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CRITICAL PERIODS OF THIRD-ORDER PLANAR HAMILTONIAN SYSTEMS

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This paper considers the critical periods of third-order planar Hamiltonian systems. It is assumed that the origin of the system is a center. With the aid of symbolic and numerical computations, we show the existence of seven local critical periods. This is the maximal number of local critical periods that a cubic Hamiltonian system can have.

Keywords: Critical periods; Hamiltonian system; center; normal form.

1. Introduction

Periodic motion or oscillation is a common phenomenon which exists in almost all disciplines of physical and engineering systems. Limit cycle is one of the source generating periodic motion, and its study plays an important role in the research of nonlinear dynamical systems. A related well-known problem to limit cycle is Hilbert's 16th problem [Hilbert, 1902], which has attracted many mathematicians and scientists. Though the problem is far away from being completely solved, some progress has been recently achieved (e.g. see the review articles [Han, 2002; Li, 2003; Yu, 2006]). To be more specific, consider the following differential equations:

$$\begin{cases} \dot{x} = P_n(x, y, \boldsymbol{\alpha}), \\ \dot{y} = Q_n(x, y, \boldsymbol{\alpha}), \end{cases} \tag{1}$$

where the dot denotes differentiation with respect to time, t, $P_n(x, y)$ and $Q_n(x, y)$ represent the nth-degree polynomials of x and y, and $\boldsymbol{\alpha} \in R^k$ is a k-dimensional parameter vector. One direction in this research is to study small-amplitude limit cycles bifurcating from Hopf critical point, based on the computation of the normal form of Hopf bifurcation (or focus value or Lyapunov constant). Suppose the origin of system (1) is a fixed point with Hopf singularity. Then, we wish to ask what is the maximal number of limit cycles which can bifurcate from the origin. Bautin [1954] proved that a general quadratic system can at most have three small-amplitude limit cycles bifurcating from an isolated Hopf critical point. Recently, with the aid of computer algebra system Maple, the method of normal forms was employed to obtain 12 small-amplitude limit cycles in cubic polynomial planar systems [Yu & Han, 2004, 2005a, 2005b].

Another interesting problem is the bifurcation of limit cycles from equilibria of center type, since the monotonicity of periods of closed orbit surrounding a center is a nondegeneracy condition

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of subharmonic bifurcation for periodically forced Hamiltonian systems [Chow & Hale, 1982]. Suppose the origin of system (1) is a fixed point and further it is a nondegenerate center. (If the Jacobian of the system does not have a double zero eigenvalue at the origin, then the origin is called a nondegenerate center.)

Let $T(h, \boldsymbol{\alpha})$ be the minimum period of closed orbit of system (1) surrounding the origin for $0 < h \ll 1$. Then the origin is said to be a weak center of finite order k of the system for the parameter value $\boldsymbol{\alpha} = \boldsymbol{\alpha}_c$ if

$$\frac{\partial T(0, \boldsymbol{\alpha}_c)}{\partial h} = \frac{\partial^2 T(0, \boldsymbol{\alpha}_c)}{\partial h^2} = \cdots = \frac{\partial^k T(0, \boldsymbol{\alpha}_c)}{\partial h^k} = 0, \quad \text{but } \frac{\partial^{k+1} T(0, \boldsymbol{\alpha}_c)}{\partial h^{k+1}} \neq 0. \tag{2}$$

The origin is called an isochronous center if $(\partial^k T(0, \boldsymbol{\alpha}_c))/(\partial h^k) = 0 \ \forall\, k \geq 1$, i.e. $T(h, \boldsymbol{\alpha}) =$ constant. A local critical period is defined as a period corresponding to a critical point of the period function $T(h, \boldsymbol{\alpha})$ which bifurcates from a weak center.

For the quadratic system:

$$\begin{cases} \dot{x} = -y + \displaystyle\sum_{i+j=2} a_{ij} x^i y^j, \\ \dot{y} = x + \displaystyle\sum_{i+j=2} b_{ij} x^i y^j. \end{cases} \tag{3}$$

Chicone and Jacobs [1989] discussed weak centers and critical periods which may bifurcate from weak centers. They showed that a general quadratic system can have maximal two local critical periods as well as other possibilities such as one critical period and isochronous center. In the same paper [Chicone & Jacobs, 1989], the authors also studied the following special Hamiltonian system:

$$\ddot{w} + V(w) = 0, \tag{4}$$

where V is a $2n$-degree polynomial of w. Let $w = x$ and $\dot{w} = y$. Then the Hamiltonian of system (4) is given by

$$H(x, y) = \frac{1}{2} y^2 + \int_0^x V(s) ds. \tag{5}$$

Chicone and Jacobs [1989] have shown that system (4) can have at most $n - 2$ critical periods bifurcating from the origin.

Later, Rousseau and Toni [1993] studied a special cubic system with third-degree homogeneous polynomials only, given by

$$\begin{cases} \dot{x} = -y + \displaystyle\sum_{i+j=3} a_{ij} x^i y^j, \\ \dot{y} = x + \displaystyle\sum_{i+j=3} b_{ij} x^i y^j. \end{cases} \tag{6}$$

They similarly discussed weak centers and bifurcation of critical periods from weak centers.

Recently, Zhang *et al.* [2000] obtained some results on cubic revertible polynomial systems. A system is said to be revertible if it is symmetric with respect to a line. Up to translation and rotation of coordinates, any revertible cubic differential systems can be written in the form (e.g. see [Zhang *et al.*, 2000]):

$$\begin{cases} \dot{x} = -y + a_{20} x^2 + a_{02} y^2 + a_{21} x^2 y + a_{03} y^3, \\ \dot{y} = x + b_{11} xy + b_{30} x^3 + b_{12} xy^2, \end{cases} \tag{7}$$

where a_{ij} and b_{ij} are constant parameters. Note that although system (7) has only seven parameters (coefficients), one can further reduce one parameter by a proper scaling. It was shown [Zhang *et al.*, 2000] that system (7) can have at most four local critical periods. However, for system (7) we have recently obtained six local critical periods, which is the maximal number of local critical periods that any cubic revertible system may have [Yu & Han, 2007].

Maosas and Villadelprat [2006] recently considered a Hamiltonian system with the following Hamiltonian function:

$$H(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{4} a x^4 + \frac{1}{6} b x^6, \tag{8}$$

where a and b are constants, and $b \neq 0$. It is shown [Maosas & Villadelprat, 2006] that system (8) can at most have one critical period. It should be noted that system (8) is not a special case of system (5), since the term $g(x) = \int_0^x V(s) ds$ in $H(x, y)$ of (5) is a $(2n + 1)$-degree polynomial.

In this paper, we will particularly consider the bifurcation of local critical periods from weak center of cubic-order Hamiltonian system with the following Hamiltonian function:

$$H(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{i+j=2}^{4} h_{ij} x^i y^j, \tag{9}$$

which contains nine coefficients h_{ij}. We will show that a system with Hamiltonian (9) can have maximal seven local critical periods. Also, we will give some conditions under which the Hamiltonian system has isochronous centers. The method used in this paper is based on normal form theory, with the aid of both symbolic and numerical computations. However, a complete solution for identifying all possibilities, in particular for isochronous center, is still open.

In the next section, the general formulas are presented. The main results for the local critical periods of cubic-order Hamiltonian systems are given in Sec. 3. A numerical example is presented in Sec. 4, and finally, the conclusion is drawn in Sec. 5.

2. General Formulation

The Hamiltonian system considered in this paper is described as

$$\frac{dx}{dt'} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt'} = -\frac{\partial H}{\partial x}, \tag{10}$$

where the Hamiltonian $H(x,y)$ is given in Eq. (9). It is noted that the Hamiltonian has nine coefficients. But in fact, one can reduce them to seven coefficients. To show this, we start from a general cubic system with a fixed point at the origin, which can be written as

$$\begin{cases} \dfrac{dx}{dt'} = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \qquad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dfrac{dy}{dt'} = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \\ \qquad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{cases} \tag{11}$$

where a_{ij}'s and b_{ij}'s are real constant coefficients (parameters). The system has a total of 18 parameters, however not all of them are independent. First, note that we may use a linear transformation such that system (11) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt'} = \beta x + \nu y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \qquad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dfrac{dy}{dt'} = \pm\nu x + \beta y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \\ \qquad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{cases} \tag{12}$$

which has 16 parameters, where β and ν (> 0) are used to represent the eigenvalues of the linearized system of (11). Note that the other coefficients in (12) should be different from that of system (11), but we use the same notation for convenience. Here, when the negative sign is taken, the origin is a focus point or a center (if $\beta = 0$); otherwise, it is a saddle point or node.

Suppose the origin of system (12) is a center (i.e. $\beta = 0$). Then we can apply a time scale, $t = \nu t'$, into system (12) to obtain

$$\begin{cases} \dot{x} = y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 \\ \qquad + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} = -x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 \\ \qquad + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{cases} \tag{13}$$

where again the same notations for the new parameters are used. Now, system (13) has only 14 parameters. Further, by a rotation we can further eliminate one more parameter [Bautin, 1952; Lloyd *et al.*, 1988] from system (13), which can be written in the general form:

$$\begin{cases} \dot{x} = y + ax^2 + (b + 2d)xy + cy^2 \\ \qquad + fx^3 + gx^2y + (h - 3p)xy^2 + ky^3, \\ \dot{y} = -x + dx^2 + (e - 2a)xy - dy^2 + lx^3 \\ \qquad + (m - h - 3f)x^2y + (n - g)xy^2 + py^3. \end{cases} \tag{14}$$

This form (14), with 13 parameters, is perhaps the simplest form in the literature for cubic systems having a linear center at the origin [Lloyd *et al.*, 1988].

Now, assume that system (14) is a Hamiltonian system. Then, it is easy to find that the Hamiltonian of the system is given by

$$\begin{aligned} H(x,y) = {} & \frac{1}{2}(x^2 + y^2) - \frac{1}{3}dx^3 + \left(a - \frac{1}{2}e\right)x^2y \\ & + dxy^2 + \frac{1}{3}cy^3 - \frac{1}{4}lx^4 \\ & - \frac{1}{3}(m - h - 3f)x^3y \\ & + \frac{1}{2}(g - n)x^2y^2 - pxy^3 + \frac{1}{4}ky^4. \end{aligned} \tag{15}$$

It is clear to see that the Hamiltonian function given in (15) has only eight parameters, because the coefficients of the terms x^3 and xy^2 are not independent. Comparing the Hamiltonian (15) with the original

Hamiltonian (9) shows that $h_{12} = -3h_{30}$. This one more parameter reduction is due to rotation (since no rotation is considered in the Hamiltonian given by Eq. (9)). In other words, one can directly apply a rotation to system (10) to reduce one more parameter. Thus, for convenience we still use Eq. (9) in the following analysis, but assume

$$h_{12} = -3h_{30}. \tag{16}$$

In this paper, we apply normal form theory to study local critical periods of system (10). There are many methods for computing normal forms (e.g. see [Marsden & McCracken, 1976; Guckenheimer & Homes, 1992; Ye, 1986; Nayfeh, 1993; Chow *et al.* 1994]). Here, we use a perturbation technique based on multiple time scales [Nayfeh, 1993; Yu, 1998], which has been proved to be an efficient computational method [Yu & Han, 2004, 2005a, 2005b]. We will not discuss the details of the approach here (interested readers may find more details in [Yu, 1998]).

Suppose the normal form of system (10) with the Hamiltonian given by Eq. (9) is obtained in polar coordinates as follows:

$$\dot{r} = r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k}), \tag{17}$$

$$\dot{\theta} = 1 + \dot{\phi} = 1 + b_0 + b_1 r^2 + b_2 r^4 + \cdots + b_k r^{2k}, \tag{18}$$

where v_k is usually called the kth-order focus value or Lyapunov constant; r and θ represent the amplitude and phase of motion, respectively; v_0 and b_0 correspond to the linear part of system (10) when it contains perturbation parameters. For our study in this paper, $v_0 = b_0 = 0$.

Equation (17) (or the focus values) can be used to determine the existence and number of small-amplitude limit cycles that system (9) can have, as what is employed in finding the small limit cycles of Hilbert's 16th problem (e.g. [Yu & Han, 2005b]). Equation (18), on the other hand, can be applied to find the period of the periodic solutions and to determine the critical periods of the solutions.

In the following, we describe how to use Eq. (18) to express the period of periodic motion and how to determine the local critical periods. For convenience, let

$$h = r^2 > 0 \quad \text{and} \quad p(h) = b_1 h + b_2 h^2 + \cdots + b_{k+1} h^{k+1}. \tag{19}$$

Then Eq. (18) can be written as

$$d\theta = (1 + p(h))dt \quad (b_0 = 0 \text{ for system (10)})$$

Let the period of motion be $T(h)$. Then integrating the above equation on both sides from 0 to 2π yields

$$2\pi = (1 + p(h))T(h),$$

which gives

$$T(h) = \frac{2\pi}{1 + p(h)} \quad \text{for } 0 < h \ll 1 \quad (\text{and so } 1 + p(h) \approx 1). \tag{20}$$

Now, the local critical periods are determined by $T'(h) \equiv (dT)/(dh) = 0$, or

$$T'(h) = \frac{-2\pi p'(h)}{(1 + p(h))^2} = 0. \tag{21}$$

Thus, for $0 < h \ll 1$ (meaning that we consider small-amplitude limit cycles), the local critical periods are determined by

$$p'(h) = b_1 + 2b_2 h + \cdots + k b_k h^{k-1} + (k+1) b_{k+1} h^k = 0. \tag{22}$$

Similar to the discussion in determining the number of small-amplitude limit cycles, we can find the sufficient conditions for the polynomial $p'(h)$ to have maximal number of zeros. If $b_1 = b_2 = \cdots = b_k = 0$, but $b_{k+1} \neq 0$, then $p'(h) = 0$ can have at most k real roots. Then $b_1, b_2, \ldots, b_k$, (remember that they are expressed in terms of the coefficients of the Hamiltonian (9)) can be perturbed appropriately to have k real roots. We give a theorem below without proof (see references [Yu & Han, 2004, 2005a, 2005b]). This theorem can be used to determine the maximal number of real roots of $p'(h) = 0$. Assume that b_i depends on k independent system parameters:

$$b_i = b_i(\alpha_1, \alpha_2, \ldots, \alpha_k), \quad i = 1, 2, \ldots, k, \tag{23}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ represent the parameters of the original system (10).

Theorem 1. *Suppose that*

$$b_i(\alpha_{1c}, \alpha_{2c}, \ldots, \alpha_{kc}) = 0, \quad i = 1, 2, \ldots, k,$$

$$b_{k+1}(\alpha_{1c}, \alpha_{2c}, \ldots, \alpha_{kc}) \neq 0, \quad \textit{and}$$

$$\det\left[\frac{\partial(b_1, b_2, \ldots, b_k)}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_k)}(\alpha_{1c}, \alpha_{2c}, \ldots, \alpha_{kc})\right] \neq 0, \tag{24}$$

where $\alpha_{1c}, \alpha_{2c}, \ldots, \alpha_{kc}$ represent critical values. Then small appropriate perturbations can be applied to the critical values such that the equation $p'(h) = 0$ has k real roots.

3. Critical Periods of Cubic Hamiltonian System

In this section, we consider the local critical periods of general cubic Hamiltonian system, described by system (10) with the Hamiltonian function given by Eq. (9) satisfying $h_{12} = -3h_{30}$. More specifically, we consider the following Hamiltonian system:

$$\begin{cases} \dot{x} = y + h_{21}x^2 - 6h_{30}xy + 3h_{03}y^2 + h_{31}x^3 \\ \qquad + 2h_{22}x^2y + 3h_{13}xy^2 + 4h_{04}y^3, \\ \dot{y} = -x - 3h_{30}x^2 - 2h_{21}xy + 3h_{30}y^2 \\ \qquad - 4h_{40}x^3 - 3h_{31}x^2y - 2h_{22}xy^2 - h_{13}y^3, \end{cases} \tag{25}$$

which has eight parameters (coefficients). However, in general, we can further reduce one more parameter. To achieve this, assume that $h_{21} \neq 0$ (in case $h_{21} = 0$, the system has only seven parameters), then we can use the following scaling:

$$\begin{aligned} &x \to \frac{x}{h_{21}}, \quad y \to \frac{y}{h_{21}}, \quad h_{30} \to a_1 h_{21}, \\ &h_{03} \to a_2 h_{21}, \quad h_{40} \to a_3 h_{21}^2, \quad h_{31} \to a_4 h_{21}^2, \\ &h_{22} \to a_5 h_{21}^2, \quad h_{13} \to a_6 h_{21}^2, \quad h_{04} \to a_7 h_{21}^2, \end{aligned} \tag{26}$$

to obtain the following new system (when $h_{21} \neq 0$):

$$\begin{cases} \dot{x} = y + x^2 - 6a_1xy + 3a_2y^2 + a_4x^3 \\ \qquad + 2a_5x^2y + 3a_6xy^2 + 4a_7y^3, \\ \dot{y} = -x - 3a_1x^2 - 2xy + 3a_1y^2 \\ \qquad - 4a_3x^3 - 3a_4x^2y - 2a_5xy^2 - a_6y^3. \end{cases} \tag{27}$$

System (27) has only seven independent parameters. In other words, h_{21} can be chosen arbitrarily (except $h_{21} = 0$) if we use the original Hamiltonian system (10). This implies that for the cubic Hamiltonian polynomial system (27) (or for the original Hamiltonian (10)), in general, the maximal number of local critical periods that the system can have is seven.

Note that the advantage of the above scaling reduces the number of system parameters by one, making computation simpler. However, we then need to consider one more possibility $h_{21} = 0$. When $h_{21} = 0$, there are only seven parameters. We may assume $h_{30} \neq 0$, and apply a similar scaling to obtain a system like (27) with only six independent parameters. This clearly show that such a "degenerate" system has less independent parameters and so in general has less number of local critical periods. By doing this, we may have four different cases:

Case (i). $h_{21} = h_{30} = h_{03} = 0$ the corresponding system is given by (no scaling)

$$\begin{cases} \dot{x} = y + a_4x^3 + 2a_5x^2y + 3a_6xy^2 + 4a_7y^3, \\ \dot{y} = -x - 4a_3x^3 - 3a_4x^2y - 2a_5xy^2 - a_6y^3, \end{cases} \tag{28}$$

where $a_3 = h_{40}, a_4 = h_{31}, a_5 = h_{22}, a_6 = h_{13}, a_7 = h_{04}$. This system is actually a special Hamiltonian system with only cubic homogeneous polynomials. Note that the advantage of not using scaling in (28) is that one does not necessarily specify one of the five parameters to be nonzero, and five parameters can be handled by computation.

Case (ii). $h_{21} = h_{30} = 0, h_{03} \neq 0$: the system is described by

$$\begin{cases} \dot{x} = y + 3y^2 + a_4x^3 + 2a_5x^2y + 3a_6xy^2 + 4a_7y^3, \\ \dot{y} = -x - 4a_3x^3 - 3a_4x^2y - 2a_5xy^2 - a_6y^3. \end{cases} \tag{29}$$

Case (iii). $h_{21} = 0, h_{30} \neq 0$: the system is given by

$$\begin{cases} \dot{x} = y - 6xy + 3a_2y^2 + a_4x^3 + 2a_5x^2y \\ \qquad + 3a_6xy^2 + 4a_7y^3, \\ \dot{y} = -x - 3x^2 + 3y^2 - 4a_3x^3 - 3a_4x^2y \\ \qquad - 2a_5xy^2 - a_6y^3. \end{cases} \tag{30}$$

Case (iv). $h_{21} \neq 0$: the system is given by Eq. (27). In order to compare with Case (i) which has only cubic terms, we consider one more special case which has only quadratic terms.

Case (v). $h_{ij} = 0, i + j = 3$: quadratic Hamiltonian system, described by the following general form (a simpler system can be directly obtained from (25) by neglecting the cubic terms):

$$\begin{cases} \dot{x} = y + h_{21}x^2 + 2h_{12}xy + 3h_{03}y^2, \\ \dot{y} = -x - 3h_{30}x^2 - 2h_{21}xy - h_{12}y^2. \end{cases} \tag{31}$$

In the following, we consider the above five cases one by one.

3.1. *Case (i)*: $h_{21} = h_{30} = h_{03} = 0$ (*no scaling*)

The system describing this case is given by (28) which has five parameters. Employing the Maple program [Yu, 1998] we easily obtain the exact expressions of the coefficients b_i. In particular,

$$b_1 = \frac{1}{2}(3a_7 + 3a_3 + a_5). \tag{32}$$

Setting $b_1 = 0$ yields

$$a_7 = -a_3 - \frac{1}{3}a_5, \tag{33}$$

and further computation gives

$$b_2 = -\frac{1}{240}[1440a_3^2 + 144a_4^2 + 160(a_5 + 3a_3)^2 + 9(5a_6 + 3a_4)^2] \le 0. \tag{34}$$

This clearly shows that the only solution satisfying $b_2 = 0$ is $a_3 = a_4 = a_5 = a_6 = 0$, and thus $a_7 = 0$, leading to a linear system. There exist infinite number of nontrivial solutions such that $b_1 = 0$, but $b_2 \neq 0$. For example, let $a_3 = a_4 = a_6 = 0, a_5 \neq 0$ and choose $a_7 = -(1/3)a_5$, then $b_1 = 0$, and $b_2 = -(2/3)a_5^2 < 0$. Then, giving a small perturbation to a_5 such that $a_5 \Rightarrow a_5 - \varepsilon$ $(0 < \varepsilon \ll 1)$, we obtain

$$b_1 = \frac{1}{2}\varepsilon, \quad b_2 = -\frac{2}{3}(a_5 - \varepsilon)^2.$$

Thus, we may choose a_5 and $\varepsilon > 0$ such that $b_1 b_2 < 0$ and $0 < b_1 \ll -b_2$. In summary, we have the following theorem for Case (i).

Theorem 2. *For the Hamiltonian system* (*28*), *with only cubic homogeneous polynomials, there exists maximal 1 local critical period bifurcating from the weak center* (*the origin*).

3.2. *Case (ii)*: $h_{21} = h_{30} = 0, h_{03} \neq 0$

The system for this case is described by (29), which, like Case (i), has only five independent parameters. However, comparing Eq. (29) with Eq. (28), there is an extra term $3y^2$ in the first equation, and thus system (29) may exhibit more critical periods. In fact, applying the Maple program results in

$$b_1 = \frac{1}{4}(6a_7 + 6a_3 + 2a_5 - 15). \tag{35}$$

Setting $b_1 = 0$ we obtain

$$a_7 = -a_3 - \frac{1}{3}a_5 + \frac{5}{2}, \tag{36}$$

and then,

$$b_2 = -\frac{1}{48}[45a_6^2 + 54a_4a_6 + 45a_4^2 + 576a_3^2 + 32a_5^2 + 192a_5a_3 + 1260a_3 + 330a_5 + 2520]. \tag{37}$$

Letting $b_2 = 0$ yields

$$a_6 = -\frac{1}{15}(3a_4 \pm \sqrt{Q}),$$
$$Q = 12\,600 - 144a_4^2 - 2880a_3^2 - 160a_5^2 - 960a_5a_3 - 6300a_3 - 1650a_5. \tag{38}$$

Having determined a_7 and a_6, b_3, b_4 and b_5 are expressed in terms of a_3, a_4, a_5 and Q:

$$b_3 = b_3(a_3, a_4, a_5, Q), \quad b_4 = b_4(a_3, a_4, a_5, Q),$$
$$b_5 = b_5(a_3, a_4, a_5, Q).$$

Eliminating Q from equations: $b_3 = b_4 = b_5 = 0$, results in $F_1 = F_2 = F_3 = 0$, where

```
F1:= 4722*a5*a3^2*a4^2+863*a5^2*a3*a4^2+132573/8*a5*a3*a4^2+48*a5*a3*a4^4
     +1344*a5*a3^3*a4^2+144*a3^2*a4^4+136/3*a5^3*a3*a4^2+328*a5^2*a3^2*a4^2
     +2880*a3^4*a4^2-166005/2*a5^2*a3+849009/128*a5*a4^2-242655/8*a5*a3^2
     -526617/64*a3*a4^2-64860075/32*a5-390913425/64*a3+104056155/64*a5*a3
     +368757585/256*a3^2-2735271/32*a4^2+18832905/64*a5^2+91035*a3^3-45115/4*a5^3
     +8145/16*a4^4+309204*a3^4-37177/24*a5^3*a3+223797/2*a5*a3^3+1407*a5^2*a4^2
     +183267/4*a3^2*a4^2+10899/2*a5^2*a3^2+826/9*a5^4*a3-4312*a5^2*a3^3+491/6*a5^3*a4^2
     -5376*a5*a3^4-392*a5^3*a3^2+99*a5*a4^4+522*a3*a4^4+13284*a3^3*a4^2-4123/12*a5^4
     +140/9*a5^5+32256*a3^5+9216*a3^6+5*a5^2*a4^4-104/9*a5^4*a3^2-112*a5^2*a3^4
     +10/3*a5^4*a4^2-416/3*a5^3*a3^3+40/27*a5^5*a3+2304*a5*a3^5+25/81*a5^6+278723025/64:
F2:= ...
F3:= ...
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Next, eliminating a_4 from the three equations $F_i = 0, i = 1, 2, 3$, we obtain the equation:

$$\begin{aligned}
&(1\,05\,55\,920 + 9\,95\,328a_5a_3 + 103\,680a_5^2 + 20\,52\,864a_5 + 1\,08\,24\,192a_3 + 29\,85\,984a_3^2)a_4^4 \\
&\quad + (69\,120a_5^4 + 9\,40\,032a_5^3a_3 + 68\,01\,408a_5^2a_3^2 + 2\,78\,69\,184a_5a_3^3 + 5\,97\,19\,680a_3^4 + 16\,96\,896a_5^3 \\
&\quad + 1\,78\,95\,168a_5^2a_3 + 9\,79\,15\,392a_5a_3^2 + 27\,54\,57\,024a_3^3 + 2\,91\,75\,552a_5^2 + 34\,36\,29\,216a_5a_3 \\
&\quad + 95\,00\,56\,128a_3^2 + 13\,75\,39\,458a_5 - 17\,06\,23\,908a_3 - 1\,77\,24\,55\,608)a_4^2 \\
&\quad + 6400a_5^6 + 30\,720a_5^5a_3 - 2\,39\,616a_5^4a_3^2 - 28\,75\,392a_5^3a_3^3 - 23\,22\,432a_5^2a_3^4 + 4\,77\,75\,744a_5a_3^5 \\
&\quad + 19\,11\,02\,976a_3^6 + 3\,22\,560a_5^5 + 19\,03\,104a_5^4a_3 - 81\,28\,512a_5^3a_3^2 - 8\,94\,13\,632a_5^2a_3^3 \\
&\quad - 11\,14\,76\,736a_5a_3^4 + 66\,88\,60\,416a_3^5 - 71\,24\,544a_5^4 - 3\,21\,20\,928a_5^3a_3 + 11\,30\,00\,832a_5^2a_3^2 \\
&\quad + 2\,32\,03\,27\,296a_5a_3^3 + 6\,41\,16\,54\,144a_3^4 - 23\,38\,76\,160a_5^3 - 1\,72\,11\,39\,840a_5^2a_3 \\
&\quad - 62\,89\,61\,760a_5a_3^2 + 1\,88\,77\,01\,760a_3^3 + 6\,10\,18\,61\,220a_5^2 + 33\,71\,41\,94\,220a_5a_3 \\
&\quad + 29\,86\,93\,64\,385a_3^2 - 42\,02\,93\,28\,600a_5 - 1\,26\,65\,59\,49\,700a_3 + 90\,30\,62\,60\,100 = 0
\end{aligned} \tag{39}$$

for determining a_4, as well as two resultant equations:

$$F_4(a_3, a_5) = 0, \quad F_5(a_3, a_5) = 0.$$

Further, eliminating a_5 from the above two equations yields solution

$$a_5 = a_5(a_3),$$

and the final resultant equation

$$F_6(a_3) = 0$$

from which we obtain four real solutions, satisfying

$$b_i = 0, \quad i = 1, 2, 3, 4, 5, \quad \text{but } b_6 \neq 0.$$

These four solutions are (numerical computation using Maple command *fsolve* is up to 100 digit points):

$$\mathrm{S}_{1,2} = \begin{cases} a_3 = -0.54661247824289067707907678468082353298508287936145\ldots \\ a_4 = \pm 4.2037413584938534243669343543896195240738731993665 8\ldots \\ a_5 = 6.0056466083347604749365999334548336361313720784586 7\ldots \\ a_6 = \mp 1.8238113367181456288175738891212368406148153343287 7\ldots \\ a_7 = 1.0447302754646371854335434735292123209412921865418 9\ldots \end{cases} \tag{40}$$

and

$$\mathrm{S}_{3,4} = \begin{cases} a_3 = 0.74628402190433633862153786736666691360178359116002\ldots \\ a_4 = \pm 0.95494883586067837884491074985604391682330401843334\ldots \\ a_5 = 1.77677679276518287286832995496947484184665887640751\ldots \\ a_6 = \mp 1.96857356633865127739656172962377003142054778517880\ldots \\ a_7 = 1.16145704717393603708901881431017480578266345003747\ldots \end{cases} \tag{41}$$

Further calculating the Jacobian given in Eq. (24) at the above four critical points shows that

$$\det\left[\frac{\partial(b_1, b_2, b_3, b_4, b_5)}{\partial(n_1, n_2, m_3, m_2, n_3)}\right]_{\mathrm{S}_{1,2}} = -94787249180.0458732067608926863846170234 28\ldots \neq 0,$$

$$\det\left[\frac{\partial(b_1, b_2, b_3, b_4, b_5)}{\partial(n_1, n_2, m_3, m_2, n_3)}\right]_{\mathrm{S}_{3,4}} = 26168947720.9520091522758703334418095977636\ldots \neq 0.$$

Therefore, there exist four solutions for Case (ii), described by system (29), to have five local critical periods. Moreover, through the above solution procedure, we did not find solutions such that the origin is an isochronous center (except for the linear center under the conditions: $a_i = 0, i = 3, 4, 5, 6, 7$). The above results are summarized in the following theorem.

Theorem 3. *For the Hamiltonian system* (*29*), *there are four solutions* $S_i, i = 1, 2, 3, 4$ *for the critical point* $(a_{3c}, a_{4c}, a_{5c}, a_{6c}, a_{7c})$ *which can be perturbed to generate five local critical periods. There are no solutions for the origin to be an isochronous center.*

Remark 1. It should be pointed out that although Theorem 3 states that there are only four solutions which give five local critical periods, there are actually infinite number of solutions since h_{03} ($\neq 0$) can be chosen arbitrarily.

3.3. *Case (iii)*: $h_{21} = 0, h_{30} \neq 0$

The system for this case is described by Eq. (30), which has six independent parameters. So it is possible to have six local critical periods. Computation here is more involved. So after determining a_7 from the equation:

$$b_1 = \frac{1}{4}(6a_7 + 6a_3 + 2a_5 - 15a_2^2 - 24), \tag{42}$$

as

$$a_7 = -a_3 - \frac{1}{3}a_5 + \frac{5}{2}a_2^2 + 4, \tag{43}$$

we apply a numerical computation scheme, built in Maple, to find a solution $(a_2, a_3, a_4, a_5, a_6)$ such that $b_i = 0, i = 2, 3, \ldots, 6$, but $b_7 \neq 0$. Hence, this case may have maximal six local critical periods.

With the built-in Maple command *fsolve*:

```
with(linalg):
Mysolution := fsolve({b2,b3,b4,b5,b6}, {a2,a3,a4,a5,a6}):
```

we obtain the following solution (up to 100 digit points):

$$a_2 = -1.9530167042546119939292193098056889089603955329148127555726984491 46\ldots$$

$$a_3 = 7.4400917989078701656061658837180529593568608754342112909425556345 40\ldots$$

$$a_4 = 25.650810451661680121160529031392015050419844748260657241662966138370\ldots$$

$$a_5 = -15.116103907142304956838024968336208374161199894167826672719928308559\ldots$$

$$a_6 = -4.8231644116499007349385547526299334038450962109238948871399111384056\ldots$$

and thus

$$a_7 = 11.134295121216764579414470136341522359176653633967864687200070569 6192\ldots$$

for which it can be shown that

$$b_1 = 0.2 \times 10^{-97}, \quad b_2 = 0.7 \times 10^{-96}, \quad b_4 = 0.6 \times 10^{-92},$$

$$b_5 = -0.2061606 \times 10^{-88}, \quad b_6 = -0.2903 \times 10^{-86}$$

$$b_7 = -455046268530.8138980917633685338370108821312913008510386104278849 2220\ldots$$

Theoretically speaking, the above $b_i, i = 1, 2, \ldots, 6$ should be exactly equal to zero. However, due to numerical computation error, they are only very close to zero, which does not affect the conclusion. The above result indicates that we can have at most six local critical periods.

Further, substituting the above critical values into the Jacobian results in

$$\det\left[\frac{\partial(b_1, b_2, b_3, b_4, b_5, b_6)}{\partial(a_2, a_3, a_4, a_5, a_6, a_7)}\right]_C = 33043073114209747379855592786.5122561239884\ldots \neq 0,$$

showing that Case (iii) can indeed have six local critical periods bifurcating from the weak center (the origin). Thus, we have the following theorem.

Theorem 4. *For the Hamiltonian system (30), there exists solution* $(a_2, a_3, a_4, a_5, a_6, a_7)$ *for the critical point such that six local critical periods bifurcate from the weak center.*

3.4. *Case (iv):* $h_{21} \neq 0$

Now, we consider the most general case $h_{21} \neq 0$, described by Eq. (27). Since the system has seven independent parameters, we thus expect to have seven possible local critical periods bifurcating from the weak center (the origin). If all the parameters are chosen free, then pure symbolic computation becomes intractable.

First, a_7 can be easily determined from $b_1 = 0$, where

$$b_1 = \frac{1}{4}(6a_7 + 6a_3 + 2a_5 - 24a_1^2 - 15a_2^2 - 6a_2 - 3), \tag{44}$$

as

$$a_7 = -a_3 - \frac{1}{3}a_5 + 4a_1^2 + \frac{5}{2}a_2^2 + a_2 + \frac{2}{3}. \tag{45}$$

Then, we employ the built-in Maple command *fsolve*:

```
with(linalg):
Mysolution := fsolve({b2,b3,b4,b5,b6,b7}, {a1,a2,a3,a4,a5,a6}):
```

to obtain (up to 100 digit points):

$$a_1 = -1.2364246734877587802632498227026198925887280270878860885964592791724\ldots$$

$$a_2 = 0.98060092507330975386269672080109864634550880040950788585325641898 45\ldots$$

$$a_3 = 1.1674913994464002559419182097394352630922889053812148773764288999199\ldots$$

$$a_4 = -1.9994919410393907175319848617740528548845338962332524879138556074393\ldots$$

$$a_5 = -2.4868197428231993132153458653932486422437698219378708298475027836364\ldots$$

$$a_6 = 1.5753469339547874695849429306415571718668005296383079840768578508091\ldots$$

and so

$$a_7 = 9.6609787683751300338229845417961206886830954895960552521122331798845 8\ldots$$

With the above solution, we have

$$b_1 = -0.13 \times 10^{-97}, \quad b_2 = -0.4 \times 10^{-98}, \quad b_3 = 0.1719 \times 10^{-93}, \quad b_4 = -0.1419 \times 10^{-91},$$
$$b_5 = 0.1787 \times 10^{-89}, \quad b_6 = -0.26853633 \times 10^{-87}, \quad b_7 = 0.9740407672794 \times 10^{-85}$$

$$b_8 = 1816997149.1442982480990691356954219768614472921335341562393779286644 14\ldots$$

Further, substituting the above critical values into the Jacobian results in

$$\det\left[\frac{\partial(b_1, b_2, b_3, b_4, b_5, b_6, b_7)}{\partial(a_1, a_2, a_3, a_4, a_5, a_6, a_7)}\right] = -47197124174913575212631166663688 03.2562879\ldots \neq 0,$$

implying that Case (iv) can indeed have seven local critical periods bifurcating from the weak center (the origin). Therefore, we have the following result.

Theorem 5. *For the Hamiltonian system (27), there exists a solution for the critical point such that seven local critical periods bifurcate from the weak center. This is the maximal number of local critical periods which can be obtained from cubic Hamiltonian systems.*

3.5. *Case (v): Quadratic Hamiltonian system*

Finally, we consider the quadratic system, given by (31). For this case, it is easy to show that

$$b_1 = -\frac{3}{4}[4(h_{30}^2 + h_{03}^2) + (h_{21} + h_{03})^2 + (h_{12} + h_{30})^2]. \tag{46}$$

This clearly indicates that $b_1 \leq 0$, and $b_1 = 0$ only if all the second order terms equal zero. This implies that the period $T(h)$ monotonically increases for $0 < h \ll 1$ and $\sum_{i+j=2} h_{ij}^2 \neq 0$. Thus, we have the following result.

Theorem 6. *For the quadratic Hamiltonian system (31), there is no critical period near the origin. More specifically, the period $T(h)$ monotonically increases for $0 < h \ll 1$.*

Note that the result given in the above theorem is only a partial result of the general conclusion: A quadratic Hamiltonian system does not have critical period and the period function monotonically increases for $h > 0$ (e.g. see [Li, 1989]).

4. A Numerical Example

In the previous sections, we have established several theorems for the properties of local critical periods and isochronous center of cubic Hamiltonian systems. In this section, we present a numerical example to demonstrate how to perturb the parameters from the critical point to obtain the exact number of local critical periods as given in the theorems.

Although Theorem 1 guarantees the existence of k local critical periods if the conditions given in the theorem are satisfied, it is not easy in practice to find a set of appropriate perturbations to obtain a numerical realization. If the parameters can be perturbed one by one separately for each of b_i's, the process is straightforward. However, when the perturbation parameters are coupled in solving equations $b_i = 0$, such as those cases considered in Secs. 3.2–3.4, it is very difficult to find such perturbations. In particular, when more parameters are coupled, like the case of seven local critical periods (Theorem 5), it is extremely difficult to obtain a numerical set of perturbations.

In the following, for an illustration, we present an example chosen from Case (ii) which has five local critical periods (see Theorem 8 given in Sec. 3.2). For this case, $h_{21} = h_{30} = 0$, while $h_{03} \neq 0$. The period $T'(h)$ for this example is given by

$$T'(h) = \frac{-2\pi p'(h)}{(1 + p(h))^2},$$

where

$$p_6'(h) = b_1 + 2b_2 h + 3b_3 h^2 + 4b_4 h^3 + 5b_5 h^4 + 6b_6 h^5, \tag{47}$$

in which the subscript 6 denotes that $p(h)$ is a sixth-degree polynomial of h.

Note that for this example the parameters a_3, a_4 and a_5 are coupled in the three equations:

$$F_1(a_3, a_4, a_5) = F_2(a_3, a_4, a_5) = F_3(a_3, a_4, a_5) = 0.$$

Although we obtain the exact expressions: $a_4(a_3, a_5)$ and $a_5 = a_5(a_3)$, we cannot treat these three parameters independently. Thus, we have to find the perturbations simultaneously for b_3, b_4 and b_5, by using a_3, a_4 and a_5. Having determined perturbations on a_3, a_4 and a_5, we can determine the perturbations on a_6 and a_7 one by one since they are separated.

It has been shown in Sec. 3.2 that we have four real solutions of a_3 for the four local critical periods. The complete set of critical values of $(a_{3c}, a_{4c}, a_{5c}, a_{6c}, a_{7c})$ are given in Eqs. (40) and (41). We choose the solution S_3 for this example, under which

$$b_1 = b_2 = b_3 = b_4 = b_5 = 0,$$
$$b_6 = 4911.5370692734385786465034701904 9\ldots > 0.$$

Thus, we need perturbations such that

$$b_5 < 0, \quad b_4 > 0, \quad b_3 < 0, \quad b_2 > 0, \quad b_1 < 0 \quad \text{and} \quad |b_i| \ll |b_{i+1}| \ll 1 \quad (i = 1, 2, \ldots, 5).$$

First, consider perturbations simultaneously on a_{3c}, a_{4c} and a_{5c} for b_5, b_4 and b_3. Following the procedure given in [Yu & Han, 2005b], we obtain (computed with up to 100 digit points, but here only list the first 30 digits for brevity):

$$\begin{aligned} a_3 &= a_{3c} + \varepsilon_1 \\ &= a_{3c} - 0.00070804685 \\ &= 0.745575975054336338621537867367, \\ a_4 &= a_{4c} + \varepsilon_2 \\ &= a_{4c} + 0.013925185 \\ &= 0.968874020860678378844910749856, \\ a_5 &= a_{5c} + \varepsilon_3 \\ &= a_{5c} - 0.0019420513 \\ &= 1.774834741465182872868329954969, \end{aligned}$$

for which Eq. (47) has three real solutions for h. Then take

$$\varepsilon_4 = -0.1 \times 10^{-13} \quad \text{and} \quad \varepsilon_5 = -0.5 \times 10^{-20},$$

respectively for a_6 and a_7 to obtain

$$\begin{aligned} a_6 &= a_{6c} + \varepsilon_4 \\ &= 1.96942703265668774552656079904 3, \\ a_7 &= a_{7c} + \varepsilon_5 \\ &= 1.16281244445726937041735214764 3. \end{aligned}$$

Under the above perturbed parameter values, we have

$$\begin{aligned} b_1 &= -0.75 \times 10^{-20}, \\ b_2 &= 0.4782636049591312994866 \times 10^{-13}, \\ b_3 &= -0.41403486321711356253114186 2 \times 10^{-7}, \\ b_4 &= 0.43122529716447837933681258310 4 \times 10^{-2}, \\ b_5 &= -42.34416187406919140276552105123 5, \\ b_6 &= 4269.68050503888007594057957976726 2, \end{aligned}$$

for which Eq. (47) has five real roots:

$$\begin{aligned} h_1 &= 0.88441121458464352512707024710 9 \times 10^{-7}, \\ h_2 &= 0.77469356861907450566288753460 5 \times 10^{-6}, \\ h_3 &= 0.70259489705672490170685661193 0 \times 10^{-5}, \\ h_4 &= 0.74327685122234213846260553602 3 \times 10^{-4}, \\ h_5 &= 0.81822895613934585215102507459 7 \times 10^{-2}, \end{aligned} \tag{48}$$

as expected.

In terms of the amplitude of periodic solution, $r = \sqrt{h}$ (see Eq. (19)), the amplitudes corresponding to the five critical points (see Eq. (48)) are

$$\begin{aligned} r_1 &= 0.0002973905201221, \\ r_2 &= 0.0008801667845466, \\ r_3 &= 0.0026506506692824, \\ r_4 &= 0.0086213505393433, \\ r_5 &= 0.0904560089844420. \end{aligned}$$

In order to show that higher order terms added to $p'_6(h)$ does not affect the number of real roots of $p'_6(h)$ for $0 < h \ll 1$, we expand $p'(h)$ up to b_9 using the above perturbed parameter values to obtain

$$\begin{aligned} p'_9(h) = &-0.75 \times 10^{-20} \\ &+ 0.4782636049591312994 9 \times 10^{-13}\, h \\ &- 0.41403486321711356253 \times 10^{-7}\, h^2 \\ &+ 0.43122529716447837934 \times 10^{-2}\, h^3 \\ &- 42.344161874069191402 77\, h^4 \\ &+ 4269.680505038880075940 58\, h^5 \\ &+ 79515.612663474628651461 40\, h^6 \\ &+ 776520.338029458484518789 55\, h^7 \\ &+ 3929787.825017610554353696 07\, h^8 \end{aligned}$$

which has the following six real roots:

$$\begin{aligned} h_1 &= -0.96952281163357789049159317415 8 \times 10^{-1}, \\ h_2 &= 0.88441121458464347567603682647 0 \times 10^{-7}, \\ h_3 &= 0.77469356862089379772978720219 5 \times 10^{-6}, \\ h_4 &= 0.70259488610281832696990105542 9 \times 10^{-5}, \\ h_5 &= 0.74328915721628212422416151033 5 \times 10^{-4}, \\ h_6 &= 0.70130197637715879754786538835 7 \times 10^{-2}. \end{aligned} \tag{49}$$

Compared to the roots of $p'_6(h)$, the positive five roots of $p'_9(h)$ are almost the same as that of $p'_6(h)$ (see Eq. (48)). The extra real root of $p'_9(h)$ is negative, which obviously does not belong to the interval $0 < h \ll 1$. This clearly shows that adding higher-order terms to $p'_6(h)$ does not change the number of local critical periods for small values of h.

5. Conclusion

In this paper we have shown that general planar cubic Hamiltonian systems can have maximal seven local critical periods which bifurcate from weak center. The methodology used in this paper is based on a perturbation technique for computing normal forms. Also some solutions are found for which the center of the system becomes an isochronous center. The approach developed in this paper can be extended to consider other dynamical systems. A complete solution for identifying all possibilities is still open.

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