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EXISTENCE CONDITIONS OF THIRTEEN LIMIT CYCLES IN A CUBIC SYSTEM*

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In this paper, we study a cubic system and obtain a concrete condition under which the cubic system has 13 limit cycles.

Keywords: Cubic; limit cycle; bifurcation; polynomial system.

1. Introduction and Main Result

As we know, the second part of the Hilbert problem is to find the maximal number and relative locations of limit cycles of polynomial systems of degree n. Let $H(n)$ denote this number, which is called the Hilbert number. Then the problem of finding $H(n)$ is divided into two parts: find an upper and lower bounds of it. For the upper bound there are important works of Écalle [1990] and IIyashenko and Yakovenko [1991]. However, if $H(n) < \infty$ holds or not is still open, even for the case $n = 2$. On the other hand, many works have been done on the lower bound, especially for quadratic and cubic systems. See [Li, 2003] for a detailed introduction to recent advancement of the problem. For example, Bautin [1952] proved $H(2) \geq 3$ by studying Hopf bifurcation. Chen and Wang [1979] and Shi [1980] separately proved $H(2) \geq 4$. Li and Huang [1987] first found a cubic system having 11 limit cycles, giving $H(3) \geq 11$. Li and Liu [1991], and Liu *et al.* [2003] respectively found more cubic systems having 11 limit cycles with the same distribution. Later, Han *et al.* [2004] and Han *et al.* [2004, 2005] used the method of stability-changing of a homoclinic loop to give more cubic systems having 11 limit cycles with two different distributions. Then Zhang *et al.* [2005] studied an asymmetric cubic system and found three different distributions of 11 limit cycles. Yu and Han [2004, 2005a, 2005b] proved further $H(3) \geq 12$ by studying Hopf bifurcation in some centrally symmetric cubic systems. Recently, Liu and Li [2008] obtained a sufficient condition for

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the existence of 13 limit cycles in this kind of cubic systems. The 13 limit cycles have the distribution: one large limit cycle bifurcated from the equator surrounds 12 small limit cycles which are bifurcated from two symmetric foci. Then in the same year, Li *et al.* [2009] considered a cubic system of the form

$$\begin{aligned}\dot{x} &= y(y^2 - k^2),\\ \dot{y} &= x(x+1)(x-\lambda)\\ &\quad + \varepsilon y(\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y^2),\end{aligned} \tag{1}$$

where $0 < \lambda < 1, k > 10$, and $\alpha_1, \alpha_2, \alpha_3$ and α_4 are parameters, and proved that the system can also have 13 limit cycles if k is sufficiently large. The limit cycles are obtained by proving the existence of zeros of Melnikov functions based on some known results, and present a new distribution. Both works of Liu and Li [2008], Li *et al.* [2009] are the best results so far for cubic systems and are very important, yielding $H(3) \geq 13$.

In this paper, motivated by the work of Li *et al.* [2009], we consider the following cubic system

$$\begin{aligned}\dot{x} &= 2y(y^2 - k^2),\\ \dot{y} &= -(x^3 + bx^2 - x)\\ &\quad - \varepsilon y(\delta_1 + \delta_2 x + \delta_3 x^2 + \delta_4 y^2),\end{aligned} \tag{2}$$

where $k > 0, b > 0$, and $\delta_1, \delta_2, \delta_3$ and δ_4 are parameters. It is easy to see that (1) and (2) are equivalent. We use the method developed in Han and Chen [2000], Han *et al.* [2008], Yang and Han [2007] to prove that system (2) can have 13 limit cycles. The main purpose is to give a concrete condition for the existence of 13 limit cycles, which can be taken as an improvement of the work by Li *et al.* [2009] in two aspects: a simple and concrete condition for the existence of 13 limit cycles is given; on the other hand, the proof method (i.e. the way to find zeros of Melnikov functions) is simpler and more direct. The condition here is also much simpler than that of Liu and Li [2008]. More precisely, we have the following main result.

Theorem 1. *System* (*2*) *has 13 limit cycles if*

$$k = 100, \quad b = \frac{7}{2}, \quad 0 < k_1\delta_4 < \delta_3 < k_2\delta_4 \tag{3}$$

where

$$k_1 \approx 0.75249999999975, \quad k_2 \approx 0.75250000000036,$$

and

$$0 < \varepsilon \ll K_0 + K_1\delta_1 - \delta_2 \ll \delta_1^* - \delta_1 \ll 1 \tag{4}$$

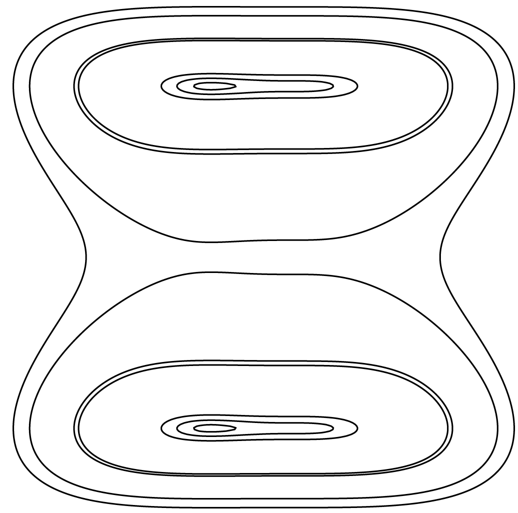

Fig. 1. The distribution of the 13 limit cycles of system (2).

with

$$\begin{aligned}K_0 &\approx \frac{775149067775642000000}{8051754528899929}\delta_4\\ &\quad + \frac{2127709682470669621}{603881589667494675}\delta_3,\\ K_1 &\approx \frac{38757455034549104}{120776317933498935}, \quad \delta_1^* = -30000\delta_4.\end{aligned}$$

In particular, if

$$\delta_3 = 20000, \quad \delta_4 = \frac{8 \times 10^6}{301} \tag{5}$$

and (*4*) *holds with*

$$\begin{aligned}K_0 &\approx \frac{265838587879011391034962400}{1038676334228090841},\\ K_1 &\approx \frac{38757455034549104}{120776317933498935}\\ \delta_1^* &= -\frac{24}{301} \times 10^{10}\end{aligned} \tag{6}$$

then system (*2*) *has 13 limit cycles.*

Remark. The distribution of the 13 limit cycles under the condition of the above theorem is just the same as obtained in Li *et al.* [2009] as shown in Fig. 1.

2. Proof of the Main Result

Consider system (2). For $\varepsilon = 0$, (2) is Hamiltonian with

$$H(x, y) = \frac{1}{2}y^4 - k^2y^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{b}{3}x^3, \tag{7}$$

and has five centers A_1, A_2, A_3, A_4, O (the origin) and four saddles S_0, S_1, S_2, S_3, where

$$\begin{aligned} A_1 &= (x_1(b), k) \quad A_2 = (x_2(b), k), \\ A_3 &= (x_1(b), -k), \quad A_4 = (x_2(b), -k), \\ S_0 &= (0, k), \quad S_1 = (x_1(b), 0), \\ S_2 &= (x_2(b), 0), \quad S_3 = (0, -k) \end{aligned} \tag{8}$$

and $x_i(b) = (-b + (-1)^i\sqrt{b^2+4})/2, i = 1, 2$. Let

$$\begin{aligned} \alpha_1 &= H(A_1) = H(A_3), \\ \alpha_2 &= H(A_2) = H(A_4), \\ h_0 &= H(S_0) = H(S_3), \\ h_1 &= H(S_1), \\ h_2 &= H(S_2). \end{aligned} \tag{9}$$

For example, for $k = 100, b = 7/2$, then (8) and (9) become

$$\begin{aligned} A_1 &= \left(-\frac{7}{4} - \frac{\sqrt{65}}{4}, 100\right), \\ A_2 &= \left(-\frac{7}{4} + \frac{\sqrt{65}}{4}, 100\right), \\ A_3 &= \left(-\frac{7}{4} - \frac{\sqrt{65}}{4}, -100\right), \\ A_4 &= \left(-\frac{7}{4} + \frac{\sqrt{65}}{4}, -100\right), \\ S_0 &= (0, 100), \\ S_1 &= \left(-\frac{7}{4} - \frac{\sqrt{65}}{4}, 0\right), \\ S_2 &= \left(-\frac{7}{4} + \frac{\sqrt{65}}{4}, 0\right), \\ S_3 &= (0, -100) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \alpha_1 &= -\frac{19200003673}{384} - \frac{455}{384}\sqrt{65}, \\ \alpha_2 &= -\frac{19200003673}{384} + \frac{455}{384}\sqrt{65}, \\ h_0 &= -50000000, \\ h_1 &= -\frac{3673}{384} - \frac{455}{384}\sqrt{65}, \\ h_2 &= -\frac{3673}{384} + \frac{455}{384}\sqrt{65} \end{aligned} \tag{11}$$

respectively. It is easy to see that $\alpha_1 < \alpha_2 < h_0 < h_1 < h_2 < 0$.

Then for each of h_1 and h_2, the equation $H(x, y) = h_j$ defines a double homoclinic loop passing through S_j, denoted by L_j, and for h_0, the equation $H(x, y) = h_0$ with $y > 0$ defines a double homoclinic loop passing through S_0, denoted by L_0. Let $L_{01} = L_0|_{x\le 0}, L_{02} = L_0|_{x\ge 0}$, $L_{11} = L_1|_{y\ge 0}, L_{12} = L_1|_{y\le 0}$, and L_{21} be the homoclinic loop passing through S_2 and surrounding a unique singular point at the origin, and L_{22} be the homoclinic loop passing through S_2 and surrounding all other singular points. Then $L_j = L_{j1} \cup L_{j2}, j = 0, 1, 2$. See Fig. 2.

Introduce

$$\begin{aligned} &L_h^{\pm}: H(x, y) = h, \quad h \in (h_0, h_1), \;\; \pm y > 0, \\ &L_{jh}^{\pm}: H(x, y) = h, \quad h \in (\alpha_j, h_0), \;\; j = 1, 2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \pm y > 0 \\ &\tilde{L}_{1h}: H(x, y) = h, \quad h \in (h_2, 0), \;\; \tilde{L}_{1h} \subset \operatorname{Int} L_{21}, \\ &\tilde{L}_{2h}: H(x, y) = h, \quad h > h_2, \;\; \operatorname{Int} \tilde{L}_{2h} \supset L_{22}, \end{aligned}$$

with $\tilde{L}_{2h}$ surrounding all the singular points. In the following, we fix $(k, b, \delta_3, \delta_4)$ satisfying (3). Then

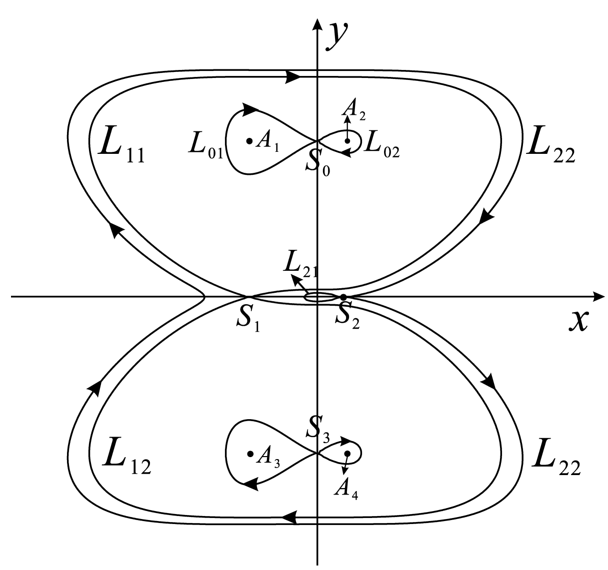

Fig. 2. Four double homoclinic loops of $(2)|_{\varepsilon=0}$.

correspondingly we have the following Melnikov functions

$$M^{\pm}(h,\delta_1,\delta_2)=\oint_{L_h^{\pm}} q(x,y,\delta_1,\delta_2)dx, \quad h\in(h_0,h_1),$$
$$M_j^{\pm}(h,\delta_1,\delta_2)=\oint_{L_{jh}^{\pm}} q(x,y,\delta_1,\delta_2)dx, \quad h\in(\alpha_j,h_0),\quad j=1,2,$$
$$\tilde{M}_1(h,\delta_1,\delta_2)=\oint_{\tilde{L}_{1h}} q(x,y,\delta_1,\delta_2)dx, \quad h_2<h<0,$$
$$\tilde{M}_2(h,\delta_1,\delta_2)=\oint_{\tilde{L}_{2h}} q(x,y,\delta_1,\delta_2)dx, \quad h>h_2, \tag{12}$$

where

$$q(x,y,\delta_1,\delta_2)=-y(\delta_1+\delta_2 x+\delta_3 x^2+\delta_4 y^2). \tag{13}$$

Since $H(x,y)$ is even in y and q is odd in y, we have (see [Li *et al.*, 2009])

$$\begin{aligned} M^+(h,\delta_1,\delta_2)&=M^-(h,\delta_1,\delta_2),\\ M_j^+(h,\delta_1,\delta_2)&=M_j^-(h,\delta_1,\delta_2), \quad j=1,2. \end{aligned} \tag{14}$$

Thus, we will only consider the zeros of M^+, M_j^+ and $\tilde{M}_j$, $j=1,2$.

For convenience, we let

$$\begin{aligned} M(h,\delta_1,\delta_2)&=M^+(h,\delta_1,\delta_2),\\ M_j(h,\delta_1,\delta_2)&=M_j^+(h,\delta_1,\delta_2), \quad j=1,2. \end{aligned} \tag{15}$$

We will prove Theorem 1 by finding zeros of the functions M, M_1, M_2 and $\tilde{M}_2$. First, on their analytical property at the endpoints of their domain, by Remark 1.4 in [Han & Chen, 2000] or Theorem 2.2 in [Han *et al.*, 2008], Theorem 1.2 in [Han, 2000] and Lemma 2.9 in [Han, 2006] we have

Lemma 1. *Let (3) be satisfied by* δ_3 *and* δ_4 *constant. Then*

$$\begin{aligned} M(h,\delta_1,\delta_2)&=c_0+2c_1(h-h_0)\ln|h-h_0|+c_2(h-h_0)\\ &\quad+O(|h-h_0|^2\ln|h-h_0|), \quad 0<h-h_0\ll 1,\\ M(h,\delta_1,\delta_2)&=\bar{c}_0+O(|h-h_1|\ln|h-h_1|), \quad 0<h_1-h\ll 1,\\ M_j(h,\delta_1,\delta_2)&=c_{0j}+c_1(h-h_0)\ln|h-h_0|+c_{2j}(h-h_0)\\ &\quad+O(|h-h_0|^2\ln|h-h_0|), \quad j=1,2,\ 0<h_0-h\ll 1,\\ M_1(h,\delta_1,\delta_2)&=b_{01}(h-\alpha_1)+O((h-\alpha_1)^2), \quad 0<h-\alpha_1\ll 1,\\ \tilde{M}_j(h,\delta_1,\delta_2)&=\tilde{c}_{0j}+O(|h-h_2|\ln|h-h_2|), \quad j=1,2,\ 0<h-h_2\ll 1,\\ \tilde{M}_1(h,\delta_1,\delta_2)&=\tilde{b}_{01}h+O(h^2), \quad 0<-h\ll 1, \end{aligned} \tag{16}$$

where

$$\begin{aligned} c_{0j}&=M_j(h_0,\delta_1,\delta_2)\\ &=\oint_{L_{0j}} q(x,y,\delta_1,\delta_2)dx, \quad j=1,2,\\ c_0&=c_{01}+c_{02},\\ c_1&=\frac{1}{200}\delta_1+150\delta_4,\\ c_2&=\bar{c}_2+O(c_1), \quad \bar{c}_2=\bar{c}_{21}+\bar{c}_{22},\\ \bar{c}_{2j}&=\oint_{L_{0j}}[q_y(x,y,\delta_1,\delta_2)-q_y(0,100,\delta_1,\delta_2)]dt, \quad j=1,2,\\ \bar{c}_0&=\oint_{L_{11}} q(x,y,\delta_1,\delta_2)dx,\\ \tilde{c}_{0j}&=\oint_{L_{2j}} q(x,y,\delta_1,\delta_2)dx, \quad j=1,2. \end{aligned} \tag{17}$$

and

$$\begin{aligned} \tilde{b}_{01}&=\frac{\sqrt{2}\pi}{100}q_y(O,\delta_1,\delta_2)=-\frac{\sqrt{2}\pi}{100}\delta_1,\\ b_{01}&=\frac{\sqrt{2}\pi q_y(A_1,\delta_1,\delta_2)}{50\sqrt{65+7\sqrt{65}}}\\ &=\frac{2\sqrt{2}\pi}{\sqrt{65+7\sqrt{65}}}\left(-\frac{1}{100}\delta_1+\frac{7+\sqrt{65}}{400}\delta_2\right.\\ &\quad\left.-\frac{57+7\sqrt{65}}{800}\delta_3-300\delta_4\right). \end{aligned} \tag{18}$$

Under the condition (3), (7) becomes

$$H(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^4 - 10000y^2 - \frac{1}{2}x^2 + \frac{7}{6}x^3.$$

By (17) we have

$$\begin{aligned} c_{0j} &= -(\delta_1 I_{j1} + \delta_2 I_{j2} + \delta_3 I_{j3} + \delta_4 I_{j4}), \quad j=1,2, \\ \bar{c}_{2j} &= -\oint_{L_{0j}} (\delta_2 x + \delta_3 x^2 + 3\delta_4(y^2 - 10^4))dt \\ &= -\oint_{L_{0j}} \frac{\delta_2 x + \delta_3 x^2 + 3\delta_4(y^2-10^4)}{2y(y^2-10^4)}dx \\ &= -(\delta_2 J_{j2} + \delta_3 J_{j3} + 3\delta_4 J_{j4}), \quad j=1,2, \end{aligned} \tag{19}$$

where

$$I_{ji} = \oint_{L_{0j}} x^{i-1} y dx = \int_{\hat{x}_j}^0 x^{i-1}(y_1 - y_2)dx, \quad j=1,2, \ \ i=1,2,3,$$

$$I_{j4} = \oint_{L_{0j}} y^3 dx = \int_{\hat{x}_j}^0 (y_1^3 - y_2^3)dx, \quad j=1,2,$$

$$\begin{aligned} J_{ji} &= \oint_{L_{0j}} \frac{x^{i-1}dx}{2y(y^2-10000)} \\ &= \int_{\hat{x}_j}^0 \left(\frac{x^{i-1}}{2y_1(y_1^2-10000)} - \frac{x^{i-1}}{2y_2(y_2^2-10000)}\right)dx, \\ &\qquad j=1,2, \ \ i=2,3, \end{aligned}$$

$$J_{j4} = \oint_{L_{0j}} \frac{1}{2y}dx = \int_{\hat{x}_j}^0 \left(\frac{1}{2y_1} - \frac{1}{2y_2}\right)dx, \quad j=1,2$$

with

$$H(\hat{x}_1, 100) = H(\hat{x}_2, 100) = h_0,$$

$$\begin{aligned} \hat{x}_1 &= -\frac{7}{3} - \frac{\sqrt{67}}{3}, \\ y_1 &= \frac{1}{6}\sqrt{360000 - 6x\sqrt{-18x^2 - 84x + 36}}, \\ \hat{x}_2 &= -\frac{7}{3} + \frac{\sqrt{67}}{3}, \\ y_2 &= \frac{1}{6}\sqrt{360000 + 6x\sqrt{-18x^2 - 84x + 36}}. \end{aligned} \tag{20}$$

By Maple we can obtain

$$\begin{aligned} I_{11} &\approx 0.19336179335719, \\ I_{12} &\approx -0.60397799779263, \\ I_{13} &\approx 2.1276842302270, \\ I_{14} &\approx 5800.8535538510, \\ I_{21} &\approx 0.42548181555552 \times 10^{-3}, \\ I_{22} &\approx 0.96408125135325 \times 10^{-4}, \\ I_{23} &\approx 0.25452243669621 \times 10^{-4}, \\ I_{24} &\approx 12.764454466315, \\ J_{12} &\approx -0.036722972448375, \\ J_{13} &\approx 0.10568693840688, \\ J_{14} &\approx -0.96680909021854 \times 10^{-5}, \\ J_{22} &\approx 0.77058591984887 \times 10^{-2}, \\ J_{23} &\approx 0.20196618711375 \times 10^{-2}, \\ J_{24} &\approx -0.21274090779586 \times 10^{-7}. \end{aligned} \tag{21}$$

By (17) again, we have

$$\begin{aligned} \bar{c}_0 &= -(\delta_1 \bar{I}_1 + \delta_2 \bar{I}_2 + \delta_3 \bar{I}_3 + \delta_4 \bar{I}_4), \\ \tilde{c}_{01} &= -(\delta_1 \tilde{I}_1 + \delta_2 \tilde{I}_2 + \delta_3 \tilde{I}_3 + \delta_4 \tilde{I}_4), \\ \tilde{c}_{02} &= -(\delta_1 \tilde{J}_1 + \delta_2 \tilde{J}_2 + \delta_3 \tilde{J}_3 + \delta_4 \tilde{J}_4), \end{aligned} \tag{22}$$

where

$$\bar{I}_i = \oint_{L_{11}} x^{i-1} y dx = \int_{\bar{x}_1}^{\bar{x}_2} x^{i-1}(\bar{y}_1 - \bar{y}_2)dx, \quad i=1,2,3,$$

$$\bar{I}_4 = \oint_{L_{11}} y^3 dx = \int_{\bar{x}_1}^{\bar{x}_2} (\bar{y}_1^3 - \bar{y}_2^3)dx,$$

$$H(\bar{x}_1, 100) = H(\bar{x}_1, 100) = h_1, \quad \bar{x}_1 \approx -120.10904980338, \quad \bar{x}_2 \approx 117.77510233465,$$

$$\bar{y}_1 = \frac{1}{12}\sqrt{1440000 + 6\sqrt{57599988981 - 1365\sqrt{65} + 576x^2 - 1344x^3 - 288x^4}},$$

$$\bar{y}_2 = \frac{1}{12}\sqrt{1440000 - 6\sqrt{57599988981 - 1365\sqrt{65} + 576x^2 - 1344x^3 - 288x^4}},$$

and

$$\begin{aligned}
&\tilde{I}_i = \oint_{L_{21}} x^{i-1} y dx = 2\int_{\tilde{x}_4}^{\tilde{x}_3} x^{i-1}\tilde{y}_2 dx, \quad \tilde{I}_4 = \oint_{L_{21}} y^3 dx = 2\int_{\tilde{x}_4}^{\tilde{x}_3} \tilde{y}_2^3 dx, \quad i = 1,2,3,\\
&\tilde{J}_i = \oint_{L_{22}} x^{i-1} y dx = 2\left(\int_{\tilde{x}_1}^{\tilde{x}_5} x^{i-1}\tilde{y}_1 dx + \int_{\tilde{x}_2}^{\tilde{x}_1} x^{i-1}\tilde{y}_2 dx + \int_{\tilde{x}_5}^{\tilde{x}_4} x^{i-1}\tilde{y}_2 dx\right), \quad i = 1,2,3,\\
&\tilde{J}_4 = \oint_{L_{22}} y^3 dx = 2\left(\int_{\tilde{x}_1}^{\tilde{x}_5} \tilde{y}_1^3 dx + \int_{\tilde{x}_2}^{\tilde{x}_1} \tilde{y}_2^3 dx + \int_{\tilde{x}_5}^{\tilde{x}_4} \tilde{y}_2^3 dx\right),\\
&H(\tilde{x}_1, 100) = H(\tilde{x}_2, 0) = H(\tilde{x}_3, 0) = H(\tilde{x}_4, 0) = H(\tilde{x}_5, 100) = h_2, \quad \tilde{x}_1 < \tilde{x}_2 < \tilde{x}_3 < \tilde{x}_4 < \tilde{x}_5,\\
&\tilde{x}_1 \approx -120.1090612, \quad \tilde{x}_2 = -\frac{7}{12} - \frac{\sqrt{65}}{4} - \frac{1}{6}\sqrt{49 + 21\sqrt{65}},\\
&\tilde{x}_3 = -\frac{7}{12} - \frac{\sqrt{65}}{4} + \frac{1}{6}\sqrt{49 + 21\sqrt{65}}, \quad \tilde{x}_4 = -\frac{7}{4} + \frac{1}{4}\sqrt{65}, \quad \tilde{x}_5 \approx 117.7751137,\\
&\tilde{y}_1 = \frac{1}{12}\sqrt{1440000 + 6\sqrt{57599988981 + 1365\sqrt{65} + 576x^2 - 1344x^3 - 288x^4}},\\
&\tilde{y}_2 = \frac{1}{12}\sqrt{1440000 - 6\sqrt{57599988981 + 1365\sqrt{65} + 576x^2 - 1344x^3 - 288x^4}}.
\end{aligned}$$

Also by Maple, we get

$$\begin{aligned}
\bar{I}_1 &\approx 26426.695431194,\\
\bar{I}_2 &\approx -30847.384030487,\\
\bar{I}_3 &\approx 93503233.017653,\\
\bar{I}_4 &\approx 594536975.78668,\\
\tilde{I}_1 &\approx -0.6165686494 \times 10^{-3},\\
\tilde{I}_2 &\approx -0.2266214100 \times 10^{-4},\\
\tilde{I}_3 &\approx -0.6263873180 \times 10^{-5},\\
\tilde{I}_4 &\approx -0.5273174788 \times 10^{-9},\\
\tilde{J}_1 &\approx 0.5285407076 \times 10^{5},\\
\tilde{J}_2 &\approx -0.6169553114 \times 10^{5},\\
\tilde{J}_3 &\approx 0.1870066096 \times 10^{9},\\
\tilde{J}_4 &\approx 0.1189074254 \times 10^{10}.
\end{aligned} \tag{23}$$

Now substituting (21) into (19) and by (17) we get

$$\begin{aligned}
c_0 &= \overline{K}_0 + \overline{K}_1\delta_1 + \overline{K}_2\delta_2,\\
c_1 &= \frac{1}{200}\delta_1 + 150\delta_4,
\end{aligned}$$

where

$$\begin{aligned}
\overline{K}_0 &= -(I_{13} + I_{23})\delta_3 - (I_{14} + I_{24})\delta_4\\
&\approx -\frac{2127709682470669621}{10^{18}}\delta_3\\
&\quad - \frac{1162723601663463}{2 \times 10^{11}}\delta_4,\\
\overline{K}_1 &= -(I_{11} + I_{21}) \approx -\frac{2422340939659319}{125 \times 10^{14}},\\
\overline{K}_2 &= -(I_{12} + I_{22}) \approx \frac{24155263586699787}{4 \times 10^{16}}.
\end{aligned}$$

Therefore we have

Lemma 2. *Let (3) be satisfied. Then $c_0 \geq 0$ if and only if*

$$\delta_2 \geq K_0 + K_1\delta_1,$$

and $c_1 \geq 0$ if and only if $\delta_1 \geq -30000\delta_4$, where

$$\begin{aligned}
K_0 = -\frac{\overline{K}_0}{\overline{K}_2} &\approx \frac{775149067775642000000}{8051754528899929}\delta_4\\
&\quad + \frac{2127709682470669621}{603881589667494675}\delta_3,
\end{aligned}$$

$$K_1 = -\frac{\overline{K}_1}{\overline{K}_2} \approx \frac{38757455034549104}{120776317933498935}. \tag{24}$$

Let $\delta_1^* = -30000\delta_4$, $\delta_2^* = K_0 + K_1\delta_1^*$. Then by (24)

$$\delta_2^* \approx -\frac{3291534008000}{8051754528899929}\delta_4 + \frac{2127709682470669621}{603881589667494675}\delta_3.$$

Next, we study the property of the functions M, M_1, M_2 and $\tilde{M}_2$ as $(\delta_1, \delta_2) = (\delta_1^*, \delta_2^*)$. In this case, by (18), (19) and (21)–(24), we have

$$\begin{aligned}
c_{01} &\approx -\frac{16007678383413707737}{402587726444996450000000000}\delta_4 + \frac{734995475623633653531524894 1}{20129386322249822500000000000000}\delta_3,\\
c_{02} &\approx \frac{16007678383413707737}{402587726444996450000000000}\delta_4 - \frac{7349954756236336535315248941}{20129386322249822500000000000000}\delta_3,\\
\bar{c}_2 &\approx \frac{69269175158411302787920692328991}{40258772644499645000000000000000000000}\delta_4 - \frac{165102008721979610637151884636 01}{30194079483374733750000000000000000}\delta_3,\\
b_{01} &\approx \frac{(28248237268876517\sqrt{65} - 4633315056457821781)\sqrt{2}}{483105271733995740000\sqrt{65+7\sqrt{65}}}\delta_3 - \frac{(8228835020\sqrt{65} + 57601845140)\sqrt{2}}{8051754528899929\sqrt{65+7\sqrt{65}}}\delta_4,\\
\bar{c}_0 &\approx 198263874.53882\,\delta_4 - 93394545.687548\delta_3,\\
\tilde{c}_{02} &\approx 396547843.8\,\delta_4 - 186789232.3\delta_3,\\
\tilde{b}_{01} &= 300\sqrt{2}\,\delta_4\pi,\\
\tilde{c}_{01} &\approx -18.49704737\,\delta_4 + 0.00008611108867\delta_3
\end{aligned} \tag{25}$$

which together with (3) follows that

$$\begin{aligned}
&c_{01} > 0, \quad \bar{c}_2 < 0, \quad b_{01} < 0, \quad \bar{c}_0 > 0,\\
&\tilde{c}_{02} > 0, \quad \tilde{c}_{01} < 0, \quad \tilde{b}_{01} > 0.
\end{aligned} \tag{26}$$

Hence, by (16) and (26) we have

$$\begin{aligned}
M(h, \delta_1^*, \delta_2^*) &< 0 \quad \text{for } 0 < h - h_0 \ll 1,\\
M(h, \delta_1^*, \delta_2^*) &> 0 \quad \text{for } 0 < h_1 - h \ll 1,\\
M_1(h, \delta_1^*, \delta_2^*) &> 0 \quad \text{for } 0 < h_0 - h \ll 1,\\
M_1(h, \delta_1^*, \delta_2^*) &< 0 \quad \text{for } 0 < h - \alpha_1 \ll 1,\\
\tilde{M}_2(h, \delta_1^*, \delta_2^*) &> 0 \quad \text{for } 0 < h - h_2 \ll 1.
\end{aligned} \tag{27}$$

Then we can prove

Lemma 3. *Let (3) be satisfied. Then*

(1) *the function $M_1(h, \delta_1^*, \delta_2^*)$ has a zero $h_0^* \in (\alpha_1, h_0)$;*
(2) *$M(h, \delta_1^*, \delta_2^*)$ has three zeros $h_j^* \in (h_0, h_1)$, $j = 3, 4, 5$, with $h_3^* < h_4^* < h_5^*$;*
(3) *$\tilde{M}_2(h, \delta_1^*, \delta_2^*)$ has a zero $\tilde{h}_0^* \in (h_2, +\infty)$;*
(4) *all of these zeros are of an odd multiplicity.*

Proof. First by (27), the function $M_1(h, \delta_1^*, \delta_2^*)$ has a zero $h_0^* \in (\alpha_1, h_0)$. Similarly, the function $M(h, \delta_1^*, \delta_2^*)$ also has a zero in (h_0, h_1). To find more zeros of M, let us calculate their values at some points.

By (12), (13) and (15) we have

$$\begin{aligned}
M(h, \delta_1, \delta_2) &= -\oint_{L_h} (\delta_1 + \delta_2 x + \delta_3 x^2 + \delta_4 y^2) y dx\\
&= -(\delta_1 \hat{I}_1 + \delta_2 \hat{I}_2 + \delta_3 \hat{I}_3 + \delta_4 \hat{I}_4),\\
&\qquad h_0 < h < h_1,\\
\tilde{M}_2(h, \delta_1, \delta_2) &= -\oint_{\tilde{L}_{2h}} (\delta_1 + \delta_2 x + \delta_3 x^2 + \delta_4 y^2) y dx\\
&= -(\delta_1 \hat{J}_1 + \delta_2 \hat{J}_2 + \delta_3 \hat{J}_3 + \delta_4 \hat{J}_4),\\
&\qquad h > h_2,
\end{aligned}$$

where

$$\hat{I}_j = \oint_{L_h} x^{j-1} y dx = \int_{x_1(h)}^{x_2(h)} x^{j-1}(\hat{y}_1 - \hat{y}_2) dx, \quad j = 1, 2, 3,$$

$$\hat{I}_4 = \oint_{L_h} y^3 dx = \int_{x_1(h)}^{x_2(h)} (\hat{y}_1^3 - \hat{y}_2^3) dx,$$

$$\hat{J}_j = \oint_{\tilde{L}_{2h}} x^{j-1} y dx = 2\left(\int_{\tilde{x}_1(h)}^{\tilde{x}_4(h)} x^{j-1}\hat{y}_1 dx + \int_{\tilde{x}_4(h)}^{\tilde{x}_3(h)} x^{j-1}\hat{y}_2 dx + \int_{\tilde{x}_2(h)}^{\tilde{x}_1(h)} x^{j-1}\hat{y}_2 dx\right), \quad j = 1, 2, 3,$$

$$\hat{J}_4 = \oint_{\tilde{L}_{2h}} y^3 dx = 2\left(\int_{\tilde{x}_1(h)}^{\tilde{x}_4(h)} \hat{y}_1^3 dx + \int_{\tilde{x}_4(h)}^{\tilde{x}_3(h)} \hat{y}_2^3 dx + \int_{\tilde{x}_2(h)}^{\tilde{x}_1(h)} \hat{y}_2^3 dx\right),$$

$$\hat{y}_1 = \frac{1}{6}\sqrt{360000 + 6\sqrt{-18x^4 - 84x^3 + 36x^2 + 3600000000 + 72h}},$$

$$\hat{y}_2 = \frac{1}{6}\sqrt{360000 - 6\sqrt{-18x^4 - 84x^3 + 36x^2 + 3600000000 + 72h}}.$$

and $x_1(h) < 0 < x_2(h)$, $\tilde{x}_1(h) < \tilde{x}_2(h) < 0 < \tilde{x}_3(h) < \tilde{x}_4(h)$ satisfy

$$H(x_i(h), 100) = h, \quad i = 1, 2, \quad h_0 < h < h_1,$$

$$H(\tilde{x}_i(h), 0) = h, \quad i = 2, 3, \quad h > h_2,$$

$$H(\tilde{x}_i(h), 100) = h, \quad i = 1, 4, \quad h > h_2.$$

By Maple 10 we have

$$\begin{aligned} M(h, \delta_1^*, \delta_2^*)|_{h=-31502882.64306} &\approx 18119541.753047\delta_4 - 24079125.253207\delta_3 \\ &\equiv M_1^*, \\ M(h, \delta_1^*, \delta_2^*)|_{h=-31502882.64304} &\approx 18119541.753057\delta_4 - 24079125.253240\delta_3 \\ &\equiv M_2^*, \\ \tilde{M}_2(h, \delta_1^*, \delta_2^*)|_{h=9\times 10^7} &\approx 410963172.382053\delta_4 - 645192646.536934\delta_3 \\ &\equiv \tilde{M}^*. \end{aligned} \tag{28}$$

By (3) and (28), we have

$$M_1^* > 0, \quad M_2^* < 0, \quad \tilde{M}^* < 0.$$

Thus, by (27), $M(h, \delta_1^*, \delta_2^*)$ has three zeros h_3^*, h_4^* and h_5^* satisfying

$$h_3^* \in (h_0, -31502882.64306),$$

$$h_4^* \in (-31502882.64306, -31502882.64304),$$

$$h_5^* \in (-31502882.64304, h_1),$$

while $\tilde{M}_2(h, \delta_1^*, \delta_2^*)$ has a zero $\tilde{h}_0^* \in (h_2, 9 \times 10^7)$. Note that all of the zeros $h_0^*, \tilde{h}_0^*, h_3^*, h_4^*$ and h_5^* are of an odd multiplicity. The proof is completed. ■

Unfortunately, we cannot find a zero of $\tilde{M}_1(h, \delta_1^*, \delta_2^*)$ for $h \in (h_2, 0)$.

By Lemmas 2 and 3 we can prove

Lemma 4. *Let (3) be satisfied by δ_3 and δ_4 constant. If*

$$0 < K_0 + K_1\delta_1 - \delta_2 \ll \delta_1^* - \delta_1 \ll 1, \tag{29}$$

where $\delta_1^* = -30000\delta_4$, and K_0 and K_1 are given by (24), *then*

(1) *$M(h, \delta_1, \delta_2)$ has five zeros in (h_0, h_1);*
(2) *$M_1(h, \delta_1, \delta_2)$ has a zero in (α_1, h_0);*
(3) *$\tilde{M}_2(h, \delta_1, \delta_2)$ has a zero in $(h_2, +\infty)$;*
(4) *all of these zeros are of an odd multiplicity.*

Proof. By Lemma 2, under (29) we have

$$0 < -c_0 \ll -c_1 \ll -c_2$$

which ensures that $M(h, \delta_1, \delta_2)$ has two new zeros $h_1^*, h_2^* \in (h_0, h_3^*)$ near h_0 both having an odd multiplicity with $h_1^* < h_2^*$. At the same time, the zeros h_3^*, h_4^* and h_5^* of $M(h, \delta_1, \delta_2)$, h_0^* of $M_1(h, \delta_1, \delta_2)$ and $\tilde{h}_0^*$ of $\tilde{M}_2(h, \delta_1, \delta_2)$ remain under (29). This ends the proof. ■

Corollary 1. *Let $k = 100, b = 7/2$ and (δ_3, δ_4) satisfy (5). Then for K_0, K_1 and δ_1^* satisfying (6) and δ_1 and δ_2 satisfying (29), the conclusions (1)–(4) of Lemma 4 hold.*

In fact, under (5) we have

$$k_1\delta_4 \approx 19999.999999993, \quad k_2\delta_4 \approx 20000.000000010.$$

Thus, (3) is satisfied. Hence, the corollary follows from Lemma 4.

On the other hand, in this case, similar to (25) and (28) we have

$$c_{01} \approx 7.3016544125820 > 0,$$

$$\bar{c}_2 \approx -108.90321618344 < 0,$$

$$b_{01} \approx -147.39575898743 < 0,$$
$$\bar{c}_0 \approx 0.34015808347891 \times 10^{13} > 0,$$
$$\tilde{c}_{02} \approx 0.680369293 \times 10^{13} > 0,$$

and

$$M(h, \delta_1^*, \delta_2^*)|_{h=-31502882.64306} \approx 0.22899937435633 > 0,$$
$$M(h, \delta_1^*, \delta_2^*)|_{h=-31502882.64304} \approx -0.16451532561954 < 0,$$
$$\tilde{M}_2(h, \delta_1^*, \delta_2^*)|_{h=9\times 10^7} \approx -1981243697993.1 < 0$$

which also yield Corollary 1 by the above discussion. Now, it is clear that Theorem 1 follows from Lemma 4, Corollary 1 and the well-known Poincare–Pontrjagin–Andronov theorem (Theorem 6.1 in [Li, 2003]).

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