 3, \ldots, k-1)$ are identical, with $g_{lj}(z)$ $(l = 1, 2;\ j = 2, 3, \ldots, k)$ and $g_{lj}(x_{11}, x_{21}, x_{12}, x_{22}, \ldots, x_{1(j-1)}, x_{2(j-1)})$ having the same form. With the assumption, we now prove that the conclusion is also true for kth order, namely, the kth order terms in the normal forms, q_{k2} in the CMR method and $\mathrm{D}_{k-1} x_{11}$ in the MTS method, are identical.

In the CMR method,

$$\begin{aligned}q_{k2}(z) = g_{1k}(z) + J_1 h_k(z) - \mathrm{D}_z h_k(z) J_1 z \\ - \sum_{j=1}^{k-2} \mathrm{D}_z h_{(k-j)}(z) q_{j2}(z).\end{aligned}$$

For the MTS method, by the first equation of (15),

$$\begin{aligned}&\mathrm{D}_{x_{11}} x_{1k} J_1 x_{11} - J_1 x_{1k} \\ &\quad = -\mathrm{D}_{k-1} x_{11} - \sum_{j=1}^{k-2} \mathrm{D}_{x_{11}} x_{1(k-j)} \mathrm{D}_j x_{11} \\ &\qquad + g_{1k}(x_{11}, 0, x_{12}, x_{22}, \ldots, x_{1(k-1)}, x_{2(k-1)}).\end{aligned}$$

That is,

$$\begin{aligned}\mathrm{D}_{k-1} x_{11} = g_{1k}(x_{11}, 0, x_{12}, x_{22}, \ldots, x_{1(k-1)}, x_{2(k-1)}) \\ + J_1 x_{1k} - \mathrm{D}_{x_{11}} x_{1k} J_1 x_{11} \\ - \sum_{j=1}^{k-2} \mathrm{D}_{x_{11}} x_{1(k-j)} \mathrm{D}_j x_{11}.\end{aligned}$$

Thus, as long as $h_k(z)$ takes the same form of $x_{1k}(x_{11})$, $q_{k2}(z)$ and $\mathrm{D}_{k-1} x_{11}$ are identical, with z corresponding to x_{11}. Then, the kth order normal forms derived using the CMR method,

$$\dot{z} = J_1 z + q_{22}(z) + q_{32}(z) + \cdots + q_{k2}(z),$$

and using the MTS method,

$$\dot{x}_{11} = J_1 x_{11} + \mathrm{D}_1 x_{11} + \mathrm{D}_2 x_{11} + \cdots + \mathrm{D}_{k-1} x_{11},$$

are identical.

Further, note that for the CMR method, with (4) and (5), it can be shown that the kth order terms on the center manifold, denoted by $l_k(z)$, satisfy

$$\begin{aligned}\mathrm{D}_z l_k(z) J_1 z + \sum_{j=1}^{k-2} [\mathrm{D}_z l_{k-j}(z) g_{1(j+1)}(z)] \\ - J_2 l_k(z) - g_{2k}(z) = 0. \quad (21)\end{aligned}$$

With the use of (13), the second equation of (15) can be rewritten as

$$\begin{aligned}\mathrm{D}_{x_{11}} x_{2k}(x_{11}) J_1 x_{11} + \sum_{j=1}^{k-2} [\mathrm{D}_{x_{11}} x_{2(k-j)}(x_{11}) \mathrm{D}_j x_{11}] \\ - J_2 x_{2k}(x_{11}) - g_{2k}(x_{11}, x_{21}, x_{12}, x_{22}, \ldots, \\ x_{1(k-1)}, x_{2(k-1)}) = 0. \quad (22)\end{aligned}$$

Obviously, $g_{jk}(z)$ $(j = 1, 2)$ is exactly in the same form as that of $g_{jk}(x_{11}, 0, x_{12}, x_{22}, \ldots, x_{1(k-1)}, x_{2(k-1)})$, with z corresponding to x_{11}, and so is Eq. (21) as that of Eq. (22), and thus $l_k(z)$ and $x_{2k}(x_{11})$ have the exact same solution with z corresponding to x_{11}.

The proof of Theorem 1 is complete. ■

Remark 3

(a) It is clear from Eq. (16) that the role of the multiple time scales is to distinguish different order terms in the solution, resulting in different order normal form terms.
(b) From the proof of Theorem 1, we can see that the MTS method combines the two steps involved in using center manifold theory and normal form theory into one unified step to obtain the normal form and nonlinear transformation simultaneously.

3. Equivalence of the MTS and CMR Methods for DDEs

In the previous section, we have shown that the MTS and CMR methods are equivalent for ODE systems associated with the semisimple n_1-Hopf–n_2-zero singularity. In this section, we turn to consider such singularity in DDE systems, and to obtain the conditions under which the normal forms obtained using the two methods are identical up to third order.

Consider the general m-dimensional delay differential equation:

$$\dot{u}(t) = N_0(\alpha)u(t) + N_1(\alpha)u(t-1) + f(u(t), u(t-1)), \tag{23}$$

where $u \in \mathrm{R}^m$ is a state vector, $\alpha \in \mathrm{R}^n$ is a parameter vector, $f \in \mathrm{C}^\infty$, $f(0) = \mathrm{D}f(0) = 0$. In general, the nonlinear function f should contain α. However, since the unfolding terms are involved in $N_0(\alpha)$ and $N_1(\alpha)$, f will be expanded around a critical point $\alpha = \alpha_c$, and thus α is not explicitly shown in f. If the equilibrium of system (23) is not a trivial solution, we can transfer the nontrivial equilibrium to the origin by a simple translation, and if the delay in system (23) is $\tau \neq 1$, we can obtain the form (23) by scaling the time delay, $t \mapsto t/\tau$. So, without loss of generality, we use system (23) in the following analysis.

Remark 4. In general, system (23) can be directly extended to involve multiple delays for the case when using the MTS method. That is, the MTS method can be used to study the following system with multiple delays,

$$\dot{u}(t) = N_0 u(t) + \sum_{j=1}^{p} N_j u(t-\tau_j) + f(u(t), u(t-\tau_1), \ldots, u(t-\tau_p)).$$

However, since the CMR method can only be able to deal with constant delays or the delays with their ratios to the maximum delay being constants [Faria, 2001], we use (23) in this section for a comparison of the two methods.

The characteristic equation of (23), evaluated at the trivial equilibrium $u = 0$, is given by

$$\det \Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) = \lambda\mathrm{I} - N_0(\alpha) - N_1(\alpha)\mathrm{e}^{-\lambda}, \tag{24}$$

where I is the $m \times m$ identity matrix. For the DDE system (23), we have the following result.

Theorem 2. *Assume that system (23) undergoes a semisimple n_1-Hopf–n_2-zero $(n_1 \geq 1, n_2 \geq 0, n = n_1 + n_2 \geq 1)$ bifurcation from the trivial equilibrium at the critical point, defined by $\alpha = \alpha_c$, where α is a parameter vector involved in system (23), at which the characteristic equation (24) has n_1 pairs of purely imaginary roots $\pm\mathrm{i}\omega_j$ $(j = 1, 2, \ldots, n_1)$ and n_2 zero roots, and all other roots have negative real part. If the second-order terms in the normal form vanish at $\alpha = \alpha_c$, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Proof. For convenience, we define the characteristic matrix $\Delta(\lambda)$ as $\Delta_c(\lambda)$ at the critical point, $\alpha = \alpha_c$, and denote $\Delta_c^*(\lambda)$ the adjoint matrix of $\Delta_c(\lambda)$. Then, let p_j $(j = 1, 2, \ldots, n_1)$ and p_l $(l = n_1 + 1, \ldots, n)$ be the eigenfunctions of $\Delta_c(\lambda)$ corresponding to the eigenvalues $\mathrm{i}\omega_j$ and 0, respectively; and p_j^* $(j = 1, 2, \ldots, n_1)$ and p_l^* $(l = n_1 + 1, \ldots, n)$ be the normalized eigenfunctions of $\Delta_c^*(\lambda)$ corresponding to the eigenvalues $-\mathrm{i}\omega_j$ and 0, respectively, satisfying the inner products,

$$\langle p_j^*, p_j \rangle = \bar{p}_j^{*\mathrm{T}} p_j = 1, \quad j = 1, 2, \ldots, n. \tag{25}$$

We take perturbation $\alpha = \alpha_c + \epsilon\alpha_\epsilon$ in (23). Substituting it into $N_0(\alpha)$ and $N_1(\alpha)$, we have the following expansions in terms of ϵ:

$$N_0(\alpha) = N_0(\alpha_c) + \epsilon N_0^{(1)}(\alpha_\epsilon) + \epsilon^2 N_0^{(2)}(\alpha_\epsilon) + \cdots,$$

$$N_1(\alpha) = N_1(\alpha_c) + \epsilon N_1^{(1)}(\alpha_\epsilon) + \epsilon^2 N_1^{(2)}(\alpha_\epsilon) + \cdots,$$

where $N_0(\alpha_c)$ and $N_1(\alpha_c)$ are the values of N_0 and N_1 evaluated at the critical point, $\alpha = \alpha_c$. Note that the so-called unfolding terms, necessary for bifurcation analysis, will come from $N_0(\alpha)$ and $N_1(\alpha)$.

Then, with the MTS method, suppose the solution of (23) is given by

$$\begin{aligned} u(t) &= \epsilon u_1(T_0, T_1, T_2, \ldots) + \epsilon^2 u_2(T_0, T_1, T_2, \ldots) \\ &\quad + \epsilon^3 u_3(T_0, T_1, T_2, \ldots) + \cdots. \end{aligned} \tag{26}$$

To deal with the terms involving delays, we expand $u_j(T_0 - 1, T_1 - \epsilon, T_2 - \epsilon^2, \ldots)$ at $u_j(T_0 - 1, T_1, T_2, \ldots)$ for $j = 1, 2, \ldots$. Ignoring the high order terms involving parameters, we obtain

$$\begin{aligned} &f(u(t), u(t-1)) \\ &\quad = \sum_{j \ge 2} \epsilon^j f_j(u_1, u_{1,1}, \ldots, u_{j-1}, u_{j-1,1}), \end{aligned}$$

where $u_{p,1} := u_p(T_0 - 1, T_1, T_2, \ldots)$, $p = 1, 2, \ldots$. Then, substituting solution (26) with the multiple scales (12) into (23) and balancing the coefficients of ϵ^j $(j = 1, 2, \ldots)$ yields a set of ordered linear differential equations (LDEs).

First, from the ϵ^1-order LDE, we have

$$\mathrm{D}_0 u_1 - N_0(\alpha_c) u_1 - N_1(\alpha_c) u_{1,1} = 0. \tag{27}$$

Since $\pm \mathrm{i}\omega_j$ $(j = 1, 2, \ldots, n_1)$ and zero (with multiplicity n_2) are the eigenvalues of the linear part of (23), the solution of (27) restricted to the center subspace can be expressed in the form of

$$\begin{aligned} u_1(T_0, T_1, T_2, \ldots) &= \sum_{j=1}^{n_1} G_j(T_1, T_2, \ldots) p_j \mathrm{e}^{\mathrm{i}\omega_j T_0} \\ &\quad + \sum_{j=1}^{n_1} \overline{G}_j(T_1, T_2, \ldots) \overline{p}_j \mathrm{e}^{-\mathrm{i}\omega_j T_0} \\ &\quad + \sum_{l=n_1+1}^{n} G_l(T_1, T_2, \ldots) p_l. \end{aligned} \tag{28}$$

Next, from the ϵ^2-order LDE, we obtain

$$\begin{aligned} &\mathrm{D}_0 u_2 - N_0(\alpha_c) u_2 - N_1(\alpha_c) u_{2,1} \\ &\quad = -\mathrm{D}_1 u_1 + N_0^{(1)}(\alpha_\epsilon) u_1 + N_1^{(1)}(\alpha_\epsilon) u_{1,1} \\ &\qquad - N_1(\alpha_c) \mathrm{D}_1 u_{1,1} + f_2(u_1, u_{1,1}). \end{aligned} \tag{29}$$

Substituting solution (28) into (29) yields the equation,

$$\begin{aligned} &\mathrm{D}_0 u_2 - N_0(\alpha_c) u_2 - N_1(\alpha_c) u_{2,1} \\ &\quad = g_2^{s,0} + \sum_{j=1}^{n_1} \xi_{2j} \mathrm{e}^{\mathrm{i}\omega_j T_0} + \sum_{j=1}^{n_1} \overline{\xi}_{2j} \mathrm{e}^{-\mathrm{i}\omega_j T_0} \\ &\qquad + \sum_{l=n_1+1}^{n} \xi_{2l} + g_2^u, \end{aligned} \tag{30}$$

where $g_2^{s,0}$ is a constant vector, representing all the terms expressed in $G_{l_1} G_{l_2} e_l$ $(l_1, l_2 = n_1 + 1, \ldots, n;$ $l = 1, \ldots, 2n_1 + n_2)$ and $G_j \overline{G}_j e_l$ $(j = 1, 2, \ldots, n_1)$, generated from $f_2(u_1, u_{1,1})$, and g_2^u denotes the remaining terms in $f_2(u_1, u_{1,1})$ that do not produce secular terms, and

$$\begin{aligned} \xi_{2j} &= -\mathrm{D}_1 G_j p_j + N_0^{(1)}(\alpha_\epsilon) G_j p_j \\ &\quad + N_1^{(1)}(\alpha_\epsilon) G_j p_j \mathrm{e}^{-\mathrm{i}\omega_j} - N_1(\alpha_c) p_j \mathrm{D}_1 G_j \mathrm{e}^{-\mathrm{i}\omega_j} \\ &\quad + g_2^{s,h_j}(u_1, u_{1,1}), \quad j = 1, 2, \ldots, n_1, \\ \xi_{2l} &= -\mathrm{D}_1 G_l p_l - N_1(\alpha_c) \mathrm{D}_1 G_l p_l \\ &\quad + (N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)) G_l p_l, \\ &\qquad l = n_1 + 1, \ldots, n, \end{aligned}$$

where $g_2^{s,h_j}(u_1, u_{1,1})$ $(j = 1, 2, \ldots, n_1)$ is a part of $f_2(u_1, u_{1,1})$ which generates secular terms in the solution, consisting of the terms $G_j G_l e_{2j-1}$ $(j = 1, 2, \ldots, n_1$, $l = n_1 + 1, \ldots, n)$.

Equation (30) is a linear nonhomogeneous equation for u_2, which has a periodic solution if and only if the so-called "solvability conditions" are satisfied [Nayfeh, 1981], that is, $\langle p_j^*, \xi_{2j} \rangle = 0$ $(j = 1, 2, \ldots, n_1)$ and $\langle p_l^*, g_2^{s,0} + \sum_{k=n_1+1}^{n} \xi_{2k} \rangle = 0$ $(l = n_1 + 1, \ldots, n)$, are satisfied. Solving these equations for $\mathrm{D}_1 G_j$ $(j = 1, 2, \ldots, n_1)$ and $(\mathrm{D}_1 G_{n_1+1}, \ldots, \mathrm{D}_1 G_n)^{\mathrm{T}}$, yields,

$$\mathrm{D}_1 G_j = \frac{\langle p_j^*, (N_0^{(1)}(\alpha_\epsilon) p_j + N_1^{(1)}(\alpha_\epsilon) \mathrm{e}^{-\mathrm{i}\omega_j} p_j) G_j + g_2^{s,h_j} \rangle}{\langle p_j^*, p_j + N_1(\alpha_c) \mathrm{e}^{-\mathrm{i}\omega_j} p_j \rangle}, \quad j = 1, 2, \ldots, n_1,$$

$$\begin{pmatrix} \mathrm{D}_1 G_{n_1+1} \\ \vdots \\ \mathrm{D}_1 G_n \end{pmatrix} = K_z \begin{pmatrix} \bar{p}^*_{n_1+1}\left(g_2^{s,0} + \sum_{k=n_1+1}^{n} [N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)] p_k G_k\right) \\ \vdots \\ \bar{p}^*_{n}\left(g_2^{s,0} + \sum_{k=n_1+1}^{n} [N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)] p_k G_k\right) \end{pmatrix},$$

where K_z is assumed to be invertible, given by

$$K_z = \begin{pmatrix} \bar{p}^*_{n_1+1}(\mathrm{I} + N_1(\alpha_c))p_{n_1+1} & \cdots & \bar{p}^*_{n_1+1}(\mathrm{I} + N_1(\alpha_c))p_n \\ \vdots & \cdots & \vdots \\ \bar{p}^*_{n}(\mathrm{I} + N_1(\alpha_c))p_{n_1+1} & \cdots & \bar{p}^*_{n}(\mathrm{I} + N_1(\alpha_c))p_n \end{pmatrix}^{-1}. \tag{31}$$

Remark 5. It is noted from the above expressions that each $\mathrm{D}_1 G_k$ contains two parts, one comes from the parameter perturbation, called unfolding, and the other part comes from the contribution of g_2^{s,h_j} and $g_2^{s,0}$, which are the second-order terms in the normal form. Under the assumption that the second-order terms in the normal form vanish at the critical point, i.e. $g_2^{s,0} = g_2^{s,h_j} = 0$, we can show that the normal forms obtained using the MTS and CMR methods are identical up to third order. In order for the consistence with the CMR method discussed next, we will still call the unfolding terms the second-order terms in the normal form.

Thus, under the assumption, setting $g_2^{s,0} = g_2^{s,h_j} = 0$ yields

$$\mathrm{D}_1 G_j = \frac{\langle p_j^*, (N_0^{(1)}(\alpha_\epsilon)p_j + N_1^{(1)}(\alpha_\epsilon)\mathrm{e}^{-\mathrm{i}\omega_j} p_j) G_j\rangle}{\langle p_j^*, p_j + N_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_j} p_j\rangle}, \quad j = 1, 2, \ldots, n_1,$$

$$\begin{pmatrix} \mathrm{D}_1 G_{n_1+1} \\ \vdots \\ \mathrm{D}_1 G_n \end{pmatrix} = K_z \begin{pmatrix} \bar{p}^*_{n_1+1}\left(\sum_{k=n_1+1}^{n} [N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)] p_k G_k\right) \\ \vdots \\ \bar{p}^*_{n}\left(\sum_{k=n_1+1}^{n} [N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)] p_k G_k\right) \end{pmatrix}. \tag{32}$$

Then, the particular solution of u_2 is obtained from (30) as

$$u_2 = \sum_{j=1}^{n_1} \zeta_{1j}\mathrm{e}^{\mathrm{i}\omega_j T_0} + \sum_{j=1}^{n_1} \zeta_{2j}\mathrm{e}^{2\mathrm{i}\omega_j T_0} + c.c. + \zeta_0, \tag{33}$$

where $\zeta_0, \zeta_{1j}, \zeta_{2j} \in \mathrm{C}^m$, and $\zeta_{1j} = \zeta_{1j}(\alpha_\epsilon)$, indicating that ζ_{1j} has relevance to the parameter vector α_ϵ, which actually represents the contribution from the unfolding terms, and *c.c.* stands for the complex conjugate of the preceding terms.

Further, from the ϵ^3-order LDE, we similarly obtain

$$\begin{aligned} \mathrm{D}_0 u_3 - N_0(\alpha_c)u_3 - N_1(\alpha_c)u_{3,1} &= -\mathrm{D}_2 u_1 - \mathrm{D}_1 u_2 + N_0^{(1)}(\alpha_\epsilon)u_2 + N_0^{(2)}(\alpha_\epsilon)u_1 \\ &\quad + N_1^{(2)}(\alpha_\epsilon)u_{1,1} - N_1^{(1)}(\alpha_\epsilon)\mathrm{D}_1 u_{1,1} \end{aligned}$$

$$+ N_1^{(1)}(\alpha_\epsilon)u_{2,1} - N_1(\alpha_c)\mathrm{D}_2u_{1,1} + \frac{1}{2}N_1(\alpha_c)\mathrm{D}_1^2u_{1,1} - N_1(\alpha_c)\mathrm{D}_1u_{2,1} + f_3(u_1, u_{1,1}, u_2, u_{2,1}, \mathrm{D}_1u_{1,1}), \tag{34}$$

where $\mathrm{D}_1^2 = \frac{\partial^2}{\partial T_1^2}$, and $f_3(u_1, u_{1,1}, u_2, u_{2,1}, \mathrm{D}_1u_{1,1})$ denotes the ϵ^3 order terms after substituting (26) into (23).

Substituting the solutions (28) and (33) into (34), we have

$$\mathrm{D}_0u_3 - N_0(\alpha_c)u_3 - N_1(\alpha_c)u_{3,1} = \sum_{j=1}^{n_1}\xi_{3j}\mathrm{e}^{\mathrm{i}\omega_jT_0} + c.c + \sum_{l=n_1+1}^{n}\xi_{3l} + g_3^{s,0} + g_3^u,$$

where $g_3^{s,0}$, consisting of the terms $G_{l1}G_{l2}G_{l3}e_l$ and $G_{l1}G_j\overline{G}_je_l$, where $l_1, l_2, l_3 = n_1+1, \ldots, n_1+n_2$; $j = 1, 2, \ldots, n_1$; $l = 2n_1+1, \ldots, 2n_1+n_2$, denotes the third-order terms in $f_3(u_1, u_{1,1}, u_2, u_{2,1}, \mathrm{D}_1u_{1,1})$ that produce constant terms, and g_3^u denotes the remaining third-order terms in $f_3(u_1, u_{1,1}, u_2, u_{2,1}, \mathrm{D}_1u_{1,1})$, and ξ_{3j} and ξ_{3l} are given by

$$\xi_{3j} = \rho_j - p_j\mathrm{D}_2G_j - N_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_j}p_j\mathrm{D}_2G_j + g_3^{s,h_j}, \quad j = 1, 2, \ldots, n_1,$$

$$\xi_{3l} = \rho_l - p_l\mathrm{D}_2G_l - N_1(\alpha_c)p_l\mathrm{D}_2G_l, \quad l = n_1+1, \ldots, n,$$

in which g_3^{s,h_j}, consisting of the terms $G_jG_{l1} \times G_{l2}e_{2j-1}$ and $G_jG_r\overline{G}_re_{2j-1}$, where $j, r = 1, 2, \ldots, n_1$; $l_1, l_2 = n_1+1, \ldots, n_1+n_2$, represents the third-order terms in $f_3(u_1, u_{1,1}, u_2, u_{2,1})$ that produce secular terms, and

$$\rho_j = -\mathrm{D}_1\zeta_{1j} + N_0^{(1)}(\alpha_\epsilon)\zeta_{1j} + N_0^{(2)}(\alpha_\epsilon)G_jp_j + N_1^{(2)}(\alpha_\epsilon)G_jp_j\mathrm{e}^{-\mathrm{i}\omega_j} - N_1^{(1)}(\alpha_\epsilon)\mathrm{D}_1G_jp_j\mathrm{e}^{-\mathrm{i}\omega_j} + N_1^{(1)}(\alpha_\epsilon)\zeta_{1j}\mathrm{e}^{-\mathrm{i}\omega_j} + \frac{1}{2}N_1(\alpha_c)\mathrm{D}_1^2G_jp_j\mathrm{e}^{-\mathrm{i}\omega_j} - N_1(\alpha_c)\mathrm{D}_1\zeta_{1j}\mathrm{e}^{-\mathrm{i}\omega_j}, \quad j = 1, 2, \ldots, n_1,$$

$$\rho_l = -\mathrm{D}_1\zeta_0 + N_0^{(1)}(\alpha_\epsilon)\zeta_0 + N_0^{(2)}(\alpha_\epsilon)G_lp_l + N_1^{(2)}(\alpha_\epsilon)G_lp_l - N_1^{(1)}(\alpha_\epsilon)\mathrm{D}_1G_lp_l + N_1^{(1)}(\alpha_\epsilon)\zeta_0 + \frac{1}{2}N_1(\alpha_c)\mathrm{D}_1^2G_lp_l - N_1(\alpha_c)\mathrm{D}_1\zeta_0, \quad l = n_1+1, \ldots, n.$$

Then, the solvability conditions are similarly given by $\langle p_j^*, \xi_{3j}\rangle = 0$ $(j = 1, 2, \ldots, n_1)$ and $\langle p_l^*, g_3^{s,0} + \sum_{k=n_1+1}^{n}\xi_{3l}\rangle = 0$ $(l = n_1+1, \ldots, n)$. Note that ρ_j and ρ_l contain the parameter terms, which are actually the unfolding terms. We ignore the higher-order terms in the expansion of parameters, and obtain the derivatives D_2G_j $(j = 1, 2, \ldots, n_1)$ and $(\mathrm{D}_2G_{n_1+1}, \ldots, \mathrm{D}_2G_n)^{\mathrm{T}}$ in the form of

$$\mathrm{D}_2G_j = \frac{\langle p_j^*, g_3^{s,h_j}\rangle}{\langle p_j^*, p_j + N_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_j}p_j\rangle}, \quad j = 1, 2, \ldots, n_1,$$
$$\begin{pmatrix}\mathrm{D}_2G_{n_1+1}\\ \vdots \\ \mathrm{D}_2G_n\end{pmatrix} = K_z\begin{pmatrix}\bar{p}_{n_1+1}^*g_3^{s,0}\\ \vdots \\ \bar{p}_n^*g_3^{s,0}\end{pmatrix}, \tag{35}$$

where K_z is given in (31).

Finally, using the backwards scaling, $G_j \mapsto G_j/\epsilon$, yields the normal form of system (23) up to the third-order terms,

$$\dot{G} = \mathrm{D}_1G + \mathrm{D}_2G, \quad \text{where } G = (G_1, G_2, \ldots, G_n)^{\mathrm{T}}, \tag{36}$$

associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the MTS method.

Now, we apply the CMR method to compute the normal form of system (23), restricted to the center manifold, near the semisimple n_1-Hopf–n_2-zero critical point: $\alpha = \alpha_c$. Define

$$\eta(\theta) = \begin{cases} N_0(\alpha_c), & \text{for } \theta = 0,\\ 0, & \text{for } \theta \in (-1, 0),\\ -N_1(\alpha_c), & \text{for } \theta = -1.\end{cases}$$

Then, the linearized equation of (23) at the trivial equilibrium can be written as

$$\dot{u}(t) = L_cu_t,$$

with $L_c\phi = \int_{-1}^0 d\eta(\theta)\phi(\theta)$, $\forall\,\phi \in \mathrm{C} \triangleq \mathrm{C}([-1, 0], \mathrm{R}^m)$, and the bilinear form on $\mathrm{C}^* \times \mathrm{C}$ (here $*$ stands for adjoint) as

$$\begin{aligned}\langle\psi(s),\phi(\theta)\rangle\\ &= \psi(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\psi(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,\end{aligned} \tag{37}$$

in which $\phi \in \mathrm{C}$, $\psi \in \mathrm{C}^*$. Thus, the phase space C is decomposed by $\Lambda = \{\pm\mathrm{i}\omega_1, \pm\mathrm{i}\omega_2, \ldots, \pm\mathrm{i}\omega_{n_1}, \overbrace{0,\ldots,0}^{n_2}\}$, as $\mathrm{C} = P \oplus Q$, where $Q = \{\varphi \in \mathrm{C} : (\psi, \varphi) = 0, \text{for all } \psi \in P^*\}$, and the bases for P and its adjoint P^* are given by

$$\begin{aligned}\Phi(\theta) = (&\varphi_1(\theta), \overline{\varphi}_1(\theta), \varphi_2(\theta), \overline{\varphi}_2(\theta), \ldots,\\ &\varphi_{n_1}(\theta), \overline{\varphi}_{n_1}(\theta), \hat{\varphi}_{n_1+1}(\theta), \ldots, \hat{\varphi}_n(\theta))\end{aligned}$$

and

$$\begin{aligned}\Psi(s) = (&\psi_1(s), \overline{\psi}_1(s), \psi_2(s), \overline{\psi}_2(s), \ldots,\\ &\psi_{n_1}(s), \overline{\psi}_{n_1}(s), \hat{\psi}_{n_1+1}(s), \ldots, \hat{\psi}_n(s))^{\mathrm{T}},\end{aligned}$$

respectively, where $\varphi_j(\theta) = \varphi_j(0)\mathrm{e}^{\mathrm{i}\omega_j\theta}$, $\hat{\varphi}_l(\theta) \equiv \varphi_l$, for $\theta \in [-1, 0]$, and $\psi_j(s) = \psi_j(0)\mathrm{e}^{-\mathrm{i}\omega_j s}$, $\hat{\psi}_l(s) \equiv \psi_l$, for $s \in [0, 1]$, where $j = 1, 2, \ldots, n_1$; $l = n_1 + 1, \ldots, n$, and $\langle\Psi(s), \Phi(\theta)\rangle = \mathrm{I}$.

We use the same bifurcation parameters, given by $\alpha = \alpha_c + \alpha_\epsilon$, where $\alpha_\epsilon = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a perturbation parameter vector. Note here that there is no explicit ϵ in the perturbation parameter. Substituting these bifurcation parameters into N_0 and N_1, we have the following expansions in terms of α_ϵ:

$$\begin{aligned}N_0(\alpha) &= N_0(\alpha_c) + \alpha_\epsilon N_0'(\alpha_c) + \cdots\\ &\triangleq N_0(\alpha_c) + N_0^{(1)}(\alpha_\epsilon) + \cdots,\\ N_1(\alpha) &= N_1(\alpha_c) + \alpha_\epsilon N_1'(\alpha_c) + \cdots\\ &\triangleq N_1(\alpha_c) + N_1^{(1)}(\alpha_\epsilon) + \cdots.\end{aligned}$$

Then, Eq. (23) can be rewritten as

$$\begin{aligned}\dot{u}(t) = N_0(\alpha)u_t &+ N_1(\alpha)u_t(-1)\\ &+ f(u(t), u(t-1), \alpha_\epsilon).\end{aligned} \tag{38}$$

We now consider the enlarged phase space BC of functions from $[-1, 0]$ to R^m, which are continuous on $[-1, 0)$ with a possible jumping discontinuity at zero. This space can be identified as $\mathrm{C} \times \mathrm{R}^m$. Thus, its elements can be written in the form $\tilde{\varphi} = \varphi + X_0 c$, where $\varphi \in \mathrm{C}$, $c \in \mathrm{R}^m$ and X_0 is an $m \times m$ matrix-valued function, defined by $X_0(\theta) = 0$ for $\theta \in [-1, 0)$ and $X_0(0) = \mathrm{I}$. In the space BC, Eq. (38) becomes an abstract ODE, described by

$$\dot{w} = Aw + X_0 F(w, \alpha_\epsilon), \tag{39}$$

where $w \in \mathrm{C}$, and A is defined by

$$A : \mathrm{C}^1 \to \mathrm{BC}, \quad Aw = \dot{w} + X_0[L_0 w - \dot{w}(0)]$$

and

$$\begin{aligned}F(w, \alpha_\epsilon) = [&N_0(\alpha)w(0) + N_1(\alpha)w(-1)\\ &- N_0(\alpha_c)w(0) - N_1(\alpha_c)w(-1)]\\ &+ f(w, \alpha_\epsilon).\end{aligned}$$

Neglecting the higher-order terms in the expansion of the perturbation parameter, we obtain

$$\begin{aligned}F(w, \alpha_\epsilon) = N_0^{(1)}(\alpha_\epsilon)w(0) &+ N_1^{(1)}(\alpha_\epsilon)w(-1)\\ &+ f_2(w, 0) + f_3(w, 0) + \cdots.\end{aligned}$$

Further, introducing the continuous projection $\pi : \mathrm{BC} \mapsto P$, $\pi(\varphi + X_0 c) = \Phi[(\Psi, \varphi) + \Psi(0)c]$, we can decompose the enlarged phase space by $\Lambda = \{\pm\mathrm{i}\omega_1, \pm\mathrm{i}\omega_2, \ldots, \pm\mathrm{i}\omega_{n_1}, 0, \ldots, 0\}$ as $\mathrm{BC} = P \oplus \operatorname{Ker}\pi$, where $\operatorname{Ker}\pi = \{\varphi + X_0 c : \pi(\varphi + X_0 c) = 0\}$, denoting the Kernel under the projection π. Let $x = (x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_{n_1}, \overline{x}_{n_1}, x_{n_1+1}, \ldots, x_n)^{\mathrm{T}}$, $y \in Q^1 := Q \cap \mathrm{C}^1 \subset \operatorname{Ker}\pi$, and A_{Q^1} be the restriction of A as an operator from Q^1 to the Banach space $\operatorname{Ker}\pi$.

In addition, denote $w = \Phi x + y$. Then, Eq. (39) is decomposed into the form of

$$\begin{aligned}\dot{x} &= Bx + \Psi(0)F(\Phi x + y, \alpha_\epsilon),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1}y + (\mathrm{I} - \pi)X_0 F(\Phi x + y, \alpha_\epsilon),\end{aligned} \tag{40}$$

where $B = \mathrm{diag}\{\mathrm{i}\omega_1, -\mathrm{i}\omega_1, \mathrm{i}\omega_2, -\mathrm{i}\omega_2, \ldots, \mathrm{i}\omega_{n_1}, -\mathrm{i}\omega_{n_1}, 0, \ldots, 0\}$.

To find the normal form, we rewrite Eq. (40) in the series form,

$$\begin{aligned}\dot{x} &= Bx + \sum_{j\geq 2} f_j^1(x, y, \alpha_\epsilon),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1}y + \sum_{j\geq 2} f_j^2(x, y, \alpha_\epsilon).\end{aligned} \tag{41}$$

Remark 6. Here, we omit the coefficient $\frac{1}{j!}$ in (41) before $f_j^1(x, y, \alpha_\epsilon)$, which is for the consistence in comparing the two methods. The coefficient $\frac{1}{j!}$ is used in [Faria & Magalhães, 1995b], which does not affect our results and conclusion.

Let $V_j^{3n_1+2n_2}(X)$ denote the linear space of jth degree homogeneous polynomials in the $2n_1$ complex variables $x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_{n_1}, \overline{x}_{n_1}$, the n_2 real variables $x_{n_1+1}, \ldots, x_n$ as well as the real parameter vector α, with coefficients in space X. Further, let M_j $(j \geq 2)$ denote the operator defined in $V_j^{3n_1+2n_2}(\mathrm{C}^{3n_1+2n_2} \times \mathrm{Ker}\,\pi)$, with the values taken from the same space, by

$$\begin{aligned} M_j(p, h) &= (M_j^1 p, M_j^2 h), \\ (M_j^1 p)(x, \alpha_\epsilon) &= D_x p(x, \alpha_\epsilon) Bx - Bp(x, \alpha_\epsilon), \\ (M_j^2 h)(x) &= D_x h(x) Bx - A_{Q^1} h(x), \end{aligned} \tag{42}$$

where $p(x, \alpha_\epsilon) \in V_j^{3n_1+2n_2}(\mathrm{C}^{3n_1+2n_2})$, $h(x)(\theta) \in V_j^{3n_1+2n_2}(\mathrm{Ker}\,\pi)$.

The above decompositions can be denoted as

$$\begin{aligned} V_j^{3n_1+2n_2}(\mathrm{C}^{3n_1+2n_2}) &= \mathrm{Im}(M_j^1) \oplus \mathrm{Im}(M_j^1)^c \\ V_j^{3n_1+2n_2}(\mathrm{C}^{3n_1+2n_2}) &= \mathrm{Ker}(M_j^1) \oplus \mathrm{Ker}(M_j^1)^c \\ V_j^{3n_1+2n_2}(\mathrm{Ker}\pi) &= \mathrm{Im}(M_j^2) \oplus \mathrm{Im}(M_j^2)^c \\ V_j^{3n_1+2n_2}(Q^1) &= \mathrm{Ker}(M_j^2) \oplus \mathrm{Ker}(M_j^2)^c. \end{aligned}$$

Now, we denote the projections associated with the above decompositions of $V_j^{3n_1+2n_2}(\mathrm{C}^{2n_1+n_2}) \times V_j^{3n_1+2n_2}(\mathrm{Ker}\ \pi)$ over $\mathrm{Im}(M_j^1) \times \mathrm{Im}(M_j^2)$ and of $V_j^{3n_1+2n_2} \times V_j^{2n_1+n_2}(Q^1)$ over $\mathrm{Ker}(M_j^1)^c \times \mathrm{Ker} \times (M_j^2)^c$ by, respectively, $P_{I,j} = (P_{I,j}^1, P_{I,j}^2)$ and $P_{K,j} = (P_{K,j}^1, P_{K,j}^2)$. The right inverse of M_j with range defined by the spaces complementary to the kernels of M_j with range defined by the spaces complementary to the kernels of M_j^i $(i = 1, 2)$, namely $M_j^{-1} = ((M_j^1)^{-1}, (M_j^2)^{-1})$ with $M_j^{-1} \circ P_{I,j} \circ M_j = P_{K,j}$.

Then, the kth order $(k \geq 2)$ normal form, derived with a recursive procedure by computing the jth order terms $1 \leq j \leq k-1$ at each step, can be expressed as

$$\begin{aligned} \dot{x} &= Bx + \sum_{j=1}^{k-1} g_j^1(x, y, \alpha_\epsilon) + \tilde{f}_k^1(x, y, \alpha_\epsilon) + \cdots, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1} y + \sum_{j=1}^{k-1} g_j^2(x, y, \alpha_\epsilon) + \tilde{f}_k^2(x, y, \alpha_\epsilon) + \cdots, \end{aligned} \tag{43}$$

with $g_1^1(x, y, \alpha_\epsilon) = g_1^2(x, y, \alpha_\epsilon) = 0$. The kth order normal form of system (43) is derived from the $(k-1)$th order normal form through a transformation of variables

$$(x, y) \to (x, y) + U_k(x),$$

where $U_k = M_k^{-1} P_{I,k} f_k(x, 0, \alpha_\epsilon)$. Actually, g_k can be solved via $g_k = \tilde{f}_k - M_k U_k$.

Therefore, repeating the above iteration procedure for $k = 2, 3, \ldots$, we obtain the normal form restricted to the center manifold arising from (23) as

$$\dot{x} = Bx + \sum_{j \geq 2} g_j^1(x, 0, \alpha_\epsilon), \tag{44}$$

associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the CMR method.

Next, we compare the two normal forms derived by using the MTS and CMR methods. Note that in the MTS method, the first-order linear solution of system (23) is u_1, given in (28), while in the CMR method, the linear solution on the center manifold is expressed by $\Phi(\theta)x(t)$, that is,

$$\begin{aligned} \Phi(\theta)x(t) = &\sum_{j=1}^{n_1} \varphi_j(0) \mathrm{e}^{\mathrm{i}\omega_j(t+\theta)} x_j(t) \\ &+ \sum_{j=1}^{n_1} \overline{\varphi}_j(0) \mathrm{e}^{-\mathrm{i}\omega_j(t+\theta)} \overline{x}_j(t) \\ &+ \sum_{l=n_1+1}^{n} \varphi_l x_l(t), \quad \theta \in [-1, 0]. \end{aligned} \tag{45}$$

In fact, if we choose $\varphi_j(0) = p_j$, $\psi_j(0) = K_j \overline{p}_j^{*\mathrm{T}}$ $(j = 1, 2, \ldots, n_1)$, where $K_j = [1 + \mathrm{e}^{-\mathrm{i}\omega_j} \overline{p}_j^{*\mathrm{T}} N_1(\alpha_c) p_j]^{-1}$, and $\varphi_l = p_l$, $\Psi_z(0) \triangleq (\psi_{n_1+1}(0), \ldots, \psi_n(0))^{\mathrm{T}} = K_z(\overline{p}_{n_1+1}^*, \ldots, \overline{p}_n^*)^{\mathrm{T}}$, where K_z is given by (31), then both the inner products (25) and (37) are normalized. Note that $t + \theta$ in (45) corresponds to T_0 in (28), thus, neglecting the difference in the notations, the two linear solutions derived by the two methods are identical, that is, $\Phi(0)x$ and $\Phi(-1)x$ in the CMR method correspond to u_1 and $u_{1,1}$ in the MTS method, respectively.

In the CMR method, for the operator M_2^1, we may choose the decomposition $V_2^{3n_1+2n_2} \times (\mathrm{C}^{2n_1+n_2}) = \mathrm{Im}(M_2^1) \oplus \mathrm{Im}(M_2^1)^c$, where the complementary space $\mathrm{Im}(M_2^1)^c$ is spanned by $\alpha_k x_j e_{2j-1}$, $\alpha_k \bar{x}_j e_{2j}$, $\alpha_k x_{l_1} e_l$, $x_j x_{l_1} e_{2j-1}$, $\bar{x}_j x_{l_1} e_{2j}$, $x_j \bar{x}_j e_l$ and $x_{l_1} x_{l_2} e_l$, where $k = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n_1$; $l_1, l_2 = n_1 + 1, \ldots, n$; $l = 2n_1 + 1, \ldots, 2n_1 + n_2$, e_p $(p = 1, 2, \ldots, 2n_1 + n_2)$ is the pth unit vector, and α_k is the kth component of α_ϵ. Therefore, the second-order terms of the normal form are given by

$$\begin{aligned} g_2^1(x, \alpha_\epsilon) = (&g_{21}^1(x, \alpha_\epsilon), \ldots, g_{2n_1}^1(x, \alpha_\epsilon), \\ &g_{2(n_1+1)}^1(x, \alpha_\epsilon), \ldots, g_{2n}^1(x, \alpha_\epsilon))^{\mathrm{T}}, \end{aligned}$$

$$\begin{aligned} g_{2j}^1(x, \alpha_\epsilon) &= \psi_j(0)[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)\mathrm{e}^{-\mathrm{i}\omega_j}] \\ &\quad \times \varphi_j(0) x_j, \quad j = 1, 2, \ldots, n_1, \\ g_{2l}^1(x, \alpha_\epsilon) &= \psi_l[N_0(\alpha_\epsilon) + N_1(\alpha_\epsilon)] \\ &\quad \times \sum_{k=n_1+1}^{n} \varphi_k x_k, \quad l = n_1 + 1, \ldots, n, \end{aligned} \tag{46}$$

where $N_0^{(1)}(\alpha_\epsilon)$ and $N_1^{(1)}(\alpha_\epsilon)$ are the first-order approximations in the parameter α_ϵ, and $g_2^1(x, 0) = 0$ due to the assumption that the second-order terms in the normal form vanish at the critical point. In order to compare the two normal forms, we can rewrite (32) for the MTS method as

$$\begin{aligned} \mathrm{D}_1 G_j &= K_j \langle p_j^*, (N_0^{(1)}(\alpha_\epsilon) p_j + N_1^{(1)}(\alpha_\epsilon)\mathrm{e}^{-\mathrm{i}\omega_j} p_j) G_j \rangle \\ &= \psi_j(0)[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)\mathrm{e}^{-\mathrm{i}\omega_j}]\varphi_j(0) G_j, \quad j = 1, 2, \ldots, n_1, \\ \begin{pmatrix} \mathrm{D}_1 G_{n_1+1} \\ \vdots \\ \mathrm{D}_1 G_n \end{pmatrix} &= K_z \begin{pmatrix} \bar{p}_{n_1+1}^* \left(\displaystyle\sum_{k=n_1+1}^{n} [N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)] p_k G_k \right) \\ \vdots \\ \bar{p}_n^* \left(\displaystyle\sum_{k=n_1+1}^{n} [N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)] p_k G_k \right) \end{pmatrix} \\ &= \Psi_z(0)(N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)) \sum_{k=n_1+1}^{n} \varphi_k(0) G_k. \end{aligned} \tag{47}$$

Note that x_j $(j = 1, 2, \ldots, n)$, used to represent the normal form in the CMR method, corresponds to G_j, used to denote the normal form in the MTS method. Thus, neglecting the difference in the notations, the two equations in (47) are identical to the last two equations in (46), implying that the normal forms obtained using the MTS method and the CMR method are actually identical up to second order.

Next, we consider the third-order terms of the normal form. Since we only consider the linear approximation of parameters in the CMR method, we ignore the higher-order (starting from the second order) approximations in the parameter α for the MTS method. By using $M_2^2 h_2(x, \alpha_\epsilon) = (\mathrm{I} - \pi) X_0 f_2(\Phi x)$, we obtain

$$\begin{aligned} &D_x h_2(x, \alpha_\epsilon)(\theta) B x - \dot{h}_2(x, \alpha_\epsilon)(\theta) \\ &\quad + X_0[\dot{h}_2(x, \alpha_\epsilon)(0) - L_c(h_2(x, \alpha_\epsilon)(\theta))] \\ &\quad = [X_0 - \Phi\Psi(0)] f_2(\Phi x), \end{aligned}$$

which can be written as

$$\begin{aligned} &D_x h_2(x, \alpha_\epsilon)(\theta) B x - \dot{h}_2(x, \alpha_\epsilon)(\theta) = -\Phi\Psi(0) f_2(\Phi x), \\ &\dot{h}_2(x, \alpha_\epsilon)(0) - L_c(h_2(x, \alpha_\epsilon)(\theta)) = f_2(\Phi x). \end{aligned} \tag{48}$$

Neglecting the higher-order terms in the expansion of the perturbation parameter, $h_2(x, 0)(\theta)$ has the following form:

$$h_2(x, 0)(\theta) = \sum_{|q|=2} h_{2,q}(\theta) x^q.$$

Since we have neglected the higher-order terms in the expansion of the perturbation parameter in the CMR method from the third-order terms, we will also neglect the higher-order terms in the expansion of the perturbation parameter in the MTS method from the third-order terms, Eq. (29) becomes

$$\begin{aligned} \mathrm{D}_0 u_2 - N_0(\alpha_c)u_2 - N_1(\alpha_c)u_{2,1} \\ = f_2(u_1, u_{1,1}). \end{aligned} \tag{49}$$

We have shown that $\Phi(0)x$ and $\Phi(-1)x$ in the CMR method correspond to u_1 and $u_{1,1}$ in the MTS method, respectively. Next, we prove that the second-order solutions in the CMR method $\Phi(0)U_2^1 + h_2(0)$ and $\Phi(-1)U_2^1 + h_2(-1)$ correspond to the second-order solutions in the MTS method $u_2(T_0, T_1, \ldots)$ and $u_2(T_0 - 1, T_1, \ldots)$, respectively. In fact,

$$\begin{aligned} &\left.\frac{\mathrm{d}\Phi(\theta)U_2^1(x)}{\mathrm{d}\theta}\right|_{\theta=0} \\ &\quad = L_c(\Phi)U_2^1 = [N_0(\alpha_c)\Phi(0) + N_1(\alpha_c)\Phi(-1)]U_2^1, \\ &\left.\frac{\mathrm{d}h_2(x,0)(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} \\ &\quad = N_0(\alpha_c)h_2(x)(0) + N_1(\alpha_c)h_2(x)(-1) \\ &\qquad + f_2(\Phi(0)x, \Phi(-1)x). \end{aligned}$$

Denote $\tilde{u}_2(\theta) = \Phi(\theta)U_2^1 + h_2(\theta)$ and $\tilde{u}_{2t}(\theta) = \tilde{u}_2(t+\theta)$, then,

$$\begin{aligned} \left.\frac{\mathrm{d}\tilde{u}_{2t}(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} &= N_0(\alpha_c)\tilde{u}_{2t}(0) + N_1(\alpha_c)\tilde{u}_{2t}(-1) \\ &\quad + f_2(\Phi(0)x(t), \Phi(-1)x(t)). \end{aligned}$$

Noting that

$$\begin{aligned} \left.\frac{\mathrm{d}\tilde{u}_{2t}(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} &= \left.\frac{\mathrm{d}\tilde{u}_2(t+\theta)}{\mathrm{d}\theta}\right|_{\theta=0} \\ &= \left.\frac{\mathrm{d}\tilde{u}_2(t+\theta)}{\mathrm{d}(t+\theta)}\right|_{\theta=0} \xrightarrow{t+\theta=T} \left.\frac{\mathrm{d}\tilde{u}_2(T)}{\mathrm{d}(T)}\right|_{T=t}, \end{aligned}$$

we can rewrite Eq. (49) as

$$\begin{aligned} &\frac{\mathrm{d}u_2(T_0, \ldots)}{\mathrm{d}T_0} \\ &\quad = N_0(\alpha_c)u_2(T_0, \ldots) + N_1(\alpha_c)u_2(T_0 - 1, \ldots) \\ &\qquad + f_2(u_1, u_{1,1}). \end{aligned}$$

Thus, $\Phi(0)U_2^1 + h_2(0)$ and $\Phi(-1)U_2^1 + h_2(-1)$ in the CMR method correspond to $u_2(T_0, T_1, \ldots)$ and $u_2(T_0 - 1, T_1, \ldots)$ in the MTS method, respectively.

It should be pointed out here that in the CMR method, the delay is taken by the function $h_{2,q}(\theta)$ in θ, while in the MTS method, taken by $T_0 = t + \theta$ in $\mathrm{e}^{(\lambda_j + \lambda_k)T_0}$. A big difference between the CMR method and the MTS method has been revealed: finding $h_{2,q}(\theta)$ needs solving of Eq. (48) which is actually a partial differential equation with boundary conditions, while solving u_2 from Eq. (49) only needs solving algebraic equations. In fact, a complete solution for $h_{2,q}(\theta)$ is not necessary for the normal form computation, which only needs the values at the two bounded points: $h_{2,q}(0)$ and $h_{2,q}(-1)$. Thus, we only need to compare Eq. (49) with the second equation of Eq. (48) for the two methods. Since the MTS method does not define a transform in function form, but rather directly defines it in the algebraic form, with the delay involved in the exponential function, this greatly simplifies the computation. Moreover, it can be seen that the CMR method cannot deal with more than one delay, due to $h_2(\theta)$ taking the boundary values, while the MTS method does not have this limit. It should be also noted that although the CMR method can deal with fixed constant delays or the delays with their ratios to the maximum delay being constants [Faria, 2001], there are difficulties for the cases in which at least one of the delays is treated as a perturbation parameter. Unfortunately, in real applications, delays are usually treated as perturbation parameters.

For the MTS method, the third-order terms in Eq. (23) are given by $f_3(u_1, u_{1,1}, u_2, u_{2,1}, \mathrm{D}_1 u_{1,1})$, since $\mathrm{D}_1 u_{1,1}|_{\alpha_\epsilon = 0} = 0$ due to the assumption that the second-order terms vanish at $\alpha = \alpha_c$, neglecting the terms involving the parameter, $f_3(u_1, u_{1,1}, u_2, u_{2,1}, \mathrm{D}_1 u_{1,1})$ can be rewritten as $f_3(u_1, u_{1,1}, u_2, u_{2,1})$. For the CMR method, the third-order terms in the first equation of (41) are written by $f_3^1(x, y) = f_3^1(\Phi x + \Phi U_2^1 + h_2) = \Psi(0) f_3(\Phi x + \Phi U_2^1 + h_2)$, which has the same form with the third-order terms, $\Psi(0) f_3(u_1, u_{1,1}, u_2, u_{2,1})$, derived by the MTS method using the solvability conditions. In fact,

$$\begin{aligned} &f_3^1(\Phi x + \Phi U_2^1 + h_2) \\ &\quad = \Psi(0)\Bigg[f_3(\Phi(0)x, \Phi(-1)x) \\ &\qquad + \frac{\partial f_2(\Phi(0)x, \Phi(-1)x)}{\partial(\Phi(0)x)}(\Phi(0)U_2^1 + h_2(x)(0)) \end{aligned}$$

$$+\frac{\partial f_2(\Phi(0)x,\Phi(-1)x)}{\partial(\Phi(-1)x)} \times(\Phi(-1)U_2^1+h_2(x)(-1))\Bigg],$$

$$\begin{aligned}\Psi(0)f_3(u_1,u_{1,1},u_2,u_{2,1}) &= \Psi(0)(f_3(u_1,u_{1,1}) + f_2(u_1+u_2,u_{1,1}+u_{2,1}))\\ &= \Psi(0)\left[f_3(u_1,u_{1,1})+\frac{\partial f_2(u_1,u_{1,1})}{\partial u_1}u_2 + \frac{\partial f_2(u_1,u_{1,1})}{\partial u_{1,1}}u_{2,1}\right].\end{aligned}$$

Thus, for the CMR method, the third-order term $f_3^1(\Phi x+\Phi U_2^1+h_2)$ has the same form as the third-order term $\Psi(0)f_3(u_1,u_{1,1},u_2,u_{2,1})$ in the MTS method.

The third-order terms of normal form given by (35) for the MTS method, taking only the linear approximation of parameters, can be written as

$$\begin{aligned}D_2G_j &= K_j\langle p_j^*, g_3^{s,h_j}\rangle\\ &= \psi_j(0)g_3^{s,h_j}, \quad j=1,2,\ldots,n_1,\\ \begin{pmatrix}D_2G_{n_1+1}\\ \vdots\\ D_2G_n\end{pmatrix} &= K_z\begin{pmatrix}\bar{p}_{n_1+1}^* g_3^{s,0}\\ \vdots\\ \bar{p}_n^* g_3^{s,0}\end{pmatrix}\\ &= \Psi_z(0)g_3^{s,0}.\end{aligned} \tag{50}$$

Note that g_3^{s,h_j} $(j=1,2,\ldots,n_1)$ contains the terms $G_jG_{l_1}G_{l_2}e_{2j-1}$ and $G_jG_r\overline{G}_re_{2j-1}$, and $g_3^{s,0}$ contains the terms $G_{l_1}G_{l_2}G_{l_3}e_l$ and $G_{l_1}G_r\overline{G}_re_l$, where $l_1,l_2,l_3=n_1+1,\ldots,n_1+n_2$; $l=2n_1+1,\ldots,2n_1+n_2$; $r=1,2,\ldots,n_1$.

Next, for operator M_3^1, we may choose the decomposition $V_3^{2n_1+n_2}(\mathrm{C}^m)=\mathrm{Im}(M_k^1)\oplus\mathrm{Im}(M_k^1)^c$ with the complementary space $\mathrm{Im}(M_3^1)^c$ spanned by $x_jx_{l_1}x_{l_2}e_{2j-1}$, $\bar{x}_jx_{l_1}x_{l_2}e_{2j}$, $x_jx_r\bar{x}_re_{2j-1}$, $\bar{x}_jx_r\bar{x}_re_{2j}$, $x_{l_1}x_{l_2}x_{l_3}e_l$ and $x_{l_1}x_r\bar{x}_re_l$, where $j=1,2,\ldots,n_1$; $l=2n_1+1,\ldots,2n_1+n_2$; $l_1,l_2,l_3=n_1+1,\ldots,n$; $r=1,2,\ldots,n_1$, and e_k $(k=1,2,\ldots,2n_1+n_2)$ is the kth unit vector, and $V_3^{2n_1+n_2}(\mathrm{C}^m)$ represents the linear space of the third-degree homogeneous polynomials in the $2n_1+n_2$ variables $(x_1,\bar{x}_1,x_2,\bar{x}_2,\ldots,x_{n_1},\bar{x}_{n_1},x_{n_1+1},\ldots,x_n)$ with coefficients in C^m. Therefore, the third-order terms of the normal form are given by

$$\begin{aligned}g_3^1(x,0) &= (g_{31}^1(x,0),\bar{g}_{31}^1(x,0),\ldots,g_{3n_1}^1(x,0),\\ &\qquad \bar{g}_{3n_1}^1(x,0),g_{3(2n_1+1)}^1(x,0),\ldots,\\ &\qquad g_{3(2n_1+n_2)}^1(x,0))^{\mathrm{T}},\\ g_{3j}^1(x,0) &= \psi_j(0)\hat{f}_{3j}, \quad j=1,2,\ldots,n_1,\\ \begin{pmatrix}g_{3(2n_1+1)}^1(x,0)\\ \vdots\\ g_{3(2n_1+n_2)}^1(x,0)\end{pmatrix} &= \Psi_z(0)\hat{f}_{3z},\end{aligned} \tag{51}$$

where $\hat{f}_{3j}$ represents all the terms expressed in $x_jx_{l_1}x_{l_2}e_{2j-1}$ and $x_jx_r\bar{x}_re_{2j-1}$, and $\hat{f}_{3z}$ denotes all the terms expressed in $x_{l_1}x_{l_2}x_{l_3}e_l$ and $x_{l_1}x_r\bar{x}_re_l$, and the index notations are the same as that used for the MTS method. Thus, if we treat x_j and G_j $(j=1,2,\ldots,n)$ just as two different notations, then g_3^{s,h_j} and $g_3^{s,0}$ in (50) have the same forms as that of $\hat{f}_{3j}$ and $\hat{f}_{3z}$ in (51), respectively. Therefore, the third-order normal forms derived by using the two methods are identical.

This completes the proof of Theorem 2. ■

In order to apply Theorem 2, first we need to compute the second-order normal form to check whether or not its part evaluated at the critical point equals zero. In the following, we give two useful results which can be used in applications to justify if this condition is satisfied.

Corollary 3.1. *Assume that system* (*23*) *undergoes a semisimple* n_1*-Hopf–*n_2*-zero* ($n_1\geq 1, n_2\geq 0, n=n_1+n_2\geq 1$) *bifurcation from the trivial equilibrium at the critical point, and all characteristic roots have nonpositive real part. If system* (*23*) *does not contain second-order terms, then the normal forms associated with the semisimple* n_1*-Hopf–*n_2*-zero singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Corollary 3.2. *Assume that system* (*23*) *undergoes a semisimple* n_1*-Hopf* ($n_1\geq 1$) *bifurcation from the trivial equilibrium at the critical point, and all characteristic roots have nonpositive real part. Then, the normal forms associated with the semisimple* n_1*-Hopf singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Remark 7

(a) In order to apply the MTS method, we have assumed that there is at least one pair of purely imaginary eigenvalues for system (23) at the critical point: $\alpha = \alpha_c$, i.e. $n_1 \geq 1$. In fact, if $n_1 = 0$, the normal form is the same as that of the abstract ODE in BC space.

(b) Since for any $w \in \mathrm{C} = P \oplus Q$, the formula $w = \Phi x + y_t$ holds, where $x = (x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_{n_1}, \overline{x}_{n_1}, x_{n_1+1}, \ldots, x_n)$, $\Phi x \in P$ and $y_t \in Q$. Thus, $\pi(w) = x_1\varphi_1 + \overline{x}_1\overline{\varphi}_1 + x_2\varphi_2 + \overline{x}_2\overline{\varphi}_2 + \cdots + x_{n_1}\varphi_{n_1} + \overline{x}_{n_1}\overline{\varphi}_{n_1} + x_{n_1+1}\varphi_{n_1+1} + \cdots + x_n\varphi_n$, implying that the construction of the project π in the CMR method is to generate the solution as a linear combination of the bases. On the other hand, in the MTS method, the expression (28) for the linear solution u_1 is indeed a linear combination of the bases. So from the view point of computation, the MTS method can be considered as a simple realization of the CMR method.

(c) From the proof of Theorem 2, it is seen that in the MTS method, it is assumed that there does not exist unstable manifold, and the two steps involved in using center manifold theory and normal form theory are combined into one unified step to obtain the normal form and nonlinear transformation simultaneously. Thus, a simpler system is directly obtained by eliminating the secular terms, compared to the CMR method for which the computation of the terms are expanded on the bases of $\mathrm{Im}(M_j)^c$. Although the CMR method can be used to deal with DDEs which involve unstable manifold [Faria & Magalhães, 1995b], the normal forms of such systems are not interesting since the solutions would quickly evolve outside of the local region where the normal forms are applicable.

(d) The characteristic equation (24) has n_1 pairs of purely imaginary roots $\pm i\omega_j$ $(j = 1, 2, \ldots, n_1)$. When $n_1 \geq 2$, a possible n_1-Hopf bifurcation with the ratio $\omega_1 : \omega_2 : \cdots : \omega_{n_1}$ appears. If there exist $m_j \in \mathrm{Z}$, $j = 1, 2, \ldots, n_1$, with at least two nonzero, such that $\sum_{j=1}^{n_1} m_j\omega_j = 0$, then n_1-Hopf bifurcation is called resonant; otherwise, it is called nonresonant. From the proof of Theorem 2, it is easily seen that both the MTS and CMR methods can deal with resonant and nonresonant cases, without altering the equivalence of the two normal forms, derived by the two methods.

(e) By a comparison between the MTS and CMR methods, we can see that when dealing with DDEs the MTS method, unlike the CMR method which involves solving differential equations, only involves algebraic manipulations with explicit algebraic formulas and simple procedure, making it easier to implement them in symbolic computation. In particular, when more than one discrete delay is involved in DDEs, the MTS method can be directly extended to consider such cases, while the CMR method has difficulty to deal with if at least one of the delays is treated as a perturbation parameter, which is usually the case in applications. Although two discrete delays have been considered in a DDE using the CMR method, it is assumed that the ratio of the two delays is fixed to be a constant and thus an equivalent single delay is actually considered [Yuan & Wei, 2007]. Therefore, the MTS method is simpler than the CMR method in computation. It should be noted however that the CMR method can deal with nonsemisimple cases, which is still open for the MTS method.

In the following three sections, we will prove the equivalence of the MTS and CMR methods for the NFDE and PFDE systems, as well as for the DDEs, NFDEs and PFDEs with distributed delays. Since some parts of the proofs are similar to that for DDEs (see the proof of Theorem 2), we will skip some detailed steps whenever possible.

4. Equivalence of the MTS and CMR Methods for NFDEs

In this section, we consider neutral functional differential equations (NFDE) or neutral delay differential equations (NDDE). The CMR method associated with NFDEs used in this paper is based on [Wang & Wei, 2008].

The MTS method can be used to study more general NFDEs with multiple delays,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t) + \sum_{j=1}^{p} M_j(\alpha)u(t-\tau_j)\right] = N_0(\alpha)u(t) + \sum_{j=1}^{p} N_j(\alpha)u(t-\tau_j)$$

$$+F(u(t), u(t-\tau_1), \ldots, u(t-\tau_p), \alpha)$$
$$+G(u(t-\tau_1), \ldots, u(t-\tau_p),$$
$$\dot{u}(t-\tau_1), \ldots, \dot{u}(t-\tau_p), \alpha). \tag{52}$$

Given that the CMR method has limitation to deal with DDEs [see Remark 7(e)], here we only consider the NFDEs with single delay for a comparison with the MTS method. Thus, without loss of generality, we shall use the following NFDE in this section for comparing the MTS and CMR methods,

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t) + M_1(\alpha)u(t-1)]$$
$$= N_0(\alpha)u(t) + N_1(\alpha)u(t-1)$$
$$+F(u(t), u(t-1), \alpha)$$
$$+G(u(t-1), \dot{u}(t-1), \alpha). \tag{53}$$

The characteristic equation of (53), evaluated at the trivial equilibrium $u = 0$, is given by

$$\det \Delta(\lambda) = 0, \quad \text{where}$$
$$\Delta(\lambda) = \lambda \mathrm{I} + \lambda M_1 \mathrm{e}^{-\lambda} - N_0 - N_1 \mathrm{e}^{-\lambda}, \tag{54}$$

with I as the $m \times m$ identity matrix. For the NFDE system (53), we have the following result.

Theorem 3. *Assume that system (53) undergoes a semisimple n_1-Hopf–n_2-zero ($n_1 \geq 1$, $n_2 \geq 0$, $n = n_1 + n_2 \geq 1$) bifurcation from the trivial equilibrium at the critical point, defined by $\alpha = \alpha_c$, and the characteristic equation (54) has n_1 pairs of purely imaginary roots $\pm \mathrm{i}\omega_j$ ($j = 1, 2, \ldots, n_1$) and n_2 zero roots, and all other roots have negative real part. If the second-order terms in the normal form vanish at $\alpha = \alpha_c$, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Proof. Similar to the proof for Theorem 2 in the previous section for general DDEs, we define the characteristic matrix $\Delta(\lambda)$ of (53) as $\Delta_c(\lambda)$ at the critical point, $\alpha = \alpha_c$, and denote $\Delta_c^*(\lambda)$ the adjoint matrix of $\Delta_c(\lambda)$. Then, let p_j ($j = 1, 2, \ldots, n_1$) and p_l ($l = n_1 + 1, \ldots, n$) be the eigenfunctions of $\Delta_c(\lambda)$ corresponding to the eigenvalues $\mathrm{i}\omega_j$ and 0, respectively; and p_j^* ($j = 1, 2, \ldots, n_1$) and p_l^* ($l = n_1 + 1, \ldots, n$) be the normalized eigenfunctions of $\Delta_c^*(\lambda)$ corresponding to the eigenvalues $-\mathrm{i}\omega_j$ and 0, respectively, satisfying the inner product (25).

The perturbation for (53) is taken the same as before: $\alpha = \alpha_c + \epsilon\alpha_\epsilon$. Substituting it into M_1, N_0 and N_1, we have the following expansions in terms of ϵ,

$$M_1(\alpha) = M_1(\alpha_c) + \epsilon M_1^{(1)}(\alpha_\epsilon) + \epsilon^2 M_1^{(2)}(\alpha_\epsilon) + \cdots,$$
$$N_0(\alpha) = N_0(\alpha_c) + \epsilon N_0^{(1)}(\alpha_\epsilon) + \epsilon^2 N_0^{(2)}(\alpha_\epsilon) + \cdots,$$
$$N_1(\alpha) = N_1(\alpha_c) + \epsilon N_1^{(1)}(\alpha_\epsilon) + \epsilon^2 N_1^{(2)}(\alpha_\epsilon) + \cdots,$$

where $M_1(\alpha_c)$, $N_0(\alpha_c)$ and $N_1(\alpha_c)$ are the values of M_1, N_0 and N_1 evaluated at the critical point, $\alpha = \alpha_c$.

With the MTS method, suppose the solution of (53) is given by

$$u(t) = \epsilon u_1(T_0, T_1, T_2, \ldots) + \epsilon^2 u_2(T_0, T_1, T_2, \ldots)$$
$$+ \epsilon^3 u_3(T_0, T_1, T_2, \ldots) + \cdots, \tag{55}$$

which, together with the multiple time scales (12), is substituted into (53) and then balancing the coefficients of ϵ^j, $j = 1, 2, \ldots$ yields a set of ordered linear differential equations (LDEs).

For the ϵ^1-order LDE, we have

$$\mathrm{D}_0 u_1 + M_1(\alpha_c)\mathrm{D}_0 u_{1,1}$$
$$- N_0(\alpha_c)u_1 - N_1(\alpha_c)u_{1,1} = 0, \tag{56}$$

where $u_{1,1} = u_1(T_0 - 1, T_1, T_2, \ldots)$. Since $\pm\mathrm{i}\omega_j$ ($j = 1, 2, \ldots, n_1$) and zero (with multiplicity n_2) are the eigenvalues of the linear part of (53), the linear solution of (56) restricted to the center subspace can be expressed in the form of (with the same reason as for the ODEs and DDEs)

$$u_1(T_0, T_1, T_2, \ldots)$$
$$= \sum_{j=1}^{n_1} G_j(T_1, T_2, \ldots) p_j \mathrm{e}^{\mathrm{i}\omega_j T_0}$$
$$+ \sum_{j=1}^{n_1} \overline{G}_j(T_1, T_2, \ldots) \overline{p}_j \mathrm{e}^{-\mathrm{i}\omega_j T_0}$$
$$+ \sum_{l=n_1+1}^{n} G_l(T_1, T_2, \ldots) p_l. \tag{57}$$

Next, from the ϵ^2-order LDE, we obtain

$$
\begin{aligned}
&\mathrm{D}_0u_2 + M_1(\alpha_c)\mathrm{D}_0u_{2,1} - N_0(\alpha_c)u_2 - N_1(\alpha_c)u_{2,1}\\
&\quad = -\mathrm{D}_1u_1 - M_1^{(1)}(\alpha_\epsilon)\mathrm{D}_0u_{1,1} - M_1(\alpha_c)\mathrm{D}_1u_{1,1} + M_1(\alpha_c)\mathrm{D}_0\mathrm{D}_1u_{1,1} + N_0^{(1)}(\alpha_\epsilon)u_1\\
&\qquad + N_1^{(1)}(\alpha_\epsilon)u_{1,1} - N_1(\alpha_c)\mathrm{D}_1u_{1,1} + f_2(u_1, u_{1,1}),
\end{aligned}
\tag{58}
$$

where $u_{2,1} = u_2(T_0 - 1, T_1, T_2, \ldots)$, and $f_2(u_1, u_{1,1})$ represents the ϵ^2-order terms in (53). Substituting solution (57) into (58), and using the solvability conditions, we obtain D_1G_j $(j = 1, 2, \ldots, n_1)$ and $(\mathrm{D}_1G_{n_1+1}, \ldots, \mathrm{D}_1G_n)^{\mathrm{T}}$ as follows:

$$
\begin{aligned}
&\mathrm{D}_1G_j = K_j\langle p_j^*, -M_1^{(1)}(\alpha_\epsilon)G_jp_j\mathrm{i}\omega_j\mathrm{e}^{-\mathrm{i}\omega_j} + N_0^{(1)}(\alpha_\epsilon)G_jp_j + N_1^{(1)}(\alpha_\epsilon)\mathrm{e}^{-\mathrm{i}\omega_j}p_jG_j\rangle,\\
&\begin{pmatrix} \mathrm{D}_1G_{n_1+1} \\ \vdots \\ \mathrm{D}_1G_n \end{pmatrix} = K_z \begin{pmatrix} \bar{p}_{n_1+1}^*\left(\displaystyle\sum_{k=n_1+1}^{n}[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)]p_kG_k\right) \\ \vdots \\ \bar{p}_n^*\left(\displaystyle\sum_{k=n_1+1}^{n}[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon)]p_kG_k\right) \end{pmatrix},
\end{aligned}
\tag{59}
$$

where $K_j = [1 + \bar{p}_j^*M_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_j}p_j(1 - \mathrm{i}\omega_j) + \bar{p}_j^*N_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_j}p_j]^{-1}$, and the assumption that the second-order terms in the normal form vanish at the critical point has been used. K_z is assumed to be invertible, given by

$$
K_z = \begin{pmatrix} \bar{p}_{n_1+1}^*(\mathrm{I} + M_1(\alpha_c) + N_1(\alpha_c))p_{n_1+1} & \cdots & \bar{p}_{n_1+1}^*(\mathrm{I} + M_1(\alpha_c) + N_1(\alpha_c))p_n \\ \vdots & \cdots & \vdots \\ \bar{p}_n^*(\mathrm{I} + M_1(\alpha_c) + N_1(\alpha_c))p_{n_1+1} & \cdots & \bar{p}_n^*(\mathrm{I} + M_1(\alpha_c) + N_1(\alpha_c))p_n \end{pmatrix}^{-1}.
\tag{60}
$$

Further, from the ϵ^3-order LDE, we similarly obtain

$$
\begin{aligned}
&\mathrm{D}_0u_3 + M_1(\alpha_c)\mathrm{D}_0u_{3,1} - N_0(\alpha_c)u_3 - N_1(\alpha_c)u_{3,1}\\
&\quad = -\mathrm{D}_2u_1 - \mathrm{D}_1u_2 - M_1^{(2)}(\alpha_\epsilon)\mathrm{D}_0u_{1,1} - M_1^{(1)}(\alpha_\epsilon)\mathrm{D}_1u_{1,1} - M_1(\alpha_c)\mathrm{D}_2u_{1,1} + M_1^{(1)}(\alpha_\epsilon)\mathrm{D}_0\mathrm{D}_1u_{1,1}\\
&\qquad + M_1(\alpha_c)\mathrm{D}_1^2u_{1,1} + M_1(\alpha_c)\mathrm{D}_0\mathrm{D}_2u_{1,1} - \frac{1}{2}M_1(\alpha_c)\mathrm{D}_0\mathrm{D}_1^2u_{1,1} - M_1^{(1)}(\alpha_\epsilon)\mathrm{D}_0u_{2,1} - M_1(\alpha_c)\mathrm{D}_1u_{2,1}\\
&\qquad + M_1(\alpha_c)\mathrm{D}_0\mathrm{D}_1u_{2,1} + N_0^{(1)}(\alpha_\epsilon)u_2 + N_0^{(2)}(\alpha_\epsilon)u_1 + N_1^{(2)}(\alpha_\epsilon)u_{1,1} - N_1^{(1)}(\alpha_\epsilon)\mathrm{D}_1u_{1,1} + N_1^{(1)}(\alpha_\epsilon)u_{2,1}\\
&\qquad - N_1(\alpha_c)\mathrm{D}_2u_{1,1} + \frac{1}{2}N_1(\alpha_c)\mathrm{D}_1^2u_{1,1} - N_1(\alpha_c)\mathrm{D}_1u_{2,1} + f_3(u_1, u_{1,1}, u_2, u_{2,1}),
\end{aligned}
\tag{61}
$$

where $u_{3,1} = u_3(T_0 - 1, T_1, T_2, \ldots)$, and $f_3(u_1, u_{1,1}, u_2, u_{2,1})$ represents the ϵ^3-order terms in (53).

Neglecting the higher-order terms in the expansion of the perturbation parameter and solving the solvability conditions yields the derivatives D_2G_j $(j = 1, 2, \ldots, n_1)$ and $(\mathrm{D}_2G_{n_1+1}, \ldots, \mathrm{D}_2G_n)^{\mathrm{T}}$, given by

$$\mathrm{D}_2G_j = K_j\langle p_j^*, g_3^{s,h_j}\rangle, \quad j = 1, 2, \ldots, n_1,$$

$$\begin{pmatrix}\mathrm{D}_2G_{n_1+1}\\ \vdots \\ \mathrm{D}_2G_n\end{pmatrix} = K_z\begin{pmatrix}\overline{p}^*_{n_1+1}g_3^{s,0}\\ \vdots \\ \overline{p}^*_n g_3^{s,0}\end{pmatrix}, \tag{62}$$

where g_3^{s,h_j} and $g_3^{s,0}$ stand for the same notations as that used for the DDE systems.

Finally, by using the backwards scaling, $G_j \mapsto G_j/\epsilon$, we obtain the normal form up to third order for system (53),

$$\dot{G} = \mathrm{D}_1G + \mathrm{D}_2G,$$
$$\text{where } G = (G_1, G_2, \ldots, G_n)^{\mathrm{T}}, \tag{63}$$

associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the MTS method.

Now, we apply the CMR method to compute the normal form of (53) restricted to the center manifold near the semisimple n_1-Hopf–n_2-zero critical point: $\alpha = \alpha_c$. Define

$$\xi(\theta) = \begin{cases} M_1(\alpha_c), & \theta = -1,\\ 0, & \theta \in (-1, 0],\end{cases}$$

$$\eta(\theta) = \begin{cases} N_0(\alpha_c), & \theta = 0,\\ 0, & \theta \in (-1, 0),\\ -N_1(\alpha_c), & \theta = -1.\end{cases}$$

Then, the linearized equation of (53) at the trivial equilibrium is

$$\frac{\mathrm{d}}{\mathrm{d}t}[Dx_t] = L_c x_t,$$

satisfying $D\varphi = \varphi(0) - \int_{-1}^0 \mathrm{d}\xi(\theta)\varphi(\theta)$, $L_c\varphi = \int_{-1}^0 \mathrm{d}\eta(\theta)\varphi(\theta)$, $\forall\, \varphi \in \mathrm{C} = \mathrm{C}([-1, 0], \mathrm{R}^m)$, and the bilinear form on $\mathrm{C}^* \times \mathrm{C}$ ($*$ stands for adjoint) is

$$\begin{aligned}&\langle\psi(s), \varphi(\theta)\rangle\\ &\quad = \psi(0)\varphi(0)\\ &\qquad - \int_{-1}^0 \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\int_0^\varsigma \psi(s - \varsigma)\mathrm{d}\xi(\theta)\varphi(s)\mathrm{d}s\right]\bigg|_{\varsigma=\theta}\\ &\qquad - \int_{-1}^0\int_0^\theta \psi(s-\theta)\mathrm{d}\eta(\theta)\varphi(s)\mathrm{d}s,\end{aligned}$$

in which $\varphi \in \mathrm{C}$, $\psi \in \mathrm{C}^*$. Then, the phase space C is decomposed by $\Lambda = \{\pm\mathrm{i}\omega_1, \pm\mathrm{i}\omega_2, \ldots, \pm\mathrm{i}\omega_{n_1}, \overbrace{0, \ldots, 0}^{n_2}\}$, as $\mathrm{C} = P \oplus Q$, where $Q = \{\varphi \in \mathrm{C} : (\psi, \varphi) = 0, \text{for all } \psi \in P^*\}$, and the bases for P and its adjoint P^* are given by

$$\begin{aligned}\Phi(\theta) = (&\varphi_1(\theta), \overline{\varphi}_1(\theta), \varphi_2(\theta), \overline{\varphi}_2(\theta), \ldots,\\ &\varphi_{n_1}(\theta), \overline{\varphi}_{n_1}(\theta), \hat{\varphi}_{n_1+1}(\theta), \ldots, \hat{\varphi}_n(\theta))\end{aligned}$$

and

$$\begin{aligned}\Psi(s) = (&\psi_1(s), \overline{\psi}_1(s), \psi_2(s), \overline{\psi}_2(s), \ldots,\\ &\psi_{n_1}(s), \overline{\psi}_{n_1}(s), \hat{\psi}_{n_1+1}(s), \ldots, \hat{\psi}_n(s))^{\mathrm{T}},\end{aligned}$$

respectively, where $\varphi_j(\theta) = \varphi_j(0)\mathrm{e}^{\mathrm{i}\omega_j\theta}$, $\psi_j(s) = \psi_j(0)\mathrm{e}^{-\mathrm{i}\omega_j s}$, $\hat{\varphi}_l(\theta) \equiv \varphi_l$ for $\theta \in [-1, 0]$, $\hat{\psi}_l(s) \equiv \psi_l$ for $s \in [0, 1]$, where $j = 1, 2, \ldots, n_1$; $l = n_1 + 1, \ldots, n$, and $\langle\Psi(s), \Phi(\theta)\rangle = \mathrm{I}$.

In the enlarged space BC, (53) becomes an abstract ODE,

$$\frac{\mathrm{d}w_t}{\mathrm{d}t} = Aw_t + X_0\tilde{F}(w_t, \alpha_\epsilon), \tag{64}$$

where $w_t \in \mathrm{C}$, and A is defined by

$$A : \mathrm{C}^1 \to \mathrm{BC}, \quad Aw_t = w_t'(\theta) + X_0[L_c w_t - Dw_t']$$

and

$$\begin{aligned}\tilde{F}(w_t, \varepsilon) = [&N_0(\alpha)w_t(0) + N_1(\alpha)w_t(-1)\\ &- N_0(\alpha_c)w_t(0) - N_1(\alpha_c)w_t(-1)]x_t\\ &+ F(w_t, \alpha_\epsilon) + G(w_t, \dot{w}_t, \alpha_\epsilon).\end{aligned}$$

Denote $w = \Phi x + y$. Then, Eq. (64) is decomposed into

$$\begin{aligned}\dot{x} &= Bx + \Psi(0)\tilde{F}(\Phi x + y, \alpha_\epsilon),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1}y + (\mathrm{I} - \pi)X_0\tilde{F}(\Phi x + y, \alpha_\epsilon),\end{aligned} \tag{65}$$

where $B = \mathrm{diag}\{\mathrm{i}\omega_1, -\mathrm{i}\omega_1, \mathrm{i}\omega_2, -\mathrm{i}\omega_2, \ldots, \mathrm{i}\omega_{n_1}, -\mathrm{i}\omega_{n_1}, 0, \ldots, 0\}$.

The remaining part of deriving the normal form by using the CMR method is similar to that in the proof for Theorem 2 in the DDE case, and hence the details are omitted here.

Similar to the proof of Theorem 2, we only need to show that (i) choosing the basis for the linear space leads to the identical linear solutions in the center subspace; and (ii) the second-order terms u_2 and $\Phi(0)U_2^1 + h_2(0)$, $u_{2,1}$ and $\Phi(-1)U_2^1 + h_2(-1)$ are identical for the NFDE (53), respectively.

Actually, we can also choose $\varphi_j(0) = p_j$, $\psi_j(0) = K_j \bar{p}_j^{*\mathrm{T}}$ $(j = 1, 2, \ldots, n_1)$, $\varphi_l = p_l$, $\Psi_l = K_z(\bar{p}^*_{n_1+1}, \ldots, \bar{p}^*_n)^{\mathrm{T}}$ $(l = n_1 + 1, \ldots, n)$, where $K_j = [1 + \bar{p}_j^* M_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_j} p_j(1 - \mathrm{i}\omega_j) + \bar{p}_j^* N_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_j} \times p_j]^{-1}$, assuming that K_z is invertible, given by (60). Then, neglecting the difference in the notations shows that the linear solution in the center subspace obtained by using the MTS and CMR methods are identical.

For the CMR method, the transformation h_2 satisfies

$$\begin{aligned} D_x h_2(x)(\theta)Bx - \dot{h}_2(x)(\theta) + X_0[\dot{h}_2(x)(0) \\ + M_1(\alpha_c)\dot{h}_2(x)(-1) - L_c(h_2(x))] \\ = [X_0 - \Phi(\theta)\Psi(0)]f_2(\Phi(\theta)x), \end{aligned}$$

which can be written as

$$\begin{aligned} &D_x h_2(x)(\theta)Bx - \dot{h}_2(x)(\theta) \\ &\quad = -\Phi(\theta)\Psi(0)f_2(\Phi(\theta)x), \\ &\dot{h}_2(x)(0) + M_1(\alpha_c)\dot{h}_2(x)(-1) - L_c(h_2(x)) \\ &\quad = f_2(\Phi(\theta)x), \end{aligned} \tag{66}$$

where f_2 represents the second-order terms in (53).

We again ignore the higher-order terms in the expansion of parameter α, and thus the ϵ^2-order LDE for the MTS method becomes

$$\begin{aligned} &\mathrm{D}_0 u_2 + M_1(\alpha_c)\mathrm{D}_0 u_{2,1} - N_0(\alpha_c)u_2 - N_1(\alpha_c)u_{2,1} \\ &\quad = f_2(u_1, u_{1,1}). \end{aligned} \tag{67}$$

Similar to proving Theorem 2, we have

$$\begin{aligned} &\left.\frac{\mathrm{d}\Phi(\theta)U_2^1(x)}{\mathrm{d}\theta}\right|_{\theta=0} \\ &\quad = -M_1(\alpha_c)\Phi(-1)U_2^1(x) + L_c(\Phi)U_2^1 \\ &\quad = -M_1(\alpha_c)\Phi(-1)U_2^1(x) \\ &\qquad + [N_0(\alpha_c)\Phi(0) + N_1(\alpha_c)\Phi(-1)]U_2^1 \\ &\left.\frac{\mathrm{d}h_2(x,0)(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} \\ &\quad = -M_1(\alpha_c)\left.\frac{\mathrm{d}h_2(x,0)(\theta)}{\mathrm{d}\theta}\right|_{\theta=-1} \\ &\qquad + N_0(\alpha_c)h_2(x)(0) + N_1(\alpha_c)h_2(x)(-1) \\ &\qquad + f_2(\Phi(0)x, \Phi(-1)x). \end{aligned}$$

Similarly denoting $\tilde{u}_2(\theta) = \Phi(\theta)U_2^1 + h_2(\theta)$ and $\tilde{u}_{2t}(\theta) = \tilde{u}_2(t+\theta)$, we obtain

$$\begin{aligned} \left.\frac{\mathrm{d}\tilde{u}_{2t}(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} &= -M_1(\alpha_c)\tilde{u}_{2t}(-1) + N_0(\alpha_c)\tilde{u}_{2t}(0) \\ &\quad + N_1(\alpha_c)\tilde{u}_{2t}(-1) \\ &\quad + f_2(\Phi(0)x(t), \Phi(-1)x(t)) \\ &= \left.\frac{\mathrm{d}\tilde{u}_2(t+\theta)}{\mathrm{d}\theta}\right|_{\theta=0} \\ &= \left.\frac{\mathrm{d}\tilde{u}_2(t+\theta)}{\mathrm{d}(t+\theta)}\right|_{\theta=0} \xrightarrow{t+\theta=T} \left.\frac{\mathrm{d}\tilde{u}_2(T)}{\mathrm{d}(T)}\right|_{T=t}. \end{aligned}$$

Equation (67) can thus be rewritten as

$$\begin{aligned} \frac{\mathrm{d}u_2(T_0,\ldots)}{\mathrm{d}T_0} &= -M_1(\alpha_c)\frac{\mathrm{d}u_2(T_0-1,\ldots)}{\mathrm{d}T_0} \\ &\quad + N_0(\alpha_c)u_2(T_0,\ldots) \\ &\quad + N_1(\alpha_c)u_2(T_0-1,\ldots) \\ &\quad + f_2(u_1, u_{1,1}), \end{aligned}$$

which clearly shows that the corresponding second-order solutions u_2 and $\Phi(0)U_2^1 + h_2(0)$, $u_{2,1}$ and $\Phi(-1)U_2^1 + h_2(-1)$ are identical, which is the same as that for the DDE systems, as expected. The remaining part of the proof is similar to that for Theorem 2, and thus omitted for brevity. ■

Corollary 4.1. *Assume that system (53) undergoes a semisimple n_1-Hopf–n_2-zero $(n_1 \geq 1, n_2 \geq 0, n = n_1 + n_2 \geq 1)$ bifurcation from the trivial equilibrium at the critical point, and all characteristic roots have nonpositive real part. If system (53) does not contain second-order terms, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Corollary 4.2. *Assume that system (53) undergoes a semisimple n_1-Hopf $(n_1 \geq 1)$ bifurcation from the trivial equilibrium at the critical point, and all characteristic roots have nonpositive real part. Then, the normal forms associated with the semisimple n_1-Hopf singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.*

5. Equivalence of the MTS and CMR Methods for PFDEs

In this section, we prove the equivalence of the MTS and CMR methods for partial functional differential equations (PFDE).

General PFDE systems with an equilibrium point at the origin, can be written in the form of

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= K(\alpha)\Delta u(x,t) + L(\alpha)(u_t(x,t)) \\ &\quad + F(u_t(x,\cdot),\alpha), \quad t>0, \\ &\qquad u \in \mathrm{R}^m, \ \ x \in \mathrm{R}^p, \end{aligned} \tag{68}$$

where $u_t(x,\cdot) = u(x,t+\theta)$, $\forall\,\theta \in [-\tau,0]$, with τ being the maximum of delays in (68).

For convenience of the proof, we first introduce some notations taken from [Faria, 2000]. Denote $\Omega \subset \mathrm{R}^p$ an open set, X a Hilbert space of functions from $\overline{\Omega}$ to R^m with the inner product $\langle\cdot,\cdot\rangle$, and $\mathfrak{C} = \mathrm{C}([-\tau,0];\mathrm{X})$ ($\tau>0$) the Banach space of continuous maps from $[-\tau,0]$ to X with the sup norm. Then, the PFDE (68) can be written in an abstract form (i.e. in the phase space $\mathfrak{C}$ [Faria, 2000]),

$$\begin{aligned}\frac{du(t)}{dt} &= K(\alpha)\Delta u(t) + L(\alpha)(u_t) \\ &\quad + F(u_t,\alpha), \quad t>0, \end{aligned} \tag{69}$$

where domain $(\Delta) \subset \mathrm{X}$, α is a parameter vector with appropriate dimension, L is a bounded linear operator from $\mathfrak{C}$ to X, and $F:\mathfrak{C}\to\mathrm{X}$ is a C^∞ function (or C^k-smooth, $k\geq 2$, for which the normal form can be obtained up to kth order) with $F(0,\alpha) = \mathrm{D}F(0,\alpha) = 0$.

Further, it is assumed that for the linearized equation about the zero equilibrium, $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = K(\alpha)\Delta u(t) + L(\alpha)(u_t)$, the following hypotheses hold.

(H1) $K(\alpha)\Delta$ generates a C_0 semigroup $\{T(t)\}|_{t\geq 0}$ on X with $|T(t)| \leq M\mathrm{e}^{\omega t}$ (for some $M\geq 1$, $\omega\in\mathrm{R}$) for all $t\geq 0$, and $T(t)$ is a compact operator for $t>0$.

(H2) The eigenfunctions $\{\beta_q(x)\}_{q=1}^{\infty}$ of $K(\alpha)\Delta$, corresponding to eigenvalues $\{\mu_q\}_{q=1}^{\infty}$, form an orthonormal basis for X, with $\lim_{q\to\infty}\times\mu_q = -\infty$.

(H3) The subspaces $\mathfrak{B}_q := \{\langle v(\cdot),\beta_q\rangle\beta_q \,|\, v\in\mathfrak{C}\}$ of $\mathfrak{C}$ satisfy $L(\mathfrak{B}_q) \subset \mathrm{span}\{\beta_q\}$.

(H4) L can be extended to a bounded linear operator from BC to X, where $\mathrm{BC} = \{\psi:[-\tau,0]\to \mathrm{X}\,|\,\psi$ is continuous on $[-\tau,0), \exists\lim_{\theta\to 0^-}\times\psi(\theta)\in\mathrm{X}\}$, with the sup norm.

Using the decomposition of X by $\{\beta_q\}$ and Hypothesis (H3), we obtain a sequence of "characteristic" equations

$$\lambda\beta_q - \mu_q\beta_q - L(\mathrm{e}^{\lambda\cdot}\beta_q) = 0, \quad q\in\mathrm{N}, \tag{70}$$

and there exists an n_0 such that all solutions of (70) satisfy $\mathrm{Re}(\lambda)<0$ for $q>n_0$.

The MTS method can be used to study more general PFDEs with multiple delays, similar to that for DDEs and NFDEs. Here, we only consider the PFDEs with single delay since in general the CMR method can only be applied to consider single delay. Thus, without loss of generality, we shall use the following more explicit PFDE in this section to prove the equivalence of the MTS and CMR methods,

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= K(\alpha)\Delta u(x,t) + N_0(\alpha)u(x,t) \\ &\quad + N_1(\alpha)u(x,t-1) + F(u_t,\alpha). \end{aligned} \tag{71}$$

For the PFDE system (71), we have the following result.

Theorem 4. *Assume (H1)–(H4) hold, and system (71), associated with some eigenfunctions β_q, undergoes a semisimple n_1-Hopf–n_2-zero ($n_1\geq 1$, $n_2\geq 0, n=n_1+n_2\geq 1$) bifurcation from the space homogeneous trivial equilibrium at the critical point, defined by $\alpha=\alpha_c$, at which the characteristic equation (70) has n_1 pairs of purely imaginary roots $\pm\mathrm{i}\omega_j$ ($j=1,2,\ldots,n_1$) and n_2 zero roots, and all other roots have negative real part. If the second-order terms in the normal form vanish at $\alpha=\alpha_c$, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Remark 8. In Theorem 4, the eigenvalues $\pm\mathrm{i}\omega_j$ ($j=1,2,\ldots,n_1$) and zero (with multiplicity n_2) can be associated with different eigenfunctions β_q, or with a unique eigenfunction. We prove the general case with different eigenfunctions, while in practical applications, they are usually associated with a unique eigenfunction (i.e. unimode oscillation).

Proof. Define the characteristic matrix $\Delta(\lambda)_q$ of the linearized equation of (71) as $\Delta_c(\lambda)_q$ at the critical point, $\alpha = \alpha_c$, and denote $\Delta_c^*(\lambda)_q$ the adjoint matrix of $\Delta_c(\lambda)_q$. Assume that the characteristic equation (70), corresponding to the eigenfunction β_q, has k_q pairs of purely imaginary eigenvalues $\pm\mathrm{i}\omega_{q,1}, \ldots, \pm\mathrm{i}\omega_{q,k_q}$ and n_q zero eigenvalues. Let $p_{q,j}$ $(j = 1, 2, \ldots, k_q)$ and $p_{q,l}$ $(l = k_q + 1, \ldots, k_q + n_q)$ be the eigenfunctions of $\Delta_c(\lambda)_q$ corresponding to the eigenvalues $\mathrm{i}\omega_{q,j}$ $(j = 1, 2, \ldots, k_q)$ and 0, respectively; and $p^*_{q,j}$ $(j = 1, 2, \ldots, k_q)$ and $p^*_{q,l}$ $(l = k_q + 1, \ldots, k_q + n_q)$ be the normalized eigenfunctions of $\Delta_c^*(\lambda)_q$ corresponding to the eigenvalues $-\mathrm{i}\omega_{q,j}$ $(j = 1, 2, \ldots, k_q)$ and 0, respectively, with $\sum_{q=1}^{n_0} k_q = n_1$ and $\sum_{q=1}^{n_0} n_q = n_2$, satisfying the inner product

$$\langle p^*_{q,k}, p_{q,k}\rangle = \bar{p}^{*\mathrm{T}}_{q,k} p_{q,k} = 1, \quad q = 1, 2, \ldots, n_0; \quad k = 1, 2, \ldots, k_q + n_q. \quad (72)$$

The perturbation on the parameter is taken as the same as before: $\alpha = \alpha_c + \epsilon\alpha_\epsilon$, which is substituted into K, N_0 and N_1 to obtain the expansions in terms of ϵ:

$$K(\alpha) = K(\alpha_c) + \epsilon K^{(1)}(\alpha_\epsilon) + \epsilon^2 K^{(2)}(\alpha_\epsilon) + \cdots,$$

$$N_0(\alpha) = N_0(\alpha_c) + \epsilon N_0^{(1)}(\alpha_\epsilon) + \epsilon^2 N_0^{(2)}(\alpha_\epsilon) + \cdots,$$

$$N_1(\alpha) = N_1(\alpha_c) + \epsilon N_1^{(1)}(\alpha_\epsilon) + \epsilon^2 N_1^{(2)}(\alpha_\epsilon) + \cdots,$$

where $K(\alpha_c)$, $N_0(\alpha_c)$ and $N_1(\alpha_c)$ represent the values of K, N_0 and N_1 evaluated at the critical point, $\alpha = \alpha_c$.

In the following, we first show the procedure of the MTS method, and then that of the CMR method, and finally prove the equivalence of the normal forms obtained using the two methods.

With the MTS method, suppose the solution of (71) is given by

$$\begin{aligned} u(x,t) = {} & \epsilon u_1(x, T_0, T_1, T_2, \ldots) \\ & + \epsilon^2 u_2(x, T_0, T_1, T_2, \ldots) \\ & + \epsilon^3 u_3(x, T_0, T_1, T_2, \ldots) + \cdots. \end{aligned} \quad (73)$$

Thus, the derivatives with respect to $t \in \mathrm{R}_+$ and $x \in \mathrm{R}^p$ now become

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots \\ &= \mathrm{D}_0 + \epsilon\mathrm{D}_1 + \epsilon^2\mathrm{D}_2 + \cdots, \end{aligned} \quad (74)$$

$$\Delta u = \epsilon\Delta u_1 + \epsilon^2\Delta u_2 + \epsilon^3\Delta u_3 + \cdots.$$

Substituting (73), with the multiple time scales (74), into (71) and then balancing the coefficients of ϵ^j, $j = 1, 2, \ldots$ for the resulting equations yields a set of ordered linear differential equations (LDEs).

For the ϵ^1-order LDE, we have

$$\begin{aligned} &\mathrm{D}_0 u_1 - K(\alpha_c)\Delta u_1 - N_0(\alpha_c)u_1 \\ &\quad - N_1(\alpha_c)u_{1,1} = 0, \end{aligned} \quad (75)$$

where $u_{1,1} = u_1(x, T_0 - 1, T_1, T_2, \ldots)$. Noticing that $\pm\mathrm{i}\omega_{q,j}$ $(q = 1, 2, \ldots, n_0;\ j = 1, 2, \ldots, k_q)$ and zero (with multiplicity n_2) are the eigenvalues of the characteristic equation (70), with Hypotheses (H1)–(H4), we can express the linear solution of (75) in the center subspace in the form of

$$\begin{aligned} &u_1(x, T_0, T_1, T_2, \ldots) \\ &\quad = \sum_{q=1}^{n_0}\sum_{j=1}^{k_q} \beta_q(x) G_{q,j}(T_1, T_2, \ldots) p_{q,j} \mathrm{e}^{\mathrm{i}\omega_{q,j}T_0} \\ &\qquad + \sum_{q=1}^{n_0}\sum_{j=1}^{k_q} \bar{\beta}_q(x) \bar{G}_{q,j}(T_1, T_2, \ldots) \bar{p}_{q,j} \mathrm{e}^{-\mathrm{i}\omega_{q,j}T_0} \\ &\qquad + \sum_{q=1}^{n_0}\sum_{l=k_q+1}^{k_q+n_q} \beta_q(x) G_{q,l}(T_1, T_2, \ldots) p_{q,l}, \end{aligned} \quad (76)$$

where $G_{q,k} = \langle p^*_{q,k}, \langle u_1|_{T_0=0}, \beta_q\rangle\rangle, q = 1, 2, \ldots, n_0$; $k = 1, 2, \ldots, k_q + n_q$.

Next, from the ϵ^2-order LDE, we obtain

$$\begin{aligned} &\mathrm{D}_0 u_2 - K(\alpha_c)\Delta u_2 - N_0(\alpha_c)u_2 - N_1(\alpha_c)u_{2,1} \\ &\quad = -\mathrm{D}_1 u_1 + K^{(1)}(\alpha_\epsilon)\Delta u_1 + N_0^{(1)}(\alpha_\epsilon)u_1 \\ &\qquad + N_1^{(1)}(\alpha_\epsilon)u_{1,1} - N_1(\alpha_c)\mathrm{D}_1 u_{1,1} \\ &\qquad + f_2(u_1, u_{1,1}), \end{aligned} \quad (77)$$

where $u_{2,1} = u_2(x, T_0 - 1, T_1, T_2, \ldots)$, and $f_2(u_1, u_{1,1})$ represents the ϵ^2-order terms in (71) with multiple time scales. Substituting solution (76) into (77), we obtain

$$\begin{aligned} &\mathrm{D}_0 u_2 - K(\alpha_c)\Delta u_2 - N_0(\alpha_c)u_2 - N_1(\alpha_c)u_{2,1} \\ &\quad = \sum_{q=1}^{n_0}\left[\sum_{j=1}^{k_q}\chi^{(2)}_{q,j}\mathrm{e}^{\mathrm{i}\omega_{q,j}T_0} + \sum_{j=1}^{k_q}\bar{\chi}^{(2)}_{q,j}\mathrm{e}^{-\mathrm{i}\omega_{q,j}T_0}\right. \\ &\qquad \left. + \sum_{l=k_q+1}^{k_q+n_q}\chi^{(2)}_{q,l} + g^{s,0}_{q,2} + g^{u}_{q,2}\right], \end{aligned}$$

where

$$\chi_{q,j}^{(2)} = [-\beta_q(x)\mathrm{D}_1G_{q,j} + K^{(1)}(\alpha_\epsilon)\Delta\beta_q(x)G_{q,j} + N_0^{(1)}(\alpha_\epsilon)\beta_q(x)G_{q,j} + N_1^{(1)}(\alpha_\epsilon)\beta_q(x)G_{q,j}\mathrm{e}^{-\mathrm{i}\omega_{q,j}}$$
$$- N_1(\alpha_c)\beta_q(x)\mathrm{D}_1G_{q,j}\mathrm{e}^{-\mathrm{i}\omega_{q,j}}]p_{q,j} + g_{q,2}^{s,h_j}, \quad j = 1,2,\ldots,k_q,$$
$$\chi_{q,l}^{(2)} = [-\beta_q(x)\mathrm{D}_1G_{q,l} + K^{(1)}(\alpha_\epsilon)\Delta\beta_q(x)G_{q,l} + N_0^{(1)}(\alpha_\epsilon)\beta_q(x)G_{q,l} + N_1^{(1)}(\alpha_\epsilon)\beta_q(x)G_{q,l}$$
$$- N_1(\alpha_c)\beta_q(x)\mathrm{D}_1G_{q,l}]p_{q,l}, \quad l = k_q+1,\ldots,k_q+n_q,$$

with $g_{q,2}^{s,h_j}$ and $g_{q,2}^{s,0}$, corresponding to the eigenfunctions $\beta_q(x)$, being the parts of $f_2(u_1,u_{1,1})$ which generate secular terms (associated with purely imaginary eigenvalues) and constant vector (associated with zero eigenvalues), respectively, and $g_{q,2}^u$ denotes the terms that do not produce secular terms. Further, using the solvability conditions, $\langle p_{q,j}^*, \langle\chi_{q,j}^{(2)},\beta_q\rangle\rangle = 0$, $(q = 1,2,\ldots,n_0;\ j = 1,2,\ldots,k_q)$ and $\langle p_{q,l}^*, \langle g_{q,2}^{s,0} + \sum_{k=k_q+1}^{k_q+n_q}\chi_{q,k}^{(2)},\beta_q\rangle\rangle = 0$, $(q = 1,2,\ldots,n_0;\ l = k_q+1,\ldots,k_q+n_q)$, and noting that $\Delta\beta_q(x) = \mu_q\beta_q(x)$, we obtain $\mathrm{D}_1G_{q,j}$ $(j = 1,2,\ldots,k_q)$ and $(\mathrm{D}_1G_{k_q+1},\ldots,\mathrm{D}_1G_{k_q+n_q})^{\mathrm{T}}$ as follows:

$$\mathrm{D}_1G_{q,j} = K_{q,j}\langle p_{q,j}^*, \mu_qK^{(1)}(\alpha_\epsilon)G_{q,j}p_{q,j} + N_0^{(1)}(\alpha_\epsilon)G_{q,j}p_{q,j} + N_1^{(1)}(\alpha_\epsilon)\mathrm{e}^{-\mathrm{i}\omega_{q,j}}p_{q,j}G_{q,j} + g_{q,2}^{s,h_j}\rangle,$$

$$\begin{pmatrix}\mathrm{D}_1G_{q,k_q+1}\\ \vdots \\ \mathrm{D}_1G_{q,k_q+n_q}\end{pmatrix} = K_{z,q}\begin{pmatrix}\bar{p}_{q,k_q+1}^*\left(g_{q,2}^{s,0} + \sum\limits_{k=k_q+1}^{k_q+n_q}[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon) + \mu_qK^{(1)}(\alpha_\epsilon)]p_{q,k}G_{q,k}\right)\\ \vdots \\ \bar{p}_{q,k_q+n_q}^*\left(g_{q,2}^{s,0} + \sum\limits_{k=k_q+1}^{k_q+n_q}[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon) + \mu_qK^{(1)}(\alpha_\epsilon)]p_{q,k}G_{q,k}\right)\end{pmatrix},$$

where $K_{q,j} = [1 + \bar{p}_{q,j}^*N_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_{q,j}}p_{q,j}]^{-1}$, here we assume that $K_{z,q}$ is invertible, given by

$$K_{z,q} = \begin{pmatrix}\bar{p}_{q,k_q+1}^*(\mathrm{I}+N_1(\alpha_c))p_{q,k_q+1} & \cdots & \bar{p}_{q,k_q+1}^*(\mathrm{I}+N_1(\alpha_c))p_{q,k_q+n_q}\\ \vdots & \cdots & \vdots \\ \bar{p}_{q,k_q+n_q}^*(\mathrm{I}+N_1(\alpha_c))p_{q,k_q+1} & \cdots & \bar{p}_{q,k_q+n_q}^*(\mathrm{I}+N_1(\alpha_c))p_{q,k_q+n_q}\end{pmatrix}^{-1}.$$

Note that u_2 is in the form of $u_2 = \sum_{k\geq 1}\beta_k(x)\eta_k$, where η_k is a coefficient, in terms of $G_{q,k}$, $k = 1,2,\ldots,n_q$, to be determined. Due to the assumption that the second-order terms in the normal form vanish at $\alpha = \alpha_c$, $g_{q,2}^{s,0} = 0$ and $g_{q,2}^{s,h_j} = 0$, and thus $\mathrm{D}_1G_{q,j}$ $(j = 1,2,\ldots,k_q)$ and $(\mathrm{D}_1G_{k_q+1},\ldots,\mathrm{D}_1G_{k_q+n_q})^{\mathrm{T}}$ are reduced to

$$\mathrm{D}_1G_{q,j} = K_{q,j}\langle p_{q,j}^*, \mu_qK^{(1)}(\alpha_\epsilon)G_{q,j}p_{q,j} + N_0^{(1)}(\alpha_\epsilon)G_{q,j}p_{q,j} + N_1^{(1)}(\alpha_\epsilon)\mathrm{e}^{-\mathrm{i}\omega_{q,j}}p_{q,j}G_{q,j}\rangle,$$

$$\begin{pmatrix}\mathrm{D}_1G_{q,n_1+1}\\ \vdots \\ \mathrm{D}_1G_{q,n}\end{pmatrix} = K_{z,q}\begin{pmatrix}\bar{p}_{q,k_q+1}^*\left(\sum\limits_{k=k_q+1}^{k_q+n_q}[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon) + \mu_qK^{(1)}(\alpha_\epsilon)]p_{q,k}G_{q,k}\right)\\ \vdots \\ \bar{p}_{q,k_q+n_q}^*\left(\sum\limits_{k=k_q+1}^{k_q+n_q}[N_0^{(1)}(\alpha_\epsilon) + N_1^{(1)}(\alpha_\epsilon) + \mu_qK^{(1)}(\alpha_\epsilon)]p_{q,k}G_{q,k}\right)\end{pmatrix}. \tag{78}$$

Further, from the ϵ^3-order LDE, we similarly obtain

$$\begin{aligned}
&\mathrm{D}_0u_3 - K(\alpha_c)\Delta u_3 - N_0(\alpha_c)u_3 - N_1(\alpha_c)u_{3,1}\\
&\quad = -\mathrm{D}_2u_1 - \mathrm{D}_1u_2 + K^{(2)}(\alpha_\epsilon)\Delta u_1\\
&\qquad + K^{(1)}(\alpha_\epsilon)\Delta u_2 + N_0^{(1)}(\alpha_\epsilon)u_2 + N_0^{(2)}(\alpha_\epsilon)u_1\\
&\qquad + N_1^{(2)}(\alpha_\epsilon)u_{1,1} - N_1^{(1)}(\alpha_\epsilon)\mathrm{D}_1u_{1,1}\\
&\qquad + N_1^{(1)}(\alpha_\epsilon)u_{2,1} - N_1(\alpha_c)\mathrm{D}_2u_{1,1}\\
&\qquad + \frac{1}{2}N_1(\alpha_c)\mathrm{D}_1^2u_{1,1} - N_1(\alpha_c)\mathrm{D}_1u_{2,1}\\
&\qquad + f_3(u_1, u_{1,1}, u_2, u_{2,1}),
\end{aligned} \tag{79}$$

where $u_{3,1} = u_3(x, T_0 - 1, T_1, T_2, \ldots)$, and $f_3(u_1, u_{1,1}, u_2, u_{2,1})$ represents the ϵ^3-order terms in (71). Substituting the solutions of u_1 and u_2 into (79) and neglecting the higher-order terms in the expansion of the perturbation parameter, we have

$$\begin{aligned}
&\mathrm{D}_0u_3 - K(\alpha_c)\Delta u_3 - N_0(\alpha_c)u_3 - N_1(\alpha_c)u_{3,1}\\
&\quad = \sum_{q=1}^{n_0}\left[\sum_{j=1}^{k_q}\chi_{q,j}^{(3)}\mathrm{e}^{\mathrm{i}\omega_{q,j}T_0} + \sum_{j=1}^{k_q}\overline{\chi}_{q,j}^{(3)}\mathrm{e}^{-\mathrm{i}\omega_{q,j}T_0}\right.\\
&\qquad \left. + \sum_{l=k_q+1}^{k_q+n_q}\chi_{q,l}^{(3)} + g_{q,3}^{s,0} + g_{q,3}^{u}\right],
\end{aligned}$$

where

$$\begin{aligned}
\chi_{q,j}^{(3)} &= [-\beta_q(x)\mathrm{D}_2G_{q,j}\\
&\quad - N_1(\alpha_c)\beta_q(x)\mathrm{D}_2G_{q,j}\mathrm{e}^{-\mathrm{i}\omega_{q,j}}]p_{q,j}\\
&\quad + g_{q,3}^{s,h_j}, \quad j = 1, 2, \ldots, k_q,\\
\chi_{q,l}^{(3)} &= [-\beta_q(x)\mathrm{D}_2G_{q,l} - N_1(\alpha_c)\beta_q(x)\mathrm{D}_2G_{q,l}]p_{q,l},\\
&\qquad l = k_q + 1, \ldots, k_q + n_q,
\end{aligned}$$

with $g_{q,3}^{s,h_j}$ and $g_{q,3}^{s,0}$, corresponding to the eigenfunctions $\beta_q(x)$, being the parts of $f_3(u_1, u_{1,1}, u_2, u_{2,1})$ which generate secular terms (associated with purely imaginary eigenvalues) and constant vector (associated with zero eigenvalues), respectively, and $g_{q,3}^{u}$ denotes the remaining terms. Similarly, using the solvability conditions, $\langle p_{q,j}^*, \langle\chi_{q,j}^{(3)}, \beta_q\rangle\rangle = 0$, ($q = 1, 2, \ldots, n_0$; $j = 1, 2, \ldots, k_q$) and $\langle(\overline{p}_{q,k_q+1}^*, \ldots, \overline{p}_{q,k_q+n_q}^*), \langle\sum_{k=k_q+1}^{k_q+n_q}\chi_{q,k}^{(3)} + g_{q,3}^{s,0}, \beta_q\rangle\rangle = 0$, ($q = 1, 2, \ldots, n_0$; $l = k_q + 1, \ldots, k_q + n_q$), and noting that $\Delta\beta_q(x) = \mu_q\beta_q(x)$, we obtain the derivatives $\mathrm{D}_2G_{q,j}$ ($j = 1, 2, \ldots, k_q$) and $(\mathrm{D}_2G_{n_1+1}, \ldots, \mathrm{D}_2G_n)^{\mathrm{T}}$ (i.e. the normal form terms) as follows:

$$\mathrm{D}_2G_{q,j} = K_{q,j}\langle p_{q,j}^*, g_{q,3}^{s,h_j}\rangle, \quad j = 1, 2, \ldots, k_q,$$

$$\begin{pmatrix}\mathrm{D}_2G_{q,n_1+1}\\ \vdots\\ \mathrm{D}_2G_{q,n}\end{pmatrix} = K_{z,q}\begin{pmatrix}\overline{p}_{q,k_q+1}^*g_{q,3}^{s,0}\\ \vdots\\ \overline{p}_{q,k_q+n_q}^*g_{q,3}^{s,0}\end{pmatrix}. \tag{80}$$

Finally, by using the backwards scaling, $G_{q,k} \mapsto G_{q,k}/\epsilon$ ($q = 1, 2, \ldots, n_0$; $k = 1, 2, \ldots, k_q + n_q$), we obtain the normal form up to third order for the PFDE system (71),

$$\dot{G}_q = \mathrm{D}_1G_q + \mathrm{D}_2G_q, \quad \text{where}$$
$$G_q = (G_{q,1}, G_{q,2}, \ldots, G_{q,k_q+n_q})^{\mathrm{T}}, \tag{81}$$

associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the MTS method.

Now, we apply the CMR method to compute the normal form of (71), restricted to the center manifold, near the semisimple n_1-Hopf–n_2-zero critical point: $\alpha = \alpha_c$. Let $\mathrm{C} := \mathrm{C}([-1, 0]; \mathrm{R})$, and for each $q \in \mathrm{N}$, define $L_q : \mathrm{C} \to \mathrm{R}$ by

$$L_q(\psi)\beta_q = L(\psi\beta_q).$$

Then, on B_q, the linear equation $\frac{\mathrm{d}}{\mathrm{d}t}u_t = K(\alpha)\Delta u + L(\alpha)(u_t)$ is equivalent to the FDE on R,

$$\dot{z}(t) = \mu_q z(t) + L_q z_t, \tag{82}$$

with the characteristic equation given by (70).

Further, for $1 \leq q \leq n_0$, define η_q by

$$\mu_q\psi(0) + L_q\psi = \int_{-1}^{0} d\eta_q(\theta)\psi(\theta), \quad \psi \in \mathrm{C},$$

and $(\cdot, \cdot)_q$, the adjoint bilinear form on $\mathrm{C}^* \times \mathrm{C}$, $\mathrm{C}^* = \mathrm{C}([0, 1]; \mathrm{R})$, by

$$\begin{aligned}
(\alpha, \beta)_q &= \alpha(0)\beta(0)\\
&\quad - \int_{-1}^{0}\int_{0}^{\theta}\alpha(\xi - \theta)d\eta_q(\theta)\beta(\xi)d\xi.
\end{aligned} \tag{83}$$

Based on the adjoint operator theory, we decompose C by $\Lambda_q := \{\lambda \in \mathrm{C} : \lambda \text{ satisfies (70) and } \mathrm{Re}\,\lambda = 0\}$ to obtain

$$\mathrm{C} = P_q \oplus Q_q, \quad P_q = \mathrm{span}\{\Phi_q\},$$
$$P_q^* = \mathrm{span}\{\Psi_q\}, \quad (\Psi_q, \Phi_q)_q = \mathrm{I},$$
$$\dim P_q = \dim P_q^* := m_q, \quad \dot{\Phi} = \Phi_q B_q,$$

where P_q is the generalized eigenspace for (82) associated with Λ_q, and B_q is an $m_q \times m_q$ constant matrix. Thus, $\mathfrak{C}$ can be decomposed by Λ as

$$\mathfrak{C} = P \oplus Q, \quad P = \operatorname{Im}\pi, \quad Q = \operatorname{Ker}\pi,$$

where $\dim P = \sum_{q=1}^{n_0} m_q = 2n_1 + n_2$ and $\pi := \mathfrak{C} \to P$ is the projection defined by $\pi\phi = \sum_{q=1}^{n_0} \Phi_q(\Psi_q, \langle \phi(\cdot), \beta_q\rangle)_q \beta_q$.

Similar to the DDE and NFDE systems, we consider the enlarged phase space BC of continuous functions from $[-1, 0]$ to C^m. Thus, the elements of BC have the form $\psi = \phi + X_0 c$, with $\phi \in \mathfrak{C}$ and $c \in \mathrm{X}$. Hence, in the space BC, (71) becomes an abstract ODE, described by

$$\dot{w} = Aw + X_0\tilde{F}(w, \alpha), \tag{84}$$

where $w \in \mathfrak{C}$, and A is defined by

$$\begin{aligned} &A : \mathfrak{C}^1 \to \mathrm{BC}, \\ &Aw = \dot{w} + X_0[L(\alpha_c)w \\ &\qquad + K(\alpha)\Delta w(0) - \dot{w}(0)] \end{aligned}$$

and

$$\begin{aligned} \tilde{F}(w, \alpha) = {} & [N_0(\alpha)w(0) + N_1(\alpha)w(-1) \\ & - N_0(\alpha_c)w(0) - N_1(\alpha_c)w(-1)] \\ & + F(w, \alpha), \end{aligned}$$

defined on $\mathfrak{C}^1 := \{\phi \in \mathfrak{C} : \dot{\phi} \in \mathfrak{C}, \phi(0) \in \operatorname{dom}(\Delta)\}$. For $c \in \mathrm{X}$ we have $\pi(X_0 c) = \sum_{q=1}^{n_0} \Phi_q \Psi_q(0) \times \langle \alpha, \beta_q\rangle \beta_q$.

In addition, denote $w = \sum_{q=1}^{n_0} \Phi_q z_q(t)\beta_q + y_t$, where $z_q(t) = (\Psi_q, \langle w(t)(\cdot), \beta_q\rangle)_q \in \mathrm{R}^{m_q}$, for $1 \le q \le n_0$. Then, Eq. (84) is further decomposed into the form:

$$\begin{aligned} \dot{z}_q &= B_q z_q + \Psi_q(0) \left\langle \tilde{F}\left(\sum_{k=1}^{n_0} \Phi_k z_k \beta_k + y\right), \beta_q \right\rangle, \\ &\qquad\qquad q = 1, 2, \ldots, n_0, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1} y + (\mathrm{I} - \pi) X_0 \tilde{F}\left(\sum_{k=1}^{n_0} \Phi_k z_k \beta_k + y\right), \end{aligned} \tag{85}$$

where $B_q = \operatorname{diag}\{\mathrm{i}\omega_{q,1}, -\mathrm{i}\omega_{q,1}, \mathrm{i}\omega_{q,2}, -\mathrm{i}\omega_{q,2}, \ldots, \mathrm{i}\omega_{q,k_q}, -\mathrm{i}\omega_{q,k_q}, \overbrace{0, \ldots, 0}^{n_q}\}$.

For convenience in the following proof, we rewrite (85) in a simpler form by considering its first n_0 equations as a union equation in $\mathrm{C}^{2n_1+n_2}$. To achieve this, define the $(2n_1 + n_2) \times (2n_1 + n_2)$ constant matrix $B = \operatorname{diag}(B_1, \ldots, B_{n_0})$, the $n_0 \times (2n_1 + n_2)$ matrix $\Phi = \operatorname{diag}(\Phi_1, \ldots, \Phi_{n_0})$ and the $(2n_1 + n_2) \times n_0$ matrix $\Psi = \operatorname{diag}(\Psi_1, \ldots, \Psi_{n_0})$. As a result, (85) becomes

$$\begin{aligned} \dot{z} &= Bz + \Psi(0) \begin{pmatrix} \left\langle \tilde{F}\left(\sum_{q=1}^{n_0} \Phi_q z_q \beta_q + y\right), \beta_1 \right\rangle \\ \vdots \\ \left\langle \tilde{F}\left(\sum_{q=1}^{n_0} \Phi_q z_q \beta_q + y\right), \beta_{n_0} \right\rangle \end{pmatrix}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1} y + (\mathrm{I} - \pi) X_0 \tilde{F}\left(\sum_{q=1}^{n_0} \Phi_q z_q \beta_q + y\right), \end{aligned} \tag{86}$$

where $z = (z_1, \ldots, z_{n_0}) \in \mathrm{C}^{2n_1+n_2}$, $y_t \in Q^1$.

To find the normal form of system (86), we introduce transformations into Eq. (86) to obtain

$$\begin{aligned} \dot{z} &= Bz + \sum_{j\ge 2} f_j^1(z, y, \alpha_\epsilon), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1} y + \sum_{j\ge 2} f_j^2(z, y, \alpha_\epsilon). \end{aligned} \tag{87}$$

Similarly, define the operators $M_j = (M_j^1, M_j^2)$, $j \ge 2$, by

$$\begin{aligned} &M_j^1 : V_j^M(\mathrm{C}^M) \to V_j^M(\mathrm{C}^M), \\ &(M_j^1 p)(z, \alpha_\epsilon) = D_z p(z, \alpha_\epsilon) Bz - Bp(z, \alpha_\epsilon), \\ &M_j^2 : V_j^M(Q^1) \subset V_j^M(\operatorname{Ker}\pi) \to V_j^M(\operatorname{Ker}\pi), \\ &(M_j^2 h)(z, \alpha_\epsilon) = D_z h(z, \alpha_\epsilon) Bz - A_{Q^1} h(z, \alpha_\epsilon). \end{aligned} \tag{88}$$

Now, suppose the above procedure has been performed up to order $k-1$, with the resulting equation given in the form of

$$\begin{aligned} \dot{z} &= Bz + \sum_{j=1}^{k-1} g_j^1(z, y, \alpha_\epsilon) + \tilde{f}_k^1(z, y, \alpha_\epsilon) + \cdots, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= A_{Q^1} y_t + \sum_{j=1}^{k-1} g_j^2(z, y, \alpha_\epsilon) + \tilde{f}_k^2(z, y, \alpha_\epsilon) + \cdots, \end{aligned} \tag{89}$$

where $g_j = \tilde{f}_j - M_j U_j$ $(j = 1, 2, \ldots, k-1)$ with $g_1 = 0$. Then, the center manifold and normal form can be obtained via a recursive procedure: computing the jth order terms $(j \geq 2)$ at each step, through a transformation of variables

$$(z, y) = (\hat{z}, \hat{y}) + (U_j^1, U_j^2),$$

with $z, \hat{z} \in \mathrm{C}^{2n_1+n_2}$, $y_t, \hat{y}_t \in Q^1$, and U_j^1 and U_j^2, defined by $U_j^1 : \mathrm{C}^{2n_1+n_2} \to \mathrm{C}^{2n_1+n_2}$ and $U_j^2 : \mathrm{C}^{2n_1+n_2} \to \mathrm{Q}^1$, are homogeneous polynomials of degree j in z.

Furthermore, to find the normal form restricted to the center manifold of (89), we introduce a formal change of variables: $z = \hat{z} + p(\hat{z})$, $y = \hat{y} + h(\hat{z})$, into (71) to obtain the normal form,

$$\dot{z} = Bz + \sum_{j \geq 2} g_j^1(z, \alpha_\epsilon), \tag{90}$$

where the hat has been dropped from $\hat{g}_j^1(z, \alpha_\epsilon)$ for simplicity, and g_j^1's are to be determined. Equation (90) is the normal form of the PFDE system (71), associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the CMR method.

The remaining part of the proof is similar to that for proving Theorem 2 in the DDE case, but we need to show that

(i) choosing the same bases for the linear space leads to the identical linear solutions in the center subspace; and
(ii) the second-order solutions u_2 and $\Phi(0)U_2^1 + h_2(0)$, $u_{2,1}$ and $\Phi(-1)U_2^1 + h_2(-1)$ are identical for the PFDE system (71), respectively, where $\Phi(\theta) = (\Phi_1(\theta), \Phi_2(\theta), \ldots, \Phi_{n_0}(\theta))$.

For convenience, we define

$$\begin{aligned}
H_q &= (p_{q,1}, \overline{p}_{q,1}, \ldots, p_{q,k_q}, \overline{p}_{q,k_q}, \\
&\qquad p_{q,k_q+1}, \ldots, p_{q,k_q+n_q}), \\
H_q^* &= (p_{q,1}^*, \overline{p}_{q,1}^*, \ldots, p_{q,k_q}^*, \\
&\qquad \overline{p}_{q,k_q}^*, p_{q,k_q+1}^*, \ldots, p_{q,k_q+n_q}^*), \\
\Psi_q(s) &= (\psi_{q,1}(s), \overline{\psi}_{q,1}(s), \ldots, \psi_{q,k_q}(s), \\
&\qquad \overline{\psi}_{q,k_q}(s), \psi_{q,k_q+1}, \ldots, \psi_{q,k_q+n_q}), \\
G_q &= (G_{q,1}, \overline{G}_{q,1}, \ldots, G_{q,k_q}, \\
&\qquad \overline{G}_{q,k_q}, G_{q,k_q+1}, \ldots, G_{q,k_q+n_q})^{\mathrm{T}},
\end{aligned}$$

$$\begin{aligned}
\Upsilon_q(t) &= (e^{i\omega_{q,1}t}, e^{-i\omega_{q,1}t}, e^{i\omega_{q,2}t}, e^{-i\omega_{q,2}t}, \ldots, \\
&\qquad e^{i\omega_{q,k_q}t}, e^{-i\omega_{q,k_q}t}, \overbrace{1, \ldots, 1}^{n_q}), \\
&(a_1, a_2, \ldots, a_p) \times (b_1, b_2, \ldots, b_p) \\
&\quad := (a_1 b_1, a_2 b_2, \ldots, a_p b_p).
\end{aligned}$$

With the above notations, the linear solution (76) for the MTS method can be written as

$$u_1 = \sum_{q=1}^{n_0} (H_q \times \Upsilon_q(T_0)) G_q \beta_q, \quad \text{where}$$
$$G_q = \langle H_q^*, \langle u_1|_{T_0=0}, \beta_q \rangle \rangle, \tag{91}$$

while the linear solution for the CMR method can be expressed as

$$z(t) = \sum_{q=1}^{n_0} (\Phi_q \times \Upsilon_q(t)) z_q \beta_q, \quad \text{where}$$
$$z_q(t) = (\Psi_q, \langle z|_{\Upsilon_q(t)=\Upsilon_q(0)}, \beta_q \rangle). \tag{92}$$

Thus, we can choose $\Phi_q(0) = H_q$, $\psi_{q,j}(0) = K_{q,j}\, \overline{p}_{q,j}^{*\mathrm{T}}$ $(j = 1, 2, \ldots, k_q)$, and $\Psi_{z,q}(0) = K_{z,q}(\overline{p}_{q,k_q+1}^*, \ldots, \overline{p}_{q,k_q+n_q}^*)^{\mathrm{T}}$, under which, by neglecting the difference in the notations, the two inner products (72) and (83), the two linear solutions (91) and (92) obtained by using the MTS and CMR methods, are identical.

For the CMR method, the transformation h_2 satisfies

$$\begin{aligned}
&D_z h_2(z)(\theta) Bz - \dot{h}_2(z)(\theta) \\
&\qquad + X_0(\theta)[\dot{h}_2(z)(0) - K(\alpha_c)\Delta h_2(z)(0) \\
&\qquad - L_c(h_2(z))] \\
&\quad = [X_0(\theta) - \Phi(\theta)\Psi(0)] f_2(\Phi(\theta)z),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&D_z h_2(z)(\theta) Bz - \dot{h}_2(z)(\theta) \\
&\qquad = -\Phi(\theta)\Psi(0) f_2(\Phi(\theta)z), \\
&\dot{h}_2(z)(0) - K(\alpha_c)\Delta h_2(z)(0) - L_c(h_2(z)) \\
&\qquad = f_2(\Phi(\theta)z).
\end{aligned} \tag{93}$$

We again ignore the higher-order terms in the expansion of parameter α, and can thus rewrite Eq. (77) for the MTS method as

$$\begin{aligned}
&\mathrm{D}_0 u_2 - K(\alpha_c)\Delta u_2 - N_0(\alpha_c) u_2 - N_1(\alpha_c) u_{2,1} \\
&\qquad = f_2(u_1, u_{1,1}).
\end{aligned} \tag{94}$$

Similar to the proof for Theorem 2, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}\Phi(\theta)U_2^1(x)}{\mathrm{d}\theta}\right|_{\theta=0}
&= K(\alpha_c)\Delta\Phi(0)U_2^1(x) + L_c(\Phi)U_2^1 \\
&= K(\alpha_c)\Delta\Phi(0)U_2^1(x) \\
&\quad + [N_0(\alpha_c)\Phi(0) + N_1(\alpha_c)\Phi(-1)]U_2^1, \\
\left.\frac{\mathrm{d}h_2(x,0)(\theta)}{\mathrm{d}\theta}\right|_{\theta=0}
&= K(\alpha_c)\Delta h_2(\theta)|_{\theta=0} + N_0(\alpha_c)h_2(x)(0) \\
&\quad + N_1(\alpha_c)h_2(x)(-1) \\
&\quad + f_2(\Phi(0)x, \Phi(-1)x).
\end{aligned}$$

Denoting $\tilde{u}_2(\theta) = \Phi(\theta)U_2^1 + h_2(\theta)$ and $\tilde{u}_{2t}(\theta) = \tilde{u}_2(t+\theta)$, we obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}\tilde{u}_{2t}(\theta)}{\mathrm{d}\theta}\right|_{\theta=0}
&= K(\alpha_c)\tilde{u}_{2t}(0) + N_0(\alpha_c)\tilde{u}_{2t}(0) \\
&\quad + N_1(\alpha_c)\tilde{u}_{2t}(-1) + f_2(\Phi(0)x(t), \Phi(-1)x(t)) \\
&= \left.\frac{\mathrm{d}\tilde{u}_2(t+\theta)}{\mathrm{d}\theta}\right|_{\theta=0} \\
&= \left.\frac{\mathrm{d}\tilde{u}_2(t+\theta)}{\mathrm{d}(t+\theta)}\right|_{\theta=0} \xrightarrow{t+\theta=T} \left.\frac{\mathrm{d}\tilde{u}_2(T)}{\mathrm{d}(T)}\right|_{T=t}.
\end{aligned}$$

Equation (94) can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}u_2(T_0,\ldots)}{\mathrm{d}T_0} &= K(\alpha_c)\Delta u_2(T_0-1,\ldots) \\
&\quad + N_0(\alpha_c)u_2(T_0,\ldots) \\
&\quad + N_1(\alpha_c)u_2(T_0-1,\ldots) \\
&\quad + f_2(u_1, u_{1,1}).
\end{aligned}$$

It is seen that the second-order solutions u_2 and $\Phi(0)U_2^1+h_2(0)$, $u_{2,1}$ and $\Phi(-1)U_2^1+h_2(-1)$ are identical, which is similar to that for the DDE systems, as expected, and thus the details of the remaining part are omitted for brevity. ■

Corollary 5.1. *Assume (H1)–(H4) hold, and system (71), associated with some eigenfunctions β_q, undergoes a semisimple n_1-Hopf–n_2-zero ($n_1 \geq 1$, $n_2 \geq 0, n = n_1 + n_2 \geq 1$) bifurcation from the space homogeneous trivial equilibrium at the critical point, defined by $\alpha = \alpha_c$, at which the characteristic equation (70) has n_1 pairs of purely imaginary roots $\pm i\omega_j$ ($j = 1, 2, \ldots, n_1$) and n_2 zero roots, and all other roots have negative real part. If system (71) does not contain second-order terms, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Corollary 5.2. *Assume (H1)–(H4) hold, and system (71), associated with some eigenfunctions β_q, undergoes a semisimple n_1-Hopf ($n_1 \geq 1$) bifurcation from the space homogeneous trivial equilibrium at the critical point, defined by $\alpha = \alpha_c$, and all characteristic roots have nonpositive real part. Then the normal forms associated with the semisimple n_1-Hopf singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.*

6. Equivalence of the MTS and CMR Methods for DDEs, NFDEs and PFDEs with Distributed Delays

Having proved the identity of the normal forms up to third order, derived by using the MTS and CMR methods for the DDE, NFDE and PFDE systems with discrete delays in previous sections, we now turn to consider the case with distributed delays, which not only has theoretical interests, but also has wide applications in solving real world problems (e.g. see [Nelson *et al.*, 2004; Goncalves *et al.*, 2011; Wu, 1996]). It has been shown that the MTS method can also be applied to study some special DDEs with distributed delays (e.g. see [Han & Song, 2012]). Therefore, in this section, we will show that the MTS and CMR methods are also equivalent in deriving the normal forms up to third order for general differential systems with distributed delays, including the DDE, NFDE and PFDE systems. To achieve this, without loss of generality, we will use the following explicit general differential equation (which includes the DDE, NFDE and PFDE systems) with distributed delays to prove the equivalence of the MTS and CMR methods,

$$\begin{aligned}\frac{\partial}{\partial t}&[u(x,t)+M_1(\alpha)u(x,t-1)]\\&=K(\alpha)\Delta u(x,t)+N_0(\alpha)u(x,t)\\&\quad+N_1(\alpha)u(x,t-1)\\&\quad+W(\alpha)\int_0^{+\infty}\kappa(s)u(x,t-s)ds\\&\quad+F(\tilde{u},\alpha)+G(\tilde{u},\dot{\tilde{u}},\alpha),\end{aligned}\tag{95}$$

where $\tilde{u}$ denotes $u(x,t)$, $u(x,t-1)$ and $\int_0^{+\infty}\kappa(s)\times u(x,t-s)ds$, and κ is called the delayed kernel.

Under Hypotheses (H1)–(H4) (see Sec. 5), we obtain a sequence of "characteristic" (eigenvalue-eigenfunction) equations of (95):

$$\left[\lambda+\lambda M_1\mathrm{e}^{-\lambda}-K\mu_q-N_0-N_1\mathrm{e}^{-\lambda}\right.\\ \left.-W\int_0^{+\infty}\kappa(s)\mathrm{e}^{-\lambda s}ds\right]\beta_q=0,\quad q\in\mathrm{N},\tag{96}$$

and assume that there exists an n_0 such that all solutions of (96) satisfy $\mathrm{Re}(\lambda)<0$ for $q>n_0$. In particular, when $K=0$, the characteristic equation (96) becomes

$$\det\left(\lambda\mathrm{I}+\lambda M_1\mathrm{e}^{-\lambda}-N_0-N_1\mathrm{e}^{-\lambda}\right.\\ \left.-W\int_0^{+\infty}\kappa(s)\mathrm{e}^{-\lambda s}ds\right)=0.\tag{97}$$

Obviously, the improper integral $\int_0^{+\infty}\kappa(s)\times\mathrm{e}^{-\lambda s}ds$ exactly defines the Laplace transform. To guarantee the existence of the transform, we assume

(H5) The kernel $\kappa(s)$ is piecewise continuous on $[0,+\infty)$ and is of exponential order for $s>S$. That is, there exist constants c, $M>0$, and $S>0$ such that $|\kappa(s)|\le M\mathrm{e}^{cs}$ for all $s>S$.

Thus, under (H5), $\int_0^{+\infty}\kappa(s)\mathrm{e}^{-\lambda s}ds=\mathfrak{L}(\kappa(s))\triangleq\mathfrak{K}(\lambda)$ exists, where $\mathfrak{L}$ represents the Laplace operator. So, for the eigenvalues of (96) [or (97)] with zero real part, we may define $\int_0^{+\infty}\kappa(s)\mathrm{e}^{-\mathrm{i}\omega_j s}ds=\mathfrak{K}(\mathrm{i}\omega)\triangleq a_j$, $\int_0^{+\infty}\kappa(s)\mathrm{e}^{\mathrm{i}\omega_j s}ds=\mathfrak{K}(-\mathrm{i}\omega)\triangleq b_j$, $j=1,2,\ldots,n_1$ and $\int_0^{+\infty}\kappa(s)ds=\mathfrak{K}(0)=c$. Then, we have the following result.

Theorem 5. *Assume (H1)–(H5) hold, and system (95), associated with some characteristic eigenfunctions (whose corresponding eigenvalues have nonpositive real part), undergoes a semisimple n_1-Hopf–n_2-zero ($n_1\ge 1, n_2\ge 0, n=n_1+n_2\ge 1$) bifurcation from the space homogeneous trivial equilibrium at the critical point, $\alpha=\alpha_c$, (in case $K=0$, Hypotheses (H1)–(H4) are not needed and the space homogeneous trivial equilibrium becomes a trivial equilibrium), at which the characteristic equation (96) [(97) if $K=0$] has n_1 pairs of purely imaginary roots $\pm\mathrm{i}\omega_j$ ($j=1,2,\ldots,n_1$) and n_2 zero roots, and all other roots have negative real part. If the second terms in the normal form vanish at $\alpha=\alpha_c$, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Proof. We adopt the notations used in the previous sections. So, for the ϵ^1-order LDE derived using the MTS method, we have

$$\begin{aligned}\mathrm{D}_0u_1&+M_1(\alpha_c)\mathrm{D}_0u_{1,1}-K(\alpha_c)\Delta u_1\\&-N_0(\alpha_c)u_1-N_1(\alpha_c)u_{1,1}\\&-W(\alpha_c)\int_0^{+\infty}\kappa(s)u_1(x,t-s)ds\\&=0.\end{aligned}\tag{98}$$

The linear solutions derived using the two methods can still be expressed by (76) and (92), respectively, since the characteristic equation (96) [or (97)] is satisfied. We also choose $\Phi_q(0)=H_q$, $\psi_{q,j}(0)=K_{q,j}\,\bar{p}^{*\mathrm{T}}_{q,j}$ ($j=1,2,\ldots,k_q$), and $\Psi_{z,q}(0)=K_{z,q}(\bar{p}^*_{q,k_q+1},\ldots,\bar{p}^*_{q,k_q+n_q})^{\mathrm{T}}$, where $K_{q,j}=[1+\bar{p}^*_{q,j}M_1(\alpha_c)\mathrm{e}^{-\mathrm{i}\omega_{q,j}}p_{q,j}(1-\mathrm{i}\omega_j)+\bar{p}^*_{q,j}N_1(\alpha_c)\times\mathrm{e}^{-\mathrm{i}\omega_{q,j}}p_{q,j}]^{-1}$ and

$$K_{z,q}=\begin{pmatrix}\bar{p}^*_{q,k_q+1}(\mathrm{I}+M_1(\alpha_c)+N_1(\alpha_c))p_{q,k_q+1}&\cdots&\bar{p}^*_{q,k_q+1}(\mathrm{I}+M_1(\alpha_c)+N_1(\alpha_c))p_{k_q+n_q}\\ \vdots&\cdots&\vdots\\ \bar{p}^*_{q,k_q+n_q}(\mathrm{I}+M_1(\alpha_c)+N_1(\alpha_c))p_{q,k_q+1}&\cdots&\bar{p}^*_{q,k_q+n_q}(\mathrm{I}+M_1(\alpha_c)+N_1(\alpha_c))p_{q,k_q+n_q}\end{pmatrix}^{-1},$$

under which, the two linear solutions in the center subspace are identical provided that the difference in the notations is ignored.

Next, we neglect the high-order terms in the expansion of parameter α, and thus the second-order terms u_2 and h_2 in the two methods need to satisfy the following equations:

$$\begin{aligned}
&\mathrm{D}_0 u_2 + M_1(\alpha_c)\mathrm{D}_0 u_{2,1} - K(\alpha_c)\Delta u_2 - N_0(\alpha_c)u_2 - N_1(\alpha_c)u_{2,1} - W(\alpha_c)\int_0^{+\infty} \kappa(s)u_2(x,t-s)ds \\
&\qquad = f_2\left(u_1, u_{1,1}, \int_0^{+\infty} \kappa(s)u_1(x,t-s)ds\right), \\
&\dot{h}_2(z)(0) + M_1\dot{h}_2(z)(-1) - K(\alpha_c)\Delta h_2(z) - L_c(h_2(z)(\theta)) - W(\alpha_c)\int_0^{+\infty} \kappa(s)h_2(z)(x,t-s)ds \\
&\qquad = f_2(\Phi(\theta)z).
\end{aligned}$$

The key step here is how to deal with the integral term with the distributed delay involved in the expression on the right-hand side of the equations. Actually, under (H5), this integral with distributed delay can be expressed, for example, using (76) in the MTS method as

$$\begin{aligned}
&\int_0^{+\infty} \kappa(s)u_1(x,T_0-s)\mathrm{d}s \\
&\quad = \sum_{q=1}^{n_0}\sum_{j=1}^{k_q} \beta_q(x)G_{q,j}(T_1,T_2,\ldots)p_{q,j}\mathrm{e}^{\mathrm{i}\omega_{q,j}T_0}\int_0^{+\infty}\kappa(s)\mathrm{e}^{-\mathrm{i}\omega_j s}ds \\
&\qquad + \sum_{q=1}^{n_0}\sum_{j=1}^{k_q} \overline{\beta}_q(x)\overline{G}_{q,j}(T_1,T_2,\ldots)\overline{p}_{q,j}\mathrm{e}^{-\mathrm{i}\omega_{q,j}T_0}\int_0^{+\infty}\kappa(s)\mathrm{e}^{\mathrm{i}\omega_j s}ds \\
&\qquad + \sum_{q=1}^{n_0}\sum_{l=k_q+1}^{k_q+n_q} \beta_q(x)G_{q,l}(T_1,T_2,\ldots)p_{q,l}\int_0^{+\infty}\kappa(s)ds \\
&\quad = \sum_{q=1}^{n_0}\sum_{j=1}^{k_q} \beta_q(x)G_{q,j}(T_1,T_2,\ldots)p_{q,j}\mathrm{e}^{\mathrm{i}\omega_{q,j}T_0}a_j \\
&\qquad + \sum_{q=1}^{n_0}\sum_{j=1}^{k_q} \overline{\beta}_q(x)\overline{G}_{q,j}(T_1,T_2,\ldots)\overline{p}_{q,j}\mathrm{e}^{-\mathrm{i}\omega_{q,j}T_0}b_j \\
&\qquad + \sum_{q=1}^{n_0}\sum_{l=k_q+1}^{k_q+n_q} \beta_q(x)G_{q,l}(T_1,T_2,\ldots)p_{q,l}c,
\end{aligned}$$

which has got rid of the integral form in the expression. Similarly, this can also be done for the CMR method. Therefore, we can follow the procedures used for the DDE, NFDE and PFDE systems to obtain the same conclusion. So the detailed derivations (similar to that in the previous sections) are omitted here. Hence, the conclusion of Theorem 5 holds, that is, the normal forms associated with the semisimple n_1-Hopf–n_2-zero bifurcation of (95), derived using the MTS and CMR methods, are identical up to third order provided that the same basis for the normal forms is chosen for the two methods. ■

Corollary 6.1. *Assume (H1)–(H5) hold, and system (95), associated with some system eigenfunctions (whose corresponding eigenvalues have nonpositive real part), undergoes a semisimple n_1-Hopf–n_2-zero ($n_1 \geq 1, n_2 \geq 0, n = n_1 + n_2 \geq 1$) bifurcation from the space homogeneous trivial equilibrium at the critical point, $\alpha = \alpha_c$, (in case $K = 0$,* *Hypotheses (H1)–(H4) are not needed and the space homogeneous trivial equilibrium becomes a trivial equilibrium), at which the characteristic equation (96) [(97) if $K = 0$] has n_1 pairs of purely imaginary roots $\pm i\omega_j$ ($j = 1, 2, \ldots, n_1$) and n_2 zero roots, and all other roots have negative real part. If system (95) does not contain second-order terms, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived using the multiple time scales and center manifold reduction methods, are identical up to third order.*

Corollary 6.2. *Assume (H1)–(H5) hold, and system (95), associated with some system eigenfunctions (whose corresponding eigenvalues have nonpositive real part), undergoes a semisimple n_1-Hopf ($n_1 \geq 1$) bifurcation from the space homogeneous trivial equilibrium at the critical point, $\alpha = \alpha_c$, (in case $K = 0$, Hypotheses (H1)–(H4) are not needed and the space homogeneous trivial equilibrium becomes a trivial equilibrium), and all characteristic roots have nonpositive real part, then the normal forms associated with the semisimple n_1-Hopf singularity, derived using the multiple time scales and center manifold reduction methods, are identical up to third order.*

In many applications, the following Γ-distribution delay kernel is often used

$$\kappa(s) = \beta^{\tilde{n}+1} \frac{s^{\tilde{n}} e^{-\beta s}}{\tilde{n}!}, \quad s \in (0, +\infty), \;\; \tilde{n} = 0, 1, \ldots.$$

Two special cases, $\tilde{n} = 0$ and $\tilde{n} = 1$, are called *weak delay kernel* and *strong delay kernel*, respectively. Obviously, hypothesis (H5) holds for the Γ-distribution delay kernel. Thus, a corollary can be directly obtained from Theorem 5 for the weak and strong delay kernels of the Γ-distribution delay kernel, which are most interesting and very useful in applications.

Corollary 6.3. *Assume (H1)–(H4) hold and κ is the Γ-distribution delay kernel with either $\tilde{n} = 0$ or $\tilde{n} = 1$, and system (95), associated with some system eigenfunctions (whose corresponding eigenvalues have nonpositive real part), undergoes a semisimple n_1-Hopf–n_2-zero ($n_1 \geq 1, n_2 \geq 0, n = n_1 + n_2 \geq 1$) bifurcation from the space homogeneous trivial equilibrium at the critical point, $\alpha = \alpha_c$, (in case $K = 0$, Hypotheses (H1)–(H4) are not needed and the space homogeneous trivial equilibrium becomes a trivial equilibrium), at which the characteristic equation (96) [(97) if $K = 0$] has n_1 pairs of purely imaginary roots $\pm i\omega_j$ ($j = 1, 2, \ldots, n_1$) and n_2 zero roots, and all other roots have negative real part. If one of the following conditions holds:*

(1) the second-order terms in the normal form vanish at $\alpha = \alpha_c$;
(2) system (95) do not contain second-order terms;
(3) $n_2 = 0$,

then the normal forms up to third order, derived using the multiple time scales and center manifold reduction methods, are identical.

Proof. Here, we give a different and independent proof, by first transforming system (95) to an equivalent differential system without distributed delays, and then directly applying Theorems 1–5.

Case 1. Weak kernel ($\tilde{n} = 0$), i.e. $\kappa_0(s) = \beta e^{-\beta s}$.

Let $v(x, t) = \int_0^{+\infty} \kappa_0(s) u(x, t - s) ds$. Then, introducing the transformation $t - s = \hat{s}$ and dropping the hat, we have $v(x, t) = \int_{-\infty}^{t} \kappa(t - s) \times u(x, t) ds$, and thus obtain $\dot{v}(x, t)(t) = \beta u(x, t) - \beta v(x, t)$. As a result, Eq. (95), corresponding to the weak delay kernel, is equivalent to the following differential system without distributed delays,

$$\begin{aligned} &\frac{\partial}{\partial t}[u(x, t) + M_1 u(x, t - 1)] \\ &\qquad = K(\alpha)\Delta u(x, t) + N_0(\alpha) u(x, t) \\ &\qquad\quad + N_1(\alpha) u(x, t - 1) + W(\alpha) v(x, t) \\ &\qquad\quad + F(\tilde{u}, \alpha) + G(\tilde{u}, \dot{\tilde{u}}, \alpha), \\ &\dot{v}(t) = \beta u(t) - \beta v(t), \end{aligned} \tag{99}$$

where $\tilde{u}$ denotes $u(x, t)$, $u(x, t - 1)$ and $v(x, t)$.

Case 2. Strong kernel ($\tilde{n} = 1$), i.e. $\kappa_1(s) = \beta^2 s e^{-\beta s}$.

Similarly, let $v_1(x, t) = \int_0^{+\infty} \kappa(s) u(x, t - s) ds = \int_{-\infty}^{t} \kappa(t - s) u(x, t) ds$ and $v_2(x, t) = \int_{-\infty}^{t} \beta^2 \times e^{-\beta(t-s)} u(x, t) ds$. Then, we obtain $\dot{v}_1(x, t) = v_2(x, t) - \beta v_1(x, t)$ and $\dot{v}_2(x, t) = \beta^2 u(x, t) - \beta v_2(x, t)$, under which Eq. (95), corresponding to the strong delay kernel, is equivalent to the following differential system without distributed delays,

$$\begin{aligned} &\frac{\partial}{\partial t}[u(x, t) + M_1 u(x, t - 1)] \\ &\qquad = K(\alpha)\Delta u(x, t) + N_0(\alpha) u(x, t) \end{aligned}$$

$$
\begin{aligned}
&\quad + N_1(\alpha)u(x,t-1) + W(\alpha)v_1(x,t) \\
&\quad + F(\tilde{u},\alpha) + G'(\tilde{u},\dot{\tilde{u}},\alpha), \\
&\dot{v}_1(x,t) = v_2(x,t) - \beta v_1(x,t), \\
&\dot{v}_2(x,t) = \beta^2 u(x,t) - \beta v_2(x,t),
\end{aligned}
\tag{100}
$$

where $\tilde{u}$ denotes $u(x,t)$, $u(x,t-1)$, $v_1(x,t)$ and $v_2(x,t)$.

Now we directly apply Theorems 1–5 to systems (99) and (100), which only contain discrete delays, to complete the proof. ■

7. Applications

In this section, we present a number of practical examples to demonstrate the application of the theoretical results obtained in previous sections. For the ODE systems, there are many articles in the literature which compare the MTS method with the CMR method in computing normal forms (e.g. see [Han & Yu, 2012]). In the following subsections, different types of differential equations (including the DDE, NFDE and PFDE systems) with single delay, multiple delays, or distributed delays are given to show how the MTS and CMR methods are used to derive the normal forms for a given system when either no delay is treated as perturbation parameter or at least one of the delays is chosen as a perturbation parameter.

7.1. *Single delay: The van der Pol–Duffing equation*

First, consider the van der Pol–Duffing equation with a nonlinear damping [Wei & Jiang, 2005]:

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = f(x), \quad (\varepsilon > 0), \tag{101}$$

where the forcing function f is a delayed feedback of position x. For different f, the equilibrium at the origin may exhibit a diversity of bifurcations, such as Hopf bifurcation [Wei & Jiang, 2005], Hopf-zero bifurcation [Wang & Jiang, 2010], and double-Hopf bifurcation [Ma *et al.*, 2008; Ding *et al.*, 2013a]. For these three types of bifurcations, we use Theorem 2 (or Corollaries 2.1 or 2.2) and the formulas obtained in Sec. 3 to show the equivalence of normal forms derived by using the MTS and CMR methods. Note that for this example, both the MTS and CMR methods are applicable.

7.1.1. *Case 1: Hopf bifurcation* $(n_1 = 1, n_2 = 0)$

With $f = \varepsilon k x(t-\tau)$ [Wei & Jiang, 2005], system (101) does not contain quadratic terms. Suppose the system undergoes a Hopf bifurcation from the trivial equilibrium at the critical point, $\tau = \tau_c = \tau_j^{\pm}$. The system formulation and the derivation for the critical time delays $\tau_j^{\pm}$ $(j = 0,1,2,\ldots)$ can be found in [Wei & Jiang, 2005], and thus the detailed linear analysis is omitted here.

First, we assume that (101) undergoes a Hopf bifurcation at the critical point $\tau = \tau_c$, and that the characteristic equation of the linearized part of (101) has a pair of purely imaginary roots $\pm\mathrm{i}\omega$, and the remaining roots have negative real part. We take perturbation as $\tau = \tau_c + \epsilon\tau_\epsilon$. Let $\dot{x} = y$, and rescale the time delay by $t \mapsto t/\tau$. Then, (101) can be rewritten as

$$
\begin{aligned}
&\dot{x} = \tau y, \\
&\dot{y} = -\tau x + \varepsilon k \tau x(t-1) - \varepsilon\tau(x^2-1)y.
\end{aligned}
\tag{102}
$$

According to the MTS method, we have

$$N_0 = \tau\begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}, \quad N_1 = \tau\begin{bmatrix} 0 & 0 \\ \varepsilon k & 0 \end{bmatrix}$$

and

$$p_1 = \begin{pmatrix} 1 \\ \mathrm{i}\omega \end{pmatrix}, \quad p_1^* = \begin{pmatrix} \dfrac{\varepsilon + \mathrm{i}\omega}{\varepsilon + 2\mathrm{i}\omega} \\ -\dfrac{1}{\varepsilon + 2\mathrm{i}\omega} \end{pmatrix}.$$

The linear solution of (102) can be expressed in the form of

$$u_1 = G p_1 \mathrm{e}^{\mathrm{i}\omega\tau_c T_0} + \overline{G}\bar{p}_1 \mathrm{e}^{-\mathrm{i}\omega\tau_c T_0}.$$

By Eq. (32), we obtain

$$\mathrm{D}_1 G = \frac{(1 + \omega^2 - \varepsilon k \mathrm{e}^{-\mathrm{i}\omega\tau_c})\tau_\epsilon G}{\varepsilon - 2\mathrm{i}\omega - \tau_c \varepsilon k \mathrm{e}^{-\mathrm{i}\omega\tau_c}}.$$

Thus, the particular solution of Eq. (29) is $u_2 = 0$. Then, by Eq. (35) we have

$$\mathrm{D}_2 G = \frac{\tau_c \mathrm{i}\omega\varepsilon G^2\overline{G}}{\varepsilon - 2\mathrm{i}\omega - \tau_c\varepsilon k \mathrm{e}^{-\mathrm{i}\omega\tau_c}}.$$

Therefore, it follows from Eq. (36) that the normal form of Hopf bifurcation derived by the MTS method is

$$\dot{G} = \frac{(1+\omega^2 - \varepsilon k e^{-i\omega\tau_c})\tau_\epsilon G}{\varepsilon - 2i\omega - \tau_c \varepsilon k e^{-i\omega\tau_c}} + \frac{\tau_c i\omega\varepsilon G^2\overline{G}}{\varepsilon - 2i\omega - \tau_c\varepsilon k e^{-i\omega\tau_c}} + \cdots. \quad (103)$$

Next, for the CMR method we choose

$$\Phi(\theta) = \begin{bmatrix} e^{i\omega\tau_c\theta} & e^{-i\omega\tau_c\theta} \\ i\omega e^{i\omega\tau_c\theta} & -i\omega e^{-i\omega\tau_c\theta} \end{bmatrix} \quad \text{and}$$

$$\Psi(s) = \begin{bmatrix} d(\varepsilon - i\omega)e^{-i\omega\tau_c s} & -d e^{-i\omega\tau_c s} \\ \bar{d}(\varepsilon + i\omega)e^{i\omega\tau_c s} & -\bar{d}e^{i\omega\tau_c s} \end{bmatrix},$$

where $d = (\varepsilon - 2i\omega - \tau_c\varepsilon k e^{-i\omega\tau_c})^{-1}$. Then, by using the Eqs. (40)–(44), we obtain the same normal form (103) associated with the Hopf bifurcation.

7.1.2. *Case 2: Hopf-zero bifurcation* ($n_1 = n_2 = 1$)

Assume $f(x) = \varepsilon g(x(t-\tau))$ [Wang & Jiang, 2010], where $g \in C^3$ is an odd function, satisfying

$$g(0) = g''(0) = 0, \quad g'(0) = k \neq 0, \quad g'''(0) = 6b \neq 0,$$

showing that system (101) does not contain quadratic terms. When the parameters satisfy

$$k = \frac{1}{\varepsilon},$$

$$\tau = \tau_0 = \begin{cases} \dfrac{1}{\sqrt{2-\varepsilon^2}}[\pi - \arcsin(\varepsilon\sqrt{2-\varepsilon^2})], & \text{for } 0 < \varepsilon < 1, \\[2ex] \dfrac{1}{\sqrt{2-\varepsilon^2}}\arcsin(\varepsilon\sqrt{2-\varepsilon^2}), & \text{for } 1 \le \varepsilon < \sqrt{2}, \end{cases}$$

the characteristic equation of system (101) with $f(x) = \varepsilon g(x(t-\tau))$ has a single zero and a pair of purely imaginary roots $\pm i\omega_0$ with $\omega_0 = \sqrt{2-\varepsilon^2}$, with the remaining roots having negative real part (see [Wang & Jiang, 2010]). Again, let $\dot{x} = y$, and rescale the time delay by $t \mapsto t/\tau$. Then, with $f(x) = \varepsilon g(x(t-\tau))$, Eq. (101) becomes

$$\begin{aligned} \dot{x} &= \tau y(t), \\ \dot{y} &= -\tau x(t) + \varepsilon\tau(kx(t-1) + bx^3(t-1)) \\ &\quad - \varepsilon\tau(x^2(t)-1)y(t) + \cdots. \end{aligned} \quad (104)$$

Similarly, for the MTS method, we choose

$$p_1 = \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$p_1^* = \begin{pmatrix} \dfrac{\varepsilon + i\omega_0}{\varepsilon + 2i\omega_0} \\[2ex] -\dfrac{1}{\varepsilon + 2i\omega_0} \end{pmatrix}, \quad p_2^* = \begin{pmatrix} 1 \\ -\dfrac{1}{\varepsilon} \end{pmatrix}.$$

Thus, the linear solution of system (104) can be expressed in the form of

$$u_1 = G_1 p_1 e^{i\omega_0\tau_0 T_0} + \overline{G}_1\bar{p}_1 e^{-i\omega_0\tau_0 T_0} + G_2 p_2.$$

It then follows from the Eqs. (32), (35) and (36) that the normal form of system (104) associated with the Hopf-zero bifurcation, obtained using the MTS method, is given by

$$\begin{aligned} \dot{G}_1 &= \overline{m}(i\varepsilon + 2\omega_0)\omega_0\mu_2 G_1 \\ &\quad - \overline{m}\varepsilon\tau_0 e^{-i\omega_0\tau_0}\mu_1 G_1 \\ &\quad - \overline{m}\varepsilon\tau_0(be^{-i\omega_0\tau_0} - i\omega_0) \\ &\quad \times (G_1G_2^2 + G_1^2\overline{G}_1) + \cdots, \\ \dot{G}_2 &= -\frac{\varepsilon\tau_0}{\varepsilon - \tau_0}\mu_1 G_2 - \frac{b\varepsilon\tau_0}{\varepsilon - \tau_0} \\ &\quad \times (G_2^2 + 6G_2G_1\overline{G}_1) + \cdots, \end{aligned} \quad (105)$$

where $m = (\varepsilon - 2i\omega_0 - \tau_0 e^{-i\omega\tau_c})^{-1}$, which is identical to that derived by using the CMR method (see [Wang & Jiang, 2010]).

7.1.3. *Case 3: Double Hopf bifurcation* ($n_1 = 2, n_2 = 0$)

A modified system of (101), given by

$$\ddot{x}(t) + \omega_0^2 x(t) - [b - \gamma x^2(t)]\dot{x}(t) + \beta x^3(t) = A[x(t-\tau) - x(t)], \quad (106)$$

which again does not contain quadratic terms, has been considered by Ding *et al.* [2013a] to show that

Eq. (106) may undergo a nonresonant double Hopf bifurcation at the critical point: $(A, \tau) = (A_c, \tau_c)$, where τ_c is given by

$$\tau_c = \tau_1^{(j)} = \tau_2^{(l)}, \quad l, j = 0, 1, 2, \ldots,$$

with

$$\tau_{1,2}^{(j)} = \begin{cases} \dfrac{1}{\omega_{1,2}}\left[\arccos\left(1 + \dfrac{\omega_0^2 - \omega_{1,2}^2}{A_c}\right) + 2j\pi\right], & \text{for } A_c > 0, \\ \dfrac{1}{\omega_{1,2}}\left[2\pi - \arccos\left(1 + \dfrac{\omega_0^2 - \omega_{1,2}^2}{A_c}\right) + 2j\pi\right], & \text{for } A_c < 0, \end{cases}$$

where $j = 0, 1, 2, \ldots$, and

$$\omega_{1,2} = \sqrt{\frac{2A_c + 2\omega_0^2 - b^2 \pm \sqrt{(b^2 - 2\omega_0^2 - 2A_c)^2 - 4(\omega_0^4 + 2A_c\omega_0^2)}}{2}},$$

and the corresponding A_c is then determined by $\tau_1^{(j)} = \tau_2^{(l)}$, $l, j = 0, 1, 2, \ldots$.

We assume that system (106) undergoes a nonresonant double-Hopf bifurcation at the critical point: $(A, \tau) = (A_c, \tau_c)$, and the characteristic equation of the linearized system of (106) has two pairs of purely imaginary roots $\pm i\omega_1$ and $\pm i\omega_2$, with the remaining roots having negative real part. We take perturbations as $(A, \tau) = (A_c, \tau_c) + \epsilon(A_\epsilon, \tau_\epsilon)$. Introducing $\dot{x} = y$ and rescaling $t \mapsto t/\tau$ into Eq. (106) yields

$$\begin{aligned} \dot{x} &= \tau y(t), \\ \dot{y} &= b\tau y(t) - \tau\omega_0^2 x(t) + A\tau[x(t-1) - x(t)] \\ &\quad - \gamma\tau x^2(t) y(t) - \beta\tau x^3(t). \end{aligned} \tag{107}$$

With the MTS method, we have

$$N_0 = \tau \begin{bmatrix} 0 & 1 \\ -\omega_0^2 - A & b \end{bmatrix}, \quad N_1 = \tau \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix},$$

and

$$p_1 = \begin{pmatrix} 1 \\ i\omega_1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 \\ i\omega_2 \end{pmatrix},$$

$$p_1^* = \begin{pmatrix} \dfrac{b + i\omega_1}{b + 2i\omega_1} \\ -\dfrac{1}{\varepsilon + 2i\omega_1} \end{pmatrix}, \quad p_2^* = \begin{pmatrix} \dfrac{b + i\omega_2}{b + 2i\omega_2} \\ -\dfrac{1}{\varepsilon + 2i\omega_2} \end{pmatrix}.$$

The linear solution of system (107) can be expressed as

$$\begin{aligned} u_1 &= G_1 p_1 e^{i\omega_1\tau_c T_0} + \overline{G}_1 \overline{p}_1 e^{-i\omega_1\tau_c T_0} \\ &\quad + G_2 p_2 e^{i\omega_2\tau_c T_0} + \overline{G}_2 \overline{p}_2 e^{-i\omega_2\tau_c T_0}. \end{aligned}$$

By using the formulas (32), (35) and (36), we obtain the normal form of Eq. (106) by using the MTS method, associated with the double Hopf bifurcation, as

$$\begin{aligned} \dot{G}_1 &= -g_1 A_\epsilon \tau_c (e^{-i\omega_1\tau_c} - 1)G_1 + g_1[\omega_1^2 + \omega_0^2 - A_c(e^{-i\omega_1\tau_c} - 1)]\tau_\epsilon G_1 \\ &\quad + g_1\tau_c(i\omega_1\gamma + 3\beta)G_1^2\overline{G}_1 + 2g_1\tau_c(i\omega_1\gamma + 3\beta)G_1 G_2 \overline{G}_2 + \cdots, \\ \dot{G}_2 &= -g_2 A_\epsilon \tau_c (e^{-i\omega_2\tau_c} - 1)G_2 + g_2[\omega_2^2 + \omega_0^2 - A_c(e^{-i\omega_2\tau_c} - 1)]\tau_\epsilon G_2 \\ &\quad + g_2\tau_c(i\omega_2\gamma + 3\beta)G_2^2\overline{G}_2 + 2g_2\tau_c(i\omega_2\gamma + 3\beta)G_1 \overline{G}_1 G_2 + \cdots, \end{aligned} \tag{108}$$

where $g_j = (b - 2i\omega_j - A_c\tau_c e^{-i\omega_j\tau_c})^{-1}$, $j = 1, 2$.

Next, for the CMR method we choose

$$\Phi(\theta) = \begin{bmatrix} e^{i\omega_1\tau_c\theta} & e^{-i\omega_1\tau_c\theta} & e^{i\omega_2\tau_c\theta} & e^{-i\omega_2\tau_c\theta} \\ i\omega_1 e^{i\omega_1\tau_c\theta} & -i\omega_1 e^{-i\omega_1\tau_c\theta} & i\omega_2 e^{i\omega_2\tau_c\theta} & -i\omega_2 e^{i\omega_2\tau_c\theta} \end{bmatrix}$$

and

$$\Psi(s) = \begin{bmatrix} g_1(b-\mathrm{i}\omega_1)e^{-\mathrm{i}\omega_1\tau_c s} & -g_1 e^{-\mathrm{i}\omega_1\tau_c s} \\ \bar{g}_1(b+\mathrm{i}\omega_1)e^{\mathrm{i}\omega_1\tau_c s} & -\bar{g}_1 e^{\mathrm{i}\omega_1\tau_c s} \\ g_2(b-\mathrm{i}\omega_2)e^{-\mathrm{i}\omega_2\tau_c s} & -g_2 e^{-\mathrm{i}\omega_2\tau_c s} \\ \bar{g}_2(b+\mathrm{i}\omega_2)e^{\mathrm{i}\omega_2\tau_c s} & -\bar{g}_2 e^{\mathrm{i}\omega_2\tau_c s} \end{bmatrix},$$

where $g_j = (b - 2\mathrm{i}\omega_j - A_c\tau_c \mathrm{e}^{-\mathrm{i}\omega_j\tau_c})^{-1},\ j = 1, 2.$ Then, applying the formulas (40)–(44) yields the same normal form given in (108). The detailed derivation for the normal form by using the CMR method can be found in [Ding *et al.*, 2013a].

7.2. *Multiple delays: A recurrent neural network model*

In this subsection, we consider a recurrent neural network model with four time delays and use the MTS method to find the normal form of this model [Ding *et al.*, 2013c]. The MTS method can be directly extended to consider such cases, while the CMR method has difficulty to deal with if at least one of the delays is treated as a perturbation parameter. This model is described by the following DDEs:

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + f(x_2(t-\tilde{\tau}_2)), \\ \dot{x}_2(t) &= -x_2(t) + u(t), \\ \dot{x}_3(t) &= -x_3(t) + af(x_1(t-\tilde{\tau}_1)) \\ &\quad + bf(x_2(t-\tilde{\tau}_3)), \\ y(t) &= f(x_3(t-\tilde{\tau}_4)), \end{aligned} \tag{109}$$

where $x_i(t)$ $(i = 1, 2, 3)$ is the state of the ith neuron, a and b are the connection weights, $\tilde{\tau}_j's$ $(j = 1, 2, 3, 4)$ are non-negative time delays. Here, $u(t) = y(t)$, $u(t)$ is the input, and $y(t)$ the output. The triggering nonlinear function of the neurons takes the hyperbolic tangent function, i.e. $f(\cdot) = \tanh(\cdot)$.

For simplicity, let $u_1(t) = x_1(t)$, $u_2(t) = x_2(t-\tilde{\tau}_2)$ and $u_3(t) = x_3(t - \tilde{\tau}_2 - \tilde{\tau}_4)$. Then, system (109) can be transformed into the following equations with only two delays:

$$\begin{aligned} \dot{u}_1(t) &= -u_1(t) + f(u_2(t)), \\ \dot{u}_2(t) &= -u_2(t) + f(u_3(t)), \\ \dot{u}_3(t) &= -u_3(t) + af(u_1(t-\tau_1)) \\ &\quad + bf(u_2(t-\tau_2)), \end{aligned} \tag{110}$$

where $\tau_1 = \tilde{\tau}_1 + \tilde{\tau}_2 + \tilde{\tau}_4$ and $\tau_2 = \tilde{\tau}_3 + \tilde{\tau}_4$.

Under certain conditions, system (110) may exhibit different types of bifurcations, such as fixed point bifurcation, Hopf bifurcation, Hopf-zero bifurcation, and nonresonant and resonant double-Hopf bifurcations. Here, we consider Hopf-zero and double-Hopf bifurcations, and take at least one of the delays as perturbation parameter. Thus, the CMR method cannot be applied here. For our purpose, we will omit the detailed linear analysis, but focus on the normal form derivation by using the MTS method.

The Taylor expansion of Eq. (110) truncated at the cubic order terms is as follows:

$$\begin{aligned} \dot{u}(t) &= N_0 u(t) + N_1 u(t-\tau_1) + N_2 u(t-\tau_2) \\ &\quad + f(u(t), u(t-\tau_1), u(t-\tau_2)), \end{aligned} \tag{111}$$

where

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}, \quad N_0 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$N_1 = a\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad N_2 = b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$f(u(t), u(t-\tau_1), u(t-\tau_2)) = \begin{pmatrix} -\dfrac{1}{3}u_2^3(t) \\ -\dfrac{1}{3}u_3^3(t) \\ -\dfrac{a}{3}u_1^3(t-\tau_1) - \dfrac{b}{3}u_2^3(t-\tau_2) \end{pmatrix},$$

which does not contain quadratic terms.

7.2.1. *Case 1: Hopf-zero bifurcation* $(n_1 = n_2 = 1)$

We treat the connection weight a and the time delay τ_2 as two bifurcation parameters. Suppose system (110) undergoes a Hopf-zero bifurcation from the trivial equilibrium at the critical point: $(a, \tau_2) = (a_c, \tau_{2c})$, and the characteristic equation of the linearized system, $\dot{u}(t) = L_c(u(t), u(t-\tau_1), u(t-\tau_{2c}))$, has a pair of purely imaginary roots $\pm\mathrm{i}\omega$ and a zero root, and other roots have negative real part. By a simple calculation, we obtain the eigenfunctions at the critical point, associated with the Hopf-zero bifurcation, as follows:

$$
\begin{aligned}
&p_1 = (1, 1+\mathrm{i}\omega, (1+\mathrm{i}\omega)^2)^{\mathrm{T}},\\
&p_2 = (1,1,1)^{\mathrm{T}},\\
&p_1^* = \left(\frac{a_c\mathrm{e}^{\mathrm{i}\omega\tau_1}}{2(1-\mathrm{i}\omega)^3 + a_c\mathrm{e}^{\mathrm{i}\omega\tau_1}}, \frac{(1-\mathrm{i}\omega)^2}{2(1-\mathrm{i}\omega)^3 + a_c\mathrm{e}^{\mathrm{i}\omega\tau_1}},\right.\\
&\qquad\left.\frac{1-\mathrm{i}\omega}{2(1-\mathrm{i}\omega)^3 + a_c\mathrm{e}^{\mathrm{i}\omega\tau_1}}\right)^{\mathrm{T}},\\
&p_2^* = \left(\frac{a_c}{a_c+2}, \frac{1}{a_c+2}, \frac{1}{a_c+2}\right)^{\mathrm{T}}.
\end{aligned}
\tag{112}
$$

The linear solution of system (110), associated with the Hopf-zero bifurcation, can be expressed as

$$
u_1 = G_1p_1\mathrm{e}^{\mathrm{i}\omega T_0} + \overline{G}_1\overline{p}_1\mathrm{e}^{-\mathrm{i}\omega T_0} + G_2p_2, \tag{113}
$$

where p_j, $j = 1, 2$, are given in Eq. (112).

By using the MTS method, we obtain the following normal form up to third-order terms associated with the Hopf-zero critical point: $(a, \tau_2) = (a_c, \tau_{2c})$,

$$
\begin{aligned}
\dot{G}_1 &= \delta_1G_1 + \delta_2G_1^2\overline{G}_1 + \delta_3G_1G_2^2 + \cdots,\\
\dot{G}_2 &= \delta_4G_2 + \delta_5G_2^3 + \delta_6G_1\overline{G}_1G_2 + \cdots,
\end{aligned}
\tag{114}
$$

where

$$
\begin{aligned}
\delta_1 &= \frac{(1+\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}\omega\tau_1}a_\epsilon - \mathrm{i}(1+\mathrm{i}\omega)^2b\tau_{2\epsilon}\omega\mathrm{e}^{-\mathrm{i}\omega\tau_{2c}}}{2(1+\mathrm{i}\omega)^3 + a_c\mathrm{e}^{-\mathrm{i}\omega\tau_1} + a_c\tau_1\mathrm{e}^{-\mathrm{i}\omega\tau_1}(1+\mathrm{i}\omega) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega\tau_{2c}}(1+\mathrm{i}\omega)^2},\\
\delta_2 &= -\frac{(1+\mathrm{i}\omega)^5(1-\mathrm{i}\omega) + (1+\mathrm{i}\omega)^6(1-\mathrm{i}\omega)^2 + a_c\mathrm{e}^{-\mathrm{i}\omega\tau_1}(1+\mathrm{i}\omega)}{2(1+\mathrm{i}\omega)^3 + a_c\mathrm{e}^{-\mathrm{i}\omega\tau_1} + a_c\tau_1\mathrm{e}^{-\mathrm{i}\omega\tau_1}(1+\mathrm{i}\omega) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega\tau_{2c}}(1+\mathrm{i}\omega)^2},\\
\delta_3 &= -\frac{2a_c(1+\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}\omega\tau_1} + (1+\mathrm{i}\omega)^4 + b(1+\mathrm{i}\omega)^2\mathrm{e}^{-\mathrm{i}\omega\tau_{2c}}}{2(1+\mathrm{i}\omega)^3 + a_c\mathrm{e}^{-\mathrm{i}\omega\tau_1} + a_c\tau_1\mathrm{e}^{-\mathrm{i}\omega\tau_1}(1+\mathrm{i}\omega) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega\tau_{2c}}(1+\mathrm{i}\omega)^2},\\
\delta_4 &= \frac{a_\epsilon}{a_c + 2 + a_c\tau_1 + b\tau_{2c}},\\
\delta_5 &= -\frac{2a_c + 1 + b}{3(a_c + 2 + a_c\tau_1 + b\tau_{2c})},\\
\delta_6 &= -\frac{2[a_c(1+\omega^2) + (1+\omega^2)^2 + a_c + b(1+\omega^2)]}{a_c + 2 + a_c\tau_1 + b\tau_{2c}},
\end{aligned}
\tag{115}
$$

in which $a_\epsilon = a - a_c$, $\tau_{2\epsilon} = \tau_2 - \tau_{2c}$.

7.2.2. *Case 2: Nonresonant double-Hopf bifurcation* ($n_1 = 2, n_2 = 0$)

Next, we consider a double-Hopf bifurcation and treat both time delays τ_1 and τ_2 as bifurcation parameters. Suppose system (110) undergoes a nonresonant double-Hopf bifurcation from the trivial equilibrium at the critical point: $(\tau_1, \tau_2) = (\tau_{1c}, \tau_{2c})$, and the characteristic equation of the linearized system, $\dot{u}(t) = L_c(u(t), u(t-\tau_{1c}), u(t-\tau_{2c}))$, has two pairs of purely imaginary roots $\pm\mathrm{i}\omega_1$ and $\pm\mathrm{i}\omega_2$, with the ratio $\frac{\omega_1}{\omega_2}$ being an irrational number, and other roots have negative real part. With a simple calculation, we obtain the eigenfunctions associated with the nonresonant double-Hopf bifurcation as follows:

$$
\begin{aligned}
&p_j = (1, 1+\mathrm{i}\omega_j, (1+\mathrm{i}\omega_j)^2)^{\mathrm{T}},\\
&p_j^* = \left(\frac{a\mathrm{e}^{\mathrm{i}\omega_j\tau_{1c}}}{2(1-\mathrm{i}\omega_j)^3 + a\mathrm{e}^{\mathrm{i}\omega_j\tau_{1c}}},\right.\\
&\qquad \frac{(1-\mathrm{i}\omega_j)^2}{2(1-\mathrm{i}\omega_j)^3 + a\mathrm{e}^{\mathrm{i}\omega_j\tau_{1c}}},\\
&\qquad \left.\frac{1-\mathrm{i}\omega_j}{2(1-\mathrm{i}\omega_j)^3 + a\mathrm{e}^{\mathrm{i}\omega_j\tau_{1c}}}\right)^{\mathrm{T}}, \quad j = 1, 2.
\end{aligned}
\tag{116}
$$

The linear solution of system (110), associated with the nonresonant double-Hopf bifurcation, can be written in the form of

$$
\begin{aligned}
u_1 &= G_1p_1\mathrm{e}^{\mathrm{i}\omega_1T_0} + \overline{G}_1\overline{p}_1\mathrm{e}^{-\mathrm{i}\omega_1T_0}\\
&\quad + G_2p_2\mathrm{e}^{\mathrm{i}\omega_2T_0} + \overline{G}_2\overline{p}_2\mathrm{e}^{-\mathrm{i}\omega_2T_0},
\end{aligned}
\tag{117}
$$

where p_j, $j = 1, 2$, are given in Eq. (116). Then, the normal form up to cubic order, associated with the nonresonant double-Hopf bifurcation, is given by

$$\dot{G}_1 = \delta_1 G_1 + \delta_3 G_1^2 \overline{G}_1 + \delta_5 G_1 G_2 \overline{G}_2 + \cdots, \quad \dot{G}_2 = \delta_2 G_2 + \delta_4 G_2^2 \overline{G}_2 + \delta_6 G_1 \overline{G}_1 G_2 + \cdots, \tag{118}$$

where

$$\begin{aligned}
\delta_j &= -\frac{\mathrm{i}\omega_j(1+\mathrm{i}\omega_j)\mathrm{e}^{-\mathrm{i}\omega_j\tau_{1c}}a\tau_{1\epsilon} + \mathrm{i}(1+\mathrm{i}\omega_j)^2 b\tau_{2\epsilon}\omega_j\mathrm{e}^{-\mathrm{i}\omega_j\tau_{2c}}}{2(1+\mathrm{i}\omega_j)^3 + a\mathrm{e}^{-\mathrm{i}\omega_j\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_j\tau_{1c}}(1+\mathrm{i}\omega_j) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_j\tau_{2c}}(1+\mathrm{i}\omega_j)^2}, \quad j = 1, 2,\\
\delta_3 &= -\frac{(1+\mathrm{i}\omega_1)^4(1+\omega_1^2)(2+\omega_1^2) + a(1+\mathrm{i}\omega_1)\mathrm{e}^{-\mathrm{i}\omega_1\tau_{1c}}}{2(1+\mathrm{i}\omega_1)^3 + a\mathrm{e}^{-\mathrm{i}\omega_1\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_1\tau_{1c}}(1+\mathrm{i}\omega_1) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_1\tau_{2c}}(1+\mathrm{i}\omega_1)^2},\\
\delta_4 &= -\frac{(1+\mathrm{i}\omega_2)^4(1+\omega_2^2)(2+\omega_2^2) + a(1+\mathrm{i}\omega_2)\mathrm{e}^{-\mathrm{i}\omega_2\tau_{1c}}}{2(1+\mathrm{i}\omega_2)^3 + a\mathrm{e}^{-\mathrm{i}\omega_2\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_2\tau_{1c}}(1+\mathrm{i}\omega_2) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_2\tau_{2c}}(1+\mathrm{i}\omega_2)^2},\\
\delta_5 &= -\frac{2(1+\mathrm{i}\omega_1)^4(1+\omega_2^2)(2+\omega_2^2) + 2a(1+\mathrm{i}\omega_1)\mathrm{e}^{-\mathrm{i}\omega_1\tau_{1c}}}{2(1+\mathrm{i}\omega_1)^3 + a\mathrm{e}^{-\mathrm{i}\omega_1\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_1\tau_{1c}}(1+\mathrm{i}\omega_1) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_1\tau_{2c}}(1+\mathrm{i}\omega_1)^2},\\
\delta_6 &= -\frac{2(1+\mathrm{i}\omega_2)^4(1+\omega_1^2)(2+\omega_1^2) + 2a(1+\mathrm{i}\omega_2)\mathrm{e}^{-\mathrm{i}\omega_2\tau_{1c}}}{2(1+\mathrm{i}\omega_2)^3 + a\mathrm{e}^{-\mathrm{i}\omega_2\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_2\tau_{1c}}(1+\mathrm{i}\omega_2) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_2\tau_{2c}}(1+\mathrm{i}\omega_2)^2},
\end{aligned} \tag{119}$$

with $\tau_{j\epsilon} = \tau_j - \tau_{jc},\ j = 1, 2.$

7.2.3. *Case 3*: 1:3 *resonant double-Hopf bifurcation* ($n_1 = 2, n_2 = 0$)

Now, for system (110), we consider a 1:3 resonant double-Hopf bifurcation and again treat the time delays τ_1 and τ_2 as two bifurcation parameters. Suppose system (110) undergoes a resonant double-Hopf bifurcation from the trivial equilibrium at the critical point: $(\tau_1, \tau_2) = (\tau_{1c}, \tau_{2c})$, and the characteristic equation of the linearized system, $\dot{u}(t) = L_c(u(t), u(t-\tau_{1c}), u(t-\tau_{2c}))$, has two pairs of purely imaginary roots $\pm\mathrm{i}\omega_1$ and $\pm\mathrm{i}\omega_2$, with $\frac{\omega_1}{\omega_2} = \frac{1}{3}$, and other roots have negative real part. Then, the linear solution of system (110), associated with the 1:3 resonant double Hopf bifurcation, can be expressed as

$$\begin{aligned} u_1 &= G_1 p_1 \mathrm{e}^{\mathrm{i}\omega_0 T_0} + \overline{G}_1 \overline{p}_1 \mathrm{e}^{-\mathrm{i}\omega_0 T_0}\\ &\quad + G_2 p_2 \mathrm{e}^{3\mathrm{i}\omega_0 T_0} + \overline{G}_2 \overline{p}_2 \mathrm{e}^{-3\mathrm{i}\omega_0 T_0}, \end{aligned} \tag{120}$$

where p_j, $j = 1, 2$, are given in Eq. (116) with $\omega_1 = \omega_0$ and $\omega_2 = 3\omega_0$.

By using the MTS method, we obtain the normal form up to cubic order, associated with the 1:3 resonant double-Hopf bifurcation, given by

$$\begin{aligned}
\dot{G}_1 &= \delta_1 G_1 + \delta_2 G_1^2 \overline{G}_1 + \delta_3 G_1 G_2 \overline{G}_2\\
&\quad + \delta_4 \overline{G}_1^2 G_2 + \cdots,\\
\dot{G}_2 &= \delta_5 G_2 + \delta_6 G_2^2 \overline{G}_2 + \delta_7 G_1 \overline{G}_1 G_2\\
&\quad + \delta_8 G_1^3 + \cdots,
\end{aligned} \tag{121}$$

where

$$\begin{aligned}
\delta_1 &= -\frac{\mathrm{i}\omega_0(1+\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}a\tau_{1\epsilon} + \mathrm{i}(1+\mathrm{i}\omega_0)^2 b\tau_{2\epsilon}\omega_0\mathrm{e}^{-\mathrm{i}\omega_0\tau_{2c}}}{2(1+\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}(1+\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{2c}}(1+\mathrm{i}\omega_0)^2},\\
\delta_2 &= -\frac{(1+\mathrm{i}\omega_0)^4(1+\omega_0^2)(2+\omega_0^2) + a(1+\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}}{2(1+\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}(1+\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{2c}}(1+\mathrm{i}\omega_0)^2},\\
\delta_3 &= -\frac{2(1+\mathrm{i}\omega_0)^4(1+9\omega_0^2)(2+9\omega_0^2) + 2a(1+\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}}{2(1+\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}(1+\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{2c}}(1+\mathrm{i}\omega_0)^2},\\
\delta_4 &= -\frac{(1-\mathrm{i}\omega_0)^2(1+3\mathrm{i}\omega_0)(1+\mathrm{i}\omega_0)^3 + (1+\mathrm{i}\omega_0)^2(1-\mathrm{i}\omega_0)^4(1+3\mathrm{i}\omega_0)^2 + a(1+\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}}{2(1+\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{1c}}(1+\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-\mathrm{i}\omega_0\tau_{2c}}(1+\mathrm{i}\omega_0)^2},
\end{aligned}$$

$$\delta_5 = -\frac{3\mathrm{i}\omega_0(1+3\mathrm{i}\omega_0)\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}a\tau_{1\epsilon} + 3\mathrm{i}(1+3\mathrm{i}\omega_0)^2 b\tau_{2\epsilon}\omega_0\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{2c}}}{2(1+3\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}(1+3\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{2c}}(1+3\mathrm{i}\omega_0)^2},$$

$$\delta_6 = -\frac{(1+3\mathrm{i}\omega_0)^4(1+9\omega_0^2)(2+9\omega_0^2) + a(1+3\mathrm{i}\omega_0)\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}}{2(1+3\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}(1+3\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{2c}}(1+3\mathrm{i}\omega_0)^2},$$

$$\delta_7 = -\frac{2(1+3\mathrm{i}\omega_0)^4(1+\omega_0^2)(2+\omega_0^2) + 2a(1+3\mathrm{i}\omega_0)\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}}{2(1+3\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}(1+3\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{2c}}(1+3\mathrm{i}\omega_0)^2},$$

$$\delta_8 = -\frac{(1+\mathrm{i}\omega_0)^3(1+3\mathrm{i}\omega_0)^3 + (1+3\mathrm{i}\omega_0)^2(1+\mathrm{i}\omega_0)^6 + a(1+3\mathrm{i}\omega_0)\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}}{3[2(1+3\mathrm{i}\omega_0)^3 + a\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}} + a\tau_{1c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{1c}}(1+3\mathrm{i}\omega_0) + b\tau_{2c}\mathrm{e}^{-3\mathrm{i}\omega_0\tau_{2c}}(1+3\mathrm{i}\omega_0)^2]}, \tag{122}$$

in which $\tau_{j\epsilon} = \tau_j - \tau_{jc}$, $j = 1, 2$.

7.3. *An NFDE example*

In this subsection, we consider a container crane model with a delayed position feedback [Ding *et al.*, 2013b] to illustrate the application of Theorem 3 (or Corollary 3.1 or 3.2). The equation of the model is described by

$$\begin{aligned}
&\ddot{\phi}(t) + \alpha_1\phi(t) + 2\mu\dot{\phi}(t) + k\ddot{\phi}(t-\tau) \\
&\quad = -\epsilon\alpha_3\phi^3(t) - \epsilon\alpha_4\phi(t)\dot{\phi}^2(t) - \epsilon\alpha_4\phi^2(t)\ddot{\phi}(t) - \epsilon k\phi(t-\tau)\dot{\phi}^2(t-\tau) \\
&\qquad - \epsilon k\alpha_5\phi^2\ddot{\phi}(t-\tau) - \frac{1}{2}\epsilon k\phi^2(t-\tau)\ddot{\phi}(t-\tau),
\end{aligned} \tag{123}$$

where ϕ is the oscillating angle, τ the time delay, μ the inherent damping coefficient, $k = -\frac{\hat{k}}{b-aR}$, in which $\hat{k}$ is the feedback gain (here, we also call k the feedback gain), ϵ is a small dimensionless parameter and $\alpha_i's$ $(i = 1, 3, 4, 5)$ are known constants, given by

$$\alpha_1 = \frac{g\hat{\alpha}_1}{4b(b-aR)^2}, \quad \alpha_3 = \frac{4g\hat{\alpha}_3}{(b-aR)^2}, \quad \alpha_4 = \frac{\hat{\alpha}_1^2 + 96(b-aR)\hat{\alpha}_5}{16b^2(b-aR)^2}, \quad \alpha_5 = \frac{3\hat{\alpha}_5}{b-aR},$$

with

$$a = \frac{d-c}{c}, \quad b = \sqrt{L^2 - \frac{1}{4}a^2c^2}, \quad \hat{\alpha}_1 = 4b^2 + 4a^2bR + a^2(1+a)c^2,$$

$$\hat{\alpha}_3 = \frac{16b^4 + 16a^2(8+12a+3a^2)b^3R + 4a^2(2+14a+15a^2+3a^2)b^2c^2 + 3a^4(1+a)^2c^4}{96b^3},$$

$$\hat{\alpha}_5 = \frac{4b^2 + 4a(2+3a)bR + 3a^2(1+a)c^2}{8b}.$$

It is seen that system (123) has cubic nonlinearity.

Let $\phi(t) = v_1(t)$ and $\dot{\phi}(t) = v_2(t)$. Then, rescale $v_i \to v_i/\sqrt{\epsilon}$ $(i = 1, 2)$ and $t \mapsto t/\tau$, and thus system (123) can be rewritten in the form of (53) with

$$u(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \quad N_0 = \tau_c\begin{pmatrix} 0 & 1 \\ -\alpha_1 & -2\mu \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$F(u_t) = \tau\begin{pmatrix} 0 \\ -\alpha_3 v_1^3(t) - \alpha_4 v_1(t)v_2^2(t) - kv_1(t-1)v_2^2(t-1) \end{pmatrix},$$

$$G(u_t, u(t-1), \dot{u}(t), \dot{u}(t-1)) = \begin{pmatrix} 0 \\ -\alpha_4 v_1^2(t)\dot{v}_2(t) - k\alpha_5 v_1^2(t)\dot{v}_2(t-1) - \dfrac{1}{2}kv_1^2(t-1)\dot{v}_2(t-1) \end{pmatrix}. \tag{124}$$

By a simple linear analysis, we can show that when $1-k^2 > 0$, $\alpha_1 - 4\mu^2 > 0$, system (123) may undergo a double Hopf bifurcation at the critical point: $(k,\tau) = (k_c, \tau_c)$, where τ_c is given by

$$\tau_c = \tau_1^{(j)} = \tau_2^{(l)}, \quad l, j = 0, 1, \ldots,$$

with

$$\tau_{1,2}^{(j)} = \begin{cases} \dfrac{1}{\omega_{1,2}}\left[\arccos\left(\dfrac{\alpha_1 - \omega_{1,2}^2}{k_c\omega_{1,2}^2}\right) + 2j\pi\right], & \text{for } k_c\mu < 0, \\ \dfrac{1}{\omega_{1,2}}\left[2(j+1)\pi - \arccos\left(\dfrac{\alpha_1 - \omega_{1,2}^2}{k_c\omega_{1,2}^2}\right)\right], & \text{for } k_c\mu > 0, \end{cases}$$

where $j = 0, 1, \ldots$, and

$$\omega_{1,2} = \sqrt{\frac{\alpha_1 - 2\mu^2 \pm \sqrt{4\mu^4 + \alpha_1^2k_c^2 - 4\alpha_1\mu^2}}{1-k_c^2}},$$

and k_c is then determined from $\tau_1^{(j)} = \tau_2^{(l)}$, $l, j = 0, 1, \ldots$.

We assume that system (123) undergoes a nonresonant double-Hopf bifurcation at the critical point: $(k,\tau) = (k_c, \tau_c)$, and the characteristic equation of the linearized system of (123) has two pairs of purely imaginary roots $\pm i\omega_1$ and $\pm i\omega_2$, with the remaining roots having negative real part. We take perturbations as $(k,\tau) = (k_c, \tau_c) + \epsilon(k_\epsilon, \tau_\epsilon)$. Then, with the MTS method, by a simple calculation, we have

$$p_j = (1, i\omega_j)^{\mathrm{T}},$$
$$p_j^* = \left(\frac{\alpha_1}{\alpha_1 + \omega_j^2}, \frac{i\omega_j}{\alpha_1 + \omega_j^2}\right)^{\mathrm{T}}, \quad j = 1, 2. \tag{125}$$

The linear solution of system (53), with M_1, N_0 and N_1 given in Eq. (124), associated with the nonresonant double-Hopf bifurcation, can be expressed as

$$u_1 = G_1p_1e^{i\omega_1\tau_cT_0} + \overline{G}_1\bar{p}_1e^{-i\omega_1\tau_cT_0} + G_2p_2e^{i\omega_2\tau_cT_0} + \overline{G}_2\bar{p}_2e^{-i\omega_2\tau_cT_0},$$

where p_j, $j = 1, 2$ are given in Eq. (124). Then, by the MTS method, the normal form up to cubic order is given by

$$\begin{aligned} \dot{G}_1 &= \beta_1G_1 + P_1G_1^2\overline{G}_1 + P_2G_1G_2\overline{G}_2 + \cdots, \\ \dot{G}_2 &= \beta_2G_2 + P_3G_2^2\overline{G}_2 + P_4G_1\overline{G}_1G_2 + \cdots, \end{aligned} \tag{126}$$

where

$$\begin{aligned} \beta_j &= \frac{2(i\omega_j\alpha_1 - \omega_j^2\mu)\tau_\epsilon - i\omega_j^3\tau_ce^{-i\omega_j\tau_c}k_\epsilon}{\alpha_1 + \omega_j^2 + \omega_j^2k_ce^{-i\omega_j\tau_c} - i\omega_j^3k_c\tau_ce^{-i\omega_j\tau_c}}, \quad j = 1, 2, \\ P_1 &= -\frac{i\omega_1\tau_c(2e^{i\omega_1\tau_c}\omega_1^2k_c\alpha_5 - 6\alpha_3 + 4\omega_1^2\alpha_4 + e^{-i\omega_1\tau_c}\omega_1^2k_c + 4e^{-i\omega_1\tau_c}\omega_1^2k_c\alpha_5)}{2(\alpha_1 + \omega_1^2 + \omega_1^2k_ce^{-i\omega_1\tau_c} - i\omega_1^3k_c\tau_ce^{-i\omega_1\tau_c})}, \\ P_2 &= -\frac{i\omega_1\tau_c[4k_c\alpha_5\omega_2^2\cos(\omega_2\tau_c) + e^{-i\omega_1\tau_c}\omega_1^2k_c + 2\alpha_4\omega_2^2 - 6\alpha_3 + 2\omega_1^2\alpha_4 + 2e^{-i\omega_1\tau_c}\omega_1^2k_c\alpha_5]}{\alpha_1 + \omega_1^2 + \omega_1^2k_ce^{-i\omega_1\tau_c} - i\omega_1^3k_c\tau_ce^{-i\omega_1\tau_c}}, \\ P_3 &= -\frac{i\omega_2\tau_c(2e^{i\omega_2\tau_c}\omega_2^2k_c\alpha_5 - 6\alpha_3 + 4\omega_2^2\alpha_4 + e^{-i\omega_2\tau_c}\omega_2^2k_c + 4e^{-i\omega_2\tau_c}\omega_2^2k_c\alpha_5)}{2(\alpha_1 + \omega_2^2 + \omega_2^2k_ce^{-i\omega_2\tau_c} - i\omega_2^3k_c\tau_ce^{-i\omega_2\tau_c})}, \\ P_4 &= -\frac{i\omega_2\tau_c[4k_c\alpha_5\omega_1^2\cos(\omega_1\tau_c) + e^{-i\omega_2\tau_c}\omega_2^2k_c + 2\alpha_4\omega_1^2 - 6\alpha_3 + 2\omega_2^2\alpha_4 + 2e^{-i\omega_2\tau_c}\omega_2^2k_c\alpha_5]}{\alpha_1 + \omega_2^2 + \omega_2^2k_ce^{-i\omega_2\tau_c} - i\omega_2^3k_c\tau_ce^{-i\omega_2\tau_c}}. \end{aligned} \tag{127}$$

Next, for the CMR method, we choose

$$\Phi(\theta) = \begin{bmatrix} e^{i\omega_1\tau_c\theta} & e^{-i\omega_1\tau_c\theta} & e^{i\omega_2\tau_c\theta} & e^{-i\omega_2\tau_c\theta} \\ i\omega_1 e^{i\omega_1\tau_c\theta} & -i\omega_1 e^{-i\omega_1\tau_c\theta} & i\omega_2 e^{i\omega_2\tau_c\theta} & -i\omega_2 e^{-i\omega_2\tau_c\theta} \end{bmatrix}$$

and

$$\Psi(s) = \begin{bmatrix} d_1 e^{-i\omega_1\tau_c s} & -\dfrac{i\omega_1 d_1}{\alpha_1} e^{-i\omega_1\tau_c s} \\ \bar{d}_1 e^{i\omega_1\tau_c s} & \dfrac{i\omega_1 \bar{d}_1}{\alpha_1} e^{i\omega_1\tau_c s} \\ d_2 e^{-i\omega_2\tau_c s} & -\dfrac{i\omega_2 d_2}{\alpha_1} e^{-i\omega_2\tau_c s} \\ \bar{d}_2 e^{i\omega_2\tau_c s} & \dfrac{i\omega_2 \bar{d}_2}{\alpha_1} e^{i\omega_2\tau_c s} \end{bmatrix},$$

where $d_j = \frac{\alpha_1}{\alpha_1+\omega_j^2+\omega_j^2 k_c e^{-i\omega_j\tau_c} - i\omega_j^3 k_c\tau_c e^{-i\omega_j\tau_c}}$, $j = 1, 2$.

We also use the same bifurcation parameters given by $(k,\tau) = (k_c,\tau_c) + (k_\epsilon,\tau_\epsilon)$, where k_ϵ and τ_ϵ are perturbation parameters, and denote $\varepsilon = (k_\epsilon,\tau_\epsilon)$. Thus, $\tilde{F}(w_t,\varepsilon)$ in Eq. (64) becomes

$$\tilde{F}(w_t,\varepsilon) = \begin{pmatrix} \tau_\epsilon v_2(t) \\ -\alpha_1\tau_\epsilon v_1(t) - 2\mu\tau_\epsilon v_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -(\tau_c+\tau_\epsilon)[\alpha_3 v_1^3(t) + \alpha_4 v_1(t)v_2^2(t) + (k_c+k_\epsilon)v_1(t-1)v_2^2(t-1)] \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ -\alpha_4 v_1^2(t)\dot{v}_2(t) - \left[k_\epsilon + (k_c+k_\epsilon)\alpha_5 v_1^2(t) + \dfrac{1}{2}(k_c+k_\epsilon)v_1^2(t-1)\right]\dot{v}_2(t-1) \end{pmatrix}.$$

Let $x = (x_1, \bar{x}_1, x_2, \bar{x}_2)$. Then, substituting $w_t = \Phi x + y_t$ into $\Psi(0)\tilde{F}(w_t,\varepsilon)$, and noting that $\mathrm{Im}(M_2^1)^c$ is spanned by $k_\epsilon x_1 e_1$, $\tau_\epsilon x_1 e_1$, $k_\epsilon \bar{x}_1 e_2$, $\tau_\epsilon \bar{x}_1 e_2$, $k_\epsilon x_2 e_3$, $\tau_\epsilon x_2 e_3$, $k_\epsilon \bar{x}_2 e_4$, $\tau_\epsilon \bar{x}_2 e_4$, and $\mathrm{Im}(M_3^1)^c$ spanned by $x_1^2\bar{x}_1 e_1$, $x_1 x_2\bar{x}_2 e_1$, $x_1\bar{x}_1^2 e_2$, $\bar{x}_1 x_2\bar{x}_2 e_2$, $x_2^2\bar{x}_2 e_3$, $x_1\bar{x}_1 x_2 e_3$, $x_2\bar{x}_2^2 e_4$, $x_1\bar{x}_1\bar{x}_2 e_4$, where e_i $(i = 1,2,3,4)$ is the ith unit vector, we obtain the same normal form (by neglecting the difference in notations) given in Eq. (126) for the NFDE (123), associated with the nonresonant double-Hopf bifurcation.

7.4. *Examples for the PFDE system*

In this subsection, we use two simple examples with different boundary conditions to illustrate the application of Theorem 4 (or Corollary 4.2).

7.4.1. *Hutchinson equation with Neumann boundary condition*

The first example is the Hutchinson equation with Neumann boundary condition [Faria, 2000], given by

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u(x,t)}{\partial x^2} - au(x,t-1)[1+u(x,t)], \\ &\qquad\qquad t > 0, \quad x \in (0,\pi), \\ &\frac{\partial u(x,t)}{\partial t} = 0, \quad x = 0, \pi, \end{aligned} \tag{128}$$

where d and a are positive parameters. By linearizing system (128) at the equilibrium $u = 0$, we obtain the characteristic equations

$$\lambda + ae^{-\lambda} + dk^2 = 0, \quad k = 0, 1, 2, \ldots. \tag{129}$$

It is easy to show that when $a < \frac{\pi}{2}$, all roots of the equations in (129) have negative real part, so the zero solution is asymptotically stable. When $a = a_c = \frac{\pi}{2}$, the equation in (129) with $k = 0$ has a unique pair of purely imaginary roots $\pm i\omega_0 = \pm\frac{\pi}{2}i$, and all the other solutions of the equations in (129) have negative real part. Assume that the equation in (129) with $k = 0$ has a pair of solutions,

$\lambda(a_c)$ and $\overline{\lambda}(a_c)$ at the critical point, $a = a_c$. Then, $\mathrm{Re}\,\lambda'(a_c) > 0$, and a Hopf bifurcation (i.e. $n_1 = 1$, $n_2 = 0$) occurs at $a = a_c$. Note that system (128) has quadratic nonlinearity, and thus Corollary 4.2 can be applied here, while Corollary 4.1 is not applicable for this case. Theorem 4 can also be applied here since normal form of Hopf bifurcation does not have even order terms.

To study the qualitative behavior of system (128) near the critical point: $a_c = \frac{\pi}{2}$, with the MTS method, we take perturbation as $a = a_c + \epsilon a_\epsilon$. Note that system (128) can be written in the form of (71) with

$$K = d, \quad N_0 = 0, \quad N_1 = -a \quad \text{and}$$
$$F(u_t) = -au(x, t-1)u(x, t).$$

By using the MTS method, from the ϵ^1-order LDE, we obtain

$$\mathrm{D}_0 u_1 - d\,\Delta u_1 + a_c u_{1,1} = 0, \tag{130}$$

where $u_{1,1} = u_1(x, T_0 - 1, T_1, \ldots)$. Thus, the solution of Eq. (130), associated with the Hopf bifurcation, can be expressed as

$$u_1(x,t) = \beta_0(x)G_0(T_1, T_2, \ldots)\mathrm{e}^{\mathrm{i}\omega_0 T_0} + \beta_0(x)\overline{G}_0(T_1, T_2, \ldots)\mathrm{e}^{-\mathrm{i}\omega_0 T_0}, \tag{131}$$

where $\beta_0(x) = \cos(kx)|_{k=0} = 1$.

Next, for the ϵ^2-order LDE, we have

$$\mathrm{D}_0 u_2 - d\,\Delta u_2 + a_c u_{2,1} = -\mathrm{D}_1 u_1 + a_c \mathrm{D}_1 u_{1,1} - a_c u_1 u_{1,1} - a_\epsilon u_{1,1}, \tag{132}$$

where $u_{2,1} = u_2(x, T_0 - 1, T_1, \ldots)$. Substituting solution (131) into Eq. (132) and using the formulas in Eq. (78), we obtain

$$\mathrm{D}_1 G_0 = \left.\frac{a_\epsilon G_0}{a_c - \mathrm{e}^{\mathrm{i}\omega_0}}\right|_{\omega_0 = a_c = \frac{\pi}{2}} = \frac{(2\pi - 4\mathrm{i})a_\epsilon G_0}{4 + \pi^2}, \tag{133}$$

which shows that $\mathrm{D}_1 G_0|_{\alpha_\epsilon = 0} = 0$, as expected. Then, solving the resulting differential equation yields the particular solution of u_2 as

$$u_2 = \sum_{k \geq 0} [\eta_{k,1} + \overline{\eta}_{k,1} + \eta_{k,2}\mathrm{e}^{2\mathrm{i}\omega_0 T_0} + \overline{\eta}_{k,2}\mathrm{e}^{-2\mathrm{i}\omega_0 T_0}]\beta_k(x).$$

Note that $\langle \beta_0 \beta_k, \beta_0 \rangle = 0$, $\forall k \geq 1$. Thus, we obtain

$$\eta_{0,1} = -\mathrm{e}^{-\mathrm{i}\omega_0} G_0 \overline{G}_0|_{\omega_0 = \frac{\pi}{2}} = \mathrm{i}G_0\overline{G}_0,$$

$$\eta_{0,2} = -\left.\frac{a_c \mathrm{e}^{-\mathrm{i}\omega_0} G_0^2}{2\mathrm{i}\omega_0 + a_c \mathrm{e}^{-2\mathrm{i}\omega_0}}\right|_{\omega_0 = a_c = \frac{\pi}{2}} = \frac{\pi \mathrm{i} G_0^2}{2\pi\mathrm{i} - \pi}.$$

Hence, using the formulas in Eq. (80) yields

$$\mathrm{D}_2 G_0 = \left.\frac{-a_c(\mathrm{e}^{-2\mathrm{i}\omega_0} + \mathrm{e}^{\mathrm{i}\omega_0})c_2 \overline{G}_0}{1 - a_c \mathrm{e}^{-\mathrm{i}\omega_0}}\right|_{\omega_0 = a_c = \frac{\pi}{2}} = \frac{\pi(1 + \mathrm{i})G_0^2 \overline{G}_0}{(2\mathrm{i} - 1)(2 + \pi\mathrm{i})}. \tag{134}$$

Finally, it follows from Eq. (81) that the normal form of the Hopf bifurcation, near the critical point: $a_c = \frac{\pi}{2}$, derived using the MTS method for the PFDE system (128) is

$$\dot{G}_0 = \frac{(2\pi - 4\mathrm{i})a_\epsilon G_0}{4 + \pi^2} + \frac{\pi(1 + \mathrm{i})G_0^2\overline{G}_0}{(2\mathrm{i} - 1)(2 + \pi\mathrm{i})} + \cdots. \tag{135}$$

In order to compare this result with that given by Faria [2000], we introduce the scaling $x \to x/\sqrt{\pi}$ to Eq. (135), together with $G_0 = \rho \mathrm{e}^{\mathrm{i}(\xi + \frac{\pi}{2}t)}$, to obtain the following system in polar coordinates:

$$\dot{\rho} = \frac{2\pi}{4 + \pi^2} a_\epsilon \rho + \frac{2 - 3\pi}{5(4 + \pi^2)}\rho^3 + O(a_\epsilon^2 \rho + |(\rho, a_\epsilon)|^4),$$
$$\dot{\xi} = -\frac{\pi}{2} + O(|(\rho, a_\epsilon)|),$$

which is identical to that derived by using the CMR method (see Eq. (5.10) in [Faria, 2000]).

7.4.2. *Hutchinson equation with Dirichlet boundary condition*

The second example is a scalar PFDE with Dirichlet boundary condition [Faria, 2000], described by

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) - au(x, t-1)[1 + u(x,t)], \quad t > 0, \quad x \in (0, \pi), \tag{136}$$
$$u(x,t) = 0, \quad x = 0, \pi, \quad t > 0,$$

where a is a positive parameter. In space X, the sequence of eigenvalues of Δ is $\{-k^2\}_{k=1}^{+\infty}$, with normalized eigenfunctions $\beta_k(x) = \sqrt{\frac{2}{\pi}}\sin kx$. By linearizing system (136) at the equilibrium $u = 0$, we obtain the characteristic equations,

$$\lambda + a\mathrm{e}^{-\lambda} + k^2 - 1 = 0, \quad k = 1, 2, \ldots. \tag{137}$$

It is easy to show that when $0 < a < \frac{\pi}{2}$, all roots of the equations in (137) have negative real part, so the zero solution is asymptotically stable. When $a = a_c = \frac{\pi}{2}$, the equation in (137) with $k = 1$ has a unique pair of purely imaginary roots $\pm \mathrm{i}\omega_1 = \pm\frac{\pi}{2}\mathrm{i}$, and all the solutions of the remaining equations in (137) have negative real part. Assume that the equation in (137) with $k = 1$ has a pair of solutions, $\lambda(a_c)$ and $\overline{\lambda}(a_c)$ at the critical point, $a = a_c$. Then, $\mathrm{Re}\,\lambda'(a_c) > 0$, and a Hopf bifurcation (i.e. $n_1 = 1$, $n_2 = 0$) occurs at $a = a_c$. Thus, Corollary 4.2 can be again applied for this example.

To study the qualitative behavior of system (136) near the critical point: $a_c = \frac{\pi}{2}$, with the MTS method, we take perturbation as $a = a_c + \epsilon a_\epsilon$. System (136) can be written in the form of (71) with

$$K = 1, \quad N_0 = 1, \quad N_1 = -a \quad \text{and}$$
$$F(u_t) = -au(x, t-1)u(x, t).$$

By using the MTS method, we obtain the ϵ^1-order LDE as

$$\mathrm{D}_0 u_1 - \Delta u_1 - u_1 + a_c u_{1,1} = 0, \tag{138}$$

where $u_{1,1} = u_1(x, T_0 - 1, T_1, \ldots)$. The solution of Eq. (138), associated with the Hopf bifurcation, can be written in the form of

$$\begin{aligned} u_1(x,t) &= \beta_1(x)G_1(T_1, T_2, \ldots)\mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ &\quad + \beta_1(x)\overline{G}_1(T_1, T_2, \ldots)\mathrm{e}^{-\mathrm{i}\omega_1 T_0}, \end{aligned} \tag{139}$$

where $\beta_1(x) = \sin(kx)|_{k=1} = \sin(x)$.

Next, for the ϵ^2-order LDE, we have

$$\begin{aligned} &\mathrm{D}_0 u_2 - \Delta u_2 - u_2 + a_c u_{2,1} \\ &\quad = -\mathrm{D}_1 u_1 + a_c \mathrm{D}_1 u_{1,1} - a_c u_1 u_{1,1} \\ &\quad\quad - a_\epsilon u_{1,1}, \end{aligned} \tag{140}$$

where $u_{2,1} = u_2(x, T_0 - 1, T_1, \ldots)$. Substituting solution (139) into Eq. (140) and using the formulas in Eq. (78), we obtain

$$\mathrm{D}_1 G_1 = \frac{(2\pi - 4\mathrm{i})a_\epsilon G_1}{4 + \pi^2}. \tag{141}$$

Then, solving the resulting differential equation yields the particular solution of u_2 as

$$\begin{aligned} u_2 = \sum_{k\ge 1} &[\eta_{k,1} + \overline{\eta}_{k,1} + \eta_{k,2}\mathrm{e}^{2\mathrm{i}\omega_0 T_0} \\ &+ \overline{\eta}_{k,2}\mathrm{e}^{-2\mathrm{i}\omega_0 T_0}]\beta_k(x). \end{aligned}$$

Noticing that

$$\begin{aligned} c_k &\triangleq \langle \beta_1\beta_k, \beta_1\rangle \\ &= \begin{cases} 0, & \text{for } k \text{ even}, \\ -\left(\dfrac{2}{\pi}\right)^{3/2} \dfrac{4}{k(k^2-4)}, & \text{for } k \text{ odd}, \end{cases} \end{aligned} \tag{142}$$

where $k \ge 1$, we have

$$\begin{aligned} u_2 &= \sum_{k\ge 1} c_k \frac{\pi\mathrm{i}}{2} \\ &\quad \times \left(\frac{G_1^2 \mathrm{e}^{2\mathrm{i}\omega_1 T_0}}{k^2 - 1 - \dfrac{\pi}{2} + \pi\mathrm{i}} - \frac{\overline{G}_1^2 \mathrm{e}^{-2\mathrm{i}\omega_1 T_0}}{k^2 - 1 - \dfrac{\pi}{2} - \pi\mathrm{i}}\right). \end{aligned}$$

Further, applying the formulas in Eq. (80) results in

$$\begin{aligned} \mathrm{D}_2 G_1 &= \frac{1}{4}\pi^2 \sum_{k\ge 1} c_k^2 \frac{(1+\mathrm{i})}{\left(k^2 - 1 - \dfrac{\pi}{2} + \pi\mathrm{i}\right)\left(1 + \dfrac{\pi}{2}\mathrm{i}\right)} \\ &\quad \times G_1^2 \overline{G}_1. \end{aligned} \tag{143}$$

Finally, it follows from Eq. (81) that the normal form of Hopf bifurcation derived using the MTS method, near the critical point: $a_c = \frac{\pi}{2}$, for the PFDE system (136) is

$$\begin{aligned} \dot{G}_1 &= \frac{(2\pi - 4\mathrm{i})a_\epsilon G_1}{4 + \pi^2} \\ &\quad + \frac{1}{4}\pi^2 \sum_{k\ge 1} c_k^2 \frac{(1+\mathrm{i})G_1^2\overline{G}_1}{\left(k^2 - 1 - \dfrac{\pi}{2} + \pi\mathrm{i}\right)\left(1 + \dfrac{\pi}{2}\mathrm{i}\right)} \\ &\quad + \cdots. \end{aligned} \tag{144}$$

For a comparison with Faria's result, let $G_1 = \rho\mathrm{e}^{\mathrm{i}(\xi + \frac{\pi}{2}t)}$, which transforms (144) to

$$\begin{aligned} \dot{\rho} &= \frac{2\pi}{4 + \pi^2} a_\epsilon \rho + K^* \rho^3 \\ &\quad + O(a_\epsilon^2 \rho + |(\rho, a_\epsilon)|^4), \\ \dot{\xi} &= -\frac{\pi}{2} + O(|(\rho, a_\epsilon)|), \end{aligned} \tag{145}$$

where

$$K^* = \frac{\pi^2}{4+\pi^2}\sum_{k\geq 1} c_k^2 A_k,$$

with

$$A_k = \frac{\left(1+\frac{\pi}{2}\right)\left(k^2-1-\frac{\pi}{2}\right)+\pi\left(1-\frac{\pi}{2}\right)}{\left(k^2-1-\frac{\pi}{2}\right)^2+\pi^2}.$$

System (145) is identical to that obtained by Faria [2000] using the CMR method (see Eq. (5.26) in [Faria, 2000]). Comparing the above procedure with that given in [Faria, 2000] shows that the MTS method is simpler than the CMR method.

7.5. *An example of DDE with distributed delays*

Finally, in this subsection, we consider the van der Pol equation with continuously distributed delay [Liao *et al.*, 2003],

$$\begin{aligned}\dot{u}_1(t) &= \int_0^{+\infty}\kappa(\tau)u_2(t-\tau)d\tau \\ &\quad - f\left(\int_0^{+\infty}\kappa(\tau)u_1(t-\tau)d\tau\right), \\ \dot{u}_2(t) &= -\int_0^{+\infty}\kappa(\tau)u_1(t-\tau)d\tau,\end{aligned} \tag{146}$$

where $f(u) = au + bu^3$. The weight function $\kappa(s)$ is a non-negative bounded function defined on $[0,+\infty)$ that describes the influence of the past states on the current dynamics. It is assumed in this model that the presence of the continuous time delay does not affect the equilibrium values. Therefore, we normalize the kernel to satisfy $\int_0^{+\infty}\kappa(s)ds = 1$.

Here, as a demonstration, we choose $\kappa(s)$ as a Γ-distribution delay kernel, and only consider the case of weak kernel,

$$\kappa(s) = \beta e^{-\beta s}, \quad \beta > 0, \tag{147}$$

since the case of strong kernel can be treated similarly.

The characteristic equation of the linearized system of (146) is

$$\lambda^4 + 2\beta\lambda^3 + \beta(\beta+a)\lambda^2 + \beta^2 = 0. \tag{148}$$

By a simple analysis, assuming $0 < a < 2$, we can show that when $\beta = \beta_c = \frac{4-a^2}{2a}$, Eq. (148) has a pair of purely imaginary roots $\pm i\omega_0$ with $\omega_0 = \sqrt{\frac{\beta_c a}{2}}$, and so system (146) undergoes a Hopf bifurcation at the critical point: $\beta = \beta_c$.

In the following, we compute the normal form of system (146) by directly applying the MTS method to this system without transforming it to a differential system having no distributed delays. First, by a simple calculation, we obtain

$$p_1 = \begin{pmatrix} 1 \\ -\dfrac{\beta_c}{i\omega_0(\beta_c+i\omega_0)}\end{pmatrix}, \quad p_1^* = \begin{pmatrix} 1 \\ -\dfrac{\beta_c d_0}{i\omega_0(\beta_c-i\omega_0)}\end{pmatrix},$$

where

$$\begin{aligned}d_0 &= 1 - \frac{\beta_c^2}{i\omega_0(\beta_c+i\omega_0)^2} - \frac{a\beta_c}{\beta_c+i\omega_0} \\ &\quad + \frac{\beta_c^2}{\omega_0^2(\beta_c+i\omega_0)^2} - \frac{\beta_c^2}{i\omega_0(\beta_c+i\omega_0)^2}.\end{aligned} \tag{149}$$

Then, with the MTS method, we take perturbation as $\beta = \beta_c + \epsilon\beta_\epsilon$. Because the nonlinearity is cubic, we seek a uniform second-order approximate solution of system (146) in powers of $\epsilon^{1/2}$ [Nayfeh, 2008], and thus obtain a set of ordered linear differential equations with respect to $\epsilon^{n/2}$ $(n = 1, 3, 5, \ldots)$. Certainly, we may seek the solution in powers of ϵ, and obtain a set of ordered linear differential equations with respect to ϵ^n $(n = 1, 2, 3, \ldots)$, and the results for these two different scalings are identical. Thus, the solution of system (146) is assumed to take the form:

$$\begin{aligned}u_1 &= \epsilon^{1/2}u_{11} + \epsilon^{3/2}u_{12} + \cdots, \\ u_2 &= \epsilon^{1/2}u_{21} + \epsilon^{3/2}u_{22} + \cdots.\end{aligned}$$

For the $\epsilon^{1/2}$-order LDE, we have

$$\begin{aligned}&D_0 u_{11} - \int_0^{+\infty}\beta_c e^{-\beta_c s}u_{21}(t-s)ds \\ &\quad + a\int_0^{+\infty}\beta_c e^{-\beta_c s}u_{11}(t-s)ds = 0, \\ &D_0 u_{21} - \int_0^{+\infty}\beta_c e^{-\beta_c s}u_{11}(t-s)ds = 0.\end{aligned} \tag{150}$$

The solution of system (150) can be expressed in the form of

$$u^{(1)} = Gp_1 e^{i\omega_0 T_0} + \overline{G}\overline{p}_1 e^{-i\omega_0 T_0}, \tag{151}$$

where $u^{(1)} = (u_{11}, u_{21})^T$.

Next, from the $\epsilon^{3/2}$-order LDE, we obtain

$$\begin{aligned}
&\mathrm{D}_0 u_{12} - \int_0^{+\infty} \beta_c \mathrm{e}^{-\beta_c s} u_{22}(t-s)ds + a\int_0^{+\infty} \beta_c \mathrm{e}^{-\beta_c s} u_{12}(t-s)ds \\
&\quad = -\mathrm{D}_1 u_{11} - \int_0^{+\infty} \beta_c \mathrm{e}^{-\beta_c s}\mathrm{D}_1 u_{21}(t-s)ds - \int_0^{+\infty} \beta_c\beta_\epsilon s \mathrm{e}^{-\beta_c s} u_{21}(t-s)ds \\
&\quad\quad + \int_0^{+\infty} \beta_\epsilon \mathrm{e}^{-\beta_c s} u_{21}(t-s)ds + a\int_0^{+\infty} \beta_c \mathrm{e}^{-\beta_c s}\mathrm{D}_1 u_{11}(t-s)ds \\
&\quad\quad + a\int_0^{+\infty} \beta_c\beta_\epsilon s \mathrm{e}^{-\beta_c s} u_{11}(t-s)ds - a\int_0^{+\infty} \beta_\epsilon \mathrm{e}^{-\beta_c s} u_{11}(t-s)ds \\
&\quad\quad - b\left(\int_0^{+\infty} \beta_c \mathrm{e}^{-\beta_c s} u_{11}(t-s)ds\right)^3, \\
&\mathrm{D}_0 u_{22} + \int_0^{+\infty} \beta_c \mathrm{e}^{-\beta_c s} u_{12}(t-s)ds \\
&\quad = -\mathrm{D}_1 u_{21} + \int_0^{+\infty} \beta_c \mathrm{e}^{-\beta_c s}\mathrm{D}_1 u_{11}(t-s)ds + \int_0^{+\infty} \beta_c\beta_\epsilon s \mathrm{e}^{-\beta_c s} u_{11}(t-s)ds \\
&\quad\quad - \int_0^{+\infty} \beta_\epsilon \mathrm{e}^{-\beta_c s} u_{11}(t-s)ds.
\end{aligned} \tag{152}$$

Substituting solution (151) into Eq. (152), and using solvability conditions, we obtain the normal form of system (146) derived using the MTS method, associated with Hopf bifurcation, as

$$\dot{G} = \frac{h_1}{d_0}G + \frac{h_2}{d_0}G^2\overline{G} + \cdots, \tag{153}$$

where d_0 is given in Eq. (149), and

$$\begin{aligned}
h_1 &= \frac{2\beta_c^3\beta_\epsilon - 2\beta_c\beta_\epsilon(\beta_c + \mathrm{i}\omega_0) + a\beta_c\beta_\epsilon(\beta_c + \mathrm{i}\omega_0)\mathrm{i}\omega_0 - a\beta_\epsilon \mathrm{i}\omega_0(\beta_c + \mathrm{i}\omega_0)^2}{\mathrm{i}\omega_0(\beta_c + \mathrm{i}\omega_0)^3}, \\
h_2 &= -\frac{b\beta_c^3}{(\beta_c + \mathrm{i}\omega_0)(\beta_c - \mathrm{i}\omega_0)}.
\end{aligned}$$

Now, for the CMR method we choose

$$\Phi(\theta) = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\omega_0\theta} & \mathrm{e}^{-\mathrm{i}\omega_0\theta} \\ -\dfrac{\beta_c}{\mathrm{i}\omega_0(\beta_c + \mathrm{i}\omega_0)}\mathrm{e}^{\mathrm{i}\omega_0\theta} & \dfrac{\beta_c}{\mathrm{i}\omega_0(\beta_c - \mathrm{i}\omega_0)}\mathrm{e}^{-\mathrm{i}\omega_0\theta} \end{bmatrix} \quad \text{and} \quad \Psi(s) = \begin{bmatrix} d_0\mathrm{e}^{-\mathrm{i}\omega_0 s} & \dfrac{d_0\beta_c}{(\beta_c + \mathrm{i}\omega_0)\mathrm{i}\omega_0}\mathrm{e}^{-\mathrm{i}\omega_0 s} \\ \bar{d}_0\mathrm{e}^{\mathrm{i}\omega_0 s} & -\dfrac{\bar{d}_0\beta_c}{(\beta_c - \mathrm{i}\omega_0)\mathrm{i}\omega_0}\mathrm{e}^{\mathrm{i}\omega_0 s} \end{bmatrix}.$$

We use the same bifurcation parameter, given by $\beta = \beta_c + \beta_\epsilon$, where β_ϵ is a perturbation. Thus, similar to the treatment for the DDEs, in the space BC, system (146) becomes an abstract ODE

$$\frac{\mathrm{d}w}{\mathrm{d}t} = Aw + X_0\tilde{F}(w_t, \beta_\varepsilon), \tag{154}$$

where $w \in \mathrm{C}$, and A is defined by

$$A : \mathrm{C}^1 \to \mathrm{BC}, \quad Aw = \dot{w} + X_0[L_0 w - \dot{w}(0)],$$

with

$$L_0 w = \begin{pmatrix} \displaystyle\int_0^{+\infty} \beta_c e^{-\beta_c s} u_2(t-s)ds - a\int_0^{+\infty} \beta_c e^{-\beta_c s} u_1(t-s)ds \\ \displaystyle -\int_0^{+\infty} \beta_c e^{-\beta_c s} u_1(t-s)ds \end{pmatrix}$$

and

$$\tilde{F}(w_t, \beta_\epsilon) = \begin{pmatrix} \displaystyle\int_0^{+\infty} (\beta_c+\beta_\epsilon) e^{-(\beta_c+\beta_\epsilon)s} u_2(t-s)ds - \int_0^{+\infty} \beta_c e^{-\beta_c s} u_2(t-s)ds \\ \displaystyle\int_0^{+\infty} \beta_c e^{-\beta_c s} u_1(t-s)ds - \int_0^{+\infty} (\beta_c+\beta_\epsilon) e^{-(\beta_c+\beta_\epsilon)s} u_1(t-s)ds \end{pmatrix}$$
$$+ \begin{pmatrix} \displaystyle a\int_0^{+\infty} \beta_c e^{-\beta_c s} u_1(t-s)ds - f\left(\int_0^{+\infty} (\beta_c+\beta_\epsilon) e^{-(\beta_c+\beta_\epsilon)s} u_1(t-s)ds\right) \\ 0 \end{pmatrix}.$$

Let $x = (x_1, \bar{x}_1)$. Further, denote $w_t = \Phi x + y_t$. Then, Eq. (154) is decomposed to

$$\begin{aligned} \frac{dx}{dt} &= Bx + \Psi(0)\tilde{F}(\Phi x + y_t, \beta_\epsilon), \\ \frac{dy_t}{dt} &= A_{Q^1} y_t + (I-\pi)X_0 \tilde{F}(\Phi x + y_t, \beta_\epsilon), \end{aligned} \tag{155}$$

where $B = \mathrm{diag}\{i\omega_0, -i\omega_0\}$.

Moreover, let M_2^1 denote the operator defined in $V_2^3(\mathrm{C}^2 \times \mathrm{Ker}^\pi)$, with

$$M_2^1 : V_2^3(\mathrm{C}^2) \mapsto V_2^3(\mathrm{C}^2),$$
$$(M_2^1 p)(x, \beta_\epsilon) = D_x p(x, \beta_\epsilon)Bx - Bp(x, \beta_\epsilon),$$

where $V_2^3(\mathrm{C}^2)$ represents the linear space of the second-degree homogeneous polynomials in the three variables $(x_1, \bar{x}_1, \beta_\epsilon)$ with coefficients in C^2. Then, one may choose the decomposition $V_2^3(\mathrm{C}^2) = \mathrm{Im}(M_2^1) \oplus \mathrm{Im}(M_2^1)^c$ with the complementary space $\mathrm{Im}(M_2^1)^c$ spanned by $\beta_\epsilon x_1 e_1$ and $\beta_\epsilon \bar{x}_1 e_2$, where e_i $(i = 1, 2)$ is the ith unit vector.

Similarly, let M_3^1 denote the operator defined in $V_3^2(\mathrm{C}^2 \times \mathrm{Ker}^\pi)$, with

$$M_3^1 : V_3^2(\mathrm{C}^2) \mapsto V_3^2(\mathrm{C}^2),$$
$$(M_3^1 p)(x, \beta_\epsilon) = D_x p(x, \beta_\epsilon)Bx - Bp(x, \beta_\epsilon),$$

where $V_3^2(\mathrm{C}^2)$ stands for the linear space of the third-degree homogeneous polynomials in the two variables $(x_1, \bar{x}_1)$ with coefficients in C^2. Then, one may choose the decomposition $V_3^2(\mathrm{C}^2) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$ with the complementary space $\mathrm{Im}(M_3^1)^c$ spanned by $x_1^2 \bar{x}_1 e_1$ and $x_1 \bar{x}_1^2 e_2$, where e_i $(i = 1, 2)$ is the ith unit vector. Thus, without giving the detailed calculations, we obtain the same normal form (by neglecting the difference in notations) given in Eq. (153) for the DDE system (146) with distributed delays, associated with the Hopf bifurcation.

8. Conclusion and Discussion

In this paper, we have considered ordinary differential equations and general delay differential equations, including delay differential equations, neutral functional differential equations and partial functional differential equations, with particular attention focused on the semisimple n_1-Hopf–n_2-zero singularity. We have applied the multiple time scales and center manifold reduction methods to derive the normal forms associated with the semisimple n_1-Hopf–n_2-zero singular point for ordinary differential equations, and rigorously proved the equivalence of the two methods, which yields the identical normal form up to any order for such a singularity. For general delay differential systems, if the second-order terms in the normal form vanish at the critical point, then the normal forms associated with the semisimple n_1-Hopf–n_2-zero singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order. This condition can be fulfilled by either that the system does not contain quadratic terms or that the semisimple n_1-Hopf bifurcation is considered. There two cases often occur in real applications. For illustrations, a number of practical examples have been used to show the application of the theoretical results, particularly associated with Hopf, Hopf-zero and double-Hopf bifurcations.

It has been shown that for differential equations with time delays, the computation using the MTS method is simpler than that of the CMR method, and can deal with multiple discrete delays, for which the CMR method cannot if at least one of the delays is chosen as a bifurcation parameter. Using the MTS method also makes it much easier to develop symbolic software by using an computer algebra system, such as Maple or Mathematica. This is particularly useful for those who use the MTS method to solve physical, engineering or biological system problems. Maple programs for general delay differential equations, associated with the semisimple case, are being developed, which only require a user to prepare a simple input file without any interaction when executing the programs. However, the extension of applying the MTS method from the semisimple case to nonsemisimple case, as well as to the case $n_1 = 0$ (i.e. without purely imaginary eigenvalues, e.g. the Bogdanov–Takens bifurcation), is not straightforward, while the CMR method can deal with these cases, for example, the Bogdanov–Takens bifurcation (double-zero bifurcation).

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