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Qualitative Theory of Dynamical Systems

Bifurcation of Limit Cycles in Cubic Integrable Z_2-Equivariant Planar Vector Fields

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Abstract In this paper, we study bifurcation of limit cycles in cubic planar integrable, non-Hamiltonian systems. The systems are assumed to be Z_2-equivariant with two symmetric centers. Particular attention is given to bifurcation of limit cycles in the neighborhood of the two centers under cubic perturbations. Such integrable systems can be classified as 11 cases. It is shown that different cases have different number of limit cycles and the maximal number is 10. The method used in the paper relies on focus value computation.

Keywords Hilbert's 16th problem · Limit cycle · Near-Hamiltonian system · Center bifurcation · Focus value

Mathematics Subject Classification (2000) Primary 34C07 · Secondary 34C23

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1 Introduction

The main task in the study of Hilbert's 16th problem [4] is to determine the number of limit cycles that a planar system can have. In general, this is a very difficult problem. Although many researchers have been working on this problem for more then one century, it seems still far away from completely solving the problem, see a survey article [6] for more details. Consider the following planar system:

$$\frac{dx}{dt} = P_n(x, y), \quad \frac{dy}{dt} = Q_n(x, y), \tag{1.1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent nth-degree polynomials of x and y. Roughly speaking, the second part of Hilbert's 16th problem is to find the upper bound on the number of limit cycles that the system can have, called Hilbert number $H(n)$ which only depends on the degree of the polynomials, n.

If the problem is restricted to a neighborhood of isolated singular points, the problem is then reduced to studying degenerate Hopf bifurcations, which give rise to fine focus points, and many results have been obtained (e.g., see [1,5,7,11]). Alternatively, this is equivalent to computing the normal form of differential equations associated with Hopf or degenerate Hopf bifurcations. Suppose the origin of system (1.1), $(x, y) = (0, 0)$, is an element center. Without loss of generality, we may assume that the eigenvalues of the Jacobian of system (1.1) evaluated at the origin are a purely imaginary pair, $\pm i$. Then Hopf bifurcation occurs and a family of limit cycles (depending on bifurcation parameter values) bifurcates from a critical point. Further, suppose the normal form associated with this Hopf singularity is given in polar coordinates (obtained by using, say, the method given in [13]):

$$\dot{\rho} = \rho \left(v_0 + v_1 \rho^2 + v_2 \rho^4 + \cdots + v_k \rho^{2k} + \cdots \right), \tag{1.2}$$

$$\dot{\theta} = 1 + t_1 \rho^2 + t_2 \rho^4 + \cdots + t_k \rho^{2k} + \cdots , \tag{1.3}$$

where ρ and θ represent, respectively, the amplitude and phase of the limit cycles, and $v_i, \ i = 0, 1, 2, \ldots,$ denote the focus values. The dynamical behavior of system (1.1) in the vicinity of the origin, in particular the Hopf bifurcation solutions and their stability, are determined by Eq. (1.2).

The basic idea of finding k small limit cycles around the origin is as follows: First, find the conditions such that $v_0 = v_1 = v_2 = \cdots = v_{k-1} = 0$, but $v_k \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. In 1952, Bautin [1] proved that a quadratic planar polynomial vector field can have a maximum of 3 small limit cycles. This result was believed to be the maximal number of limit cycles in quadratic systems for more than 20 years thereafter, until concrete examples were constructed by the end of the 1970s, showing the existence of 4 limit cycles in some quadratic systems [2,12]. Thus, $H(2) \geq 4$, which, after another 30 years, was still the best result for quadratic systems. Whether or not this is the upper bound for quadratic systems remains an open question today.

In the past few years, great progress has been achieved in obtaining better estimations of the lower bounds of $H(n)$ for $n \geq 3$. For cubic systems, it has been shown that $H(3) \geq 12$ [14,15]. Such a cubic system is assumed to be Z_2-invariant with two symmetric fine focus points, and the 12 limit cycles are distributed in the neighborhood of the two points. Many more results for systems with $n \geq 4$ have also been obtained (e.g., see the book [10] and references therein). Recently, it has been proved that such Z_2-equivariant system can have a 13th limit cycle at infinity [9]. Also, another cubic system with different structure has been reported to exhibit 13 limit cycles [8]. In [10], 11 cases have been specified to be integrable systems, among which one is a Hamiltonian system, and all other ten cases are integrable, non-Hamiltonian systems.

In this paper, we are interested in finding that for the remaining 10 cases (integrable, non-Hamiltonian systems) how many limit cycles can bifurcate from the two symmetric centers under cubic perturbations. The focus value computation is employed to show that these 10 cases can have 4, 8 or 10 limit cycles. We leave the bifurcation of large limit cycles near homoclinic loops in future study.

The rest of the paper is organized as follows. In the next section, we briefly describe the method and present general mathematical formulas for computing the focus values near a center. In Sect. 3, we present three typical systems which show 4, 8 and 10 limit cycles, respectively. Conclusion is drawn in Sect. 4.

2 Z_2-Equivariant Cubic Planar Systems

For convenience, we summarize some results taken from [10] about Z_2-equivariant systems. A cubic system with Z_2 symmetry, having two finite elementary focuses at $(\pm 1, 0)$, can be written in the following standard form:

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\delta}{2}x - (a_1 + 1)y + \frac{\delta}{2}x^3 + a_1 x^2 y + a_2 x y^2 + a_3 y^3, \\ \frac{dy}{dt} &= -\frac{1}{2}x + (\delta - a_4)y + \frac{1}{2}x^3 + a_4 x^2 y + a_5 x y^2 + a_6 y^3, \end{aligned} \tag{2.1}$$

where δ and a_i's are parameters. Letting

$$x = u + 1, \quad y = v, \tag{2.2}$$

system (2.1) becomes

$$\begin{aligned} \frac{du}{dt} &= \delta u - v + \frac{3\delta}{2}u^2 + 2a_1 uv + a_2 v^2 + \frac{\delta}{2}u^3 + a_1 u^2 v + a_2 u v^2 + a_3 v^3, \\ \frac{dv}{dt} &= u + \delta v + \frac{3}{2}u^2 + 2a_4 uv + a_5 v^2 + \frac{1}{2}u^3 + a_4 u^2 v + a_5 u v^2 + a_6 v^3. \end{aligned} \tag{2.3}$$

When $\delta = 0$, systems (2.2) and (2.3) are reduced to

$$\begin{aligned} \frac{dx}{dt} &= -(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3, \\ \frac{dy}{dt} &= -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \frac{du}{dt} &= -v + 2a_1uv + a_2v^2 + a_1u^2v + a_2uv^2 + a_3v^3, \\ \frac{dv}{dt} &= u + \frac{3}{2}u^2 + 2a_4uv + a_5v^2 + \frac{1}{2}u^3 + a_4u^2v + a_5uv^2 + a_6v^3, \end{aligned} \tag{2.5}$$

respectively. It is obvious that the first focus value of system (2.4) at the two symmetric singular points $(\pm 1, 0)$ is $v_0 = \delta = 0$. The next six focus values at the two singular points are all given in terms of parameters a_i's. It has been shown [10] that the six focus values v_i, $i = 1, 2, \ldots, 6$ at the singular points $(\pm 1, 0)$ are all zero, if and only if one of the following conditions is satisfied:

$$\begin{aligned}
&(C_1): \ a_4 = 0, \ \ a_1 = -a_5, \ \ a_6 = -\tfrac{1}{3}a_2; \\
&(C_2): \ a_4 = 0, \ \ a_1 + a_5 \neq 0, \ \ a_2 = a_6 = 0; \\
&(C_3): \ a_1 + a_5 \neq 0, \ \ a_6 = -\tfrac{1}{3}\,[a_2(1+2a_1+2a_5) - 2a_4(1+a_5)], \\
&\qquad 2\,(1+a_1)\,(a_1+a_5)^2 - a_4^2(1+2a_1+2a_5) = 0, \\
&\qquad 3\,(a_1+a_5)\,[2(1+a_1)\,(1+a_5) - a_3] \\
&\qquad\quad -2a_4\,[2(1+a_5)a_4 + (2+a_1+a_5)\,a_2] = 0; \\
&(C_4): \ a_3 = 2(1+a_1)(1+a_5), \ \ a_6 = -\tfrac{1}{3}\,[a_2(1+2a_1+2a_5) - 2a_4(1+a_5)], \\
&\qquad 2(1+a_5)a_4 + (2+a_1+a_5)\,a_2 = 0; \\
&(C_5): \ a_4 \neq 0, \ \ a_1 = -\tfrac{1}{2}(2-3a_4^2), \ \ a_2 = a_4, \ \ a_3 = a_4^2(1-a_4^2+a_5), \\
&\qquad a_6 = a_4(1-a_4^2); \\
&(C_6): \ a_4 \neq 0, \ \ a_1 = -\tfrac{1}{8}\Big(8-5a_4^2\Big), \ \ a_2 = \tfrac{1}{2}a_4, \ \ a_5 = -\tfrac{1}{8}\Big(8+a_4^2\Big), \\
&\qquad a_3 = -\tfrac{5}{32}a_4^4, \ \ a_6 = \tfrac{1}{4}\,a_4(2-a_4^2); \\
&(C_7): \ a_4 \neq 0, \ \ a_1 = -\tfrac{1}{32}\Big(32+15a_4^2\Big), \ \ a_2 = \tfrac{1}{4}\,a_4, \ \ a_3 = \tfrac{1}{512}a_4^2(64+15a_4^2), \\
&\qquad a_5 = -\tfrac{1}{32}\Big(96+17a_4^2\Big), \ \ a_6 = -\tfrac{3}{16}a_4\Big(4+a_4^2\Big); \\
&(C_8): \ a_4 \neq 0, \ \ a_1 = -\tfrac{1}{50}\Big(50+21a_4^2\Big), \ \ a_2 = \tfrac{1}{5}a_4, \ \ a_3 = \tfrac{1}{1250}a_4^2\Big(250+63a_4^2\Big), \\
&\qquad a_5 = -\tfrac{1}{50}\Big(200+39a_4^2\Big), \ \ a_6 = -\tfrac{1}{25}a_4\Big(35+9a_4^2\Big); \\
&(C_9): \ a_4 \neq 0, \ \ a_1 = -\tfrac{1}{9}\Big(9+4a_4^2\Big), \ \ a_2 = a_3 = 0, \ \ a_6 = \tfrac{2}{3}a_4(1+a_5); \\
&(C_{10}): a_4 \neq 0, \ \ a_1 = -\tfrac{1}{8}\Big(8+3a_4^2\Big), \ \ a_2 = -\tfrac{1}{2}a_4, \ \ a_3 = \tfrac{3}{16}a_4^2(4+a_4^2+4a_5), \\
&\qquad a_6 = \tfrac{1}{8}a_4(4-a_4^2+8a_5); \\
&(C_{11}): a_4 \neq 0, \ \ a_1 = -\tfrac{1}{32}\Big(32+15a_4^2\Big), \ \ a_2 = -\tfrac{1}{4}a_4, \ \ a_3 = \tfrac{1}{512}a_4^2\Big(832+495a_4^2\Big), \\
&\qquad a_5 = 0\tfrac{1}{32}(160+111a_4^2), \ \ a_6 = \tfrac{1}{16}a_4\Big(76+45a_4^2\Big).
\end{aligned}$$

It is further shown in [10] by using integrating factors and other approaches that all the 11 cases are integrable systems. In particular, case (C_1) is a Hamiltonian system. Therefore, for all the 11 cases, the two symmetric singular points become centers. Since case (C_1) has been studied in a separate paper, we shall only consider cases (C_2)–(C_{11}) in this paper.

In order to consider the limit cycles bifurcating from the two symmetric centers $(\pm 1, 0)$ under cubic perturbations, we add cubic perturbations to system (2.4) to obtain the following perturbed system:

$$\begin{aligned}\frac{dx}{dt} &= -(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3 + \varepsilon p(x, y, a_{ij}),\\ \frac{dy}{dt} &= -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3 + \varepsilon q(x, y, b_{ij}),\end{aligned} \tag{2.6}$$

where

$$p(x, y, a_{ij}) = \sum_{i+j=1,\,3} a_{ij}\, x_i\, y_j, \quad q(x, y, b_{ij}) = \sum_{i+j=1,\,3} b_{ij}\, x_i\, y_j, \tag{2.7}$$

and a_{ij} and b_{ij} are parameters. Unlike Hamiltonian systems, here it requires to set $p(\pm 1, 0, a_{ij}) = q(\pm 1, 0, b_{ij}) = 0$, yielding

$$a_{30} = -a_{10}, \quad b_{30} = -b_{10}. \tag{2.8}$$

Note that such constraints do not affect the number of limit cycles bifurcating from the two centers.

The corresponding system to (2.5) becomes

$$\begin{aligned}\frac{du}{dt} &= -\,v + 2\,a_1\,u\,v + a_2\,v^2 + a_1\,u^2\,v + a_2\,u\,v^2 + a_3\,v^3\\ &\quad + \varepsilon\,\big[-2a_{10}\,u + (a_{01}+a_{21})v - 3a_{10}\,u^2 + 2a_{21}u\,v\\ &\qquad -a_{10}u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3\big],\\ \frac{dv}{dt} &= u + \frac{3}{2}\,u^2 + 2\,a_4\,u\,v + a_5\,v^2 + \frac{1}{2}\,u^3 + a_4\,u^2\,v + a_5\,u\,v^2 + a_6\,v^3\\ &\quad + \varepsilon\,\big[-2b_{10}\,u + (b_{01}+b_{21})v - 3b_{10}\,u^2 + 2b_{21}u\,v\\ &\quad -b_{10}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3\big].\end{aligned} \tag{2.9}$$

The focus values of system (2.9) at the origin correspond to the focus values of system (2.6) at the centers $(\pm 1, 0)$. It should be noted that since setting $\varepsilon = 0$ in (2.6) gives the integrable system (2.4), the focus values are given in the form of

$$v_i = \varepsilon\tilde{v}_i + O(\varepsilon^2), \quad i = 0, 1, 2, \ldots \tag{2.10}$$

Table 1 The number of limit cycles around $(\pm 1, 0)$

Case	Number of limit cycles	Type of system
(C_1)	$5 \times 2 = 10$	Hamiltonian
(C_2)	$5 \times 2 = 10$	Integrable
(C_3)	$5 \times 2 = 10$	Integrable
(C_4)	$5 \times 2 = 10$	Integrable
(C_5)	$4 \times 2 = 8$	Integrable
(C_6)	$2 \times 2 = 4$	Integrable
(C_7)	$4 \times 2 = 8$	Integrable
(C_8)	$5 \times 2 = 10$	Integrable
(C_9)	$4 \times 2 = 8$	Integrable
(C_{10})	$5 \times 2 = 10$	Integrable
(C_{11})	$5 \times 2 = 10$	Integrable

Thus, for sufficiently small ε, we can use $\tilde{v}_i$ to determine the bifurcation of small limit cycles.

Remark 2.1 The Melnikov function method (e.g., see [3]) can also be used to determine the limit cycles bifurcating from centers. For non-Hamiltonian, integrable systems, the advantage of using focus value computation is able to simplifying the computation. It does not need to transfer the integrable system (2.4) to a Hamiltonian system (i.e., no integrating factor is needed). In general, finding an integrating factor is not an easy job, and yet the Taylor expansion of the transferred Hamiltonian system around a center is an infinite series and thus increases the computation complexity substantially. The focus value computation, on the other hand, can be directly performed using system (2.9).

Remark 2.2 It should be noted that all the results presented in this paper are based on the first-order terms in the focus values, i.e., $\tilde{v}_i$. When all the first-order terms $\tilde{v}_i$ equal zero, one may further consider second-order (equivalently, second-order Melnikov function), third-order, etc. focus values. High order focus values are not considered in this paper.

3 Bifurcation of Limit Cycles in the Integrable System (2.6)

To study bifurcation of limit cycles for the perturbed system (2.6) around the two centers $(\pm 1, 0)$, we equivalently consider the bifurcation of limit cycles of system (2.9) around the origin. We first summarize the results for the 11 cases and then choose 3 typical cases from (C_2)–(C_{11}) to prove the existence of limit cycles (Table 1).

3.1 Case (C_6): 4 Limit Cycles

First, we consider the simplest case (C_6) which can have only 4 limit cycles. That is, in the neighborhood of the origin of system (2.9), there can exist 2 limit cycles.

For this case, the focus value v_1 is given by

$$\tilde{v}_1 = \frac{1}{32}\Big[2a_4^3 a_{01} + a_4^2(67 - 11a_4^2)a_{10} + 2a_4(2 + a_4^2)a_{21} \\ -4(3 - a_4^2)a_{12} - a_4^2(1 - 2a_4)b_{10} - 4a_4 b_{12} + 12b_{03}\Big]. \tag{3.1}$$

Setting

$$a_{01} = \frac{1}{2a_4^2}\Big[a_4^2(67 - 11a_4^2)a_{10} + 2a_4(2 + a_4^2)a_{21} - 4(3 - a_4^2)a_{12} \\ -a_4^2(1 - 2a_4)b_{10} - 4a_4 b_{12} + 12b_{03}\Big] \tag{3.2}$$

yields $\tilde{v}_1 = 0$ and then

$$\begin{aligned}\tilde{v}_2 = {} & \frac{1}{96}(40 - 13a_4^2)\, a_{03} + \frac{1}{768}\, a_4^2(4288 - 2048\, a_4^2 + 213\, a_4^4)\, a_{10} \\ & + \frac{1}{384}\, a_4(128 + 24\, a_4^2 - 13\, a_4^4)\, a_{21} - \frac{1}{192}(192 - 144\, a_4^2 + 23\, a_4^4)\, a_{12} \\ & + \frac{1}{1536}\, a_4^3(256 - 56\, a_4^2 - 5\, a_4^4)\, b_{10} - \frac{1}{768}\, a_4^2(8 - 3\, a_4^2)(8 - 5a_4^2)\, b_{01} \\ & - \frac{1}{384}\, a_4(128 - 40\, a_4^2 - 5\, a_4^4)\, b_{12} + \frac{1}{192}(192 - 80\, a_4^2 + 15\, a_4^4)\, b_{03}.\end{aligned} \tag{3.3}$$

Further, setting $\tilde{v}_2 = 0$ results in $\tilde{v}_3 = \cdots = \tilde{v}_6 = \cdots = 0$. Thus, for this case, we can choose a_{01} such that $\tilde{v}_1 = 0$, but $\tilde{v}_2 \neq 0$. Therefore, there exists at most 2 limit cycles in the vicinity of the origin of system (2.9). Moreover, by properly perturbing δ and a_{01}, we can obtain 2 limit cycles. This implies that the original perturbed system (2.6) can have 4 limit cycles in the vicinity of the two centers $(\pm 1, 0)$.

An example for this case when $a_4 = 1$ is shown in Fig. 1.

3.2 Case (C_9): 8 Limit Cycles

For this case, we have

$$\begin{aligned}\tilde{v}_1 = {} & \frac{1}{4}\, a_{01} - \frac{1}{324}(81 - 81a_5 - 108a_4^2 - 72a_5 a_4^2 - 162a_5^2 + 64a_4^4)\, a_{10} \\ & + \frac{1}{4} a_4 a_{21} - \frac{1}{72}(9 - 18a_5 + 8a_4^2)a_{12} + \frac{1}{2}(1 - a_5)b_{10} + \frac{1}{4}(1 + a_5)b_{01} \\ & - \frac{1}{4}\, a_4 b_{12} + \frac{3}{8}\, b_{03}.\end{aligned} \tag{3.4}$$

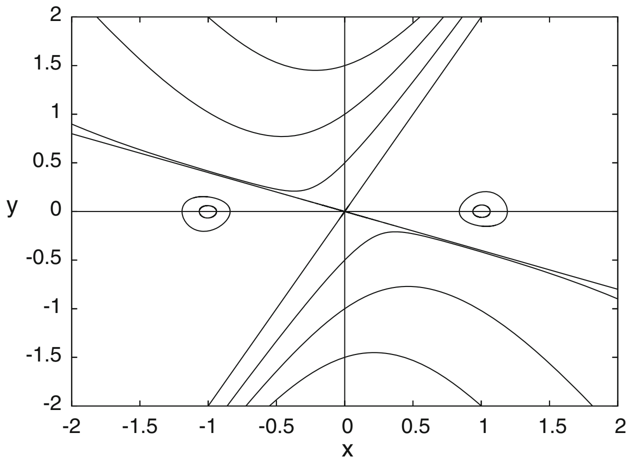

Fig. 1 A phase portrait of system (2.6) for case (C_6) when $a_4 = 1$

Letting $\tilde{v}_1 = 0$ results in

$$a_{01} = \frac{1}{81}(81 - 81a_5 - 108a_4^2 - 72a_5a_4^2 - 162a_5^2 + 64a_4^4)\,a_{10} - a_4a_{21} + \frac{1}{9}(9 - 18a_5 + 8a_4^2)a_{12} - 2(1 - a_5)b_{10} - (1 + a_5)b_{01} + a_4b_{12} - \frac{3}{2}b_{03} \tag{3.5}$$

and then further setting $\tilde{v}_2 = 0$ yields

$$\begin{aligned} a_{21} = {} & \frac{1}{39366a_4(1 + a_5)} \\ & \times \Big\{\Big[2(196830 - 17496a_4^2 - 5184a_4^4 + 12672a_4^6 + 2048a_4^8) \\ & - 18a_5(6561a_5 + 4374a_5^2 - 3888a_4^2 - 7776a_5a_4^2 - 2736a_4^4 \\ & - 1440a_5a_4^4 + 512a_4^6 - 1944a_5^2a_4^2)\Big]\,a_{10} \\ & + 9(2187 - 2187a_5 + 3888a_4^2 - 4374a_5^2 + 5832a_5a_4^2 - 1008a_4^4 \\ & + 1944a_5^2a_4^2 - 256a_4^6)a_{12} \\ & - 729a_4(9 - 36a_5 + 16a_4^2)a_{03} + 8748a_4(1 + a_5)(9 + 9a_5 - 4a_4^2)\,b_{10} \\ & - 4374\,(1 + a_5)(9 + 9a_5 - 8a_4^2)\,b_{01} + 39366\,a_4(1 + a_5)\,b_{12} \\ & - 59049(1 + a_5)b_{03}\Big\}. \end{aligned} \tag{3.6}$$

Next, letting $\tilde{v}_3 = 0$ gives

$$a_{03} = -\frac{2}{729\,(135+90a_5+16a_4^2)} \times \Big\{ 2(63+18a_5+16a_4^2)(2025+3240a_5+648a_4^2 + 1215a_5^2+576a_5a_4^2+64a_4^4)\,a_{10} + 9(5265+7695a_5+1512a_4^2+2430a_5^2+1152a_5a_4^2+128a_4^4)\,a_{12} \Big\}. \tag{3.7}$$

Having determined a_{01}, a_{21} and a_{03}, we obtain

$$\begin{aligned}
\tilde{v}_4 &= \frac{14\,a_4^6}{177147\,(135+90a_5+16a_4^2)}\,(1+a_5)\,(9+18a_5-8a_4^2)\,(63+18a_5+16a_4^2) \\
&\quad \times \Big[2\,(18+9a_5+8a_4^2)\,a_{10}+9a_{12}\Big], \\
\tilde{v}_5 &= \frac{7\,a_4^6}{258280326\,(135+90a_5+16a_4^2)}\,(1+a_5)\,(9+18a_5-8a_4^2)\,(63+18a_5+16a_4^2) \\
&\quad \times \Big[2\,(18+9a_5+8a_4^2)\,a_{10}+9a_{12}\Big] \\
&\quad \times (22437+9558a_5+14436a_4^2-648a_5^2+6192a_5a_4^2+1024a_4^4), \\
\tilde{v}_6 &= \frac{7\,a_4^6}{258280326\,(135+90a_5+16a_4^2)}\,(1+a_5)\,(9+18a_5-8a_4^2)\,(63+18a_5+16a_4^2) \\
&\quad \times \Big[2\,(18+9a_5+8a_4^2)\,a_{10}+9a_{12}\Big] \\
&\quad \times (6884280153+5663704518a_5+9128132676a_4^2+683321589a_5^2 \\
&\quad -220475844a_5^3+7847880372a_4^2a_5+3651638436a_4^4+1407704832a_5^2a_4^2 \\
&\quad +15142788a_5^4+2936960208a_4^4a_5+571651344a_4^4a_5^2-143455536a_5^3a_4^2 \\
&\quad +419416704a_4^6+188810496a_4^6a_5+14697472a_4^8), \\
&\;\;\vdots
\end{aligned}$$

which indicates that

$$\tilde{v}_4 = 0 \quad \Longrightarrow \quad \tilde{v}_5 = \tilde{v}_6 = \cdots = 0.$$

Thus, for this case, we can choose parameters a_{01}, a_{21} and a_{03} such that $\tilde{v}_1 = \tilde{v}_2 = \tilde{v}_3 = 0$, but $\tilde{v}_4 \neq 0$. Therefore, there exist at most 4 limit cycles in the vicinity of the origin of system (2.9). Further, it is seen that by proper perturbations to a_{01}, a_{21} and a_{03} we can have 4 limit cycles. Therefore, for this case the perturbed system (2.6) can have 8 limit cycles in the neighborhood of the two centers $(\pm 1, 0)$. As an example, letting $a_4 = a_5 = 1$ yields the phase portrait as shown in Fig. 2.

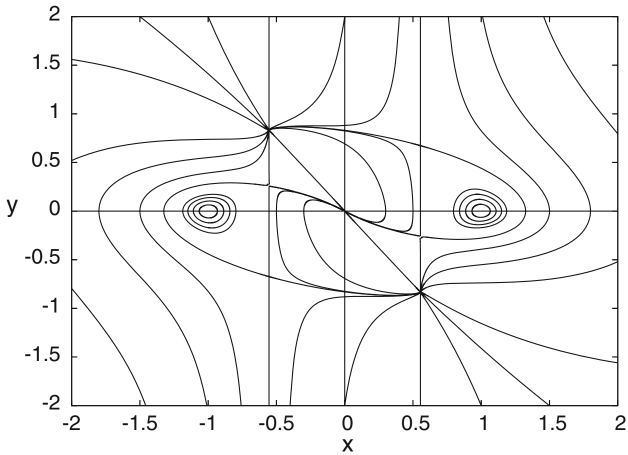

Fig. 2 A phase portrait of system (2.6) for case (C_9) when $a_4 = a_5 = 1$

3.3 Case (C_8): 10 Limit Cycles

Finally, we consider Case (C_8) which exhibits 10 limit cycles. Similarly, we obtain

$$\tilde{v}_1 = \frac{3}{1000}\,(2250 + 785a_4^2 - 12a_4^4)\,a_{10} - \frac{3}{50}\,a_4^3 a_{01} + \frac{1}{100}\,a_4(5 - 6a_4^2)\,a_{21} - \frac{3}{40}\,(15+4a_4^2)\,a_{12} + \frac{3}{200}\,(50+13a_4^2)\,(2a_4 b_{10} - b_{01}) - \frac{1}{5}\,a_4 b_{12} + \frac{3}{8} b_{03}. \tag{3.8}$$

Thus, we let

$$a_{01} = \frac{1}{60a_4}\Big[3\,(2250+785a_4^2 - 12a_4^4)\,a_{10} + 10\,a_4(5 - 6a_4^2)\,a_{21} - 75\,(15 + 4a_4^2)\,a_{12} + 15\,(50 + 13a_4^2)\,(2a_4 b_{10} - b_{01}) - 200\,a_4 b_{12} + 375\,b_{03}\Big] \tag{3.9}$$

to have $\tilde{v}_1 = 0$, and set

$$\begin{aligned} a_{03} = \frac{1}{6250a_4(11 + 3a_4^2)}\Big[&3(900000 + 688250a_4^2 + 159385a_4^4 + 10503a_4^6)\,a_{10} \\ &+ 40\,a_4(500 + 75a_4^2 - 9a_4^4)\,a_{21} - 15(18000 + 10415a_4^2 + 1527a_4^4)\,a_{12} \\ &+ 4\,a_4(25 + 9a_4^2)(6000 + 2195a_4^2 + 147a_4^4)\,b_{10} \\ &- 10\,a_4(8000 + 3825a_4^2 + 441a_4^4)\,b_{12} \\ &- (300000 + 217750a_4^2 + 43395a_4^4 + 1701a_4^6)\,b_{01} \\ &+ 25\,(6000 + 2405a_4^2 + 189a_4^4)\,b_{03}\Big] \end{aligned} \tag{3.10}$$

to yield $\tilde{v}_2 = 0$. Then

$$\tilde{v}_3 = \frac{(25+7\,a_4^2)\,(25+9\,a_4^2)}{187500000(11+3a_4^2)}\Big[225(144375+93225a_4^2+15338a_4^4)\,a_{12}$$
$$-10\,a_4(144375-118525a_4^2-35916a_4^4)a_{21}$$
$$-9(21656250+15764375a_4^2+2738775a_4^4+11346a_4^6)\,a_{10}$$
$$-6\,a_4(7218750+4613125a_4^2+570975a_4^4-42768a_4^6)\,b_{10}$$
$$+3\,(7218750+4613125a_4^2+328425a_4^4-108918a_4^6)\,b_{01}$$
$$+40\,a_4(144375+54725a_4^2+3774a_4^4)\,b_{12}$$
$$-75\,(144375+54725a_4^2+2598a_4^4)\,b_{03}\Big]. \qquad (3.11)$$

So, choosing

$$a_{12} = \frac{1}{225(144375+93225a_4^2+15338a_4^4)}$$
$$\times\Big[10\,a_4(144375-118525a_4^2-35916a_4^4)a_{21}$$
$$+9(21656250+15764375a_4^2+2738775a_4^4+11346a_4^6)\,a_{10}$$
$$+6\,a_4(7218750+4613125a_4^2+570975a_4^4-42768a_4^6)\,b_{10}$$
$$-3\,(7218750+4613125a_4^2+328425a_4^4-108918a_4^6)\,b_{01}$$
$$+40\,a_4(144375+54725a_4^2+3774a_4^4)\,b_{12}$$
$$+75\,(144375+54725a_4^2+2598a_4^4)\,b_{03}\Big] \qquad (3.12)$$

we have $\tilde{v}_3 = 0$, and then

$$\tilde{v}_4 = -\,\frac{8\,(25+7\,a_4^2)\,(25+9\,a_4^2)}{87890625(144375+93225a_4^2+15338a_4^4)}$$
$$\times\Big[5\,a_4(1075025+192473a_4^2)\,a_{21}$$
$$+9(8903125+7659650a_4^2+1325453a_4^4)\,a_{10}$$
$$+3\,a_4(11331250+8069425a_4^2+1308231a_4^4)\,b_{10}$$
$$-40\,a_4(139825+40189a_4^2)\,b_{12}$$
$$-3\,(5665625+4789925a_4^2+839556a_4^4)\,b_{01}$$
$$+1875(6475+2083a_4^2)\,b_{03}\Big]. \qquad (3.13)$$

To have $\tilde{v}_4 = 0$, we may take

$$a_{21} = -\frac{1}{5\,a_4(1075025 + 192473a_4^2)} \times \Big[9(8903125 + 7659650a_4^2 + 1325453a_4^4)\,a_{10} + 3\,a_4(11331250 + 8069425a_4^2 + 1308231a_4^4)\,b_{10} - 40\,a_4(139825 + 40189a_4^2)\,b_{12} - 3\,(5665625 + 4789925a_4^2 + 839556a_4^4)\,b_{01} + 1875(6475 + 2083a_4^2)\,b_{03}\Big]. \quad (3.14)$$

Finally, we obtain

$$\tilde{v}_5 = \frac{384384\,a_4^8\,(25 + 7a_4^2)\,(25 + 9a_4^2)}{244140625\,(1075025 + 192473a_4^2)} \times \Big[(3300 + 1213a_4^2)\,a_{10} + a_4\,(1400 + 447a_4^2)\,b_{10} - (700 + 267a_4^2)\,b_{01} - 175\,a_4 b_{12} + 500 b_{03}\Big],$$

$$\tilde{v}_6 = -\frac{18304\,a_4^8\,(25 + 7a_4^2)\,(25 + 9a_4^2)}{457763671875\,(1075025 + 192473a_4^2)} \times \Big[(3300 + 1213a_4^2)\,a_{10} + a_4\,(1400 + 447a_4^2)\,b_{10} - (700 + 267a_4^2)\,b_{01} - 175\,a_4 b_{12} + 500 b_{03}\Big] \times (558125 + 352025a_4^2 + 62334a_4^4),$$

$$\vdots$$

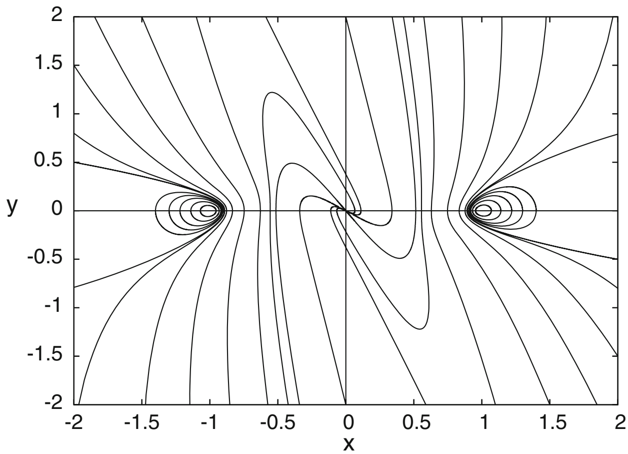

Fig. 3 A phase portrait of system (2.6) for case (C_8) when $a_4 = 1$

This implies that setting $\tilde{v}_5 = 0$ leads to $\tilde{v}_6 = \cdots = 0$. Therefore, for this case we can perturb the parameters a_{01}, a_{03} a_{12} and a_{21} to obtain 5 limit cycles around the origin of system (2.9). In other words, system (2.6) can have 10 limit cycles in the vicinity of the two centers.

To end this section, we give an example for this case by taking $a_4 = 1$. The phase portrait is depicted in Fig. 3.

4 Conclusion

In this paper, we have studied bifurcation of limit cycles in cubic planar systems with Z_2 symmetry. The attention is given to integrable, non-Hamiltonian cases. Ten such cases are investigated in this paper for limit cycles which may exist in the neighborhood of two symmetric centers under cubic perturbations. It has been shown that different cases have different number of limit cycles and the maximal number is 10. Future study is needed on the existence of limit cycles near possible homoclinic or heteroclinic loops.

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