International Journal of Bifurcation and Chaos, Vol. 28, No. 6 (2018) 1850078 (31 pages)

DOI: 10.1142/S0218127418500785

An Improvement on the Number of Limit Cycles Bifurcating from a Nondegenerate Center of Homogeneous Polynomial Systems

Pei Yu*,†,§ and Maoan Han*,‡
*Department of Mathematics, Shanghai Normal University, Shanghai 200234, P. R. China
†Department of Applied Mathematics, Western University, London, Ontario, N6A 5B7, Canada
‡School of Mathematics Sciences, Qufu Normal University, Qufu 273165, P. R. China

Jibin Li
School of Mathematical Sciences, Huaqiao University, Quanzhou, Fujian 362021, P. R. China

Received March 26, 2018

In the two articles in *Appl. Math. Comput.*, J. Giné [2012a, 2012b] studied the number of small limit cycles bifurcating from the origin of the system: $\dot{x} = -y + P_n(x, y)$, $\dot{y} = x + Q_n(x, y)$, where P_n and Q_n are homogeneous polynomials of degree n. It is shown that the maximal number of the small limit cycles, denoted by $M_h(n)$, satisfies $M_h(n) \geq 2n - 1$ for $n = 4, 5, 6, 7$; and $M_h(8) \geq 13$, $M_h(9) \geq 16$. It seems that the correct answer for their case $n = 9$ should be $M_h(9) \geq 15$. In this paper, we apply Hopf bifurcation theory and normal form computation, and perturb the isolated, nondegenerate center (the origin) to prove that $M_h(n) \geq 2n$ for $n = 4, 5, 6, 7$; and $M_h(n) \geq 2(n - 1)$ for $n = 8, 9$, which improve Giné's results with one more limit cycle for each case.

Keywords: Homogeneous polynomial system; nondegenerate center; Hopf bifurcation; limit cycle; normal form.

1. Introduction

The second part of the well-known Hilbert's 16th problem [Hilbert, 1902] is to find an upper bound on the number of limit cycles that the following planar polynomial vector fields can have,

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ with real coefficients represent polynomial functions in x and y. This upper bound is called Hilbert number, denoted as $H(n)$, a function of the degree of the polynomial functions P and Q. A modern version of this problem was later formulated by Smale, chosen as one of his 18 most challenging mathematical problems for the 21st century [Smale, 1988]. Although many results have been obtained, this problem is not even completely solved for quadratic systems. Four limit cycles were found for quadratic systems almost 40 years ago [Shi, 1979, 1980; Chen & Wang, 1979], but $H(2) = 4$ is still open. More references can be found from the review article [Li, 2003] and the book [Han & Yu, 2012].

§Author for correspondence

Later, Arnold [1983] posed the so-called weak Hilbert's 16th problem, related to the following near-Hamiltonian system [Han, 2006; Han *et al.*, 2018]:

$$\begin{aligned} \dot{x} &= H_y(x,y) + \varepsilon p_n(x,y), \\ \dot{y} &= -H_x(x,y) + \varepsilon q_n(x,y), \end{aligned} \tag{2}$$

where $H(x,y)$, $p_n(x,y)$ and $q_n(x,y)$ are all polynomial functions in x and y, and $0 < \varepsilon \ll 1$ denotes a small perturbation. Then, the geometric problem of finding bifurcating limit cycles is transferred to an algebraic problem of studying the zeros of the Abelian integral or the (first-order) Melnikov function, given in the form of

$$M(h,\delta) = \oint_{H(x,y)=h} q_n(x,y)dx - p_n(x,y)dy, \tag{3}$$

where $H(x,y) = h$ for $h \in (h_1, h_2)$ defines a closed orbit, and δ represents the parameters (or coefficients) involved in the polynomial functions $p_n(x,y)$ and $q_n(x,y)$.

For cubic planar polynomial systems, many results have been obtained on the lower bound of $H(3)$, and the best result obtained so far is $H(3) \geq 13$ [Li & Liu, 2010; Li *et al.*, 2009]. Note that in [Li & Liu, 2010], the authors considered a cubic system with Z_2 symmetry and obtained 13 limit cycles with the distribution $1 \supset (6+6)$, i.e. 12 small ones around two symmetric foci and a large one at infinity; while in [Li *et al.*, 2009], the authors studied perturbing a cubic Hamiltonian system with nine singular points to obtain 13 limit cycles with the distribution $2 \times (1,5) + 1$.

If the problem is restricted to the vicinity of an isolated fixed point, which is either an elementary center or an elementary focus, then it is equivalent to studying generalized Hopf bifurcations. This problem is usually called local bifurcation of limit cycles, and the number of bifurcating small-amplitude limit cycles is denoted by $M(n)$. In [Giné, 2007, 2012a, 2012b], Giné considered the limit cycles bifurcating from the origin of the following polynomial system:

$$\begin{aligned} \dot{x} &= -y + P(x,y), \\ \dot{y} &= x + Q(x,y), \end{aligned} \tag{4}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials starting from second-order terms. For system (4), Giné [2007, 2012a, 2012b] conjectured an upper bound for the number of functionally independent focal values, given by

Conjecture 1.1. *The number of functionally independent focal values of system (4) at the origin, i.e. the minimum number of ideal generators is* $M(n) = n^2 + 3n - 7$, *where* n *is the degree of the polynomial differential system. In the case that* P *and* Q *are homogeneous polynomials of degree* n, $M_h(n) = 2n - 1$.

Conjecture 1.1 implies that if one perturbs system (4), for P and Q being homogeneous polynomials of degree n, inside the class of the homogeneous systems with the same degree n, one can obtain at most $2n-1$ small-amplitude limit cycles. Similarly, if one perturbs system (4), for P and Q being polynomials of degree n, inside the class of the general systems with the same degree n, one can obtain at most n^2+3n-7 limit cycles. More discussions and relative references can be found in [Giné, 2012a, 2012b].

When P and Q are nth-degree homogeneous polynomials, the best-known result for $n = 2$ obtained by Bautin [1952] is $M(2) = M_h(2) = 3$. For $n = 3$, it has been shown in [Sibirskii, 1965; Blows & Lloyd, 1984; Żołądek, 1994] that $M_h(3) = 5$, indicating that the conjecture is true. Recently, Giné showed in [Giné, 2012a, 2012b] that $M_h(n) \geq 2n-1$ for $n = 4,5,6,7$, and $M_h(8) \geq 13$, $M_h(9) \geq 16$. However, we will show in Sec. 3 that for Giné's case of $M_h(9)$, the correct result should be $M_h(9) \geq 15$. In this paper, we will use the systems given in [Giné, 2012a, 2012b] to prove $M_h(n) \geq 2n$ for $n = 4,5,6,7$, indicating that *for homogeneous polynomial systems, Conjecture 1.1 can be improved at least for* $n = 4,5,6,7$. Moreover, we will show that $M_h(8) \geq 14$, $M_h(9) \geq 16$.

When P and Q are general nth-degree polynomials, many results have been obtained for cubic systems, which can be classified into two categories: one is to perturb an isolated focus and the other to perturb an isolated center. For the former when perturbing an isolated focus, nine small-amplitude limit cycles are obtained in two different systems [Yu & Corless, 2009; Chen *et al.*, 2013; Lloyd & Pearson, 2012] using purely symbolic computation. For the latter when perturbing an isolated center, there are also a few results obtained in the past two decades. In 1995, Żołądek [1995] first used a rational Darboux integral and Melnikov functions up to second-order to claim the existence of 11 small limit cycles around a center. After more than ten years, another two cubic systems were constructed to show 11 limit cycles [Christopher, 2006; Bondar & Sadovskii, 2008]. The system considered in [Żołądek, 1995] was reinvestigated by Yu and Han [2011] using the method of focus value computation, and only nine small limit cycles were obtained. Recently, Tian and Yu [2016] found the mistakes in [Żołądek, 1995] and showed that the example given in [Żołądek, 1995] indeed only has nine limit cycles using up to second-order Melnikov functions. In a very recently published paper [Tian & Yu, 2018], the authors applied high-order analysis to prove that the example given by Żołądek [1995] indeed can have 11 small-amplitude limit cycles if at least seventh-order analysis (equivalent to seventh-order Melnikov function method) is used. These results seem to indicate that Conjecture 1.1 is true for $n = 3$, i.e. $M(3) \geq 11$. However, we recently used the system given in [Christopher, 2006] to prove that 12 limit cycles can exist, i.e. $M(3) \geq 12$, indicating that *for general polynomial systems, Conjecture 1.1 can be improved at least for* $n = 3$. It has been noted that Giné also proved [Giné, 2012b] $M(4) \geq 21 = 4^2 + 3 \times 4 - 7$, and $M(5) \geq 26$ which is however still quite less than $5^2 + 3 \times 5 - 7 = 33$.

In this paper, we consider system (4) and focus on the bifurcation of small-amplitude limit cycles from the origin when P and Q are nth-degree homogeneous polynomials. More precisely, consider the following system:

$$\begin{aligned} \dot{x} &= -y + P_n(x, y), \\ \dot{y} &= x + Q_n(x, y), \end{aligned} \tag{5}$$

where P_n and Q_n ($n \geq 2$) are nth-degree homogeneous polynomials.

In [Giné, 2012a, 2012b], the author added perturbations to system (5) to obtain the following perturbed system:

$$\begin{aligned} \dot{x} &= -y + P_n(x, y) + \varepsilon p_n(x, y), \\ \dot{y} &= x + Q_n(x, y) + \varepsilon q_n(x, y), \end{aligned} \tag{6}$$

where p_n and q_n are nth-degree homogeneous polynomials. Then, Giné computed the Poincaré–Lyapunov constants of the perturbed system (6) and used the independent linear parts and maybe quadratic parts in the Poincaré–Lyapunov constants to prove the existence of small-amplitude limit cycles bifurcating from the center (the origin).

Our method used in this paper is different, based on the normal form computation for generalized Hopf bifurcations. Since the systems used in [Giné, 2012a, 2012b] are all integrable systems, the Hopf bifurcations occur at the center by introducing perturbation polynomials of the same degree. To achieve this, we add perturbation polynomials to system (5) to obtain

$$\begin{aligned} \dot{x} &= -y + P_n(x, y) + \sum_{k \geq 1} \varepsilon^k \sum_{i+j=n} a_{ijk} x^i y^j, \\ \dot{y} &= x + Q_n(x, y) + \sum_{k \geq 1} \varepsilon^k \sum_{i+j=n} b_{ijk} x^i y^j, \end{aligned} \tag{7}$$

where $0 < \varepsilon \ll 1$. When $\varepsilon = 0$, the above system is integrable with a center at the origin. Note that unlike system (6), here in (7) we introduce the perturbations in different orders of ε, but all of them are nth-degree homogeneous polynomials. The basic idea of our method and how to prove the existence of multiple limit cycles bifurcating from a single singular point will be discussed in the next section. Our main result is given in the following theorem.

Theorem 1. *For system* (5), *the number of small-amplitude limit cycles bifurcating from a nondegenerate center* (*the origin*) *satisfies* $M_h(n) \geq 2n$ *for* $n = 4, 5, 6, 7$; *and* $M_h(n) \geq 2(n-1)$ *for* $n = 8, 9$.

We will prove Theorem 1 for the cases $n = 4, 5, 6, 7$ in Sec. 3, and the cases $n = 8, 9$ in Sec. 4. Conclusion is drawn in Sec. 5.

2. Computation of Focus Values and Bifurcation of Limit Cycles

In this paper, the basic idea for proving the existence of limit cycles is based on normal form or focus value computation. For the general system (4), the normal form can be obtained using computer algebra systems (e.g. see [Yu, 1998; Tian & Yu, 2013, 2014; Han & Yu, 2012]) as given in the polar coordinates:

$$\begin{aligned} \dot{r} &= r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k} + \cdots), \\ \dot{\theta} &= \omega_c + \tau_0 + \tau_1 r^2 + \tau_2 r^4 + \cdots + \tau_k r^{2k} + \cdots, \end{aligned} \tag{8}$$

where r and θ represent the amplitude and phase of motion, respectively. v_k ($k = 0, 1, 2, \ldots$) is called the kth-order focus value. v_0 and τ_0 are obtained from linear analysis. The first equation of (8) can be used for studying the bifurcation of limit cycles and stability of bifurcating limit cycles. To find k small-amplitude limit cycles bifurcating from the origin, we first solve the k equations: $v_0 = v_1 = \cdots = v_{k-1} = 0$ such that $v_k \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. The following lemma gives sufficient conditions for proving the existence of k small-amplitude limit cycles. (The proofs can be found in [Yu & Han, 2005].)

Lemma 1. *Suppose that the focus values depend on k parameters, ν_j, $j = 1, 2, \ldots, k$, expressed as*

$$v_j = v_j(\nu_1, \nu_2, \ldots, \nu_k), \quad j = 0, 1, \ldots, k, \tag{9}$$

satisfying

$$v_j(\nu_{1c}, \ldots, \nu_{kc}) = 0, \quad j = 0, 1, \ldots, k-1,$$
$$v_k(\nu_{1c}, \ldots, \nu_{kc}) \neq 0 \quad \text{and}$$
$$\det\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(\nu_1, \nu_2, \ldots, \nu_k)}\right]_{(\nu_1,\ldots,\nu_k)=(\nu_{1c},\ldots,\nu_{kc})} \neq 0. \tag{10}$$

Then, for any given $\nu^* > 0$, there exist $\nu_1, \nu_2, \ldots, \nu_k$ and $\delta > 0$ with $|\nu_j - \nu_{jc}| < \nu^*$, $j = 1, 2, \ldots, k$ such that the equation $\dot{r} = 0$ has exactly k real positive roots r [i.e. system (4) has exactly k limit cycles] in a δ-ball with the center at the origin.

Now consider the perturbed integral system (7). To give a more clear view, we consider the following near-integral polynomial systems, described in the form of [Tian & Yu, 2018]

$$\begin{aligned}\dot{x} &= M^{-1}(x, y, \mu)H_y(x, y, \mu) + \varepsilon p(x, y, \varepsilon, \delta),\\ \dot{y} &= -M^{-1}(x, y, \mu)H_x(x, y, \mu) + \varepsilon q(x, y, \varepsilon, \delta),\end{aligned} \tag{11}$$

where $0 < \varepsilon \ll 1$, μ and δ are vector parameters; $H(x, y, \mu)$ is an analytic function in x, y and μ; $p(x, y, \varepsilon, \delta)$ and $q(x, y, \varepsilon, \delta)$ are polynomials in x and y, and analytic in δ and ε. $M(x, y, \mu)$ is an integrating factor of the unperturbed system $(11)|_{\varepsilon=0}$.

Suppose the unperturbed system $(11)|_{\varepsilon=0}$ has an elementary center. Then, considering limit cycle bifurcation in system (11) around the center, we may use the normal form theory to obtain the first equation of (8) as follows:

$$\begin{aligned}\dot{r} = r[&v_0(\varepsilon) + v_1(\varepsilon)r^2 + v_2(\varepsilon)r^4\\ &+ \cdots + v_i(\varepsilon)r^{2i} + \cdots],\end{aligned} \tag{12}$$

where

$$v_i(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k V_{ik}, \quad i = 0, 1, 2, \ldots,$$

in which V_{ik} denotes the ith ε^k-order focus value. Note that $v_i(\varepsilon) = O(\varepsilon)$ since the unperturbed system $(11)|_{\varepsilon=0}$ is an integral system. Further, because system (11) is analytic in ε, we can rearrange the terms in (12), and obtain

$$\dot{r} = V_1(r)\varepsilon + V_2(r)\varepsilon^2 + \cdots + V_k(r)\varepsilon^k + \cdots, \tag{13}$$

where

$$V_k(r) = \sum_{i=0}^{\infty} V_{ik} r^{2i+1}, \quad k = 1, 2, \ldots. \tag{14}$$

Note from the above discussions that there are two *orders* in the above formulas: one is the order in ε, and the other is in V_{ik} for a fixed k. The former is equivalent to the order of Melnikov functions, while the latter is to the order of focus values at ε^k-order.

Also note that besides the perturbation parameters δ involved in $p(x, y, \varepsilon, \delta)$ and $q(x, y, \varepsilon, \delta)$, there are also parameters μ included in the Hamiltonian function $H(x, y, \mu)$, which can also be used to increase the number of bifurcating limit cycles. In the following, we first show the equivalence of our method and the Melnikov function method [Tian & Yu, 2016, 2018], and then show why the free parameter involved in the Hamiltonian function can be used to get more limit cycles.

2.1. *The order idea and the equivalence between our method and the Melnikov function method*

By the method of normal forms, we can obtain the second differential equation in (8) for system (11), given by [Tian & Yu, 2018]

$$\dot{\theta} = T_0(r) + O(\varepsilon),$$

with $T_0(0) \neq 0$, and thus

$$\frac{dr}{d\theta} = \frac{V_1(r)\varepsilon + V_2(r)\varepsilon^2 + \cdots + V_k(r)\varepsilon^k + \cdots}{T_0(r) + O(\varepsilon)}. \tag{15}$$

Assume that the solution $r(\theta, \rho, \varepsilon)$ of (15), satisfying the initial condition $r(0, \rho, \varepsilon) = \rho$, is given in the form of

$$r(\theta,\rho,\varepsilon) = r_0(\theta,\rho) + r_1(\theta,\rho)\varepsilon + r_2(\theta,\rho)\varepsilon^2 + \cdots + r_k(\theta,\rho)\varepsilon^k + \cdots,$$

with $0 < \rho \ll 1$. Then, $r_0(0,\rho) = \rho$ and $r_k(0,\rho) = 0$, for $k \geq 1$.

If there exists a positive integer K such that $V_k(r) \equiv 0$, $1 \leq k < K$, and $V_K(r) \not\equiv 0$, then it follows from (15) that

$$r_0(\theta,\rho) = \rho, \quad r_k(\theta,\rho) = 0,$$

$$1 \leq k < K \quad \text{and} \quad r_K(\theta,\rho) = \frac{V_K(\rho)}{T_0(\rho)}\theta.$$

Thus, the displacement function $d(\rho)$ of system (15) can be written as

$$\begin{aligned} d(\rho) &= r(2\pi,\rho,\varepsilon) - \rho \\ &= 2\pi\frac{V_K(\rho)}{T_0(\rho)}\varepsilon^K + O(\varepsilon^{K+1}). \end{aligned} \tag{16}$$

Therefore, if we want to determine the number of small-amplitude limit cycles bifurcating from the center in system (11), we only need to study the number of isolated zeros of $V_K(\rho)$ for $0 < \rho \ll 1$, and have to obtain the expression of the first nonzero coefficient $V_K(r)$ in (13) by computing V_{iK}, for $i \geq 0$.

The above discussions show that the basic idea of using focus values of system (11) is actually the same as that of the Melnikov function method. Using $H(x,y) = h$ to parameterize the section (i.e. the Poincaré map), we obtain the displacement function of (11), given by

$$d(h) = M_1(h)\varepsilon + M_2(h)\varepsilon^2 + \cdots + M_k(h)\varepsilon^k + \cdots, \tag{17}$$

where

$$\begin{aligned} M_1(h) = \oint_{H(x,y,\mu)=h} & M(x,y,\mu)[q(x,y,0,\delta)dx \\ & - p(x,y,0,\delta)dy], \end{aligned} \tag{18}$$

evaluated along closed orbits $H(x,y,\mu) = h$ for $h \in (h_1,h_2)$. Then, we can study the first nonzero Melnikov function $M_k(h)$ in (17) to determine the number of limit cycles in system (11).

Remark 2.1. We give remarks on the comparison of computations for Melnikov functions and focus values.

(i) Let $H = h$, $0 < h - h_1 \ll 1$ define closed orbits around the center of system $(11)|_{\varepsilon=0}$. It is easy to see that for any integer $K \geq 1$, Eq. (16) holds if and only if $M_k(h) \equiv 0$, $1 \leq k < K$ and $M_K(h) \not\equiv 0$ in (17). Moreover, $V_K(\rho)$ for $0 < \rho \ll 1$ and $M_K(h)$ for $0 < h - h_1 \ll 1$ have the same maximum number of isolated zeros. So V_k $(k \geq 1)$ is equivalent to the kth-order Melnikov function.

(ii) As we can see, $V_k(r)$ can be obtained by the computation of normal forms or focus values.

(iii) In particular, when the original system is not a Hamiltonian system but an integral system, then even computing the coefficients of the first-order Melnikov function is much more involved than the computation of using the method of normal forms.

(iv) However, the method of normal forms (or focus values) is restricted to Hopf and generalized Hopf bifurcations, while the Melnikov function method can also be applied to study bifurcation of limit cycles from homoclinic/heteroclinic loops or any closed orbits.

Therefore, when using the focus value computation, if $V_1(r) \equiv 0$, we can then apply $V_2(r)$ to determine the existence of limit cycles; and further, if $V_2(r) \equiv 0$, then we use $V_3(r)$, and so on.

2.2. *The parameters in the Hamiltonian function used to get more limit cycles*

Next, we show that the free parameters involved in the Hamiltonian function can be used to get more limit cycles. The basic idea is discussed in [Han *et al.*, 2009] and [Han & Yu, 2012]. Suppose we study a C^∞ system of the form

$$\dot{x} = H_y + \varepsilon p(x,y,\delta), \quad \dot{y} = -H_x + \varepsilon q(x,y,\delta), \tag{19}$$

where $H(x,y)$, $p(x,y,\delta)$, $q(x,y,\delta)$ are C^∞ functions, $\varepsilon \geq 0$ is small and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact.

When $\varepsilon = 0$, system (19) becomes

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \tag{20}$$

which is a Hamiltonian system, and thus Eq. (19) is called a near-Hamiltonian system. Suppose that (20) has an elementary center at the origin, namely the function H satisfies $H_x(0,0) = H_y(0,0) = 0$, and

$$\det \frac{\partial(H_y, -H_x)}{\partial(x,y)}(0,0) > 0.$$

Therefore, without loss of generality, we may suppose that the expansion of H at the origin can be written as

$$H(x,y) = \frac{\omega}{2}(y^2 + x^2) + \sum_{i+j\geq 3} h_{ij}x^i y^j, \quad \omega > 0. \tag{21}$$

Then, the Hamiltonian system (20) has a family of periodic orbits, given by

$$L_h : H(x,y) = h, \quad h \in (0,\beta)$$

such that L_h approaches the origin as $h \to 0$. Then, we have the following results (the proofs can be found in [Han *et al.*, 2009] or [Han & Yu, 2012]).

Lemma 2. *Let (21) hold. Then $M(h,\delta)$ is C^∞ in $0 \leq h \ll 1$ with*

$$M(h,\delta) = h \sum_{l\geq 0} b_l(\delta) h^l \tag{22}$$

formally for $0 \leq h \ll 1$. Moreover, if (19) is analytic, so is M.

Lemma 3. *Under the condition of Lemma 2, if there exist $k \geq 1$, $\delta_0 \in D$ such that $b_k(\delta_0) \neq 0$ and*

$$b_j(\delta_0) = 0, \quad j = 0, 1, \ldots, k-1,$$

$$\det \frac{\partial(b_0, \ldots, b_{k-1})}{\partial(\delta_1, \ldots, \delta_k)}(\delta_0) \neq 0,$$

where $\delta = (\delta_1, \ldots, \delta_m)$, $m \geq k$, then there exist a constant $\varepsilon_0 > 0$ and a neighborhood V of the origin such that for all $0 < |\varepsilon| < \varepsilon_0$ and $|\delta - \delta_0| < \varepsilon_0$, (19) has at most k limit cycles in V. Moreover, for any neighborhood V_1 of the origin there exists (ε, δ) near $(0, \delta_0)$ such that system (19) has k limit cycles in V_1. In other words, system (19) has Hopf cyclicity k for all (ε, δ) near $(0, \delta_0)$.

In many cases, the Hamiltonian system (20) contains some constants. If we take them as parameters and change them suitably we can find more limit cycles. More precisely, suppose $H(x,y,a)$ with $a \in \mathbb{R}^n$ satisfies (21) where the coefficients h_{ij} depend on a. Then by Lemma 2, in this case we have

$$M(h,\delta,a) = h \sum_{l\geq 0} b_l(\delta,a) h^l. \tag{23}$$

For simplicity, suppose the functions p and q in (19) are linear in δ. Then the coefficients $b_l(\delta,a)$ are linear in δ. Assume that there exist an integer $k > 0$, $\delta_0 \in \mathbb{R}^m$ and $a_0 \in \mathbb{R}^n$ such that

$$b_j(\delta_0, a_0) = 0, \quad j = 0, \ldots, k-1,$$
$$\det \frac{\partial(b_0, \ldots, b_{k-1})}{\partial(\delta_1, \ldots, \delta_k)}(a_0) \neq 0. \tag{24}$$

Then the linear equations $b_j = 0$, $j = 0, \ldots, k-1$, of δ have a unique solution of the form

$$(\delta_1, \ldots, \delta_k) = \varphi(\delta_{k+1}, \ldots, \delta_m, a)$$

for a near a_0. Obviously, φ is linear in $\delta_{k+1}, \ldots, \delta_m$. Further, let

$$b_{k+j}|_{(\delta_1,\ldots,\delta_k)=\varphi(\delta_{k+1},\ldots,\delta_m,a)} = V_j(\delta_{k+1}, \ldots, \delta_m)\Delta_j(a), \quad j = 0, \ldots, n. \tag{25}$$

We have the following lemma.

Lemma 4. *Consider the near-Hamiltonian system (19), where $H(x,y,a)$ with $a \in \mathbb{R}^n$ satisfies (21) and the functions p and q are linear in $\delta \in \mathbb{R}^m$. Suppose there exist integer $k > 0$ and $\delta_0 = (\delta_{10}, \ldots, \delta_{m0}) \in \mathbb{R}^m$ and $a_0 \in \mathbb{R}^n$ such that (24) and (25) hold with*

$$V_j(\delta_{k+1,0}, \ldots, \delta_{m0}) \neq 0, \quad j = 0, \ldots, n,$$
$$\Delta_j(a_0) = 0, \quad j = 0, \ldots, n-1, \quad \Delta_n(a_0) \neq 0 \tag{26}$$

and

$$\det \frac{\partial(\Delta_0, \ldots, \Delta_{n-1})}{\partial(a_1, \ldots, a_n)}(a_0) \neq 0. \tag{27}$$

Then for all (ε, δ, a) near $(0, \delta_0, a_0)$ (19) has at most $k+n$ limit cycles near the origin, and for some (ε, δ, a) near $(0, \delta_0, a_0)$ (19) can have $k+n$ limit cycles near the origin.

Proof. We fix $(\delta_{k+1}, \ldots, \delta_m) = (\delta_{k+1,0}, \ldots, \delta_{m0})$ so that

$$V_j(\delta_{k+1}, \ldots, \delta_m) = V_j(\delta_{k+1,0}, \ldots, \delta_{m0}) \equiv V_{j0} \neq 0.$$

Then noting that $b_j = 0$ for $j = 0, \ldots, k-1$ as $(\delta_1, \ldots, \delta_k) = \varphi(\delta_{k+1}, \ldots, \delta_m, a)$, by (23)–(25), we have

$$M(h, \delta, a)|_{(\delta_1,\ldots,\delta_k)=\varphi(\delta_{k+1},\ldots,\delta_m,a)} = h^{k+1} \sum_{j\geq 0} V_{j0}\Delta_j(a)h^j \equiv \tilde{M}(h, a). \quad (28)$$

By (27), we can change a near a_0 such that

$$V_{i0}V_{i+1,0}\Delta_i\Delta_{i+1} < 0, \quad |\Delta_i| \ll |\Delta_{i+1}|, \quad i = 0, \ldots, n-1 \quad (29)$$

which implies that the function $\tilde{M}$ in (28) has n positive simple zeros $h_n^* < \cdots < h_1^*$ near $h = 0$. Having obtained a satisfying (29), by (24) we can change $(\delta_1, \ldots, \delta_k)$ near $\varphi(\delta_{k+1,0}, \ldots, \delta_{m0}, a)$ such that

$$b_j b_{j+1} < 0, \quad |b_j| \ll |b_{j+1}|, \quad j = 0, \ldots, k-1, \quad (30)$$

which implies that the function M given by (23) has k simple zeros in the interval $(0, h_n^*)$. Clearly, under (30) the zeros $h_n^*, \ldots, h_1^*$ remain to exist. Thus, under (29) and (30) the function M has $n+k$ positive simple zeros altogether. Finally, by (24)–(26), we have

$$b_j(\delta_0, a_0) = 0, \quad j = 0, \ldots, n+k-1,$$

$$b_{n+k}(\delta_0, a_0) = V_{n0}\Delta_n(a_0) \neq 0.$$

Following the proof of Lemma 2, one can show that system (19) can have $n + k$ limit cycles near the origin. The proof is complete. ∎

Remark 2.2

(1) The above proof is for the first-order Melnikov function. Similarly, one can prove that it works for the second-order Melnikov function if the first-order Melnikov function identically equals zero, and so on. Similarly, in using focus values, the process starts from ε-order analysis (V_1), and if $V_1 \equiv 0$, then goes to ε^2-order analysis (V_2), and so on.

(2) The idea used in Lemma 4 (combination of the parameters in the Hamiltonian function and perturbation functions) was discussed by Iliev [2000] to prove the existence of more limit cycles.

(3) We used the above methods to obtain 12 limit cycles in a cubic polynomial system around a single singular point [Yu & Tian, 2014]. This cubic integral system is described in the form of

$$\begin{aligned}\dot{x} &= (32a^2 - 75)10x(-6 - 9x - 3x^2 + 8axy - 12y^2),\\ \dot{y} &= (32a^2 - 75)(24a - 16ax + 90y + 15xy\\ &\quad - 16axy^2 + 60y^3), \quad (32a^2 - 75 \neq 0),\end{aligned} \quad (31)$$

which was constructed by Christopher [2006] to prove the existence of 11 limit cycles around an isolated center with a fixed value $a = 2$. We let the parameter a be free and perturb system (31) with the ε-order cubic polynomials,

$$\varepsilon p = \varepsilon \sum_{i+j=1}^{3} a_{ij1}x^i y^j, \quad \varepsilon q = \varepsilon \sum_{i+j=1}^{3} b_{ij1}x^i y^j,$$

to obtain the focus values: v_{1j}, $j = 1, 2, \ldots$. (Here, the notation v_{1j}, instead of v_{j1}, was used in [Yu & Tian, 2014].) Then we use the 11 coefficients, b_{031}, b_{121}, b_{211}, b_{301}, b_{021}, b_{111}, b_{201}, b_{101}, a_{301}, a_{211} and a to solve the first 11 focus values to obtain six sets of solutions such that $v_{j1} = 0$, $j = 1, 2, \ldots, 11$ and $v_{j12} \neq 0$. Since the solution procedure given in [Yu & Tian, 2014] is one by one, i.e. at each step, using one coefficient to solve one focus value, for example, using b_{031} to solve $v_{11} = 0$, b_{121} to solve $v_{12} = 0$, and so on. Thus, it does not need to check the determinant given in (10). In fact, we can obtain

$$\det\left[\frac{\partial(v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{110}, v_{111})}{\partial(b_{031}, b_{121}, b_{211}, b_{301}, b_{021}, b_{111}, b_{201}, b_{101}, a_{301}, a_{211}, a)}\right] = \frac{\mathrm{C}(32a^2 - 75)^{167}F_N(a^2)}{a^{66}(8a^2 + 25)^{77}(4a^2 - 5)^{67}(16384a^6 - 14400a^4 + 165000a^2 + 84375)^{12}F_D(a^2)},$$

where C represents a big integer, and F_N and F_D are 60th- and 28th-degree polynomials in a^2, respectively. It is easy to verify that for the solution given in [Yu & Tian, 2014, p. 2697], the above determinant is nonzero. Thus, by Lemma 4 and plus a linear perturbation (i.e. in addition, perturbing the zero-order focus value v_{10}), the existence of 12 small-amplitude limit cycles are obtained by perturbing the cubic polynomial system (31).

2.3. *Methods for proving that all ε^k-order focus values vanish*

Unlike the Melnikov function method, when using higher ε^k-order focus values to consider the existence of limit cycles, a common difficulty is to prove that all ε^k-order focus values vanish. This certainly cannot be done by checking an infinite number of ε^k-order focus values. Without proving this, theoretically one cannot use ε^{k+1}-order focus values to prove the existence of limit cycles. Here, for near-integrable differential systems, we introduce two approaches for proving the vanishing of all ε^k-order focus values, one of which depends upon integrating factor and corresponding first integral, and the other only depends on integrating factor. We rewrite (11) as

$$\begin{aligned} \dot{x} &= P(x, y, \mu) + \varepsilon p(x, y, \varepsilon, \delta), \\ \dot{y} &= Q(x, y, \mu) + \varepsilon q(x, y, \varepsilon, \delta), \end{aligned} \tag{32}$$

where

$$\dot{x} = P(x, y, \mu), \quad \dot{y} = Q(x, y, \mu), \tag{33}$$

is the unperturbed system which is integrable. Suppose the integrating factor for the unperturbed system (33) is $M(x, y, \mu)$, then

$$\begin{aligned} P(x, y, \mu) &= M^{-1}(x, y, \mu)H_y(x, y, \mu), \\ Q(x, y, \mu) &= -M^{-1}(x, y, \mu)H_x(x, y, \mu), \end{aligned} \tag{34}$$

where $H(x, y, \mu)$ is an analytic function in x, y and μ, which is usually called the first integral of system (33). [When system (33) is multiplied by the integrating factor M, it becomes a Hamiltonian system and then $H(x, y, \mu)$ is called the Hamiltonian function of the system.]

Suppose we have obtained ε^k-order focus values for the perturbed system (32), and found the conditions on the perturbed parameter δ such that $V_{ik} = 0$, $i = 1, 2, \ldots, j$, where j is finite. Now, we want to prove that $V_{ik} = 0$ for any integer $i \geq 1$.

Assume that for system (32) we have an ε^k-order first integral $H_k(x, y, \mu)$, then it is easy to get

$$(P + \varepsilon p)\frac{\partial H_k}{\partial x} + (Q + \varepsilon q)\frac{\partial H_k}{\partial y} = O(\varepsilon^{k+1}). \tag{35}$$

This result can be easily proved by using the closed contour $H_k = h$ as the parameter to express the displacement function. Thus, proving the vanishing of all ε^k-order focus values is equivalent to proving the existence of such an analytic function H_k.

However, sometimes even for the unperturbed system we can easily obtain an integrating factor, but it is very difficult to find the first integral. In this case, system (32) can be rewritten as

$$(Q + \varepsilon q)dx - (P + \varepsilon p)dy = 0. \tag{36}$$

If there exists an ε^k-order integrating factor $M_k(x, y, \mu)$ such that system (36) has an ε^k-order first integral, then we have the equation,

$$M_k(Q + \varepsilon q)dx - M_k(P + \varepsilon p)dy = O(\varepsilon^{k+1}), \tag{37}$$

which has the following property:

$$\frac{\partial[M_k(P + \varepsilon p)]}{\partial x} + \frac{\partial[M_k(Q + \varepsilon q)]}{\partial y} = O(\varepsilon^{k+1}), \tag{38}$$

under which all ε^k-order focus values vanish. Using this method, the proof only needs to find an M_k satisfying the above equation.

Note that the above two methods are equivalent, since based on the integrating factor M_k, we can find the first integral H_k using the following formula:

$$\frac{\partial H_k}{\partial x} = M_k(Q + \varepsilon q), \quad \frac{\partial H_k}{\partial y} = -M_k(P + \varepsilon p),$$

which obviously does not change the order of ε, but the integration of finding H_k is sometimes not easy.

3. Proof of Theorem 1 for $n = 4, 5, 6, 7$

In [Giné, 2012a], the author used the independent linear parts in Poincaré–Lyapunov constants to show that $M_h(4) \geq 6$, $M_h(5) \geq 9$, $M_h(6) \geq 9$ and $M_h(7) \geq 13$, and further in [Giné, 2012b] the author applied the independent quadratic parts in Poincaré–Lyapunov constants to prove that $M_h(4) \geq 7$ and $M_h(6) \geq 11$. These results agree with the formula given in Conjecture 1.1: $M_h(n) \geq 2n - 1$ for $n = 4, 5, 6, 7$. In this section, we will show that $M_h(n) \geq 2n$ for $n = 4, 5, 6, 7$, and thus Conjecture 1.1 can be improved at least for $n = 4, 5, 6, 7$. We start from the two simple cases, $n = 5$ and $n = 7$, which only need the ε-order focus values, then consider the case $n = 4$, which requires up to ε^2-order focus values, and finally the case $n = 6$, which even needs up to ε^3-order focus values.

3.1. $M_h(5) \geq 10$ *for the case* $n = 5$

The fifth-degree homogeneous polynomial system considered in [Giné, 2006, 2012a] is given by the following equations without ε-order terms:

$$\begin{aligned}\dot{x} &= -y + 2k_1(k_1+k_2)x^5 + 2(3-5k_1^2-3k_1k_2)x^4y \\ &\quad - 4(2+k_1^2+5k_1k_2)x^3y^2 - 4(2-5k_1^2 \\ &\quad + k_1k_2)x^2y^3 + 2(4-3k_1^2+5k_1k_2)xy^4 \\ &\quad + 2(1-k_1^2+k_1k_2)y^5 + \varepsilon p_5(x,y,\varepsilon), \\ \dot{y} &= x - 2(1-k_1^2-k_1k_2)x^5 - 2(4-3k_1^2 \\ &\quad - 5k_1k_2)x^4y + 4(2-5k_1^2-k_1k_2)x^3y^2 \\ &\quad + 4(2+k_1^2-5k_1k_2)x^2y^3 - 2(3-5k_1^2 \\ &\quad + 3k_1k_2)xy^4 - 2k_1(k_1-k_2)y^5 + \varepsilon q_5(x,y,\varepsilon),\end{aligned} \tag{39}$$

where $k_1 = \cos\phi$ and $k_2 = \sin\phi$ with arbitrary $\phi \in [0, 2\pi]$. We do not need to find the first integral of system (39), but following the form of (7), we add the ε-order polynomial perturbation to system (39), given by

$$\begin{aligned} p_5 = p_{51} &= \sum_{i+j=5} a_{ij1}x^iy^j, \\ q_5 = q_{51} &= \sum_{i+j=5} b_{ij1}x^iy^j. \end{aligned} \tag{40}$$

In [Giné, 2012a], the author used independent linear parts in Poincaré–Lyapunov constants to show that $M_h(5) \geq 9$. By using our method, we will show that the ε-order focus values are enough to prove $M_h(5) \geq 10$.

In this case, the nonzero focus values are V_{2i}, $i = 1, 2, \ldots$. We let $k_2 = \sqrt{1-k_1^2}$ (the case $k_2 = -\sqrt{1-k_1^2}$ can be similarly proved). Then we use the eight parameters: a_{ij1} $(i+j=5)$, b_{501} and b_{411}, to linearly solve the first eight ε-order focus value equations: $V_{2i1} = 0$, $i = 1, 2, \ldots, 8$, and obtain

$$\begin{aligned} V_{181} &= \frac{5C_{51}}{36C_{50}}k_1(1-k_1^2)(1-4k_1^2)^3(3-4k_1^2)F_{51}, \\ V_{201} &= \frac{C_{51}}{216C_{50}}k_1(1-k_1^2)(1-4k_1^2)^3(3-4k_1^2)G_{51}, \end{aligned}$$

where C_{50} is a 58th-degree polynomial in k_1, C_{51} is given by

$$\begin{aligned} C_{51} &= -2k_1(1-k_1^2)[C_{51}^1b_{321} + C_{51}^2b_{231} \\ &\quad + C_{51}^3b_{141} + C_{51}^4b_{051}] + \sqrt{1-k_1^2}[C_{51}^5b_{321} \\ &\quad + C_{51}^6b_{231} + C_{51}^7b_{141} + C_{51}^8b_{051}], \end{aligned}$$

where C_{51}^j, $j = 1, 2, \ldots, 8$, are polynomials in k_1, and

$$\begin{aligned} F_{51} &= 1015283712k_1^{24} - 6091702272k_1^{22} + 15990718464k_1^{20} - 23470137344k_1^{18} + 20165216256k_1^{16} \\ &\quad - 9138921984k_1^{14} + 542018048k_1^{12} + 1857404640k_1^{10} - 1187474796k_1^8 + 373427914k_1^6 \\ &\quad - 58506927k_1^4 + 2445696k_1^2 - 7920, \\ G_{51} &= -2k_1(1-k_1^2)(27639000994066303706726 4k_1^{60} + \cdots - 14000496840000) \\ &\quad + \sqrt{1-k_1^2}(5419411959620843864064 0k_1^{62} + \cdots + 45159846197400). \end{aligned}$$

Without loss of generality, we may set $b_{321} = b_{231} = b_{141} = 0$ and $b_{051} = 1$. Moreover, it can be shown that

$$\begin{aligned} \mathrm{Det}_{51} &= \det\left[\frac{\partial(V_{21}, V_{41}, V_{61}, V_{81}, V_{101}, V_{121}, V_{141}, V_{161}, V_{181})}{\partial(a_{501}, a_{411}, a_{321}, a_{231}, a_{141}, a_{051}, b_{501}, b_{411}, k_1)}\right] \\ &= \frac{25k_1^4(1-k_1^2)^3(1-4k_1^2)^6(3-4k_1^2)^4}{1183529778020352\sqrt{1-k_1^2}} \times \frac{\mathrm{Det}_{5\mathrm{N}}(k_1^2)}{\mathrm{Det}_{5\mathrm{D}}(k_1^2)}, \\ &= \frac{25k_1^4(1-k_1^2)^3(1-4k_1^2)^6(3-4k_1^2)^4}{1183529778020352\sqrt{1-k_1^2}} \times \frac{\mathrm{Det}_{5\mathrm{N}}^1(k_1^2) + k_1\sqrt{1-k_1^2}\,\mathrm{Det}_{5\mathrm{N}}^2(k_1^2)}{\mathrm{Det}_{5\mathrm{D}}(k_1^2)}, \end{aligned}$$

where $\mathrm{Det}_{5\mathrm{N}}^1$, $\mathrm{Det}_{5\mathrm{N}}^2$ and $\mathrm{Det}_{5\mathrm{D}}$ are 46th-, 45th- and 28th-degree polynomials in k_1^2, respectively.

Finally, solving $F_{51} = 0$ yields 12 real solutions for $k_1 \in (-1, 1)$:

$$\begin{aligned} k_1 = &\pm 0.9599707067\cdots, \quad \pm 0.8942014961\cdots, \quad \pm 0.8347897513\cdots, \\ &\pm 0.7225594278\cdots, \quad \pm 0.2374112789\cdots, \quad \pm 0.0594117447\cdots, \end{aligned}$$

under which $G_{51} \neq 0$ and $\text{Det}_{51} \neq 0$. Note that since $k_2 = \pm\sqrt{1-k_1^2}$, we actually have a total of 24 sets of solutions. For example, taking $k_1 = 0.2374112789\cdots$ and $k_2 = \sqrt{1-k_1^2}$, we obtain

$$V_{k1} = 0, \quad k = 2, 4, \ldots, 18,$$

$$V_{201} = -0.0665293532\cdots \neq 0,$$

$$\text{Det}_{51} = 0.1827344716\cdots \times 10^{-7} \neq 0.$$

This, by Lemma 4, implies that there exist parameter solutions for system (40) to have nine small-amplitude limit cycles bifurcating from the origin. Further, we use a linear perturbation to obtain one more small limit cycle, giving a total of ten limit cycles around the origin, i.e. $M_h(5) \geq 10$.

We can process the above procedure to ε^2-order focus values to again obtain ten small-amplitude limit cycles bifurcating from the origin.

3.2. $M_h(7) \geq 14$ *for the case* $n = 7$

The seventh-degree homogeneous polynomial system was proposed in [Giné, 2006] and studied in [Giné, 2012a] to prove $M_h(7) \geq 13$ by using the independent linear parts in the Poincaré–Lyapunov constants. The system is described by the following equations with $\varepsilon = 0$:

$$\begin{aligned}
\dot{x} &= -y + \frac{4}{3}k_1(k_1+k_2)x^7 + \frac{2}{3}(7 - 11k_1^2 - 7k_1k_2)x^6y - \frac{4}{3}(2 + 2k_1^2 + 11k_1k_2)x^5y^2 \\
&\quad - \frac{2}{3}(3 - 15k_1^2 + 11k_1k_2)x^4y^3 - \frac{4}{3}(4 + 7k_1^2 + 5k_1k_2)x^3y^4 - \frac{2}{3}(7 - 23k_1^2 + k_1k_2)x^2y^5 \\
&\quad + \frac{4}{3}(6 - 4k_1^2 + 7k_1k_2)xy^6 + 2(1 - k_1^2 + k_1k_2)y^7 + \varepsilon \sum_{i+j=7} a_{ij1}x^iy^j, \\
\dot{y} &= x - 2(1 - k_1^2 - k_1k_2)x^7 - \frac{4}{3}(6 - 4k_1^2 - 7k_1k_2)x^6y + \frac{2}{3}(7 - 23k_1^2 - k_1k_2)x^5y^2 \\
&\quad + \frac{4}{3}(4 + 7k_1^2 - 5k_1k_2)x^4y^3 + \frac{2}{3}(3 - 15k_1^2 - 11k_1k_2)x^3y^4 + \frac{4}{3}(2 + 2k_1^2 - 11k_1k_2)x^2y^5 \\
&\quad - \frac{2}{3}(7 - 11k_1^2 + 7k_1k_2)xy^6 - \frac{4}{3}k_1(k_1 - k_2)y^7 + \varepsilon \sum_{i+j=7} b_{ij1}x^iy^j,
\end{aligned} \tag{41}$$

where

$$k_1 = \cos\phi \quad \text{and} \quad k_2 = \sin\phi$$

with arbitrary $\phi \in [0, 2\pi]$. The result obtained in [Giné, 2012a] by using the independent linear parts in Poincaré–Lyapunov constants implies that we only need to use ε-order perturbations.

We use the 12 parameters: a_{ij1} $(i+j=7)$, b_{701}, b_{611}, b_{521} and b_{431} to linearly solve the first 12 focus value equations: $V_{3i1} = 0$, $i = 1, 2, \ldots, 12$. Then, V_{391}, V_{421} and V_{451} become

$$V_{391} = \frac{-k_1k_2(3 - 12k_1^2 + 12k_1^4 - 4k_1^2k_2^2)}{1070802284241315225600 0C_{70}}F_{71},$$

$$V_{421} = \frac{k_1k_2(3 - 12k_1^2 + 12k_1^4 - 4k_1^2k_2^2)}{353364753799634024448000C_{70}}G_{71},$$

$$V_{451} = \frac{k_1k_2(3 - 12k_1^2 + 12k_1^4 - 4k_1^2k_2^2)}{14387599315705898939424768000C_{70}}H_{71},$$

where C_{70} is a polynomial in k_1 and k_2, while F_{71}, G_{71} and H_{71} are polynomials linearly in b_{341}, b_{251}, b_{161} and b_{071} with polynomial coefficient in k_1 and k_2. Thus, we solve the equation $F_{71} = 0$ for b_{341} under which G_{71} and H_{71} are reduced to

$$G_{71} = -2k_1^2(1 - k_1^2 - k_2^2)G_{71}^rG_{71}^*,$$

$$H_{71} = 6k_1^2(1 - k_1^2 - k_2^2)H_{71}^rH_{71}^*,$$

where G_{71}^r and H_{71}^r are rational functions in k_1 and k_2, while G_{71}^* and H_{71}^* are polynomials in b_{251}, b_{161}, b_{071}, k_1 and k_2. Since

$$k_1^2 + k_2^2 = \cos^2\phi + \sin^2\phi = 1,$$

we have $G_{71} = H_{71} = 0$ and so $V_{421} = V_{451} = 0$. Therefore, the best result we can obtain is the solutions such that $V_{3i1} = 0$, $i = 1, 2, \ldots, 13$, but $V_{421} \neq 0$ by solving $F_{71} = 0$, which may yield

14 limit cycles. In order to find the solution, we let $k_2 = \sqrt{1-k_1^2}$ (the case $k_2 = -\sqrt{1-k_1^2}$ can be similarly proved) and then obtain

$$F_{71} = \frac{23469053426330905909333751886368816400000000000000000(1-k_1^2)(1-4k_1^2)^3(3-4k_1^2)C_{70}^*}{555165k_1^8 - 59808k_1^6 + 232562k_1^4 - 155256k_1^2 + 17161} \times [2k_1(1-k_1^2)(339k_1^2+13) - \sqrt{1-k_1^2}(309k_1^4 + 590k_1^2 - 131)]F_{71}^{**},$$

$$G_{71} = \frac{36879941098519995000381610107150997200000000000000000(1-k_1^2)(1-4k_1^2)^3(3-4k_1^2)C_{70}^*}{555165k_1^8 - 59808k_1^6 + 232562k_1^4 - 155256k_1^2 + 17161} G_{71}^{**},$$

where

$$\begin{aligned} C_{70}^* = {} & (21087k_1^8 - 86679k_1^6 + 13143k_1^4 + 9179k_1^2 + 262)b_{341} + (103497k_1^8 - 710k_1^6 - 36148k_1^4 \\ & - 6378k_1^2 + 1179)b_{251} - (19383k_1^8 - 92759k_1^6 + 34999k_1^4 - 3845k_1^2 - 786)b_{161} \\ & + (555165k_1^8 - 59808k_1^6 + 232562k_1^4 - 155256k_1^2 + 17161)b_{071} - k_1\sqrt{1-k_1^2}[(2241k_1^6 + 33381k_1^4 \\ & - 48333k_1^2 + 10663)b_{341} - (24069k_1^6 - 72631k_1^4 - 9537k_1^2 + 3827)b_{251} \\ & - (102999k_1^6 + 20099k_1^4 - 6027k_1^2 - 2383)b_{161}] \end{aligned}$$

and F_{71}^{**} is a 39th-degree polynomial in k_1^2. It can be shown that the factor in the square bracket in F_{71} does not yield solutions for the existence of 14 limit cycles. Thus, we only need to consider the solutions from the polynomial F_{71}^{**}. It is noted that F_{71}^{**} and G_{71}^{**} have no common solutions. Thus, the solutions solved from $F_{71}^{**} = 0$ do not render $G_{71}^{**} = 0$. Solving $F_{71}^{**} = 0$ yields 30 real solutions for $k_1 \in (-1, 1)$:

$$\begin{aligned} k_1 = {} & \pm 0.9964817201\cdots, \quad \pm 0.9896671578\cdots, \quad \pm 0.9558178228\cdots, \quad \pm 0.9120186923\cdots, \\ & \pm 0.9022317704\cdots, \quad \pm 0.8245906156\cdots, \quad \pm 0.8112084855\cdots, \quad \pm 0.7324854562\cdots, \\ & \pm 0.6190075998\cdots, \quad \pm 0.5708227869\cdots, \quad \pm 0.4256589331\cdots, \quad \pm 0.3706595580\cdots, \\ & \pm 0.2233323665\cdots, \quad \pm 0.1008102067\cdots, \quad \pm 0.0776411547\cdots. \end{aligned}$$

Again, due to $k_2 = \pm\sqrt{1-k_1^2}$, we have a total of 60 sets of solutions. Moreover, we can show that

$$\begin{aligned} \mathrm{Det}_{71} & = \det\left[\frac{\partial(V_{31}, V_{61}, V_{91}, V_{121}, V_{151}, V_{181}, V_{211}, V_{241}, V_{271}, V_{301}, V_{331}, V_{361}, V_{391})}{\partial(a_{701}, a_{611}, a_{521}, a_{431}, a_{341}, a_{251}, a_{161}, a_{071}, b_{701}, b_{611}, b_{521}, b_{431}, k_1)}\right] \\ & = \frac{-5k_1^6(1-k_1^2)^3(1-4k_1^2)^6(3-4k_1^2)^6}{363031503772174707128620908000488935451029321310257472073492601316617632884981 76} \\ & \quad \times [P_{52}(k_1^2) + k_1\sqrt{1-k_1^2}P_{51}(k_1^2)], \end{aligned}$$

where P_{52} and P_{51} are respectively 52nd- and 51st-degree polynomials in k_1^2. It can be easily shown that $\mathrm{Det}_{71} \neq 0$ for the roots of F_{71}^{**}. For example, by taking $k_1 = -0.9558178228\cdots$, and setting $b_{071} = 1$, $b_{161} = b_{251} = b_{341} = 0$, we obtain

$$V_{31} = V_{61} = \cdots = V_{391} = 0,$$

$$V_{421} = 0.00028790238\cdots \neq 0$$

and

$$\mathrm{Det}_{71} = 0.1354103578\cdots \times 10^{-22} \neq 0.$$

Then, by Lemma 4 and a linear perturbation, we can conclude that system (41) can have $13 + 1 = 14$ small-amplitude limit cycles bifurcating from the origin, i.e. $M_h(7) \geq 14$.

Remark 3.1. It should be noted from the above discussed cases, $n = 5$ and $n = 7$, that the coefficients k_1 and k_2 in the unperturbed systems have a nonlinear relation: $k_1^2 + k_2^2 = 1$. We will see in the next section that this nonlinear relation makes a difference.

3.3. $M_h(4) \geq 8$ *for the case* $n = 4$

For the case $n = 4$, Giné studied two systems [Giné, 2012a, 2012b], one taken from a system for case (iii) in Theorem 9 of [Giné, 2006], and the other from a system given in Sec. 3 of [Chavarriga *et al.*, 2002]. We will show that the first system can have seven small-amplitude limit cycles, while the second system can have eight small-amplitude limit cycles, bifurcating from the origin.

3.3.1. *System A*

The system is described by

$$\begin{aligned} \dot{x} &= -y - k_1x^3y + k_2y^2(2x^2 - y^2), \\ \dot{y} &= x + k_2xy^3 + k_1x^2(x^2 - 2y^2), \end{aligned} \tag{42}$$

where k_1 and k_2 are arbitrary real constants. System (42) is integrable with a center at the origin. The integrating factor is given by

$$\begin{aligned} M_{40}(x, y) = [1 &+ 2(k_1x^3 + k_2y^3) \\ &+ (k_1x^3 - k_2y^3)^2]^{-\frac{7}{6}}. \end{aligned} \tag{43}$$

In [Giné, 2012a], the author used the independent linear parts in Poincaré–Lyapunov constants to show that $M_h(4) \geq 5$, and later in [Giné, 2012b] the same author used both independent linear and quadratic parts in Poincaré–Lyapunov constants to prove that $M_h(4) \geq 7$. We add perturbations up to ε^4-order to system (42) to obtain the following perturbed system:

$$\begin{aligned} \dot{x} &= -y - k_1x^3y + k_2y^2(2x^2 - y^2) + \varepsilon p_4(x, y, \varepsilon), \\ \dot{y} &= x + k_2xy^3 + k_1x^2(x^2 - 2y^2) + \varepsilon q_4(x, y, \varepsilon), \end{aligned} \tag{44}$$

where

$$\begin{aligned} p_4 &= p_{41} + \varepsilon p_{42} + \varepsilon^2 p_{43} + \varepsilon^3 p_{44} \\ &= \sum_{i+j=4} a_{ij1}x^iy^j + \varepsilon a_{ij2}x^iy^j \\ &\quad + \varepsilon^2 a_{ij3}x^iy^j + \varepsilon^3 a_{ij4}x^iy^j, \\ q_4 &= q_{41} + \varepsilon q_{42} + \varepsilon^2 q_{43} + \varepsilon^3 q_{44} \\ &= \sum_{i+j=4} b_{ij1}x^iy^j + \varepsilon b_{ij2}x^iy^j \\ &\quad + \varepsilon^2 b_{ij3}x^iy^j + \varepsilon^3 b_{ij4}x^iy^j. \end{aligned} \tag{45}$$

For the ε-order focus values, we obtain $V_{3i-21} = V_{3i-11} = 0$, $V_{3i1} \neq 0$, for $i = 1, 2, 3, \ldots$. Using the parameters: $a_{401}, a_{311}, a_{221}, a_{131}$ to linearly solve the focus value equations: $V_{31} = V_{61} = V_{91} = V_{121} = 0$, we obtain the solution: $\mathrm{S}_{4\mathrm{A}1} = (a_{401}, a_{311}, a_{221}, a_{131})$, and then

$$\begin{aligned} V_{151} &= \frac{1323k_1^2k_2^4(k_1^2 - k_2^2)}{11534336000(158816k_1^6 - 1034082k_1^4k_2^2 - 1087243k_1^2k_2^4 - 14889k_2^6)}[k_1(56k_1^2 + k_2^2)a_{041} - 8k_1^2k_2b_{401} \\ &\quad + 2k_2(49k_1^2 + k_2^2)b_{041} + 32k_1^3b_{311} + k_1(56k_1^2 + k_2^2)b_{131} - 4k_1^2k_2b_{221}]F_{4A1}, \\ V_{181} &= \frac{-147k_1^2k_2^4(k_1^2 - k_2^2)}{7843348480000(158816k_1^6 - 1034082k_1^4k_2^2 - 1087243k_1^2k_2^4 - 14889k_2^6)}[k_1(56k_1^2 + k_2^2)a_{041} \\ &\quad - 8k_1^2k_2b_{401} + 2k_2(49k_1^2 + k_2^2)b_{041} + 32k_1^3b_{311} + k_1(56k_1^2 + k_2^2)b_{131} - 4k_1^2k_2b_{221}]G_{4A1}, \\ V_{211} &= \frac{7k_1^2k_2^4(k_1^2 - k_2^2)}{7951272955084800000(158816k_1^6 - 1034082k_1^4k_2^2 - 1087243k_1^2k_2^4 - 14889k_2^6)}[k_1(56k_1^2 + k_2^2)a_{041} \\ &\quad - 8k_1^2k_2b_{401} + 2k_2(49k_1^2 + k_2^2)b_{041} + 32k_1^3b_{311} + k_1(56k_1^2 + k_2^2)b_{131} - 4k_1^2k_2b_{221}]H_{4A1}, \end{aligned}$$

where

$$\begin{aligned} F_{4A1} &= 90957k_1^4 + 91570k_1^2k_2^2 + 90957k_2^4, \\ G_{4A1} &= 9420015381k_1^6 + 18132723551k_1^2k_2^4 + 18138559311k_1^4k_2^2 + 8554104741k_2^6, \\ H_{4A1} &= 341382716662754999 1k_1^8 + 11217196809430171012k_1^6k_2^2 + 14217938640483374394k_1^4k_2^4 \\ &\quad + 10721178041458206852k_1^2k_2^6 + 2924449724303431335k_2^8. \end{aligned}$$

Obviously, except for $(k_1, k_2) = (0, 0)$, there are no real solutions such that $V_{151} = 0$, but $V_{181} \neq 0$. Therefore, for the best result with an infinite number of solutions we can obtain $V_{31} = V_{61} = V_{91} = V_{121} = 0$, but $V_{151} \neq 0$. Moreover, for the solution $\mathrm{S}_{4\mathrm{A}1}$,

$$\begin{aligned}\mathrm{Det}_{4\mathrm{A}1} &= \det\left[\frac{\partial(V_{31}, V_{61}, V_{91}, V_{121})}{\partial(a_{401}, a_{311}, a_{221}, a_{131})}\right]_{\mathrm{S}_{4\mathrm{A}1}} \\ &= -\frac{147}{6192449487634432000} k_1^4 k_2^2 (k_1^2 - k_2^2)^2 (158816 k_1^6 - 1034082 k_1^4 k_2^2 - 1087243 k_1^2 k_2^4 - 14889 k_2^6) \\ &\neq 0,\end{aligned}$$

as long as k_1 and k_2 are taken to satisfy $k_1 k_2 \neq 0$, $k_1 \neq \pm k_2$ and $k_2 \neq \pm 0.3667704 \cdots k_1$. Hence, with proper perturbations on the solution $\mathrm{S}_{4\mathrm{A}1}$, we can obtain at least four small-amplitude limit cycles around the origin. Finally, adding a linear perturbation yields one more limit cycle. Therefore, based on ε-order focus values, we obtain at least five small-amplitude limit cycles.

Next, we want to use ε^2-order focus values to consider the bifurcation of limit cycles from the origin of system (44). So we solve the common factor in V_{151}, V_{181} and V_{211} for a_{041} to obtain the critical condition $\mathrm{C}_{4\mathrm{A}1}$, defined by

$$\mathrm{C}_{4\mathrm{A}1}: \begin{cases} a_{401} = -\dfrac{k_2[(14b_{221} + 49b_{041} + 28b_{401})k_1 + 2b_{311}k_2]}{56k_1^2 + k_2^2}, \\[2ex] a_{311} = \dfrac{(28b_{221} + 98b_{041}k_1^2 - 28b_{311}k_1k_2 + (14b_{041}2 + 7b_{401} + 4b_{221}k_2^2))}{56k_1^2 + k_2^2}, \\[2ex] a_{221} = \dfrac{(56b_{311} + 112b_{131})k_1^2 + (84b_{401} + 147b_{041} + 42b_{221})k_1k_2 + (2b_{131}2 + 7b_{311})k_2^2}{56k_1^2 + k_2^2}, \\[2ex] a_{131} = -\dfrac{28b_{041}k_1^2 - 56b_{311}k_1k_2 + (14b_{401} + 25b_{041} + 7b_{221}k_2^2)}{56k_1^2 + k_2^2}, \\[2ex] a_{041} = -\dfrac{8(7b_{131} + 4b_{311})k_1^3 - 2(2b_{221} + 4b_{401} - 49b_{041})k_1^2k_2 + b_{131}k_1k_2^2 + 2b_{041}k_2^3}{56k_1^2 + k_2^2}. \end{cases}$$

Then, under the critical condition $\mathrm{C}_{4\mathrm{A}1}$, we wish to use (38) to show that $V_{3i1} = 0$ for any positive integer i. To achieve this, we assume the ε-order integrating factor is given in the form of

$$M_{41}(x, y, \delta) = M_{40}(x, y) + \varepsilon M_{41}^*(x, y, \delta),$$

where $\delta = (b_{401}, b_{311}, b_{22}, b_{131}, b_{041})$. Then, by using (38) we obtain

$$\begin{aligned} M_{41}^* = \frac{2}{56k_1^2 + k_2^2}\{ & b_{401}k_2[(k_1x^3 - k_2y^3)(k_2x^3 + 36k_1x^2y - 6k_2xy^2 + 8k_1y^3) + k_2x^3 \\ & - 48k_1x^2y + 6k_2xy^2 - 8k_1y^3] - b_{311}[(k_1x^3 - k_2y^3)(4k_1k_2x^3 - 3(8k_1^2 + k_2^2)x^2y \\ & - 24k_1k_2xy^2 + 32k_1^2y^3) + 4k_1k_2x^3 - 3(8k_1^2 - k_2^2)x^2y + 24k_1k_2xy^2 - 32k_1^2y^3] \\ & - b_{221}[(k_1x^3 - k_2y^3)(28k_1^2x^3 - 18k_1k_2x^2y + 3k_2^2xy^2 - 4k_1k_2y^3) + 28k_1^2x^3 \\ & + 24k_1k_2x^2y - 3k_2^2xy^2 + 4k_1k_2y^3] + b_{131}y^3(56k_1^2 + k_2^2)(1 - k_1x^3 + k_2y^3) \\ & - b_{041}[(k_1x^3 - k_2y^3)(98k_1^2x^3 - 63k_1k_2x^2y + 12(7k_1^2 + k_2^2)xy^2 - 14k_1k_2y^3) \\ & + 98k_1^2x^3 + 84k_1k_2x^2y + 3(28k_1^2 - 3k_2^2)xy^2 + 14k_1k_2y^3]\}, \end{aligned}$$

for which (38) holds for $k = 1$. Thus, all the ε-order focus values vanish under the critical condition $\mathrm{C}_{4\mathrm{A}1}$.

Now we assume that the critical condition C_{4A1} holds and proceed to ε^2-order focus values V_{3i2}, $i = 1, 2, \ldots$. First, we use the five parameters a_{402}, a_{312}, a_{222}, a_{132}, a_{042} to linearly solve the five focus value equations: $V_{32} = V_{62} = V_{92} = V_{122} = V_{152} = 0$. Then, V_{182}, V_{212} and V_{242} become

$$V_{182} = \frac{49k_2(k_1^2+k_2^2)}{136902082560k_1(56k_1^2+k_2^2)^2(90957k_1^4+91570k_1^2k_2^2+90957k_2^4)}F_{4A2},$$

$$V_{212} = -\frac{7k_2}{43370579755008000k_1(56k_1^2+k_2^2)^2(90957k_1^4+91570k_1^2k_2^2+90957k_2^4)}G_{4A2},$$

$$V_{242} = \frac{7k_2}{172372032178303795200000k_1(56k_1^2+k_2^2)^2(90957k_1^4+91570k_1^2k_2^2+90957k_2^4)}H_{4A2},$$

where F_{4A2}, G_{4A2} and H_{4A2} are quadratic polynomials in b_{401}, b_{311}, b_{221} and b_{041}, which do not contain b_{402}, b_{312}, b_{222}, b_{132}, b_{042} and b_{131}. Solving b_{401} from the equation $F_{4A2} = 0$ we obtain $b_{401} = b_{401}^{\pm}(b_{401}, b_{311}, b_{221}, b_{041})$ for which G_{4A2} and H_{4A2} are reduced to

$$\begin{aligned} G_{4A2} = {} & \frac{-4320}{\begin{array}{l}92154426k_1^{10}+1248750173k_1^8k_2^2+1442811524k_1^6k_2^4+1427786006k_1^4k_2^6\\ \quad +1157457906k_1^2k_2^8+46077213k_2^{10}\end{array}} \\ & \times b_{041}^2k_1^2k_2^4(k_1^4-k_2^4)(56k_1^2+k_2^2)^2(90957k_1^4+91570k_1^2k_2^2+90957k_2^4)G_{4A2}^*, \\ H_{4A2} = {} & \frac{-25920}{\begin{array}{l}92154426k_1^{10}+1248750173k_1^8k_2^2+1442811524k_1^6k_2^4+1427786006k_1^4k_2^6\\ \quad +1157457906k_1^2k_2^8+46077213k_2^{10}\end{array}} \\ & \times b_{041}^2k_1^2k_2^4(k_1^4-k_2^4)(56k_1^2+k_2^2)^2(90957k_1^4+91570k_1^2k_2^2+90957k_2^4)H_{4A2}^*, \end{aligned}$$

where G_{4A2}^* and H_{4A2}^* are respectively 12th- and 14th-degree homogeneous polynomials in k_1 and k_2, given by

$$\begin{aligned} G_{4A2}^* = {} & 6447886448367k_1^{12}-42075685854722k_1^{10}k_2^2-51682411730095k_1^8k_2^4-29591854046844k_1^6k_2^6 \\ & -51682411730095k_1^4k_2^8-42075685854722k_1^2k_2^{10}+6447886448367k_2^{12}, \\ H_{4A2}^* = {} & 1335162927440272831173k_1^{14}-743916003843897161062 5k_1^{12}k_2^2 \\ & -19090977605930884606283k_1^{10}k_2^4-16386375933202185058681k_1^8k_2^6 \\ & -16613997039570403362681k_1^6k_2^8-18991989902511041214283k_1^4k_2^{10} \\ & -6939173149427942554625k_1^2k_2^{12}+1268723905476299263173k_2^{14}. \end{aligned}$$

Let $k_2 = k_rk_1$. Then, we obtain

$$\begin{aligned} \mathrm{Det}_{4A2} = {} & \det\left[\frac{\partial(V_{32},V_{62},V_{92},V_{122},V_{152},V_{182},V_{212})}{\partial(a_{402},a_{312},a_{222},a_{132},a_{042},b_{401},k_1)}\right] \\ = {} & -\frac{466948881k_1^{45}(k_r^2-1)^4(K_r^2+1)^2k_r^{12}b_{041}|k_1k_rb_{041}|}{10225990232612599095106110400007541285191680000000000} \\ & \times\sqrt{\frac{92154426k_r^{10}+1248750173k_r^8+1442811524k_r^6+1427786006k_r^4+1157457906k_r^2+46077213}{46077213k_r^{10}+1157457906k_r^8+1427786006k_r^6+1442811524k_r^4+1248750173k_r^2+92154426}} \\ & \times(6447886448367k_r^{12}-42075685854722k_r^{10}-51682411730095k_r^8-29591854046844k_r^6 \\ & -51682411730095k_r^4-42075685854722k_r^2+6447886448367), \end{aligned}$$

which however equals zero when $G^*_{4A2} = 0$. Therefore, by using ε^2-order focus values, we can only obtain seven limit cycles. So we carry out the above procedure to ε^3-order focus values and find that based on ε^3-order focus, only five limit cycles can be obtained, like the case in ε-order analysis. Thus, we continue to use ε^4-order focus values and can show the existence of seven limit cycles, like the case in ε^2-order analysis. Hence, we conclude that seven limit cycles can be obtained around the origin of the fourth-degree homogeneous system (42) by using up to ε^4-order analysis.

Remark 3.2. Here, it is noted from the above ε^2-order analysis that the relation between the coefficients k_1 and k_2 is linear: $k_2 = k_r k_1$. If we replace $k_1 = k_r k_1$ and use the transform $x \to (k_1)^{(1/3)} x$, $y \to (k_1)^{(1/3)} y$ in (42), we obtain the following system,

$$\dot{x} = -y - x^3 y + k_r y^2(2x^2 - y^2),$$
$$\dot{y} = x + k_r x y^3 + x^2(x^2 - 2y^2),$$

which now has only one independent parameter k_r. This is why we cannot get eight limit cycles for this system.

3.3.2. *System B*

The second fourth-degree homogeneous polynomial system is described in [Giné, 2006, 2012a, 2012b]

$$\begin{aligned}\dot{x} &= -y + 2(1-c^2)x^4 - 2s(3-5c)x^3 y \\ &\quad - 6(1-c)(1+3c)x^2y^2 + 2s(5-7c)xy^3 \\ &\quad + 4c(1-c)y^4, \\ \dot{y} &= x - 2(1-c^2)x^3 y - 6s(1+c)x^2y^2 \\ &\quad + 2(3-4c-3c^2)xy^3 + 2s(1+c)y^4,\end{aligned} \tag{46}$$

where $c = \cos\phi$ and $s = \sin\phi$ with arbitrary $\phi \in [0, 2\pi]$. This system is integrable with a polynomial inverse integrating factor [Giné, 2012a]. In [Giné, 2012b], the author used both independent linear and quadratic parts in Poincaré–Lyapunov constants to show that system (46) can have at least $2 \times 4 - 1 = 7$ small-amplitude limit cycles. We will show the existence of $2 \times 4 = 8$ limit cycles, by adding the perturbations up to ε^2-order as that used in (44), to the above system, yielding

$$\begin{aligned}\dot{x} &= -y + 2(1-c^2)x^4 - 2s(3-5c)x^3 y \\ &\quad - 6(1-c)(1+3c)x^2y^2 + 2s(5-7c)xy^3 \\ &\quad + 4c(1-c)y^4 + \varepsilon \sum_{i+j=4} a_{ij1}x^i y^j \\ &\quad + \varepsilon^2 \sum_{i+j=4} a_{ij2}x^i y^j, \\ \dot{y} &= x - 2(1-c^2)x^3 y - 6s(1+c)x^2y^2 \\ &\quad + 2(3-4c-3c^2)xy^3 + 2s(1+c)y^4 \\ &\quad + \varepsilon \sum_{i+j=4} b_{ij1}x^i y^j + \varepsilon^2 \sum_{i+j=4} b_{ij2}x^i y^j,\end{aligned} \tag{47}$$

where the perturbations p_4 and q_4 are given in (45) up to ε^2-order. Since $c^2 + s^2 = 1$, we let $s = \sqrt{1-c^2}$. The case $s = -\sqrt{1-c^2}$ can be similarly proved.

First, we consider the nonzero ε-order focus values V_{3i1}, $i = 1, 2, \ldots$. We use the five parameters: a_{401}, a_{311}, a_{221}, a_{131} and a_{041} to linearly solve the first five focus value equations: $V_{3i1} = 0$, $i = 1, 2, \ldots, 5$, one by one. Then, V_{181} and V_{211} become

$$V_{181} = \frac{9C_{41}}{761600 C^1_{40}} F_{4B1},$$

$$V_{211} = \frac{3C_{41}}{8377600000 C^1_{40}} G_{4B1},$$

where C^1_{40} is a 20th-degree polynomial in c, and C_{41} is given by

$$\begin{aligned}C_{41} &= (c-3)(c-1)^4(c+1)^5\{3(c^3 - 8c^2 \\ &\quad + 7c + 20)b_{401} + 3(c+2)(1-c)^2 b_{041} \\ &\quad + (1-c)(3c^2 - c + 6)b_{221} - 3\sqrt{1-c^2} \\ &\quad \times [(c^2 - 4c + 7)b_{311} + (1-c^2)b_{131}]\}\end{aligned}$$

and F_{4B1} and G_{4B1} are respectively 21st- and 25th-degree polynomials in c without common roots, i.e. for the roots of F_{4B1}, $G_{4B1} \neq 0$. Therefore, we may have solutions such that $V_{3i1} = 0$, $i = 1, 2, \ldots, 6$, but $V_{211} \neq 0$. In particular, F_{4B1} is given by

$$\begin{aligned}F_{4B1} &= 1003633799168000000c^{21} - 31122741535621120000c^{20} + 414313795760263168000c^{19} \\ &\quad - 3333528572432008804000c^{18} + 19166046235346814378400c^{17} - 86464362690342066276210c^{16} \\ &\quad + 319330192507926213748900c^{15} - 973679617249578717708930c^{14} + 2436257820865500815253940c^{13}\end{aligned}$$

$$- 4963937371367105564153336c^{12} + 8194486637848526968111140c^{11}$$
$$- 10928056271302255002535566c^{10} + 11745621526832094976851164c^{9}$$
$$- 10142870426136344691078455c^{8} + 7002303359147270372294434c^{7} - 3834975554547064230274947c^{6}$$
$$+ 1646862432384731468798422c^{5} - 544761720370801493159110c^{4} + 134903091752996001259444c^{3}$$
$$- 23778812089168363412288c^{2} + 2690767560170594112488c - 148403053350284019122,$$

which has only three real solutions for $c \in (-1, 1)$:

$$c = 0.4839427334\cdots, \quad 0.7229504505\cdots, \quad 0.8227464856\cdots,$$

satisfying $F_{4B1} = 0$, namely $V_{181} = 0$ for which $V_{211} \neq 0$. Since the parameters are used one by one to solve the focus value equations, perturbations can be taken to yield (including the linear perturbation) seven small-amplitude limit cycles around the origin. Alternatively, we can show that for the solution $c = 0.4839427334\cdots$, with $b_{401} = 1$, $b_{311} = b_{221} = b_{131} = b_{041} = 0$,

$$\text{Det}_{4B1} = \det\left[\frac{\partial(V_{31}, V_{61}, V_{91}, V_{121}, V_{151}, V_{181})}{\partial(a_{401}, a_{311}, a_{221}, a_{131}, a_{041}, c)}\right] = 0.1214996166\cdots \neq 0.$$

In order to find eight limit cycles around the origin of system (47), we need to consider ε^2-order focus values. But first we have to find the condition under which all the ε-order focus values vanish. This condition can be obtained by solving $F_1 = 0$, giving solution for b_{401}, and thus we can, together with the above solutions obtained from solving the focus values, define the critical condition C_{4B1} as

$$\mathrm{C}_{4B1}: \begin{cases} a_{401} = \dfrac{-1}{\sqrt{1-c^2}(c^3-8c^2+7c+20)}\{3(c+1)(c+2)(1-c)^2 b_{041} + (1-c^2)(3c^2-c+6)b_{221} \\ \qquad - \sqrt{1-c^2}[(2c^3-c^2+2c+1)b_{311} + 3(1-c)(1+c)^2 b_{131}]\}, \\ a_{311} = \dfrac{-1}{3(1-c^2)(c^3-8c^2+7c+20)}\{3(1-c)^2(22c^3+19c^2+9c+16)b_{041} \\ \qquad + (1-c)(51c^4+52c^3-63c^2-12c+28)b_{221} - 3\sqrt{1-c^2}[(11c^4+c^3-29c^2 \\ \qquad + 19c+6)b_{311} + (1-c^2)(19c^2+5c-22)b_{131}]\}, \\ a_{221} = \dfrac{-1}{(1+c)(c^3-8c^2+7c+20)}\{3(1-c)(14c^3-13c^2+7c+42)b_{041} - (1-c)(28c^3+37c^2 \\ \qquad - 21c-6)b_{221} + 3\sqrt{1-c^2}[(6c^3+3c^2-18c+13)b_{311} + (1-c)(11c^2+12c-1)b_{131}]\}, \\ a_{131} = \dfrac{-1}{(1+c)(c^3-8c^2+7c+20)}\{(34c^4-49c^3-10c^2+165c+80)b_{041} - (19c^4+40c^3-35c^2 \\ \qquad - 64c-20)b_{221} - 6\sqrt{1-c^2}[(2c^3+5c^2-6c-5)b_{311} + (1-c^2)(4c+5)b_{131}]\}, \\ a_{041} = \dfrac{-1}{3\sqrt{1-c^2}(c^3-8c^2+7c+20)}\{3(11c^4-38c^3+31c^2+66c-30)b_{041} \\ \qquad - (3c-5)(5c^3+15c^2+2c-2)b_{22} - 3\sqrt{1-c^2}[(3c-5)(c^2+4c-1)b_{311} \\ \qquad + (1-c)(7c^2+4c-5)b_{131}]\}, \\ b_{401} = \dfrac{-1}{3(c^3-8c^2+7c+20)}\{3(c+2)(1-c)^2 b_{041} + (1-c)(3c^2-c+6)b_{221} \\ \qquad - 3\sqrt{1-c^2}[(c^2-4c+7)b_{311} + (1-c^2)b_{131}]\}. \end{cases}$$

Then, we can similarly show that there exists an ε-order integrating factor M_{41} such that (38) is satisfied. This implies that all the ε-order focus values vanish under the condition C_{4B1}.

Now suppose the condition C_{4B1} holds and we consider the ε^2-order focus values V_{3i2}, $i = 1, 2, \ldots$. We use the six parameters a_{402}, a_{312}, a_{222}, a_{132}, a_{042} and b_{402} to linearly solve the first six focus value equations: $V_{3i2} = 0$, $i = 1, 2, \ldots, 6$, one by one, and then V_{212} and V_{242} become

$$V_{212} = \frac{(c-3)(1+c)^5 C_{42}}{57446400\sqrt{1-c^2}(c^3-8c^2+7c+20)^2 C_{40}^2} F_{4B2},$$

$$V_{242} = \frac{(c-3)(1+c)^5 C_{42}}{1321267200000\sqrt{1-c^2}(c^3-8c^2+7c+20)^2 C_{40}^2} G_{4B2},$$

where C_{40}^2 is a 21st-degree polynomial in c, and

$$\begin{aligned} C_{42} = {} & 4[3(1+c)(4c-9)b_{041} - (2c^3 - 6c^2 \\ & + 7c + 7)b_{221}]^2 + 9(1-c^2)[(c^2-4c+7)b_{311} \\ & + (1-c^2)b_{131}]^2 - 12\sqrt{1-c^2}[(c^2-4c+7)b_{311} \\ & + (1-c^2)b_{131}][3(1+c)(4c-9)b_{041} \\ & - (2c^3-6c^2+7c+7)b_{221}]. \end{aligned}$$

F_{4B2} and G_{4B2} are respectively 30th- and 34th-degree polynomials in c, and they do not have common roots. Thus, we may obtain solutions such that $V_{3i2} = 0$, $i = 1, 2, \ldots, 7$, but $V_{242} \neq 0$. In fact, we find four real roots of F_{4B2} for $c \in (-1, 1)$, given by

$$c = -0.6322034214\cdots, \quad 0.1611981508\cdots,$$
$$0.6367798200\cdots, \quad 0.8325994702\cdots,$$

for which $V_{212} = 0$, but $V_{242} \neq 0$. Therefore, together with $s = \pm\sqrt{1-c^2}$, we have eight sets of solutions. To verify the existence of eight limit cycles, we choose $c = 0.1611981508\cdots$, and, without of loss of generality, set

$$b_{312} = b_{222} = b_{132} = b_{042} = 0,$$
$$b_{221} = b_{131} = b_{041} = 0, \quad b_{311} = 0.01,$$

to obtain

$$V_{32} = V_{62} = \cdots = V_{212} = 0,$$
$$V_{242} = 0.0189982065\cdots \neq 0$$

and

$$\begin{aligned} \mathrm{Det}_{4B2} &= \det\left[\frac{\partial(V_{32}, V_{62}, V_{92}, V_{122}, V_{152}, V_{182}, V_{212})}{\partial(a_{402}, a_{312}, a_{222}, a_{132}, a_{042}, b_{402}, c)}\right] \\ &= 2.5111634224\cdots \neq 0. \end{aligned}$$

The above results, by Lemma 4, clearly indicate that there exist seven limit cycles around the origin of system (47) by applying ε-order and ε^2-order focus values. Further, a linear perturbation can yield one more limit cycle, leading a total of eight limit cycles. If we proceed further to ε^3-order focus values under the condition $C_{42} = 0$, we can show that the ε^3-order analysis can yield eight limit cycles. So we know that system (47) can have at least eight small limit cycles bifurcating from the nondegenerate center (the origin) by using the focus values up to ε^2-order, i.e. $M_h(4) \geq 8$.

3.4. $M_h(6) \geq 12$ *for the case* $n = 6$

In this section, we consider two sixth-degree homogeneous polynomial systems. We have a similar situation as that which occurs in the case $n = 4$: for the first system we can only get 11 limit cycles due to the linear relation between the two coefficients k_1 and k_2: $k_1 = k_r k_1$; while for the second system, we obtain 12 limit cycles since the two coefficients c and s have a nonlinear relation: $c^2 + s^2 = 1$.

3.5. *System A*

The first system was studied in [Giné, 2006, 2012a], given by

$$\begin{aligned} \dot{x} &= -y - k_1 x^5 y + 2k_2 x^2 y^4 - k_2 y^6, \\ \dot{y} &= x + k_1 x^6 - 2k_1 x^4 y^2 + k_2 x y^5, \end{aligned} \tag{48}$$

where k_1 and k_2 are arbitrary real constants. The system has an integrating factor,

$$\begin{aligned} M_{60}(x, y) = {} & [1 + 2(k_1 x^5 + k_2 y^5) \\ & + (k_1 x^5 - k_2 y^5)^2]^{-\frac{9}{10}}. \end{aligned} \tag{49}$$

In [Giné, 2012a], the author used the independent linear parts in the Poincaré–Lyapunov constants to prove $M_h(6) \geq 9$. Later in [Giné, 2012b] the author constructed another sixth-degree homogeneous polynomial system, derived from the fourth-degree system (46), to show that $M_h(6) \geq 11$. In this section, we first consider system (48) and then discuss the second system, and show that system (48) can only have 11 limit cycles even by using analysis up to ε^3-order; while for the second system we obtain 12 limit cycles and so $M_h(6) \geq 12$.

Adding perturbations to system (48) we have the following perturbed system:

$$\begin{aligned} \dot{x} &= -y - k_1 x^5 y + 2k_2 x^2 y^4 - k_2 y^6 + \varepsilon p_6(x, y), \\ \dot{y} &= x + k_1 x^6 - 2k_1 x^4 y^2 + k_2 x y^5 + \varepsilon q_6(x, y), \end{aligned} \tag{50}$$

where

$$\begin{aligned} p_6 &= \sum_{i+j=6} a_{ij1} x^i y^j + \varepsilon a_{ij2} x^i y^j + \varepsilon^2 a_{ij3} x^i y^j, \\ q_6 &= \sum_{i+j=6} b_{ij1} x^i y^j + \varepsilon b_{ij2} x^i y^j + \varepsilon^2 b_{ij3} x^i y^j. \end{aligned} \tag{51}$$

In the following, we will show that by using ε- and ε^2-order focus values, only nine small-amplitude limit cycles can be obtained, but with the ε^3-order focus values, we obtain 11 limit cycles.

Note that the nonzero focus values are given in the form of V_{5i}, $i = 1, 2, \ldots$. First, we use the eight parameters a_{ij1} $(i+j = 6)$ and b_{601} to linearly solve the first eight focus value equations: $V_{5i1} = 0$, $i = 1, 2, \ldots, 8$. Under these solutions, V_{451} and V_{501} become

$$V_{451} = \frac{-135}{549152820451446089450487021568 C_{601}} b_{061} k_2^3 (k_1^2 - k_2^2) F_{61},$$

$$V_{501} = \frac{27}{1541111547028090479990088925899380490240 C_{601}} b_{061} k_2^3 (k_1^2 - k_2^2) G_{61},$$

where C_{601} is a 16th-degree homogeneous polynomial in k_1^2 and k_2^2. F_{61} and G_{61} are respectively 22nd- and 23rd-degree homogeneous polynomials in k_1^2 and k_2^2, and have no common roots. So letting $k_2 = k_r k_1$ yields $F_{61} = k_1^{44} F_{61}^*$, where

$$\begin{aligned} F_{61}^* = {} & 48078005182526481459015626134319745719434087110715671417 07 k_r^{44} \\ & + 460748287741659450806189214496581836634020886539122728232277 k_r^{42} \\ & - 1633002402556897471577344699056991763849616341444086795037 0703 k_r^{40} \\ & - 7131375721086833218898548548791390957179643977296430659176 2586 k_r^{38} \\ & - 1755841223104126596589861036234610855387324796176835207673 25051 k_r^{36} \\ & + 882131146292877903598123035840296547939739427192174144463284 1937 k_r^{34} \\ & + 1635559324661064314360378304811679372897382561105999783135189 0567 k_r^{32} \\ & + 1083917279604081607533800420243052759362234930746589019196038 792 k_r^{30} \\ & - 5204920701285321664639705999312601254453258233583119373090623 0914 k_r^{28} \\ & - 1229172961290550987888009063914272111399864777212599560137822 03174 k_r^{26} \\ & - 1683275229910859007492529464157762694937922042500346777219834 61958 k_r^{24} \\ & - 1555103962561933844801619832162494946160184246582056028974983 81788 k_r^{22} \\ & - 1683275229910859007492529464157762694937922042500346777219834 61958 k_r^{20} \\ & - 1229172961290550987888009063914272111399864777212599560137822 03174 k_r^{18} \\ & - 5204920701285321664639705999312601254453258233583119373090623 0914 k_r^{16} \\ & + 1083917279604081607533800420243052759362234930746589019196038 792 k_r^{14} \\ & + 1635559324661064314360378304811679372897382561105999783135189 0567 k_r^{12} \\ & + 882131146292877903598123035840296547939739427192174144463284 1937 k_r^{10} \end{aligned}$$

$$
\begin{aligned}
&- 1755841223104126596589861036234610855387324796176835207673250 51k_r^8\\
&- 7131375721086833218898548548791390957179643977296430659176258 6k_r^6\\
&- 1633002402556897471577344699056991763849616341444086795037070 3k_r^4\\
&+ 4607482877416594508061892144965818366340208865391227282322 77k_r^2\\
&+ 4807800518252648145901562613431974571943408711071567141707.
\end{aligned}
$$

Moreover, we can show that

$$
\mathrm{Det}_{6\mathrm{A}1} = \det\left[\frac{\partial(V_{51}, V_{101}, V_{151}, V_{201}, V_{251}, V_{301}, V_{351}, V_{401}, V_{451})}{\partial(a_{601}, a_{511}, a_{421}, a_{331}, a_{241}, a_{151}, a_{061}, b_{601}, k_1)}\right] = C_{60} b_{061} k_1^{80} k_r^{13} (1 - k_r^2)^5 F_{61}^*,
$$

where C_{60} is a positive integer. Thus, $F_{61}^* = 0$ results in $\mathrm{Det}_{61} = 0$, implying that we cannot get ten limit cycles, but only nine limit cycles around the origin of system (48) by using the ε-order focus values.

Next, we let $b_{061} = 0$ (which can be seen from the expressions of V_{451} and V_{501}), which yields $V_{5i1} = 0$, $i = 1, 2, \ldots, 10$. So define the critical condition:

$$
\mathrm{C}_{61}: \begin{cases}
a_{601} = 0, \quad b_{061} = 0, \quad a_{511} = \dfrac{1}{4k_1}[k_1(2b_{421} + 3b_{241}) + k_2 b_{511}], \quad a_{421} = 6b_{331} + 5b_{511},\\[2ex]
a_{331} = \dfrac{1}{6k_1}(k_1 b_{241} - 5k_2 b_{511}), \quad a_{241} = 2b_{151} - 9b_{331} - 8b_{511}, \quad a_{151} = \dfrac{k_2}{k_1} b_{511},\\[2ex]
a_{061} = -\dfrac{1}{9k_1^2}(9b_{151}k_1^2 + 6b_{241}k_1k_2 + 2b_{511}k_2^2),\\[2ex]
b_{601} = -\dfrac{1}{36k_1k_2}[16(9b_{331} + 8b_{511})k_1^2 + 9(3b_{241} + 2b_{421})k_1k_2 + 9b_{511}k_2^2],
\end{cases}
$$

under which we can show that there exists an ε-order integrating factor M_{61} such that Eq. (38) holds for $k = 1$, and thus all the ε-order focus values vanish when the critical condition C_{61} is satisfied. In fact, we solve (38) for $k = 1$ to obtain

$$
\begin{aligned}
M_{61} &= M_{60} + \varepsilon M_{61}^*, \quad M_{61}^* = -\frac{1}{18}(s_1 b_{511} + s_2 b_{241} + s_3 b_{331} + s_4 b_{421} + s_5 b_{151}),\\
s_1 &= x[(k_1x^5 - k_2y^5)(9k_2 * x^4 - 20k_1x^3y + 10k_2x^2y^2 + 160k_1xy^3 - 20k_2y^4)\\
&\quad + 9k_2x^4 - 20k_1x^3y + 10k_2x^2y^2 - 160k_1xy^3 + 20k_2y^4],\\
s_2 &= 18k_1(1 + k_1x^5 - k_2y^5)x^5, \quad s_3 = -180k_1(1 - k_1x^5 + k_2y^5)x^2y^3,\\
s_4 &= 3k_1(1 + k_1x^5 - k_2y^5)x^3(9x^2 + 10y^2), \quad s_5 = -36k_1(1 - k_1x^5 + k_2y^5)y^5.
\end{aligned}
\tag{52}
$$

Now suppose the critical condition C_{61} is valid, we process to ε^2-order values. Similar to the analysis for the ε-order focus values, we may use the eight parameters a_{ij2} $(i + j = 6)$ and b_{602} to linearly solve the first eight focus value equations: $V_{5i2} = 0$, $i = 1, 2, \ldots, 8$. Under these solutions, V_{452} and V_{502} are reduced to

$$
\begin{aligned}
V_{452} &= \frac{-5C_{62}}{6589833845417353073405844258816k_1^3 C_{601}} k_2^3(k_1^2 - k_2^2) F_{61},\\
V_{502} &= \frac{C_{62}}{1849333856433708575988106711079256588288 0k_1^3 C_{601}} k_2^3(k_1^2 - k_2^2) G_{61},
\end{aligned}
$$

where C_{601}, F_{61} and G_{61} are the same as that used in the ε-order analysis, and C_{62} is given by

$$C_{62} = 324k_1^3 b_{062} + (3k_1 b_{241} + k_2 b_{511})(9k_1 b_{241} + 11k_2 b_{511}). \tag{53}$$

Thus, from the analysis for the ε-order focus values we know that using the ε^2-order focus values can also only yield nine limit cycles for system (48) around its origin.

Thus, in order to find more limit cycles, we need to use ε^3-order focus values. To achieve this, we solve b_{062} from the equation $C_{62} = 0$ and then use the solutions to define the critical condition as follows:

$$\mathrm{C}_{62}: \begin{cases} a_{602} = \dfrac{1}{81k_2}(27b_{331} + 28b_{511})(9b_{331} + 8b_{511}), \\ a_{512} = \dfrac{1}{864k_1^3k_2}\{216k_1^2k_2[k_2b_{512} + k_1(2b_{422} + 3b_{242})] - k_2(9k_2^2 + 352k_1^2)b_{511}^2 + 972k_1^2k_2b_{331}^2 \\ \quad - 27k_1^2(96k_1b_{331} + 13k_2b_{241})b_{241} - 36k_1[64b_{241}k_1^2 - (16b_{331} + 6b_{151})k_1k_2 \\ \quad - (3b_{421} - 4b_{241})k_2^2]b_{511}\}, \\ a_{422} = \dfrac{1}{27k_1^2k_2}\{27k_1^2k_2(5b_{512} + 6b_{332}) + 5(9k_2^2 - 128k_1^2)b_{511}^2 - 1215k_1^2b_{331}^2 \\ \quad - 45k_1(40k_1b_{331} - 3k_1b_{241})b_{511}\}, \\ a_{332} = -\dfrac{1}{1296k_1^3}\{216k_1^2(5k_2b_{512} - k_1b_{242}) - 5(256k_1^2 - 39k_2^2)b_{511}^2 + 180k_1b_{511}[2k_1(8b_{331} + 3b_{151}) \\ \quad + k_2(3b_{421} + 4b_{241})] + 405k_1^2(12b_{331}^2 + b_{241}^2)\}, \\ a_{242} = \dfrac{1}{27k_1^2k_2}\{27k_1^2k_2(2b_{152} - 8b_{512} - 9b_{332}) + (256k_1^2 - 123k_2^2)b_{511}^2 + 972k_1^2b_{331}^2 \\ \quad + 9k_1(128b_{331} - 39b_{241})b_{511} - 54k_2(k_2b_{511} + 3k_1b_{241})b_{331}\}, \\ a_{152} = \dfrac{1}{216k_1^3}\{216k_1^2k_2b_{512} + (896k_1^2 + 65k_2^2)b_{511}^2 + 27k_1^2(36b_{331}^2 + b_{241}^2) \\ \quad + 12k_1[6k_1(26b_{331} + 3b_{151}) + k_2(9b_{421} + 17b_{241})]b_{511}\}, \\ a_{062} = \dfrac{-1}{972k_1^4}\{108k_1^2(9k_1^2b_{152} + 6k_1k_2b_{242} + 2k_2^2b_{512}) + k_2(848k_1^2 + 45k_2^2)b_{511}^2 \\ \quad + 27k_1^2k_2(36b_{331}^2 + 5b_{241}^2) + 72k_1k_2(26k_1b_{331} + 3k_2b_{421})b_{511} - 324k_1^2(8k_1b_{331} - k_2b_{421})b_{241} \\ \quad - 180k_1(14k_1^2 - k_2^2)b_{511}b_{241} + 216k_1^2(3k_1b_{241} + 2k_2b_{511})b_{151}\}, \\ b_{602} = \dfrac{1}{7776k_1^3k_2^2}\{-216k_1^2k_2[(128k_1^2 + 9k_2^2)b_{512} + 144k_1^2b_{332} + 27k_1k_2b_{242} + 18k_1k_2b_{422}] \\ \quad + (32768k_1^4 - 13344k_1^2k_2^2 + 81k_2^4)b_{511}^2 + 243k_1^2[4(128k_1^2 - 9k_2^2)b_{331}^2 + 13k_2^2b_{241}^2] \\ \quad + 36[3k_1k_2(128k_1^2 - 9k_2^2)b_{421} - 12k_1k_2(4k_1^2 - 3k_2^2)b_{241} + 8k_1^2(512k_1^2 - 51k_2^2)b_{331} \\ \quad + 6k_1^2(128k_1^2 - 9k_2^2)b_{151}]b_{511} + 5184k_1^3k_2(3b_{421} + 5b_{241})b_{331} + 31104k_1^4b_{331}b_{151}\}, \\ b_{062} = \dfrac{-1}{324k_1^3}[27k_1^2b_{241}^2 + 42k_1k_2b_{241}b_{511} + 11k_2^2b_{511}^2]. \end{cases}$$

Then, when the critical conditions C_{61} and C_{62} hold, we can show that there exists an ε^2-order integrating factor M_{62} to satisfy (38), and then all the ε- and ε^2-order focus values vanish under these conditions. To prove this, we actually find

$$M_{62} = M_{60} + \varepsilon M_{61} + \varepsilon^2 M_{62}^*,$$

where M_{60} and M_{61} are given in (49) and (52), respectively, and M_{62}^* is given by

$$\begin{aligned} M_{62}^* = &-\frac{1}{18k_1}(s_1 b_{512} + s_2 b_{422} + s_3 b_{332} + s_4 b_{242} + s_5 b_{152}) + \frac{1}{3888k_1^3 k_2}[s_6 b_{511}^2 + s_7 b_{421}^2 + s_8 b_{331}^2 + s_9 b_{241}^2 \\ &+ s_{10} b_{151}^2 + b_{511}(s_{11} b_{331} + s_{12} b_{241} + s_{13} b_{151} + s_{14} b_{421}) + b_{151}(s_{15} b_{331} + s_{16} b_{241} + s_{17} b_{421}) \\ &+ b_{331}(s_{18} b_{241} + s_{19} b_{421}) + s_{20} b_{421} b_{241}], \end{aligned}$$

where s_i, $i = 1, 2, \ldots, 5$, are given in (52), and

$$\begin{aligned} s_6 = &-20480k_1^4 x^7 y(x^2 - 2y^2) - 16k_1^3 k_2 x^2(594x^8 + 245x^6y^2 + 80x^4y^4 - 6080x^2y^6 + 2560y^8) \\ &- 8k_1^2k_2^2 xy(135x^8 + 450x^6y^2 - 3192x^4y^4 + 1760x^2y^6 + 2240y^8) + 6k_1k_2^3(54x^{10} + 105x^8y^2 \\ &- 200x^4y^6 + 2040x^2y^8 - 672y^{10}) - 3k_2^4 xy^5(27x^4 + 30x^2y^2 + 260y^4) + 10240k_1^3x^2y(7x^2 - 4y^2) \\ &- 32k_1^2k_2 x(297x^4 + 880x^2y^2 + 560y^4) + 192k_1k_2^2y^3(50x^2 - 21y^2) + 3k_2^3 x(27x^4 + 30x^2y^2 - 140y^4), \\ s_7 = &\ 972k_1^3k_2x^{10}, \\ s_8 = &-972k_1^2x[-80k_1xy(x^2 - 2y^2) + k_2(9x^4 + 10y^2x^2 + 20y^4) + 40k_1^2x^6y(x^2 - 4y^2) \\ &+ k_1k_2x(9x^8 + 10x^6y^2 - 20x^4y^4 - 140x^2y^6 + 160y^8) - k_2^2y^5(9x^4 + 10x^2y^2 - 20y^4)], \\ s_9 = &\ 27k_1^2k_2x[2k_1x^5(99x^4 + 155x^2y^2 + 80y^4) - k_2y^5(117x^4 + 130x^2y^2 + 60y^4) + 117x^4 + 130x^2y^2 + 60y^4], \\ s_{10} = &\ 3888k_1^3k_2y^{10}, \\ s_{11} = &-72k_1[-160k_1^2x^2y(13x^2 - 16y^2) + 8k_1k_2x(33x^4 + 70x^2y^2 + 65y^4) + 48k_2^2y^5 + 160k_1^3x^7y(5x^2 - 16y^2) \\ &+ 4k_1^2k_2x^2(66x^8 + 50x^6y^2 - 55x^4y^4 - 800x^2y^6 + 640y^8) - k_1k_2^2xy^3(135x^6 + 462x^4y^2 \\ &- 100x^2y^4 - 520y^6) + 48k_2^3y^{10}], \\ s_{12} = &\ 18k_1[640k_1^2x^5 + 32k_1k_2y^3(50x^2 - 19y^2) + 8k_2^2x(9x^4 + 10x^2y^2 - 5y^4) + 640k_1^3x^{10} \\ &- 4k_1^2k_2x^5y(45x^4 + 150x^2y^2 - 392y^4) + k_1k_2^2(153x^10 + 260x^8y^2 + 80x^6y^4 - 200x^4y^6 \\ &+ 1840x^2y^8 - 608y^{10}) - 8k_2^3xy^5(9x^4 + 10x^2y^2 + 20y^4)], \\ s_{13} = &-216k_1^2k_2x[k_1x(9x^8 + 10x^6y^2 - 20x^4y^4 + 20x^2y^6 - 160y^8) - 2k_2y^5(9x^4 + 10y^2x^2 - 20y^4) \\ &+ 9x^4 + 10x^2y^2 + 20y^4], \\ s_{14} = &-108k_1k_2x[y(20k_1^2x^6(x^2 - 8y^2) - k_2^2y^4(9x^4 + 10x^2y^2 - 20y^4)) + k_2(9x^4 + 10x^2y^2 + 20y^4)], \\ s_{15} = &\ 38880k_1^3k_2x^2y^8, \quad s_{16} = 648k_1^3k_2x^3y^5(9x^2 + 10y^2), \quad s_{17} = 3888k_1^3k_2x^5y^5, \\ s_{18} = &\ 648k_1^2[20k_1x^5 - 16k_2y^5 + 20k_1^2x^{10} + k_1k_2x^5(45x^2y^3 + 46y^5) - 16k_2^2y^{10}], \\ s_{19} = &\ 19440k_1^3k_2x^7y^3, \quad s_{20} = 324k_1^3k_2x^8(9x^2 + 10y^2). \end{aligned}$$

With the above conditions, (38) holds for $k = 2$.

We now assume that the critical conditions C_{61} and C_{62} hold and proceed to ε^3-order focus values. We may use the nine parameters a_{ij3} $(i+j=6)$, b_{603} and b_{063} to linearly solve the first nine focus value equations: $V_{5i3}=0$, $i=1,2,\ldots,9$. Under these solutions, V_{503}, V_{553} and V_{603} become

$$V_{503}=\frac{13}{22418455320849446819121130334307722723328k_1^5k_2^2C_{603}}F_{63},$$

$$V_{553}=\frac{-1}{48208646321954650439838078670895326944244531200k_1^5k_2^2C_{603}}G_{63},$$

$$V_{603}=\frac{1}{43194947104471366794094918489122212942043099955200000k_1^5k_2^2C_{603}}H_{63},$$

where C_{603} is a 22nd-degree homogeneous polynomial in k_1^2 and k_2^2, and F_{63}, G_{63} and H_{63} are 3rd-degree homogeneous polynomials with respect to b_{511}, b_{331} and b_{241}. Therefore, we let

$$b_{511}=B_{511}b_{331},\quad b_{241}=B_{241}b_{331},$$

under which V_{503}, V_{553} and V_{603} are reduced to

$$V_{503}=b_{331}^3V_{503a},\quad V_{553}=b_{331}^3V_{553a},$$
$$V_{603}=b_{331}^3V_{603a},$$

where V_{503a}, V_{553a} and V_{603a} are third-degree homogeneous polynomials in B_{511} and B_{241} with the coefficients in terms of k_1^2 and k_2^2. Now, eliminating B_{511} from the two polynomial equations, $V_{503a}=V_{553a}=0$, we obtain a solution

$$B_{511}=-\frac{1}{16}k_2(9k_2+24k_1B_{241})$$

and a resultant:

$$R_1=k_1k_2(k_1^2-k2^2)(8k_1B_{241}-3k_2)$$
$$\times R_{12}(k_1,k_2)R_{1a}(k_1,k_2),$$

where R_{12} and R_{1a} are respectively 22nd- and 38th-degree homogeneous polynomials in k_1^2 and k_2^2. Similarly, eliminating B_{511} from the two polynomial equations, $V_{503a}=V_{603a}=0$, yields the same solution B_{511} given above, and another resultant:

$$R_2=k_1k_2(k_1^2-k2^2)(8k_1B_{241}-3k_2)$$
$$\times R_{12}(k_1,k_2)R_{2a}(k_1,k_2),$$

where R_{2a} is a 39th-degree homogeneous polynomial in k_1^2 and k_2^2. Since the two polynomials R_{1a} and R_{2a} do not have common roots, in order to have more than 12 limit cycles, we must have solutions such that $R_1=R_2=0$, which comes from the common factors:

$$k_1k_2(k_1^2-k2^2)(8k_1B_{241}-3k_2)R_{12}(k_1,k_2).$$

However, it can be shown that $k_1k_2(k_1^2-k_2^2)=0$ does not give feasible solutions, and that letting $(8k_1B_{241}-3k_2)R_{12}(k_1,k_2)=0$ yields $V_{5i3}=0$, $i=1,2,\ldots$. This indicates that there do not exist solutions such that $V_{503}=V_{553}=V_{603}=0$, but $V_{653}\neq 0$, implying that system (48) cannot have 13 limit cycles bifurcating from the origin. Thus, the best result we can obtain is $V_{503}=V_{553}=0$ but $V_{603}\neq 0$, meaning that system (48) can have at most 12 limit cycles around the origin. To find the solutions for 12 limit cycles, we only need to solve the 38th-degree homogeneous polynomial equation $R_{1a}=0$. However, $R_{1a}=0$ yields a zero divisor for the solution of b_{063}. If we change to use other parameters, for example b_{512}, the same situation appears. Therefore, we cannot let $R_{1a}=0$ and so 12 limit cycles are not possible to be obtained via the ε^3-order analysis. Even continuing to ε^4-order analysis, the best result we can obtain is 11 limit cycles.

Remark 3.3. The above results show a similar situation as that for System A in case $n=4$, due to the linear relation between the coefficients $k_2=k_rk_1$.

3.6. *System B*

Now we discuss another sixth-degree homogeneous polynomial system, which was introduced in [Giné, 2012b] in which the author used the fourth-degree system (46) and take $c=\cos\phi=\frac{1}{3}$ and so $s=\sin\phi=\frac{2\sqrt{2}}{3}$. Then taking the polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ into system (46) yields the system

$$\dot{r}=\frac{1}{18}r^4f_4(\theta),\quad \dot{\theta}=1+\frac{1}{18}r^3g_4(\theta),\qquad (54)$$

where

$$\begin{aligned} f_4(\theta) &= 12\sqrt{2}\sin\theta - 18\sqrt{2}\sin(3\theta) \\ &\quad + 2\sqrt{2}\sin(5\theta) + 6\cos(\theta) \\ &\quad + 9\cos(3\theta) + 17\cos(5\theta), \\ g_4(\theta) &= 6\sin\theta + 5\sin(3\theta) - 17\sin(5\theta) \\ &\quad - 12\sqrt{2}\cos(\theta) + 12\sqrt{2}\cos(3\theta) \\ &\quad + 2\sqrt{2}\cos(5\theta). \end{aligned} \tag{55}$$

Next, using the change $R = r^{3/5}$ in (54) yields

$$\dot{R} = \frac{1}{30}R^6 f_4(\theta), \quad \dot{\theta} = 1 + \frac{1}{18}R^5 g_4(\theta), \tag{56}$$

which corresponds to a sixth-degree homogeneous polynomial system in polar coordinates. Finally, taking the Cartesian coordinates $x = R\cos\theta$, $y = R\sin\theta$ gives the following sixth-degree homogeneous polynomial system with a linear center at the origin:

$$\begin{aligned} \dot{x} &= -y + \frac{16}{15}x^6 - \frac{16\sqrt{2}}{15}x^5 y - \frac{104}{45}x^4 y^2 \\ &\quad + \frac{112\sqrt{2}}{45}x^3 y^3 - \frac{128}{15}x^2 y^4 + \frac{128\sqrt{2}}{45}xy^5 + \frac{8}{9}y^6, \\ \dot{y} &= x - \frac{112}{45}x^5 y - \frac{208\sqrt{2}}{45}x^4 y^2 + \frac{24}{5}x^3 y^3 \\ &\quad - \frac{128\sqrt{2}}{45}x^2 y^4 + \frac{56}{45}xy^5 + \frac{16\sqrt{2}}{15}y^6. \end{aligned} \tag{57}$$

In [Giné, 2012b], the author obtained the Poincaré–Lyapunov constants of system (57) and applied independent linear and quadratic parts in the Poincaré–Lyapunov constants to prove the existence of 11 small limit cycles around the origin, i.e. $M_h(6) \geq 11$.

Here, we want to find 12 limit cycles and thus need one more parameter. To achieve this, we do not choose particular values for c and s, but let them be free with the restriction $c^2 + s^2 = 1$. Following the above procedure, we can deduce the following general system from (46) with up to ε^2-order perturbations added:

$$\begin{aligned} \dot{x} &= -y + \frac{6}{5}s^2 x^6 - \frac{6}{5}s(3 - 5c)x^5 y \\ &\quad - \frac{4}{5}(1 - c)(1 - 10c)x^4 y^2 + \frac{4}{5}s(3 + 5c)x^3 y^3 \\ &\quad - \frac{2}{5}(21 + 16c - 45c^2)x^2 y^4 + \frac{2}{5}s(23 - 37c)xy^5 \\ &\quad + 4c(1 - c)y^6 + \varepsilon \sum_{i+j=6} (a_{ij1}x^i y^j + \varepsilon a_{ij2}x^i y^j), \\ \dot{y} &= x - \frac{14}{5}s^2 x^5 y - \frac{2}{5}s(9 + 25c)x^4 y^2 \\ &\quad + \frac{4}{5}(9 - 4c - 15c^2)x^3 y^3 - \frac{4}{5}s(7 - 5c)x^2 y^4 \\ &\quad + \frac{2}{5}(9 - 16c - 5c^2)xy^5 + \frac{6}{5}s(1 + c)y^6 \\ &\quad + \varepsilon \sum_{i+j=6} (b_{ij1}x^i y^j + \varepsilon a_{ij2}x^i y^j). \end{aligned} \tag{58}$$

The nonzero focus values of system (58) are V_{5i}, $i = 1, 2, \ldots$. We use the nine parameters: a_{ij1} $(i + j = 6)$, b_{601} and b_{511} to linearly solve the first nine nonzero ε-order focus value equations: $V_{5i1} = 0$, $i = 1, 2, \ldots, 9$, and then V_{501} and V_{551} become

$$\begin{aligned} V_{501} &= \frac{3}{20234854081361920000000000C_{60}} C_{61}F_{61}, \\ V_{551} &= \frac{3}{14601470705110761472000000000000C_{60}} \\ &\quad \times C_{61}G_{61}, \end{aligned} \tag{59}$$

where C_{60}, F_{61} and G_{61} are respectively 85th-, 100th- and 104th-degree polynomials in c, and C_{61} is given by

$$\begin{aligned} C_{61} &= 9(1 - c)(405c^4 + 41c^3 - 1703c^2 \\ &\quad + 1331c + 2310)b_{421} + 6(2565c^5 + 572c^4 \\ &\quad - 8598c^3 + 3718c^2 + 6473c - 4570)b_{241} \\ &\quad - 5(6855c^5 - 6436c^4 - 4512c^3 + 13126c^2 \\ &\quad - 2623c - 6538)b_{061} + 9\sqrt{1 - c^2} \\ &\quad \times [(935c^4 + 472c^3 - 3586c^2 \\ &\quad + 1672c + 795)b_{331} - 2(1225c^4 + 872c^3 \\ &\quad - 3520c^2 - 952c + 2615)b_{151}]. \end{aligned}$$

It is noted that F_{61} and G_{61} have no common roots. Thus, the solutions solved from $F_{61} = 0$ may give at most 11 limit cycles. As a matter of fact,

$F_{61} = 0$ yields eight real solutions for $c \in (-1, 1)$:

$$c = -0.4256471772\cdots, \quad 0.0092200993\cdots, \quad 0.4384943994\cdots, \quad 0.6611352222\cdots,$$
$$0.7190150115\cdots, \quad 0.7427713930\cdots, \quad 0.7661298173\cdots, \quad 0.9184750287\cdots.$$

Taking $c = 0.4384943994\cdots$ and letting $b_{331} = b_{241} = b_{151} = b_{061} = 0$ and $b_{421} = 1$, we obtain

$$V_{5i1} = 0, \quad i = 1, 2, \ldots, 10, \quad V_{551} = -0.0291358324\cdots \neq 0$$

and

$$\text{Det}_{6\text{B}1} = \det\left[\frac{\partial(V_{51}, V_{101}, V_{151}, V_{201}, V_{251}, V_{301}, V_{351}, V_{401}, V_{451}, V_{501})}{\partial(a_{601}, a_{511}, a_{421}, a_{331}, a_{241}, a_{151}, a_{061}, b_{601}, b_{511}, c)}\right] = -0.1211756888\cdots \times 10^{-6} \neq 0.$$

Thus, based on the ε-order focus values, by Lemma 4 and a linear perturbation we have shown that system (46) can have 11 limit cycles around the origin.

In order to obtain 12 limit cycles around the origin of system (46), we proceed to ε^2-order focus values. But we first need all ε-order focus values to vanish, which can be reached under the condition solved from $C_{61} = 0$, yielding the following critical condition (with the above obtained solutions):

$$\text{C}_{61} : (a_{ij1}(i + j = 6), b_{601}, b_{511}, b_{421}),$$

for which we can similarly prove that there exists an ε-order integrating factor such that (38) holds, and thus all ε-order focus values vanish under the critical condition C_{61}. We then use the ten parameters: a_{ij1} $(i + j = 6)$, b_{602}, b_{512} and b_{422} to linearly solve the first ten ε^2-order focus value equations: $V_{5i2} = 0$, $i = 1, 2, \ldots, 10$, and then V_{552} and V_{602} become

$$\begin{aligned} V_{552} &= \frac{-\sqrt{1 - c^2}}{33655250604817121280000000(1 - c)C_{60}^*} C_{62}^2 F_{62}, \\ V_{602} &= \frac{\sqrt{1 - c^2}}{2272923335840361573777408000000000000(1 - c)C_{60}^*} C_{62}^2 G_{62}, \end{aligned} \tag{60}$$

where C_{60}^*, F_{62} and G_{62} are respectively 97th-, 112nd- and 116th-degree polynomials in c, and C_{62} is given by

$$\begin{aligned} C_{62} = {} & 6(1 + c)(85c^2 - 109c + 150)b_{241} + 4(450c^4 - 935c^3 - 60c^2 + 881c - 84)b_{061} \\ & + 9\sqrt{1 - c^2}[(15c^3 - 7c^2 + 13c + 75)b_{331} + 2(15c^3 - 47c^2 - 57c + 9)b_{151}]. \end{aligned}$$

It is noted that F_{62} and G_{62} have no common roots, and $F_{62} = 0$ yields eight real solutions for $c \in (-1, 1)$:

$$c = -0.7920476237\cdots, \quad 0.3305253257\cdots, \quad 0.5898851253\cdots, \quad 0.6991452236\cdots,$$
$$0.7268420405\cdots, \quad 0.7410753485\cdots, \quad 0.7640594569\cdots, \quad 0.9137473453\cdots.$$

Taking $c = 0.3305253257\cdots$ and letting $b_{332} = b_{242} = b_{152} = b_{062} = b_{241} = b_{151} = b_{061} = 0$ and $b_{331} = 0.01$, we obtain

$$V_{5i2} = 0, \quad i = 1, 2, \ldots, 11, \quad V_{601} = 0.2153861925\cdots \neq 0$$

and

$$\text{Det}_{6\text{B}2} = \det\left[\frac{\partial(V_{52}, V_{102}, V_{152}, V_{202}, V_{252}, V_{302}, V_{352}, V_{402}, V_{452}, V_{502}, V_{552})}{\partial(a_{602}, a_{512}, a_{422}, a_{332}, a_{242}, a_{152}, a_{062}, b_{602}, b_{512}, b_{422}, c)}\right] = -0.0001010852\cdots \neq 0,$$

which, by Lemma 4 plus a linear perturbation, implies that system (46) can have 12 limit cycles around the origin by using up to ε^2-order analysis.

Summarizing the results obtained in this section for $n = 6$ shows $M_h(6) \geq 12$.

4. Proof of Theorem 1 for $n = 8, 9$

In this section, we prove Theorem 1 for the cases $n = 8, 9$.

4.1. $n = 8$

In [Giné, 2012b], with the same idea and procedure presented in the previous section for the second sixth-degree homogeneous polynomial system, Giné used system (46) with $c = \frac{1}{3}$ and $\frac{2\sqrt{2}}{3}$ to obtain the following eighth-degree homogeneous polynomial system:

$$\begin{aligned}\dot{x} &= -y + \frac{16}{21}x^8 - \frac{16\sqrt{2}}{21}x^7y + \frac{8}{63}x^6y^2 \\ &\quad + \frac{128\sqrt{2}}{63}x^5y^3 - \frac{88}{9}x^4y^4 + \frac{16\sqrt{2}}{3}x^3y^5 \\ &\quad - \frac{520}{63}x^2y^6 + \frac{160\sqrt{2}}{63}xy^7 + \frac{8}{9}y^8, \\ \dot{y} &= x - \frac{176}{63}x^7y - \frac{272\sqrt{2}}{63}x^6y^2 + \frac{232}{63}x^5y^3 \\ &\quad - \frac{48\sqrt{2}}{7}x^4y^4 + \frac{64}{9}x^3y^5 - \frac{16\sqrt{2}}{9}x^2y^6 \\ &\quad + \frac{40}{63}xy^7 + \frac{16\sqrt{2}}{21}y^8\end{aligned} \tag{61}$$

and used linear parts and quadratic parts in Poincaré–Lyapunov constants to show $M_h(8) \geq 13$.

We want to prove $M_h(8) \geq 14$ for this example. In order to do this, we follow the same procedure to obtain the following eighth-degree homogeneous polynomial system with up to ε^2-order perturbations (and with free c and s satisfying $c^2 + s^2 = 1$):

$$\begin{aligned}\dot{x} &= -y + \frac{6}{7}s^2x^8 - \frac{6}{7}s(3-5c)x^7y + \frac{2}{7}(5-18c+13c^2)x^6y^2 - \frac{2}{7}s(3-41c)x^5y^3 \\ &\quad - \frac{2}{7}(31+38c-85c^2)x^4y^4 + \frac{2}{7}s(37-27c)x^3y^5 - \frac{2}{7}(33+6c-55c^2)x^2y^6 \\ &\quad + \frac{2}{7}s(31-53c)xy^7 + 4c(1-c)y^8 + \varepsilon\sum_{i+j=8}(a_{ij1}x^iy^j + \varepsilon a_{ij2}x^iy^j), \\ \dot{y} &= x - \frac{22}{7}s^2x^7y - \frac{2}{7}s(9+41c)x^6y^2 + \frac{2}{7}(19-4c-43c^2)x^5y^3 - \frac{2}{7}s(31+15c)x^4y^4 \\ &\quad + \frac{2}{7}(39-24c-55c^2)x^3y^5 - \frac{2}{7}s(19-29c)x^2y^6 + \frac{2}{7}(9-120c-c^2)xy^7 + \frac{6}{7}s(1+c)y^8 \\ &\quad + \varepsilon\sum_{i+j=8}(b_{ij1}x^iy^j + \varepsilon b_{ij2}x^iy^j).\end{aligned} \tag{62}$$

Let $s = \sqrt{1-c^2}$ (the case $s = -\sqrt{1-c^2}$ can be similarly proved). We obtain the nonzero focus values V_{7i}, $i = 1, 2, \ldots$. We first consider ε-order focus values V_{7i1}, $i = 1, 2, \ldots$, and use the 11 parameters: a_{ij1} $(i+j = 8)$, b_{801} and b_{711} to linearly solve the first 11 ε-order focus value equations: $V_{7i1} = 0$, $i = 1, 2, \ldots, 11$, and then V_{841} and V_{911} become

$$\begin{aligned}V_{841} &= \frac{-9(3-c)(1-c^2)^4(5-4c)^2C_{81}}{186087290653596853993472000000000C_{80}}F_{81}, \\ V_{911} &= \frac{-3(3-c)(1-c^2)^4(5-4c)^2C_{81}}{900885791512193089553196646400000000000C_{80}}G_{81},\end{aligned} \tag{63}$$

where C_{80} is a 130th-degree polynomial in c, and C_{81} is given by

$$\begin{aligned}C_{81} &= -3(14164920c^8 - 8591219c^7 + 54727190c^6 - 210314419c^5 - 126007094c^4 + 526970143c^3 \\ &\quad - 37635382c^2 - 310524921c + 94902686)b_{261} + 18(944328c^8 + 4443467c^7 - 901166c^6 - 42094535c^5\end{aligned}$$

$$+ 28491844c^4 + 51660973c^3 - 37212734c^2 - 12778801c + 7296096)b_{441} - 9(314776c^8 + 4083121c^7 - 1510642c^6 - 36141035c^5 + 40085982c^4 + 28447139c^3 - 42582342c^2 + 1500695c + 5501250)b_{621} + 4(19830888c^8 - 124064521c^7 + 476916874c^6 - 592201589c^5 - 636274384c^4 + 1402757237c^3 - 19513262c^2 - 657824727c + 127939948)b_{081} + \sqrt{1-c^2}[7(19272c^7 + 170875c^6 - 120984c^5 - 1247329c^4 + 1500664c^3 + 456081c^2 - 975400c + 173013)b_{531} - 2(224840c^7 + 511763c^6 - 291192c^5 - 5101977c^4 + 3504136c^3 + 5249929c^2 - 4689048c + 459837)b_{351} + (944328c^7 - 2034725c^6 + 3153016c^5 - 2173361c^4 - 8966536c^3 + 6888881c^2 + 6124488c - 4425307)b_{171}] \tag{64}$$

and F_{81} and G_{81} are respectively 136th- and 140th-degree polynomials in c. Note that F_{81} and G_{81} have no common roots. Therefore, we may have solutions for c such that $V_{841} = 0$ but $V_{911} \neq 0$, implying that 13 limit cycles may exist in system (62) around the origin. Actually, solving $V_{841} = 0$ gives 11 real solutions for $c \in (-1, 1)$:

$$c = -0.6216821257\cdots, \quad -0.1893161285\cdots, \quad 0.1143053953\cdots, \quad 0.6425455377\cdots, \quad 0.7003068466\cdots, \quad 0.7272898514\cdots, \quad 0.7447074714\cdots, \quad 0.7734471745\cdots, \quad 0.8037660909\cdots, \quad 0.9435242040\cdots, \quad 0.9844282851\cdots,$$

under which

$$\mathrm{Det}_{81} = \frac{\partial(V_{71}, V_{141}, V_{211}, V_{281}, V_{351}, V_{421}, V_{491}, V_{561}, V_{631}, V_{701}, V_{771})}{\partial(a_{801}, a_{711}, a_{621}, a_{531}, a_{441}, a_{351}, a_{261}, a_{171}, a_{081}, b_{801}, b_{711})}(3-c)(1-c)^8(1+c)^{11} D_{112}(c) \neq 0,$$

where $D_{112}(c)$ is a 112nd-degree polynomial in c. This implies that based on the ε-order analysis, system (54) can have 13 limit cycles bifurcating from the origin.

To find more limit cycles, we continue to use ε^2-order focus values. But we first need to find the conditions under which all the ε-order focus values vanish. To achieve this, solving $C_{81} = 0$ for b_{621} and then simplifying the solutions yields the following critical condition:

$$\mathrm{C}_{81} : (a_{ij1}(i+j=8), b_{801}, b_{711}, b_{621}),$$

with which we can similarly show that all ε-order focus values vanish.

Now, assume the critical condition C_{81} holds, we proceed to ε^2-order focus values. Similarly, we can use the 12 parameters: a_{ij2} $(i+j=8)$, b_{802}, b_{712}, b_{622} to linearly solve the first 12 focus value equations: $V_{7i2} = 0$, $i = 1, 2, \ldots, 12$, and then

$$V_{912} = \frac{C_1 F_1}{F_{90}} F_{92}, \qquad V_{982} = \frac{C_2 F_1}{F_{90}} F_{98},$$

where C_1 and C_2 are two integers, F_{90}, F_{92} and F_{98} are respectively 8th-, 164th- and 168th-degree polynomials in c, and F_{92} and F_{98} have no common roots. F_1 is given by

$$F_1 = -9(1-c^2)(79282c^7 - 56973c^6 - 397957c^5 + 876294c^4 - 791674c^3 + 4641c^2 + 267157c + 58542)b_{441} + 3(1-c^2)(1009890c^7 - 386981c^6 - 3388217c^5 + 2854944c^4 + 2294752c^3 - 4533117c^2 + 1078471c + 1035986)b_{261} - 2(1-c)(3965766c^8 - 454937c^7 + 4602836c^6 - 21585266c^5 + 2287846c^4 + 11452183c^3 - 12059516c^2 + 11032044c + 5566196)b_{081} + 9\sqrt{1-c^2}[(19943c^8 + 5880c^7 - 181538c^6 + 351518c^5 - 450722c^4 + 55508c^3 + 516230c^2 - 207274c - 133737)b_{531} - (176400c^8 - 24717c^7$$

$$- 1021062c^6 + 1292737c^5 - 118080c^4 - 1225091c^3 + 1330410c^2 - 184049c - 266868)b_{351}$$
$$+ (536795c^8 + 36001c^7 - 1944936c^6 + 19569c^5 + 2010630c^4 + 142787c^3 - 828024c^2$$
$$- 116373c + 65599)b_{171}].$$

It can be shown that $F_{90}F_{98} \neq 0$ for the solutions of $F_{92} = 0$, and $F_1 \neq 0$ for almost all real values of $b_{531}, b_{441}, b_{351}, b_{261}, b_{171}$ and b_{081}. Therefore, there exist solutions such that $V_{7i2} = 0$, $i = 13$, but $V_{982} \neq 0$, implying the existence of at most 14 limit cycles. In fact, solving $F_{92} = 0$ yields 12 real solutions for $c \in (-1, 1)$:

$$c = -0.8417985399\cdots,\quad 0.2413684054\cdots,\quad 0.4438551874\cdots,\quad 0.5610726865\cdots,$$
$$0.7086336549\cdots,\quad 0.7250049804\cdots,\quad 0.7311867037\cdots,\quad 0.7415580800\cdots,$$
$$0.7758984507\cdots,\quad 0.7979656089\cdots,\quad 0.9443072033\cdots,\quad 0.9788036986\cdots.$$

For example, taking $c = -0.8417985399\cdots$ and letting $b_{53j} = b_{44j} = b_{35j} = b_{26j} = b_{17j} = 0$, $j = 1, 2$, $b_{082} = 0$ and $b_{081} = 0.00001$, we obtain

$$a_{801} = -0.00000879\cdots,\quad a_{711} = -0.00001434\cdots,\quad a_{621} = -0.00000347\cdots,$$
$$a_{531} = -0.00001627\cdots,\quad a_{441} = 0.00002299\cdots,\quad a_{351} = 0.00012434\cdots,$$
$$a_{261} = 0.00004539\cdots,\quad a_{171} = 0.00006857\cdots,\quad a_{081} = 0.00001161\cdots,$$
$$b_{801} = -0.00005500\cdots,\quad b_{711} = -0.00001611\cdots,\quad b_{621} = 0.00001268\cdots,$$
$$a_{802} = 1248715.66\cdots,\quad a_{712} = 4998544.13\cdots,\quad a_{622} = -7548790.03\cdots,$$
$$a_{532} = -13180993.2\cdots,\quad a_{442} = -11839536.6\cdots,\quad a_{352} = -4325747.70\cdots,$$
$$a_{262} = 4681194.52\cdots,\quad a_{172} = 3868980.88\cdots,\quad a_{082} = 993219.036\cdots,$$
$$b_{802} = 1080390.20\cdots,\quad b_{712} = -3989523.50\cdots,\quad b_{622} = -7365171.82\cdots,$$

under which

$$V_{7i1} = 0,\quad i = 1, 2, \ldots,\quad V_{7i2} = 0,\quad i = 1, 2, \ldots, 13,\quad V_{982} = -0.0094576211\cdots \neq 0$$

and

$$\mathrm{Det}_{82} = \frac{\partial(V_{72}, V_{142}, V_{212}, V_{282}, V_{352}, V_{422}, V_{492}, V_{562}, V_{632}, V_{702}, V_{772}, V_{842}, V_{912})}{\partial(a_{802}, a_{712}, a_{622}, a_{532}, a_{442}, a_{352}, a_{262}, a_{172}, a_{082}, b_{802}, b_{712}, b_{622}, c)}$$
$$= -0.8048954611 \times 10^{74} \neq 0,$$

which, by Lemma 4 and plus a linear perturbation, implies that system (54) indeed can have 14 small-amplitude limit cycles bifurcating from the origin, i.e. $M_h(8) \geq 14$.

4.2. $n = 9$

In [Giné, 2012b], based on system (48) with $k_1 = \cos\phi = \frac{1}{3}$ and $k_2 = \sin\phi = \frac{2\sqrt{2}}{3}$, Giné used the similar procedure described in the precious section to derive the following ninth-degree homogeneous polynomial system (with $\varepsilon = 0$):

$$\dot{x} = -y + \frac{1 + 2\sqrt{2}}{9}x^9 + \frac{2(15 - 4\sqrt{2})}{9}x^8 y - \frac{2(1 + 12\sqrt{2})}{9}x^7 y^2 + \frac{2(11 - 10\sqrt{2})}{9}x^6 y^3$$

$$
\begin{aligned}
&- \frac{4(7+3\sqrt{2})}{3}x^5y^4 - \frac{2(5+2\sqrt{2})}{3}x^4y^5 - \frac{2(7-4\sqrt{2})}{9}x^3y^6 - \frac{2(3-2\sqrt{2})}{9}x^2y^7 \\
&+ \frac{67+18\sqrt{2}}{3}xy^8 + \frac{4(4+\sqrt{2})}{3}y^9 + \varepsilon\sum_{i+j=9}(a_{ij1}x^iy^j + \varepsilon a_{ij2}x^iy^j + \varepsilon^2 a_{ij3}x^iy^j), \\
\dot{y} = x &- \frac{4(4-\sqrt{2})}{9}x^9 - \frac{67-18\sqrt{2}}{9}x^8y + \frac{2(3+2\sqrt{2})}{9}x^7y^2 + \frac{2(7+4\sqrt{2})}{9}x^6y^3 \\
&+ \frac{2(5-2\sqrt{2})}{3}x^5y^4 + \frac{4(7-3\sqrt{2})}{3}x^4y^5 - \frac{2(11+10\sqrt{2})}{9}x^3y^6 + \frac{2(1-12\sqrt{2})}{9}x^2y^7 \\
&- \frac{2(15+4\sqrt{2})}{3}xy^8 - \frac{1-2\sqrt{2}}{3}y^9 + \varepsilon\sum_{i+j=9}(b_{ij1}x^iy^j + \varepsilon b_{ij2}x^iy^j + \varepsilon^2 b_{ij3}x^iy^j)
\end{aligned} \tag{65}
$$

and applied the independent linear and quadratic parts in Poincaré–Lyapunov constants to prove the existence of 16 limit cycles around the origin. However, using our approach we can show that when k_1 and k_2 are fixed, system (65) can only yield 15 limit cycles around the origin.

In fact, we have applied perturbations up to ε^3-order, as shown in system (65) to prove that for each order perturbations, only 15 limit cycles can be obtained. That is, using higher-order perturbations does not increase the number of limit cycles, which agrees with that observed in cases $n = 5$ and $n = 7$, where the independent linear parts in Poincaré–Lyapunov constants (equivalently only ε-order focus values need to be considered) are enough to prove the existence of limit cycles. For system (65), the nonzero focus values are V_{4i}, $i = 1, 2, \ldots$. For the ε-order analysis, we use the 14 parameters: a_{ij1} $(i+j=9)$, b_{901}, b_{811}, b_{721} and b_{631} to linearly solve the first 14 focus value equations: $V_{4i1} = 0$, $i = 1, 2, \ldots, 14$, and then obtain

$$V_{601} = C_{601}F_{91}, \quad V_{641} = C_{641}F_{91},$$
$$V_{681} = C_{681}F_{91},$$

where C_{4i1}, $i = 15, 16, 17$, are constants, and

$$
\begin{aligned}
F_{91} = {} & 553643615793b_{541} + 5625067284633b_{451} \\
& - 4809393630279b_{361} + 5224428329831b_{271} \\
& + 117207537361114b_{181} + 76817108218558b_{091} \\
& - 2\sqrt{2}(396008410537b_{541} \\
& - 1240822332696b_{451} + 1398861695777b_{361} \\
& - 947945405697b_{271} - 1192716284987b_{181}).
\end{aligned}
$$

This clearly shows that the best result we can obtain is the solution such that $V_{4i1} = 0$, $i = 1, 2, \ldots, 14$, but $V_{601} \neq 0$, implying that system (65) can have only 15 small limit cycles around the origin.

Next, solving $F_{91} = 0$ for b_{541}, with the above obtained solutions, yields the critical condition:

$$\mathrm{C}_{91}: \ (a_{ij1}(i+j=9), b_{901}, b_{811}, b_{721}, b_{631}, b_{541}),$$

in terms of $b_{451}, b_{361}, b_{271}, b_{181}$ and b_{091}. Under the critical condition C_{92}, we can show that all the ε-order focus values vanish. Now, with the condition C_{91}, we use the 14 parameters: a_{ij2}, $i+j=9$, and b_{902}, b_{812}, b_{722}, b_{632}, to linearly solve the first 14 focus value equations: $V_{4i2} = 0$, $i = 1, 2, \ldots, 14$, and then obtain

$$V_{602} = C_{602}F_{92}, \quad V_{642} = C_{642}F_{92},$$
$$V_{682} = C_{682}F_{92},$$

where C_{4i2}, $i = 15, 16, 17$, are constants, and F_{92} is a polynomial, linearly in $b_{542}, b_{452}, b_{362}, b_{272}, b_{182}, b_{092}$, and quadratically in $b_{451}, b_{361}, b_{271}, b_{181}, b_{091}$.

Similarly, we may solve $F_{92} = 0$ for b_{542} to define the critical condition:

$$\mathrm{C}_{92}: \ (a_{ij2}, i+j=9, b_{902}, b_{812}, b_{722}, b_{632}, b_{542}),$$

in terms of $b_{45j}, b_{36j}, b_{27j}, b_{18j}, b_{09j}$, $j = 1, 2$. We can also show that under the critical conditions C_{91} and C_{92}, all the ε- and ε^2-order focus values vanish. This indicates that in using ε^2-order focus values, no quadratic terms can be used to get more limit cycles, and so only 15 limit cycles can be obtained from the ε^2-order analysis.

Finally, we proceed to ε^3-order focus values, and use the 14 parameters: a_{ij3} $(i+j=9)$, b_{903}, b_{813}, b_{723} and b_{633} to linearly solve the first 14 focus value equations: $V_{4i3}=0$, $i=1,2,\ldots,14$, and then obtain

$$V_{603}=C_{603}F_{93}, \quad V_{643}=C_{643}F_{93},$$
$$V_{683}=C_{683}F_{93},$$

where C_{4i3}, $i=15,16,17$, are constants, and F_{93} is a polynomial, linearly in $b_{543}, b_{453}, b_{363}, b_{273}, b_{183}, b_{093}$, and cubically in $b_{45j}, b_{36j}, b_{27j}, b_{18j}, b_{09j}$, $j=1,2$. Similarly, we can show that even by using ε^3-order focus values, no quadratic or cubic terms can be used to get more limit cycles, and so only 15 limit cycles can be obtained from the ε^3-order analysis.

The above analysis has shown that for $n=9$, using independent linear parts in Poincaré–Lyapunov constants is enough to prove the existence of limit cycles around the origin, that is, equivalently using the ε-order focus values is enough. In order to obtain 16 limit cycles for this case, similarly we let k_1 and k_2 be free and apply the similar procedure used in the fifth-degree system (54) to obtain the following ninth-degree homogeneous polynomial system with ε-order perturbation:

$$\begin{aligned}
\dot{x} &= -y + k_1(k_1+k_2)x^9 + 2(2-3k_1^2-2k_1k_2)x^8y - 2k_1(k_1+6k_2)x^7y^2 + 2(1+2k_1^2-5k_1k_2)x^6y^3 \\
&\quad - 2(4+6k_1^2+9k_1k_2)x^5y^4 - 6(1-4k_1^2+k_1k_2)x^4y^5 - 2k_1(7k_1-2k_2)x^3y^6 \\
&\quad - 2(1-6k_1^2-k_1k_2)x^2y^7 + (8-5k_1^2+9k_1k_2)xy^8 + 2(1-k_1^2+k_1k_2)y^9 + \varepsilon\sum_{i+j=9} a_{ij1}x^iy^j, \\
\dot{y} &= x - 2(1-k_1^2-k_1k_2)x^9 - (8-5k_1^2-9k_1k_2)x^8y + 2(1-6k_1^2+k_1k_2)x^7y^2 + 2k_1(7k_1+2k_2)x^6y^3 \\
&\quad + 6(1-4k_1^2-k_1k_2)x^5y^4 + 2(4+6k_1^2-9k_1k_2)x^4y^5 - 2(1+5k_1k_2+2k_1^2)x^3y^6 \\
&\quad + 2k_1(k_1-6k_2)x^2y^7 - 2(2-3k_1^2+2k_1k_2)xy^8 - k_1(k_1-k_2)y^9 + \varepsilon\sum_{i+j=9} b_{ij1}x^iy^j,
\end{aligned} \tag{66}$$

where $k_1=\cos\phi$ and $k_2=\sin\phi$ with arbitrary $\phi\in[0,2\pi]$.

Let $k_2=\sqrt{1-k_1^2}$ (the case $k_2=-\sqrt{1-k_1^2}$ can be similarly proved). We similarly use the 14 parameters: a_{ij1} $(i+j=9)$, b_{901}, b_{811}, b_{721} and b_{631} to linearly solve the first 14 focus value equations: $V_{4i1}=0$, $i=1,2,\ldots,14$, and then obtain

$$V_{601}=\frac{-5k_1(1-k_1^2)(3-4k_1^2)(1-4k_1^2)^3C_{91}}{411844608C_{90}}F_{91a}F_{91b},$$
$$V_{641}=\frac{-5k_1(1-k_1^2)(3-4k_1^2)(1-4k_1^2)^3C_{91}}{54363488256C_{90}}G_{91},$$

where C_{90} is a 119th-degree polynomial in k_1^2, and C_{91} is a linear function in b_{541}, b_{451}, b_{361}, b_{271}, b_{181} and b_{091} with coefficients involving k_1. F_{91a} is a function involving $\sqrt{1-k_1^2}$, while F_{91b} is a 54th-degree polynomial in k_1^2 and G_{91} is a 243rd-degree polynomial in k_1. It can be shown that there exist 24 real solutions solved from $F_{91b}=0$ for $k_1\in(-1,1)$ as

$$\begin{aligned}
k_1 = {} & \pm 0.0476828554\cdots, \quad \pm 0.0812093313\cdots, \quad \pm 0.1696914143\cdots, \quad \pm 0.2957590710\cdots, \\
& \pm 0.6794021147\cdots, \quad \pm 0.7686199440\cdots, \quad \pm 0.8225603147\cdots, \quad \pm 0.8411988944\cdots, \\
& \pm 0.8888817498\cdots, \quad \pm 0.9037696460\cdots, \quad \pm 0.9383113584\cdots, \quad \pm 0.9751611858\cdots.
\end{aligned}$$

Choosing $k_1=0.6794021147\cdots$ and taking $b_{451}=b_{361}=b_{271}=b_{181}=b_{091}=0$ and $b_{541}=1$, we obtain

$$V_{4i1}=0, \quad i=1,2,\ldots,15, \quad V_{641}=-0.0000108665\cdots\neq 0$$

and

$$\text{Det}_{91} = \frac{\partial(V_{41}, V_{81}, V_{121}, V_{161}, V_{201}, V_{241}, V_{281}, V_{321}, V_{361}, V_{401}, V_{441}, V_{481l}, V_{521}, V_{561}, V_{601})}{\partial(a_{901}, a_{811}, a_{721}, a_{631}, a_{541}, a_{451}, a_{361}, a_{271}, a_{181}, a_{091}, b_{901}, b_{811}, b_{721}, b_{631}, k_1)}$$
$$= 0.1849908882 \cdots \times 10^{-25} \neq 0,$$

which, by Lemma 4 and a linear perturbation, clearly indicates that system (66) indeed has 16 small-amplitude limit cycles bifurcating from the origin, i.e. $M_h(9) \geq 16$.

The above procedure can continue to ε^2-order focus values and it can be shown that no more limit cycles can be obtained, that is, by using even ε^2-order focus values, we can still use the parameters a_{ij2} and part of b_{ij2} to linearly solve the focus value equations to obtain 16 limit cycles around the origin.

5. Conclusion

In this paper, we have applied the method of normal forms to show that nth-degree homogeneous polynomial systems with an isolated, nondegenerate center can have small-amplitude limit cycles $M(n) \geq 2n$ for $n = 4, 5, 6, 7$ and $M(n) \geq 2(n-1)$ for $n = 8, 9$, which improve the conjecture proposed in [Giné, 2012a, 2012b]. Moreover, for such systems, the following has been observed.

(1) When n is odd, the coefficients in the unperturbed systems can be used to increase the number of limit cycles. It may need only the ε-order focus values, as shown for cases $n = 5$, $n = 7$ and $n = 9$.
(2) When n is even, maximal number of limit cycles cannot be obtained by using only ε-order focus values. ε^2-order or even ε^3-order focus values may be needed. Whether quadratic or even cubic terms, in addition to linear terms, focus values are required to get more limit cycles depending upon the system equations. Moreover, it has been observed that if the two coefficients in the unperturbed system have linear relation, it cannot be used to increase the number of limit cycles, as indicated by System A in cases $n = 4$ and $n = 6$; but can be used to increase the number of limit cycles if the relation is nonlinear, as we have seen from System B in cases $n = 4$ and $n = 6$, as well as the system given in case $n = 8$.
(3) For $n = 8, 9$, new systems need to be constructed to prove $M_h(n) \geq 2n$. The problem is far from completely solved for $n \geq 10$.

We propose a new conjecture as given below.

Conjecture 5.1. *For system* (5), *the number of small limit cycles bifurcating from a nondegenerate center* (*the origin*) *is given by* $M_h(n) \geq 2n$.

Acknowledgments

This research was partially supported by the Natural Sciences and Engineering Research Council of Canada, NSERC No. R2686A02 (P. Yu), and the National Natural Science Foundation of China, NNSF No. 11431008 (M. Han) and Nos. 11571318 and 11162020 (J. Li).

References

Arnold, V. I. [1983] *Geometric Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, NY).

Bautin, N. [1952] "On the number of limit cycles appearing from an equilibrium point of the focus or center type under varying coefficients," *Matem. Sb.* **30**, 181–196.

Blows, T. R. & Lloyd, N. G. [1984] "The number of limit cycles of certain polynomial differential equations," *Proc. Roy. Soc. Edinburgh Sect. A* **98**, 215–239.

Bondar, Yu. L. & Sadovskii, A. P. [2008] "On a Żołądek theorem," *Diff. Eqs.* **44**, 274–277.

Chavarriga, J., Giné, J. & Grau, M. [2002] "Integrable systems via polynomial inverse integrating factors," *Bull. Sci. Math.* **126**, 315–331.

Chen, L. S. & Wang, M. S. [1979] "The relative position, and the number, of limit cycles of a quadratic differential system," *Acta. Math. Sin.* **22**, 751–758.

Chen, C., Corless, R., Maza, M., Yu, P. & Zhang, Y. [2013] "A modular regular chains method and its application to dynamical systems," *Int. J. Bifurcation and Chaos* **23**, 1350154-1–21.

Christopher, C. [2006] *Estimating Limit Cycle Bifurcation from Centers*, *in Trends in Mathematics*: *Differential Equations with Symbolic Computation* (Birkhäuser), pp. 23–35.

Giné, J. [2006] "The center problem for a linear center perturbed by homogeneous polynomials," *Acta Math. Sin.* **22**, 1613–1620.

Giné, J. [2007] "On the number of algebraically independent Poincaré–Lyapunov constants," *Appl. Math. Comput.* **188**, 1870–1877.

Giné, J. [2012a] "Limit cycle bifurcations from a non-degenerate center," *Appl. Math. Comput.* **218**, 4703–4709.

Giné, J. [2012b] "Higher order limit cycle bifurcations from non-degenerate centers," *Appl. Math. Comput.* **218**, 8853–8860.

Han, M. [2006] *Bifurcation of Limit Cycles of Planar Systems*, *Handbook of Differential Equations*, *Ordinary Differential Equations*, Vol. 3, eds. Canada, A., Drabek, P. & Fonda, A. (Elsevier).

Han, M., Yang, J. M. & Yu, P. [2009] "Hopf bifurcations for near-Hamiltonian systems," *Int. J. Bifurcation and Chaos* **19**, 4117–4130.

Han, M. & Yu, P. [2012] *Normal Forms*, *Melnikov Functions*, *and Bifurcation of Limit Cycles* (Springer-Verlag, London).

Han, M., Sheng, L. & Zhang, X. [2018] "Bifurcation theory for finitely smooth planar differential system," *J. Diff. Eqs.* **264**, 3596–3618.

Hilbert, D. [1902] "Mathematical problems," (M. Newton, Transl.) *Bull. Am. Math. Soc.* **8**, 437–479.

Iliev, I. D. [2000] "On the limit cycles available from polynomial perturbations of the Bogdanov–Takens Hamiltonian," *Israel J. Math.* **115**, 269–284.

Li, J. [2003] "Hilbert's 16th problem and bifurcations of planar polynomial vector fields," *Int. J. Bifurcation and Chaos* **13**, 47–106.

Li, C., Liu, C. & Yang, J. [2009] "A cubic system with thirteenth limit cycles," *J. Diff. Eqs.* **246**, 3609–3619.

Li, J. & Liu, Y. [2010] "New results on the study of Z_q-equivariant planar polynomial vector fields," *Qual. Th. Dyn. Syst.* **9**, 167–219.

Lloyd, N. & Pearson, J. [2012] "A cubic differential system with nine limit cycles," *J. Appl. Anal. Comput.* **2**, 293–304.

Shi, S. [1979] "An example for quadratic systems (E_2) to have at least four limit cycles," *Sci. Sin.* **11**, 1051–1056 (in Chinese).

Shi, S. [1980] "A concrete example of the existence of four limit cycles for plane quadratic systems," *Sci. Sin.* **23**, 153–158.

Sibirskii, K. S. [1965] "On the number of limit cycles in the neighborhood of a singular point," *Diff. Eqs.* **1**, 36–47.

Smale, S. [1988] "Mathematical problems for the next century," *Math. Intell.* **20**, 7–15.

Tian, Y. & Yu, P. [2013] "An explicit recursive formula for computing the normal form and center manifold of n-dimensional differential systems associated with Hopf bifurcation," *Int. J. Bifurcation and Chaos* **23**, 1350104-1–18.

Tian, Y. & Yu, P. [2014] "An explicit recursive formula for computing the normal forms associated with semisimple cases," *Commun. Nonlin. Sci. Numer. Simulat.* **19**, 2294–2308.

Tian, Y. & Yu, P. [2016] "Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with cubic polynomials," *J. Diff. Eqs.* **260**, 971–990.

Tian, Y. & Yu, P. [2018] "Bifurcation of small limit cycles in cubic integrable systems using higher-order analysis," *J. Diff. Eqs.* **264**, 5950–5976.

Yu, P. [1998] "Computation of normal forms via a perturbation technique," *J. Sound Vib.* **211**, 19–38.

Yu, P. & Han, M. [2005] "Small limit cycles bifurcating from fine focus points in cubic-order Z_2-equivariant vector fields," *Chaos Solit. Fract.* **24**, 329–348.

Yu, P. & Corless, R. [2009] "Symbolic computation of limit cycles associated with Hilbert's 16th problem," *Commun. Nonlin. Sci. Numer. Simulat.* **14**, 4041–4056.

Yu, P. & Han, M. [2011] "A study on Żołądek's example," *J. Appl. Anal. Comput.* **1**, 143–153.

Yu, P. & Tian, Y. [2014] "Twelve limit cycles around a singular point in a planar cubic-degree polynomial system," *Commun. Nonlin. Sci. Numer. Simulat.* **19**, 2690–2705.

Żołądek, H. [1994] "On certain generalization of Bautins theorem," *Nonlinearity* **7**, 273–279.

Żołądek, H. [1995] "Eleven small limit cycles in a cubic vector field," *Nonlinearity* **8**, 843–860.