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Eighteen limit cycles around two symmetric foci in a cubic planar switching polynomial system

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Abstract

In this paper, we present a cubic planar switching polynomial system with Z_2-symmetry, and prove that such a system can exhibit at least 9 small-amplitude limit cycles around each of two symmetric foci, giving a total 18 limit cycles. This is a new lower bound for the number of limit cycles bifurcating in cubic switching polynomial systems around foci, simultaneously obtained around the same time when more limit cycles are achieved by perturbing a cubic switching integral system.

Keywords: Switching system; Z_2-equivariant system; Lyapunov constant; Bi-center; Limit cycle

1. Introduction

The well-known self-sustained oscillation, leading to limit cycles, can often exist in almost all fields of science and engineering. Thus, developing limit cycle theory is not only theoretically significant, but also practically important. This phenomenon is closely related to bifurcation theory, and was studied by Poincaré one hundred years ago with his developed qualitative theory for differential equations. Nowadays, limit cycle theory has been extensively studied in planar

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vector fields, (e.g. see [18,20,25]), stimulated by the Hilbert's 16th problem [24]. In particular, the second part of Hilbert's 16th problem is to find an upper bound on the number of limit cycles that planar polynomial differential systems can have. This number is called Hilbert number, denoted by $H(n)$, where n is the degree of the polynomials. So far, the best result obtained for quadratic systems is four limit cycles, i.e. $H(2) \ge 4$ [7,33,34]. For cubic systems, many results have been obtained on the lower bound of the number and the best result is $H(3) \ge 13$ [26,27]. Although Ilyashenko [22,23] and Écalle [12] independently proved in 1990's that the number of limit cycles is finite for given planar polynomial vector fields, the problem is not even completely solved for general quadratic systems, i.e. $H(2) = 4$ is still open.

Recently, increasing interest has been attracted to discontinuous or non-smooth systems, which often arise in modeling physical and engineering problems. Practical examples include impact and dry frictions in mechanical engineering [4,10], switching circuits in power electronics [2], and feedback systems in control theory [1,3], etc. One type of the discontinuous systems, so-called switching system, is defined by different continuous vector fields in at least two different regions divided by switching lines or curves. The simple switching system can be described by the following equations:

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \begin{cases} \big(F^+(x, y, \mu),\, G^+(x, y, \mu)\big), & \text{for } y > 0, \\ \big(F^-(x, y, \mu),\, G^-(x, y, \mu)\big), & \text{for } y < 0, \end{cases} \tag{1.1}$$

where $F^{\pm}(x, y, \mu)$ and $G^{\pm}(x, y, \mu)$ are analytic functions in x and y. It is seen that system (1.1) actually includes two systems: the first equation is called the upper system, defined for $y > 0$, and the second is called the lower system, defined for $y < 0$. Note that $y = 0$ (i.e., the x-axis) is a switching line.

The investigation of limit cycle bifurcations for switching systems started 50 years ago (e.g. see [13,31,32]). In particular, Filippov [13] established some basic qualitative theory for switching equations and defined three types of pseudo-focus singular points: focus-focus (FF), parabolic-focus (PF) and parabolic-parabolic (PP). Coll et al. [9] derived the formulas for computing the first three Lyapunov quantities associated with the three types of singularities, and proved that at least 4 limit cycles can bifurcate from the weak focus in the FF-type case. Later, Gasull and Torregrosa [15] obtained 5 limit cycles in a quadratic switching system, two more than that of general smooth quadratic systems. It is well known that smooth linear systems cannot have limit cycles. However, for piecewise linear systems, Han and Zhang [21] proved that 2 limit cycles can bifurcate from a focus of either FF, FP or PP type. Chen and Du [5] constructed a switching Bautin system (i.e., both the upper and the lower systems are in the Bautin form) and proved that 9 limit cycles can bifurcate from a center of the system. Chen et al. [8] constructed a class of discontinuous quadratic Bautin system and showed that at least 5 and 8 limit cycles can bifurcate from weak foci and centers, respectively. Recently, Tian and Yu [36] also considered the switching Bautin system and gave a complete classification on the conditions of a singular point to be a center, and constructed an example to prove the existence of 10 limit cycles bifurcating from the center, which is much larger than 3 limit cycles obtained from an isolated center of smooth quadratic systems. Recently, a planar quadratic switching system has been constructed to obtain more limit cycles [11] by using the averaging approach up to ε^2 order, which is equivalent to second-order Melnikov function method. It is shown in [11] that 8 limit cycles are obtained from the ε-order perturbation, while 16 limit cycles are obtained from the ε^2-order perturbation, which is surprisingly high with a jump of 8 limit cycles from the ε-order.

However, very fewer results have been obtained for cubic switching systems. Llibre et al. [29] obtained 12 limit cycles in a family of isochronous cubic polynomial systems, bifurcating from periodic orbits. Li et al. [28] constructed a switching Z_2 cubic system to show the existence of 15 limit cycles. Recently, Tian and Han studied bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems [35]. Also, Han et al. [19] developed a more general bifurcation theory for finitely smooth planar systems. Very recently, the method presented in [14] and the "normal form" of switching linear systems have been used by Gouveia and Torregrosa [16] to find 24 limit cycles in a cubic switching polynomial system, which is associated with a rational first integral, by perturbing a single Darboux center. The authors first obtain two limit cycles by using two parameters in the linear switching system: one by breaking the trace and the other by sliding. The sliding feature corresponds to the so-called "Pseudo-Hopf bifurcation". Then, they proved that the linear terms of the 23 Lyapunov quantities are linearly independent, but only 22 hyperbolic crossing limit cycles are obtained, yielding a total 2+22=24 limit cycles. We have used the method presented in this paper as well as our Maple program to confirm the existence of the 24 limit cycles by computing the ε-order and ε^2-order Lyapunov quantities.

The above results show that perturbing centers can generate more limit cycles than perturbing foci does, like what was done in [17] in which perturbing two foci yields 16 limit cycles while perturbing bi-centers gives 18 limit cycles. As a matter of fact, perturbing foci to find limit cycles is much harder than perturbing center to find limit cycles, since the former solves nonlinear algebraic equations while the later solves linear equations. So far, for continuous planar cubic polynomial systems, the best result for the limit cycles around a singular focus is 9 [6,37,30], while around a center such a cubic system can have 12 limit cycles [38].

The purpose of this paper is to find maximal number of limit cycles by perturbing foci, rather than perturbing centers. Moreover, using symmetry may make it easy to generate more limit cycles. Therefore, in this paper, we consider bifurcation of limit cycles in a cubic switching Z_2-equivariant system and show that the system can have 18 limit cycles bifurcating from two symmetric singular foci. It should be noted that those 18 limit cycles given in [17] were obtained by perturbing two symmetric centers using a center condition for the unperturbed system, while our 18 limit cycles obtained in this paper are around two symmetric fine foci. In fact, the authors of [17] proved the existence of only 16 limit cycles around two symmetric foci.

For the general dynamical system:

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{1.2}$$

it is Z_2-equivariant if and only if

$$P(-x, -y) = -P(x, y), \quad \text{and} \quad Q(-x, -y) = -Q(x, y), \tag{1.3}$$

regardless whether the system is smooth or discontinuous. Therefore, the switching system (1.1) is Z_2-equivariant if and only if

$$F^+(-x, -y, \mu) = -F^-(x, y, \mu) \quad \text{and} \quad G^+(-x, -y, \mu) = -G^-(x, y, \mu). \tag{1.4}$$

Hence, without loss of generality, a piecewise cubic Z_2-equivariant system can be written in the form of

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \quad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \\ \quad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{for } y > 0, \\ \begin{pmatrix} -a_{00} + a_{10}x + a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 \\ \quad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ -b_{00} + b_{10}x + b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2 \\ \quad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{for } y < 0, \end{cases} \tag{1.5}$$

where a_{ij} and b_{ij} are real coefficients. Further, assume that the system (1.5) has two symmetric Hopf-type singular points at $(\pm 1, 0)$, which yields

$$a_{00} = -a_{20} = \frac{1}{2}b_{11}, \quad a_{10} = -a_{30} = \frac{1}{2}(b_{01} + b_{21}), \quad b_{20} = -b_{00}, \quad b_{30} = -b_{10}. \tag{1.6}$$

In order to overcome the difficulty in the computation of the Lyapunov constants of system (1.5), we simplify the system by setting

$$a_{00} = 0, \quad a_{10} = -\frac{1}{2}\delta, \tag{1.7}$$

and in addition, let

$$b_{00} = -4b_{02}, \quad b_{10} = 2(a_{01} + a_{21}), \quad a_{11} = -2b_{02}. \tag{1.8}$$

It should be noted that the sliding feature described in [14] for switching linear system, associated with Pseudo-Hopf bifurcation, can generate one more limit cycle besides the crossing limit cycle due to breaking trace. For system (1.5) to have sliding feature, the condition $a_{00} \neq 0$ must hold. However, for $a_{00} \neq 0$, we found that computing the Lyapunov quantities and solving the multivariate polynomials become extremely difficult and it is not possible to obtain 18 limit cycles. Thus, in this paper we do not consider Pseudo-Hopf bifurcation due to sliding, which requires $a_{00} \neq 0$.

Now, under the conditions given in (1.6), (1.7) and (1.8), and using $b_{01} + \frac{1}{2}\delta$ for b_{01}, system (1.5) becomes

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{\delta}{2}(-x + x^3) + a_{01}y - 2b_{02}xy + a_{02}y^2 \\ \quad + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ \frac{\delta}{2}y - 4b_{02} + 2(a_{01} + a_{21})x + b_{01}y + 4b_{02}x^2 + b_{02}y^2 \\ \quad -2(a_{01} + a_{21})x^3 - b_{01}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{for } y > 0, \\ \begin{pmatrix} \frac{\delta}{2}(-x + x^3) + a_{01}y + 2b_{02}xy - a_{02}y^2 \\ \quad + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ \frac{\delta}{2}y + 4b_{02} + 2(a_{01} + a_{21})x + b_{01}y - 4b_{02}x^2 - b_{02}y^2 \\ \quad -2(a_{01} + a_{21})x^3 - b_{01}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{for } y < 0, \end{cases} \tag{1.9}$$

where the small perturbation parameter δ is added in order to yield one more small-amplitude limit cycle from a linear perturbation (i.e., breaking the trace). Then, the critical eigenvalues of the upper and lower systems of (1.9), evaluated at the symmetric singular points $(\pm 1, 0)$ when $\delta = 0$, are given by $\pm i\,\omega_{c+}$ and $\pm i\,\omega_{c-}$, respectively, where

$$\omega_{c+} = 2|a_{01} + a_{21} + 2b_{02}|, \quad \omega_{c-} = 2|a_{01} + a_{21} - 2b_{02}|. \tag{1.10}$$

However, in order to have a closed orbit which consists of an upper plane orbit and a lower plane orbit, it requires that the upper plane orbit and the lower plane orbit must rotate in the same direction. Therefore, the following condition must hold:

$$(a_{01} + a_{21} + 2b_{02})(a_{01} + a_{21} - 2b_{02}) > 0, \tag{1.11}$$

where the subscript "c" indicates the critical value of $\omega_{\pm}$ at $\delta = 0$. Note that in general $\omega_{c+} \neq \omega_{c-}$ unless $b_{02} \equiv 0$. In almost all published articles related to the topic discussed in this paper, authors usually, for convenience, made choices on parameters such that $\omega_{c+} = \omega_{c-}$, which guarantees that the upper and lower systems have a same frequency for the bifurcating limit cycles. However, this is not necessary, and choosing $\omega_{c+} \neq \omega_{c-}$ may increase the possibility of having bifurcation of more limit cycles. It should be noted that due to the condition (1.11), we are not allowed to choose

$$a_{01} + a_{21} = 0, \quad b_{02} \neq 0, \tag{1.12}$$

since this would yield

$$(a_{01} + a_{21} + 2b_{02})(a_{01} + a_{21} - 2b_{02}) = -4\,b_{02}^2 < 0,$$

violating the condition (1.11).

In addition, we can show that when $a_{21} = 0$, only 16 limit cycles can bifurcate from $(\pm 1, 0)$. Therefore, we assume $a_{21} \neq 0$ and show that system (1.9) can have 9 small-amplitude limit cycles around each of the two singular points, giving a total 18 limit cycles, a new lower bound for cubic switching systems around fine foci, rather than the 18 limit cycles obtained from perturbing the bi-centers under a center condition [17].

The rest of this paper is organized as follows. In the next section, we present some results and formulas which are needed to prove the main result in Section 3. Then, in Section 3, we investigate bifurcation of limit cycles in system (1.9) and show the existence of 18 limit cycles. The conditions under which the two singular points of (1.9) become bi-centers are also discussed. Finally, conclusion is drawn in Section 4.

2. Preliminary

In this section, for the convenience of the readers, we present some results and formulas which will be used for computing the Lyapunov constants of the switching system (1.1). The results and formulas are mainly taken from [28]. We omit the detailed proofs, which can be found in [28]. We start from the following switching system:

$$\Big(\frac{dx}{dt},\ \frac{dy}{dt}\Big)=\begin{cases}\big(-y+F^{+}(x,y,\mu),\ x+G^{+}(x,y,\mu)\big), & \text{for}\ \ y>0,\\ \big(-y+F^{-}(x,y,\mu),\ x+G^{-}(x,y,\mu)\big), & \text{for}\ \ y<0,\end{cases} \tag{2.1}$$

which has a fixed point at the origin.

First, we introduce the classical method to compute the Lyapunov constants and period constants of the following general differential system,

$$\begin{aligned}\frac{dx}{dt}&=\delta x-y+\sum_{k=2}^{n}X_k(x,y,\mu)\equiv X(x,y,\mu),\\ \frac{dy}{dt}&=x+\delta y+\sum_{k=2}^{n}Y_k(x,y,\mu)\equiv Y(x,y,\mu).\end{aligned} \tag{2.2}$$

With the polar coordinates transformation, $x=r\cos\theta,\ \ y=r\sin\theta$, system (2.2) can be rewritten as

$$\frac{dr}{dt}=r\Big(\delta+\sum_{k=2}^{n}\varphi_{k+2}(\theta)r^k\Big),\qquad \frac{d\theta}{dt}=1+\sum_{k=2}^{n}\psi_{k+2}(\theta)r^k, \tag{2.3}$$

where $\varphi_k(\theta)$, $\psi_k(\theta)$ are polynomials of $\cos\theta$ and $\sin\theta$. From equation (2.3) we have

$$\frac{dr}{d\theta}=\frac{r(\delta+\sum_{k=2}^{n}\varphi_{k+2}(\theta)r^k)}{1+\sum_{k=2}^{n}\psi_{k+2}(\theta)r^k}, \tag{2.4}$$

whose expansion around $r=0$ can be expressed in the form of

$$\frac{dr}{d\theta}=r\sum_{k=1}^{\infty}R_k(\theta)r^k. \tag{2.5}$$

By the Poincaré–Lindstedt method, the general solution of (2.5) can be obtained as

$$r=\tilde{r}(\theta,h)=\sum_{k=1}^{\infty}v_k(\theta)h^k,$$

where $v_1(0)=1,\ v_k(0)=0$ form all $k\ge 2$. Now, submitting the above solution $r=\tilde{r}(\theta,h)$ into (2.5) results in a set of differential equations, which are then solved for finding the solutions $v_1(\theta),\ v_2(\theta),\ \ldots$ Then, we define the difference map (or successive function) as

$$\Delta(h)=\tilde{r}(2\pi,h)-h, \tag{2.6}$$

which in turn gives the condition to define a center as $r(2\pi,h)=h$.

Now we consider the computation of Lyapunov constants for system (2.1). Since the above classical procedure cannot be directly applied to a switching system due to discontinuity, we may slightly modify the condition to define the Poincaré map. To achieve this, note that the polar coordinates expression for (2.1) can be written as

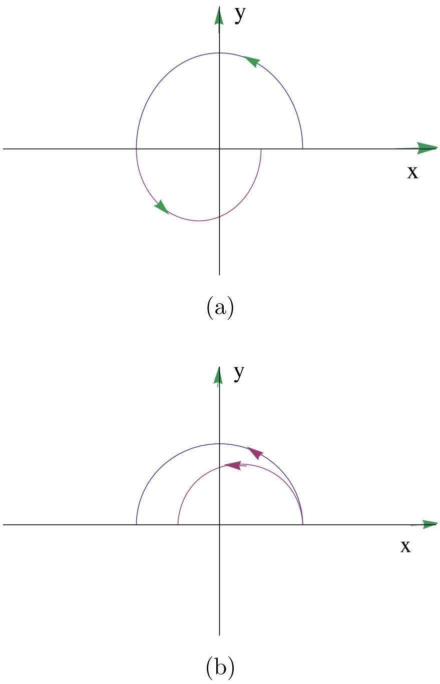

Fig. 1. The Poincaré map: (a) for (2.1); and (b) for (2.8).

$$\begin{aligned} &(R^+(r,\theta), 1+\Theta^+(r,\theta)), \quad \theta \in [0,\pi], \\ &(R^-(r,\theta), 1+\Theta^-(r,\theta)), \quad \theta \in [\pi,2\pi]. \end{aligned} \tag{2.7}$$

Also note that although a return map cannot be simply defined for (2.1) like that for continuous systems, we may follow the approach presented in [15] to define half-return maps for the upper and lower systems of (2.1). Equivalently, we may introduce the transformation $y \to -y$ and with a time reversing $t \to -t$ to change the lower half system into upper system, and thus compute two positive half-return maps for

$$\Big(\frac{dx}{dt}, \frac{dy}{dt}\Big) = \begin{cases} \big(-y + F^+(x,y,\mu),\ x + G^+(x,y,\mu)\big), & \text{for } y > 0, \\ \big(-y - F^-(x,-y,\mu),\ x + G^-(x,-y,\mu)\big), & \text{for } y < 0. \end{cases} \tag{2.8}$$

The idea discussed above is illustrated in Fig. 1.

Now suppose the solutions for the two systems in (2.8) are respectively given by

$$r_1 = \tilde{r}_1(\theta,h) = \sum_{k=1}^{\infty} u_k(\theta) h^k \quad \text{and} \quad r_2 = \tilde{r}_2(\theta,h) = \sum_{k=1}^{\infty} v_k(\theta) h^k,$$

satisfying $u_1(0) = v_1(0) = 1$, $u_k(0) = v_k(0) = 0$ for all $k \geq 2$. Then we can thus define the following Poincaré maps:

$$\Delta_1(h) = \tilde{r}_1(\pi,h) - h \quad \text{and} \quad \Delta_2(h) = \tilde{r}_2(\pi,h) - h,$$

for the two systems in (2.8), respectively. Finally, the Poincaré map for the switching system (2.1) can be defined as

$$\begin{aligned}\Delta(h) &= \Delta_1(h) - \Delta_2(h) = \tilde{r}_1(\pi, h) - \tilde{r}_2(\pi, h) \\ &= \sum_{k=1}^{n} \big(u_k(\pi) - v_k(\pi)\big) h^k = \sum_{k=1}^{n} L_k h^k,\end{aligned} \tag{2.9}$$

where L_k is called the kth-order Lyapunov constant (or focal value) of the switching system (2.1).

Obviously, the symmetry principle for continuous systems cannot be used to prove the center conditions of switching systems. The following lemmas give the sufficient conditions for the origin of system (2.1) to be a center.

Lemma 2.1. [28] *If the upper and lower systems of* (2.1) *have the first integrals* $H^+(x, y)$ *and* $H^-(x, y)$ *near the origin, respectively, and either both* $H^+(x, y)$ *and* $H^-(x, y)$ *are even functions in* x *or* $H^+(x, 0) \equiv H^-(x, 0)$*, then the origin of system* (2.1) *is a center.*

Lemma 2.2. [28] *Assuming that* $\delta = 0$*, if system* (2.1) *is symmetric with respect to the* x*-axis, i.e. the functions on the right-hand side of system* (2.1) *satisfy*

$$F^+(x, y, \mu) = -F^-(x, -y, \mu), \quad G^+(x, y, \mu) = G^-(x, -y, \mu), \quad y > 0, \tag{2.10}$$

or if system (2.1) *is symmetric with respect to the* y*-axis, i.e. the functions on the right-hand side of system* (2.1) *satisfy*

$$\begin{aligned} &F^+(x, y, \mu) = F^+(-x, y, \mu), \;\; y > 0; \qquad F^-(x, y, \mu) = F^-(-x, y, \mu), \;\; y < 0; \\ &G^+(x, y, \mu) = -G^+(-x, y, \mu), \;\; y > 0; \quad G^-(x, y, \mu) = -G^-(-x, y, \mu), \;\; y < 0, \end{aligned} \tag{2.11}$$

then the origin of system (2.1) *is a center.*

The following lemma gives sufficient conditions for the number of limit cycles bifurcating from a singular point of system (1.1).

Lemma 2.3. [36] *Suppose that there exists a sequence of Lyapunov constants of system* (1.1)*,* $V_{i_0}(\varepsilon)$, $V_{i_1}(\varepsilon)$, $\ldots$, $V_{i_k}(\varepsilon)$*, at a singular point with* $1 = i_0 < i_1 < \cdots < i_{k-1} < i_k$*, such that* $V_j = O(|(V_{i_0}, \ldots, V_{i_l})|)$ *for any* $i_l < j < i_{l+1}$*. Further, at the singular point, if* $V_{i_0}(0) = V_{i_1}(0) = \cdots = V_{i_{k-1}}(0) = 0$*,* $V_{i_k}(0) \neq 0$*, and*

$$\det\left[\frac{\partial(V_{i_0}, V_{i_1}, \cdots, V_{i_{k-1}})}{\partial(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)}(0, \cdots, 0)\right] \neq 0,$$

then system (1.1) *has exactly* k *limit cycles in a* δ*-ball with its center at the singular point.*

3. 18 limit cycles in the cubic-order switching system (1.9)

In this section, we consider bifurcation of limit cycles around the two symmetric singular points $(\pm 1, 0)$ of the Z_2-equivariant system (1.9), and prove that there exist 18 small-amplitude limit cycles around $(\pm 1, 0)$. In the next section, we will derive the conditions under which $(\pm 1, 0)$ may become centers. It has been shown in the introduction section that the Lyapunov constants may belong to one of the four categories, given in (1.11). In the following analysis, we present the details for the first category, i.e., we assume

$$(a_{01} + a_{21} + 2b_{02})(a_{01} + a_{21} - 2b_{02}) > 0. \tag{3.1}$$

The Lyapunov constants for the other three categories can be similarly obtained.

Since the vector field is symmetric with the origin, we only need to study the singular point $(1, 0)$. To achieve this, we first shift $(1, 0)$ of system (1.9) to the origin. Introducing $x = 1 + x_1$, $y = x_2$ into system (1.9), we obtain

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} \begin{pmatrix} \dfrac{\delta}{2} x_1(1+x_1)(2+x_1) + (a_{01}+a_{21}-2b_{02})x_2 \\ \quad +2(a_{21}-b_{02})\, x_1x_2 + (a_{02}+a_{12})x_2^2 \\ \quad +a_{21}x_1^2x_2 + 2a_{12}x_1x_2^2 + a_{03}x_2^3 \\ \delta x_2 - 4(a_{01}+a_{21}-2b_{02})x_1 \\ \quad -2(3a_{01}+3a_{21}-2b_{02})x_1^2 \\ \quad -2b_{01}x_1x_2 + (b_{02}+b_{12})x_2^2 - 2(a_{01}+a_{21})x_1^3 \\ \quad -b_{01}x_1^2x_2 + 2b_{12}x_1x_2^2 + b_{03}x_2^3 \end{pmatrix}, & (x_2 > 0), \\ \begin{pmatrix} \dfrac{\delta}{2} x_1(1+x_1)(2+x_1) + (a_{01}+a_{21}+2b_{02})x_2 \\ \quad +2(a_{21}+b_{02})\, x_1x_2 - (a_{02}-a_{12})x_2^2 \\ \quad +a_{21}x_1^2x_2 + 2a_{12}x_1x_2^2 + a_{03}x_2^3 \\ \delta x_2 - 4(a_{01}+a_{21}+2b_{02})x_1 \\ \quad -2(3a_{01}+3a_{21}+2b_{02})x_1^2 \\ \quad -2b_{01}x_1x_2 - (b_{02}-b_{12})x_2^2 - 2(a_{01}+a_{21})x_1^3 \\ \quad -b_{01}x_1^2x_2 + 2b_{12}x_1x_2^2 + b_{03}x_2^3 \end{pmatrix}, & (x_2 < 0). \end{cases} \tag{3.2}$$

Thus, the origin of system (3.2) corresponds to the singular point $(1, 0)$ of system (1.9).

We have the following result.

Theorem 3.1. *There exist* 9 *small-amplitude limit cycles bifurcating from the origin of system* (3.2)*, and thus system* (1.9) *has at least* 18 *small-amplitude limit cycles with* 9 *enclosing each of the two symmetric singular points* $(\pm 1, 0)$.

Proof. In order to apply the method and formula (2.9) presented in the previous section to compute the Lyapunov constants L_k $(k \geq 1)$ associated with the origin of system (3.2), we first need to transform the linear part of system (3.2) into the Jordan canonical form. To achieve this, introducing the following transformations:

$$x_1 = -X, \quad x_2 = 2Y$$

into (3.2) yields

$$\begin{pmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\omega_{c-}Y + \frac{\delta}{2}X(1-X)(2-X) \\ \quad +2\big[2(a_{21}-b_{02})XY - 2(a_{02}+a_{12})Y^2 \\ \quad -a_{21}X^2Y + 4a_{12}XY^2 - 4a_{03}Y^3\big] \\ \omega_{c-}X + \delta Y - (3a_{01}+3a_{21}-2b_{02})X^2 \\ \quad +2b_{01}XY + 2(b_{02}+b_{12})Y^2 + (a_{01}+a_{21})X^3 \\ \quad -b_{01}X^2Y - 4b_{12}XY^2 + 4b_{03}Y^3 \end{pmatrix}, & (Y>0), \\ \begin{pmatrix} -\omega_{c+}Y + \frac{\delta}{2}X(1-X)(2-X) \\ \quad +2\big[2(a_{21}+b_{02})XY + 2(a_{02}-a_{12})Y^2 \\ \quad -a_{21}X^2Y + 4a_{12}XY^2 - 4a_{03}Y^3\big] \\ \omega_{c+}X + \delta Y - (3a_{01}+3a_{21}+2b_{02})X^2 \\ \quad +2b_{01}XY - 2(b_{02}-b_{12})Y^2 + (a_{01}+a_{21})X^3 \\ \quad -b_{01}X^2Y - 4b_{12}XY^2 + 4b_{03}Y^3 \end{pmatrix}, & (Y<0). \end{cases} \tag{3.3}$$

Now, for system (3.3), the zero-order Lyapunov constant is $L_0 = 2\pi\delta$. So letting $\delta = 0$ yields $L_0 = 0$. It should be noted that when we compute the Lyapunov constants L_i, $i \geq 2$ for the upper and lower half vector fields under the condition $\delta = 0$, we need to make a scaling such that $\omega_{c\pm} \to 1$, and then after obtaining the Lyapunov constants for the scaled systems we multiply the Lyapunov constants by $\omega_{c\pm}$. Therefore,

$$L_1 = -\frac{32}{3\omega_{c+}\omega_{c-}}\, b_{02}(a_{01} - a_{21} - 2b_{12}). \tag{3.4}$$

So we set $b_{02} = 0$ under which the condition $L_1 = 0$ and $\omega_{c+} = \omega_{c-} = 2(a_{01} + a_{21}) \neq 0$. Then, we obtain

$$L_2 = \frac{\pi\left[6(a_{01}+a_{21})b_{03} + 2(a_{01}-a_{21}-2b_{12})a_{12} + (a_{01}+a_{21}-b_{12})b_{01}\right]}{2(a_{01}+a_{21})}.$$

Since when $b_{02} = 0$, $a_{01} + a_{21} \neq 0$ due to the restriction (1.11), we may solve b_{03} from the equation $L_2 = 0$ to obtain

$$b_{03} = -\frac{a_{12}(a_{01}-a_{21}-2b_{12}) + b_{01}(a_{01}+a_{21}-b_{12})}{6(a_{01}+a_{21})}, \tag{3.5}$$

under which

$$L_3 = \frac{32\,a_{02}\big[16a_{12}(a_{21}+b_{12}) + b_{01}(3a_{01}+a_{21}-2b_{12})\big]}{45(a_{01}+a_{21})^2}.$$

Suppose $a_{21}+b_{12}\neq 0$, we solve $L_3=0$ for a_{12}, yielding

$$a_{12}=-\frac{b_{01}(3a_{01}+a_{21}-2b_{12})}{16(a_{21}+b_{12})}. \tag{3.6}$$

Next, solving $L_4=0$ for a_{03} we obtain

$$\begin{aligned} a_{03}= & \frac{1}{96(a_{21}+b_{12})^2(a_{01}+a_{21})(a_{01}+3a_{21}+2b_{12})} \\ & \times\big\{6\big[24(a_{21}+b_{12})^2-5b_{01}^2\big]a_{01}^3 \\ & \quad +\big[16(14a_{21}-b_{12})(b_{12}+a_{21})^2-(59a_{21}-31b_{12})b_{01}^2\big]a_{01}^2 \\ & \quad +2\big[24(3a_{21}+2b_{12})(a_{21}-b_{12})(b_{12}+a_{21})^2 \\ & \qquad -(16a_{21}^2-47a_{21}b_{12}-18b_{12}^2)b_{01}^2+240(a_{21}+b_{12})^2a_{02}^2\big]a_{01} \\ & \quad +3(a_{21}-b_{12})(a_{21}+6b_{12})(3a_{21}+2b_{12})b_{01}^2 \\ & \quad +480a_{21}(a_{21}+b_{12})^2a_{02}^2\big\}, \end{aligned} \tag{3.7}$$

for which $a_{01}+3a_{21}+2b_{12}\neq 0$ is assumed. Summarizing the above results shows that we can solve the first 5 Lyapunov constants equations $L_i=0$, $i=0,1,2,3,4$ one by one using one parameter for each, namely, $\delta=0$ for L_0, $b_{02}=0$ for L_1, b_{03} (given in (3.5)) for L_2, a_{12} (given in (3.7)) for L_3 and a_{03} (given in (3.7)) for L_4, with the assumption:

$$(a_{01}+a_{21})(a_{21}+b_{12})(a_{01}+3a_{21}+2b_{12})\neq 0.$$

Starting from L_5, the Lyapunov constant equations cannot be linearly solved like that in the above process. However, it is noted that L_5 is linear with respect to b_{01}^2, and so we solve $L_5(b_{01}^2)=0$ to obtain $b_{01}^2=\dfrac{B_{01\mathrm{n}}}{B_{01\mathrm{d}}}$, where

$$\begin{aligned} B_{01\mathrm{n}}= & \ 8(a_{01}+a_{21})(a_{21}+b_{12})^2\big\{645a_{01}^3+2(713a_{21}+68b_{12})a_{01}^2 \\ & +5\big[(3a_{21}+2b_{12})(71a_{21}-58b_{12})+324a_{02}^2\big]a_{01}+1620a_{21}a_{02}^2\big\}, \\ B_{01\mathrm{d}}= & \ 3\big\{265a_{01}^4+20(39a_{21}-14b_{12})a_{01}^3+30(25a_{21}^2-40a_{21}b_{12}-12b_{12}^2)a_{01}^2 \\ & -4(a_{21}^3+438a_{21}^2b_{12}+108a_{21}b_{12}^2-64b_{12}^3)a_{01} \\ & -5(3a_{21}+2b_{12})(17a_{21}^3+42a_{21}^2b_{12}-36a_{21}b_{12}^2-8b_{12}^3)\big\}. \end{aligned} \tag{3.8}$$

Thus, under the restriction:

$$(a_{01}+a_{21})(a_{21}+b_{12})(a_{01}+3a_{21}+2b_{12})B_{01\mathrm{d}}\neq 0, \quad \frac{B_{01\mathrm{n}}}{B_{01\mathrm{d}}}>0, \tag{3.9}$$

the higher Lyapunov constants are simplified as

$$
\begin{aligned}
L_6 &= -\frac{5\pi b_{01}}{2304(a_{01}+a_{21})^4 B_{01\mathrm{d}}^2}\, L_{6a},\\
L_7 &= \frac{b_{01}}{30481920(a_{01}+a_{21})^5 B_{01\mathrm{d}}^2}\, L_{7a},\\
L_8 &= \frac{b_{01}}{97542144(a_{01}+a_{21})^6 B_{01\mathrm{d}}^3}\, L_{8a},\\
L_9 &= -\frac{b_{01}}{32188907520(a_{01}+a_{21})^7 B_{01\mathrm{d}}^3}\, L_{9a},
\end{aligned}
\tag{3.10}
$$

where $b_{01} \neq 0$, and L_{6a}, L_{7a}, L_{8a} and L_{9a} are respectively 12th, 13th, 18th and 34th-degree homogeneous polynomials in a_{01}, a_{02}, a_{21} and b_{12}. Hence, in general it is not possible to have solutions for the four parameters such that $L_{6a} = L_{7a} = L_{8a} = L_{9a} = 0$ and so bifurcation of 10 limit cycles from the origin of (3.2) is not possible. The next best result we can obtain is 9 limit cycles. Further, it can be shown that if $a_{02} = 0$, then the maximal number of limit cycles bifurcating from the origin of (3.2) is 8. Therefore, assuming $a_{02} \neq 0$ and letting

$$
a_{01} = A_{01}a_{02}, \quad a_{21} = A_{21}a_{02}, \quad b_{12} = B_{12}a_{02}, \tag{3.11}
$$

under which

$$
\begin{aligned}
L_6 &= -\frac{5\pi b_{01} a_{02}^{12}}{2304(a_{01}+a_{21})^4 B_{01\mathrm{d}}^2}\, L_{6b},\\
L_7 &= \frac{b_{01} a_{02}^{13}}{30481920(a_{01}+a_{21})^5 B_{01\mathrm{d}}^2}\, L_{7b},\\
L_8 &= \frac{b_{01} a_{02}^{18}}{97542144(a_{01}+a_{21})^6 B_{01\mathrm{d}}^3}\, L_{8b},
\end{aligned}
$$

where L_{6b}, L_{7b} and L_{8b} are polynomials in A_{01}, A_{21} and B_{12}. Eliminating B_{12} from the equations $L_{6b} = L_{7b} = L_{8b} = 0$ yields a solution $B_{12} = B_{12}(A_{01}, A_{21})$, and two resultants:

$$
\begin{aligned}
\mathrm{R}_1 &= (A_{01}+A_{21})\,\mathrm{R}_0\,\mathrm{R}_{1\mathrm{a}},\\
\mathrm{R}_2 &= A_{01}(A_{01}+A_{21})\,\mathrm{R}_0\,\mathrm{R}_{2\mathrm{a}}\,\mathrm{R}_{2\mathrm{b}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{R}_0 = {} & 158375A_{01}^3(16A_{01}^2-1125)A_{21}^4\\
& -50A_{01}^2(571496A_{01}^4-64311525A_{01}^2+1447083225)A_{21}^3\\
& -30A_{01}(1383992A_{01}^6-127265400A_{01}^4+4863651075A_{01}^2+3424842000)A_{21}^2\\
& +2\,(2001448A_{01}^8+1399647555A_{01}^6-47313284625A_{01}^4\\
& \qquad -85384854000A_{01}^2+5978711250)A_{21}\\
& +A_{01}(14413856A_{01}^8+787844115A_{01}^6-17103105450A_{01}^4\\
& \qquad -66536413200A_{01}^2+11957422500),\\
\mathrm{R}_{2\mathrm{a}} = {} & (3A_{01}+5A_{21})(10125A_{21}^3+22599A_{21}^2A_{01}+14643A_{21}A_{01}^2+2329A_{01}^3),
\end{aligned}
$$

and $\mathrm{R_{1a}}$ and $\mathrm{R_{2b}}$ are two lengthy polynomials in A_{01} and A_{21}, with 470 and 9867 terms, respectively.

Further, it can be verified that the roots of the polynomial $\mathrm{R_{2a}}$ satisfy $L_{6b} = L_{7b} = 0$ but $L_{8b} \neq 0$. The procedure is as follows: since $\mathrm{R_{2a}}$ is a homogeneous polynomial in A_{01} and A_{21}, we first solve $\mathrm{R_{2a}}$ for A_{21} in terms of A_{01}, given in the form of $A_{21} = CA_{01}$, and then substitute each of roots into equation $\mathrm{R_{1a}} = 0$ to obtain solutions for A_{01}, and finally use these solutions to verify if they satisfy $L_{6b} = L_{7b} = L_{8b} = 0$. Therefore, the roots of the polynomial $\mathrm{R_{2a}}$ do not yield solutions for 9 limit cycles. So the only possible solutions for getting maximal number of limit cycles are from the equations $\mathrm{R_{1a}} = \mathrm{R_{2b}} = 0$. Eliminating A_{21} from these two equations we obtain the resultant

$$\mathrm{R_{12}} = C_1 \pi^{31} A_{01}^{594} (8753 A_{01}^2 - 388800) \mathrm{R_{12a}}(\mathrm{A}_{01}^2)\, \mathrm{R_{12b}}(\mathrm{A}_{01}^2),$$

where $\mathrm{R_{12a}}$ is a 54th-degree polynomial in A_{01}^2, and $\mathrm{R_{12b}}(\mathrm{A}_{01}^2)$ is an extremely lengthy polynomial. It can be shown that the roots of $\mathrm{R_{12b}}(\mathrm{A}_{01}^2)$ satisfy $L_{6b} = L_{7b} = 0$, but $L_{8b} \neq 0$, and thus these solutions are not what we want. Further, it is easy to check that $A_{01}^2 = \dfrac{388800}{8753}$ is not a solution satisfying $\mathrm{R_{1a}} = \mathrm{R_{2b}} = 0$.

Solving $\mathrm{R_{12a}} = 0$ for A_{01}^2 yields 15 real positive solutions, and then it follows from $\mathrm{R_{1a}} = \mathrm{R_{2a}} = 0$ to get 15 solutions for A_{21}. Since A_{01} can take positive or negative values, we obtain a total of 30 sets of real solutions (A_{01}, A_{21}), which yields 30 corresponding solutions for B_{12}. However, 10 of them yield $b_{01}^2 < 0$ and so only 20 of them are feasible solutions. Further, we use the equations $L_{6b} = L_{7b} = L_{8b} = 0$ to verify that these 20 solutions are indeed feasible solutions, given by

$$\begin{aligned}
a_{02} \neq 0, \quad (A_{01}, A_{21}) &= (\,\pm 248.79821330\cdots,\ \mp 233.57405054\cdots\,),\\
&= (\,\pm 150.92502137\cdots,\ \mp 65.91409014\cdots\,),\\
&= (\,\pm 27.58838890\cdots,\ \pm 1267.58985568\cdots\,),\\
&= (\,\mp 15.21386288\cdots,\ \pm 15.92266649\cdots\,),\\
&= (\,\mp 3.94963193\cdots,\ \pm 16.82017893\cdots\,),\\
&= (\,\pm 3.74421627\cdots,\ \mp 1.55641859\cdots\,),\\
&= (\,\pm 1.74784016\cdots,\ \pm 107.71189435\cdots\,),\\
&= (\,\mp 1.51708021\cdots,\ \pm 11.40541080\cdots\,),\\
&= (\,\pm 0.62554539\cdots,\ \pm 2.27540356\cdots\,),\\
&= (\,\mp 0.42994608\cdots,\ \pm 0.43805436\cdots\,).
\end{aligned}$$

It should be noted that all the computations and solutions obtained in this paper are symbolic, except, in the last step, for solving the roots of polynomial $\mathrm{R_{12a}}$, for which exact solutions have an infinite number of decimal points. The above numerical expression of these solutions are given in a form for a convenient presentation, which represent the exact solutions. One may use the so-called "interval computation" (a built-in package in Maple) to isolate each of the solutions in an interval as small as one wishes. For example, the solution $A_{01} = 0.62554539\cdots$ can be expressed as

$$A_{01} = 0.62554539 \cdots \in \left(\frac{25663}{41025} - 10^{-100}, \ \frac{25663}{41025} + 10^{-100} \right).$$

It is now seen at the critical values under which $b_{02} = 0$ that

$$\omega_{c\pm} = 2|a_{01} + a_{21} \pm 2b_{02}| = 2|a_{01} + a_{21}| = 2|(A_{01} + A_{21})a_{02}|,$$

which satisfies the condition (3.1). For example, we choose the last second solution with $a_{02} = -1$ for which the original parameters take the following critical values (besides $a_{00} = a_{10} = 0$):

$$\begin{aligned} & b_{00} = a_{11} = 0, \ \ a_{02} = -1, && b_{10} = 5.80189793 \cdots, \\ & \delta = 0, \quad b_{02} = 0, && b_{03} = 0.65924104 \cdots, \\ & a_{12} = 0.25224850 \cdots, && a_{03} = -0.97925833 \cdots, \\ & b_{01} = 6.89286156 \cdots, && a_{01} = 0.62554539 \cdots, \\ & a_{21} = 2.27540356 \cdots, && b_{12} = 3.87732206 \cdots, \end{aligned}$$

under which

$$L_i = 0, \ i = 0, 1, \cdots, 8, \quad L_9 = -0.05995729 \cdots, \quad \omega_{c\pm} = 5.80189792 \cdots.$$

Moreover, a direct computation shows that

$$\det \left[\frac{\partial(L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8)}{\partial(b_{02}, b_{03}, a_{12}, a_{03}, b_{01}, a_{01}, a_{21}, b_{12})} \right] = -0.06223110 \cdots \neq 0,$$

implying, by Lemma 2.3, that 8 small-amplitude limit cycles can bifurcate from the origin of system (3.2). Further, a linear perturbation on δ for L_0 yields one more small-amplitude limit cycle, and thus system (1.9) has $9 \times 2 = 18$ small-amplitude limit cycles around the two symmetric fine foci $(\pm 1, 0)$.

It should be noted that although $\omega_{c+} = \omega_{c-}$ at the critical values, $\omega_+ \neq \omega_-$ since $b_{02} \neq 0$ under perturbation.

This finishes the proof for Theorem 3.1. □

4. Conditions for $(\pm 1, 0)$ of system (1.9) to be bi-centers

Having proved the existence of 18 limit cycles in the previous section, we now turn to consider the conditions under which the two symmetric singular points $(\pm 1, 0)$ become bi-centers. Based on the Lyapunov constants L_k, we obtain the following theorem.

Theorem 4.1. *When one of the following four conditions is satisfied, the symmetric singular points* $(\pm 1, 0)$ *of system* (1.9) *are bi-center:*

$$\begin{aligned} &(1) \quad \delta = b_{02} = b_{03} = b_{01} = a_{12} = 0, \quad (a_{01} + a_{21} \neq 0), \\ &(2) \quad \delta = b_{02} = 3b_{03} + a_{12} = b_{01} = a_{21} + b_{12} = 0, \quad (a_{01} + a_{21} \neq 0, \ a_{12} \neq 0), \\ &(3) \quad \delta = b_{02} = a_{01} = a_{02} = a_{03} = b_{01} - 2a_{12} = b_{03}a_{21} - a_{12}b_{12} = 0, \quad (a_{21} \neq 0) \\ &(4) \quad \delta = b_{02} = a_{01} = a_{02} = a_{03} = b_{01} = b_{03} = a_{21} + 2b_{12} = 0, \quad (a_{21} \neq 0). \end{aligned} \tag{4.1}$$

Proof. To prove that the four conditions are necessary for $(\pm 1, 0)$ of system (3.9) to be bi-centers, it is suffice to show that each condition satisfies $L_k = 0,\ k = 1, 2, 3, \ldots$. First note that the condition given in (3.1):

$$(a_{01} + a_{21} - 2b_{02})(a_{01} + a_{21} + 2b_{02}) > 0,$$

are needed to guarantee that the two singular points $(\pm 1, 0)$ are elementary centers, and that the trajectories in the upper half plane and the lower half plane move in the same direction of rotation. To find the four conditions, we consider the first Lyapunov constant L_1, which is given in (3.4),

$$L_1 = -\frac{8\,b_{02}(a_{01} - a_{21} - 2b_{12})}{3(a_{01} + a_{21} - 2b_{02})(a_{01} + a_{21} + 2b_{02})},$$

and the second Lyapnov constant L_2, given by

$$\begin{aligned} L_2 = {} & \frac{1}{36(a_{01} + a_{21} - 2b_{02})^2(a_{01} + a_{21} + 2b_{02})^2} \\ & \times \big\{ -64b_{02}(a_{01} - a_{21} - 2b_{12})\big[(a_{01} - a_{21} - 2b_{12})^2 \\ & \quad - 4(a_{01} - b_{12} - b_{02})(a_{01} - b_{12} + b_{02})\big] \\ & \quad + 9\pi\big[-16b_{02}(a_{21} + b_{12})(a_{01} + a_{21})a_{02} \\ & \qquad + 6(a_{01} + a_{21})(a_{01} + a_{21} + 2b_{02})(a_{01} + a_{21} - 2b_{02})b_{03} \\ & \qquad + 2\big((a_{01} - a_{21} - 2b_{12})(a_{01} + a_{21})^2 - 4b_{02}^2(a_{01} + 3a_{21} + 2b_{12})\big)a_{12} \\ & \qquad + \big(-4b_{02}^2 b_{12} + (a_{01} - a_{21} - 2b_{12})^2(5a_{01} + a_{21} - 5b_{12}) \\ & \qquad - 4(a_{01} - a_{21} - 2b_{12})(a_{01} - b_{12})(3a_{01} - 4b_{12}) \\ & \qquad + 4(a_{01} - b_{12})^2(2a_{01} - 3b_{12})\big)b_{01}\big]\big\}. \end{aligned} \tag{4.2}$$

It is seen from L_1 that $L_1 = 0$ has the solution, either $b_{02} = 0$ $(a_{01} + a_{21} \neq 0)$ or $a_{01} - a_{21} - 2b_{12} = 0$ $(b_{02} \neq 0)$. Actually, the first solution generates four conditions given in (4.1), while the second solution yields the following three conditions:

$$\begin{aligned} &(1)' \quad \delta = a_{01} - b_{12} = a_{21} + b_{12} = b_{01} = a_{12} + 3b_{03} = 0, && (b_{02} \neq 0), \\ &(2)' \quad \delta = a_{01} = a_{21} = b_{12} = a_{02} = a_{03} = 0, && (b_{02} \neq 0), \\ &(3)' \quad \delta = a_{01} = a_{21} = b_{12} = b_{01} + 4b_{03} = a_{12} + 2b_{03} = 0, && (b_{02} \neq 0). \end{aligned}$$

First consider the case $b_{02} = 0$ $(a_{01} + a_{21} \neq 0)$. It is obvious that $L_2 = 0$ if in addition $b_{03} = a_{12} = b_{01} = 0$ under which $L_k = 0,\ k = 2, 3, \ldots, 9$, yielding the condition (1). Now suppose $b_{03} = a_{01} = 0$ and $b_{01} = 0$ $(a_{12} \neq 0)$, then $L_2 = 0$ yields $a_{21} + 2b_{12} = 0$ for which $L_3 = \dfrac{32a_{02}a_{12}}{45b_{12}^2}$ and so $a_{02} = 0$. Then $L_4 = \dfrac{5\pi a_{03}a_{12}}{12b_{12}^2}$ and $L_5 = -\dfrac{35\pi a_{03}a_{12}}{72b_{12}^2}$, yielding $a_{03} = 0$, under which $L_k = 0,\ k = 6, 7, 8, 9$. This gives the condition (4).

If $b_{03}=a_{01}=0$ but $b_{01}\neq 0$, then

$$L_2=\frac{\pi}{4a_{21}^2}\big[(b_{01}-2a_{12})(a_{21}-b_{12})+6(b_{03}a_{21}-a_{12}b_{12}).\big].$$

Letting $b_{01}-2a_{12}=0$ and $b_{03}a_{21}-a_{12}b_{12}=0$, we have (with $b_{12}=\dfrac{b_{03}a_{21}}{a_{12}}$)

$$L_3=\frac{32a_{02}(2b_{03}+3a_{12})}{15a_{21}^2}.$$

Setting $L_3=0$ yields $a_{02}=0$ or $2b_{03}+3a_{12}=0$. If $a_{02}=0$ we have

$$L_4=\frac{\pi a_{03}(2b_{03}+3a_{12})}{a_{21}^2}.$$

Thus setting $a_{03}=0$ leads to $L_k=0,\ k=4,5,\ldots,9$, which is the condition (3). If $2b_{03}+3a_{12}=0$ ($a_{02}\neq 0$), we then obtain $L_4=\dfrac{5\pi a_{02}^2a_{12}}{4a_{21}^2}$ and so setting $L_4=0$ gives $a_{12}=0$, which leads to a subcase of (2).

Now assume $b_{01}=0$. Then

$$\begin{aligned}
L_2&=\frac{\pi}{2(a_{01}+a_{21})^2}\big[(3b_{03}+a_{12})(a_{01}+a_{21})-2a_{12}(a_{21}+b_{12})\big],\\
L_3&=\frac{-1}{180(a_{01}+a_{21})^3}\Big\{(3b_{03}+a_{12})\big[75\pi(3a_{01}-a_{21}-4b_{12})(a_{01}-a_{21}-2b_{12})\\
&\qquad+512a_{02}(a_{01}-3a_{21}-4b_{12})\big]\\
&\qquad+6b_{03}(a_{21}+b_{12})\big[75\pi(3a_{01}-a_{21}-4b_{12})+1024a_{02}\big]\Big\}.
\end{aligned}$$

Hence, setting $3b_{03}+a_{12}=0$ and $a_{21}+b_{12}=0$ we obtain $L_k=0,\ k=2,3,\ldots,9$. This proves the condition (2).

Next, consider the case $a_{01}-a_{21}-2b_{12}=0$ ($b_{02}\neq 0$). First, it is easy to see that $L_2=0$ if $b_{01}=0$ and $a_{21}+b_{12}=0$ under which we have

$$\begin{aligned}
L_3&=-\frac{32a_{02}(a_{12}+3b_{03})}{45b_{02}^2},\\
L_4&=-\frac{(a_{12}+3b_{03})\big\{25\pi\big[28a_{02}a_{12}+9b_{02}(2a_{03}+b_{12})\big]-1024a_{02}b_{02}\big\}}{1440b_{02}^3},
\end{aligned}$$

which indicates that $a_{12}+3b_{03}=0$ should be chosen for $L_3=L_4=0$, and then we indeed have all $L_k=0,\ k=3,4,\ldots,9$. This leads to condition $(1)'$.

Note that in the condition $(1)'$, $b_{01}=0$ but a_{01} is free. Now suppose $a_{01}=0$ and b_{01} is free. Thus, $a_{21}+2b_{12}=0$ under which we obtain

$$L_2=\frac{-\pi b_{12}}{16(b_{02}^2-b_{12}^2)^2}\big\{8b_{02}\big[a_{02}b_{12}-b_{02}(a_{12}+2b_{03})\big]+(b_{02}^2+3b_{12}^2)(b_{01}+4b_{03})\big\}.$$

Choosing $b_{12}=0$, and so $a_{21}=0$, we further have

$$L_3=\frac{-8a_{02}}{45b_{02}^2}\left[4(a_{12}+2b_{03})+(b_{01}+4b_{03})\right],$$
$$L_4=\frac{-1}{11520b_{02}^3}\left\{(b_{01}+4b_{03})\left[3a_{02}(1400\pi a_{12}-75\pi b_{01}-2048b_{02})+2700\pi a_{03}b_{02}\right]\right.$$
$$\left.+2(2a_{12}-b_{01})\left[a_{02}(1400\pi a_{12}-175\pi b_{01}-2048b_{02})+900\pi a_{03}b_{02}\right]\right\}.$$

It is easy to see from L_3 and L_4 that there exist two solutions such that $L_3=L_4=0$: one is $a_{02}=a_{03}=0$, and the other is $a_{12}+2b_{03}=b_{01}+4b_{03}=0$, which is equivalent to $2a_{12}-b_{01}=b_{01}+4b_{03}=0$. Further it can be shown that under these two solutions, $L_k=0,\ k=5,6,\ldots,9$. This gives the conditions $(2)'$ and $(3)'$.

However, it is easy to see from the conditions $(1)'$, $(2)'$ and $(3)'$ that

$$(a_{01}+a_{21}+2b_{02})(a_{01}+a_{21}-2b_{02})=-4b_{02}^2<0,$$

violating the condition given in (3.1), implying that the trajectories in the upper half plane and the lower half plane of system (1.9) move in the opposite direction of rotation. Thus, the three conditions $(1)'$, $(2)'$ and $(3)'$ are not conditions for $(\pm1,0)$ of system (1.9) to be bi-center. Note that we compute the Lyapunov constants under the condition (3.1). Now if the condition is not satisfied, the Lyapunov constants are actually not equal to zero.

Finally, we prove that the four conditions given in (4.1) are also sufficient. First note that besides the four conditions, the following conditions hold for all cases:

$$a_{00}=a_{10}=b_{00}-2a_{11}=b_{10}-2(a_{01}+a_{21})=a_{11}+2b_{02}=\delta=0. \tag{4.3}$$

First, consider the condition (1) under which system (1.9) becomes

$$\begin{pmatrix}\dfrac{dx}{dt}\\ \dfrac{dy}{dt}\end{pmatrix}=\begin{cases}\begin{pmatrix}a_{01}y+a_{02}y^2+a_{21}x^2y+a_{03}y^3\\ 2(a_{01}+a_{21})x-2(a_{01}+a_{21})x^3+b_{12}xy^2\end{pmatrix}, & \text{for } y>0,\\ \begin{pmatrix}a_{01}y-a_{02}y^2+a_{21}x^2y+a_{03}y^3\\ 2(a_{01}+a_{21})x-2(a_{01}+a_{21})x^3+b_{12}xy^2\end{pmatrix}, & \text{for } y<0.\end{cases} \tag{4.4}$$

It is easy to see that system (4.4) is symmetric with respect to the line $y=0$ (the x-axis). Introducing the transformation $x=1+X,\ y=Y$ into (4.4), we have a new system which is still symmetric with respect to the line $Y=0$ (the X-axis). Thus, by Lemma 2.2, the origin $(X,Y)=(0,0)$ of the new system is a center. This shows that the condition (1) is sufficient for $(\pm1,0)$ of system (1.9) to be bi-center.

Next, consider the condition (2) under which system (1.9) can be rewritten as

$$\begin{pmatrix}\dfrac{dx}{dt}\\ \dfrac{dy}{dt}\end{pmatrix}=\begin{cases}\begin{pmatrix}a_{01}y+a_{02}y^2-b_{12}x^2y+a_{12}xy^2+a_{03}y^3\\ 2(a_{01}-b_{12})x+b_{12}xy^2-2(a_{01}-b_{12})x^3-\dfrac{a_{12}}{3}y^3\end{pmatrix}, & \text{for } y>0,\\ \begin{pmatrix}a_{01}y-a_{02}y^2-b_{12}x^2y+a_{12}xy^2+a_{03}y^3\\ 2(a_{01}-b_{12})x+b_{12}xy^2-2(a_{01}-b_{12})x^3-\dfrac{a_{12}}{3}y^3\end{pmatrix}, & \text{for } y<0.\end{cases} \tag{4.5}$$

It can be shown that both upper and lower systems are Hamiltonian systems, with the first integrals given by

$$H_2^{\pm}(x,y) = -(a_{01}-b_{12})x^2 + \frac{1}{2}a_{01}y^2 \pm \frac{1}{3}a_{02}y^3 + \frac{1}{2}(a_{01}-b_{12})x^4 - \frac{1}{2}b_{12}x^2y^2 + \frac{1}{3}a_{12}xy^3 + \frac{1}{4}a_{03}y^4.$$

Similarly by using the transformation $x = 1 + X$, $y = Y$ we obtain

$$\begin{aligned}\tilde{H}_2^{\pm}(X,Y) = & -\frac{1}{2}(a_{01}-b_{12}) + 2(a_{01}-b_{12}X^2 + \frac{1}{2}(a_{01}-b_{12})Y^2 \\ & +2(a_{01}-b_{12}X^3 - b_{12}XY^2 + \frac{1}{3}(a_{12} \pm a_{02})Y^3 \\ & +\frac{1}{2}(a_{01}-b_{12}X^4 - \frac{1}{2}b_{12}X^2Y^2 + \frac{1}{3}a_{12}XY^3 + \frac{1}{4}a_{03}Y^4,\end{aligned}$$

which satisfies $\tilde{H}_2^{+}(X,0) \equiv \tilde{H}_2^{-}(X,0)$. Thus, by Lemma 2.1, we know that the origin of the transformed system of (4.5) (under $x = 1 + X$, $y = Y$) is a center and so the condition (2) is sufficient for $(\pm 1, 0)$ of system (1.9) to be bi-center.

Now we consider the conditions (3) and (4), for which system (1.9) becomes smooth. Under the condition (3), system (1.9) is described by the following equations:

$$\begin{aligned}\frac{dx}{dt} &= a_{21}x^2y + a_{12}xy^2, \\ \frac{dy}{dt} &= 2a_{21}x + 2a_{12}y - 2a_{21}x^3 - 2a_{12}x^2y + b_{12}xy^2 + \frac{a_{12}b_{12}}{a_{21}}y^3,\end{aligned} \tag{4.6}$$

which has elementary centers at $(\pm 1, 0)$. The system is integrable with the first integral, given by

$$H_3 = \begin{cases} x^{-\frac{2b_{12}}{a_{21}}}\left(\dfrac{a_{21}}{b_{12}} + \dfrac{a_{21}}{a_{21}-b_{12}}x^2 + \dfrac{1}{2}y^2\right), & \text{for } a_{21}b_{12} \neq 0,\ b_{12} \neq a_{21}, \\ x^2 + \dfrac{1}{2}y^2 - \ln x^2, & \text{for } a_{21} \neq 0,\ b_{12} = 0, \\ \dfrac{1}{x^2} + \dfrac{y^2}{2x^2} + \ln x^2, & \text{for } b_{12} = a_{21} \neq 0, \end{cases}$$

with the integrating factors:

$$\begin{aligned}&\frac{1}{x^{1+\frac{2b_{12}}{a_{21}}}(a_{21}x + a_{12}y)}, && \text{for } a_{21} \neq 0,\ b_{12} \neq a_{21}, \\ &\frac{1}{x^3(a_{12}x + a_{21}y)}, && \text{for } b_{12} = a_{21} \neq 0.\end{aligned}$$

So $(\pm 1, 0$ are centers of system (4.6), implying that the two singular points $(\pm 1, 0)$ of system (1.9) are centers under the condition (3).

With the condition (4), system (1.9) becomes

$$\begin{aligned} \frac{dx}{dt} &= -2b_{12}x^2y + a_{12}xy^2, \\ \frac{dy}{dt} &= -4b_{12}x + 4b_{12}x^3 + b_{12}xy^2, \end{aligned} \tag{4.7}$$

which has two elementary centers at $(\pm 1, 0)$. It can be shown that with the integrating factor $\mu = \frac{1}{x}$, one can find the first integral of system (4.7), given by

$$H_4 = \frac{1}{3}\big(12b_{12}x - 4b_{12}x^3 - 3b_{12}xy^2 + a_{12}y^3\big). \tag{4.8}$$

This shows that the two elementary centers $(\pm 1, 0)$ of (4.7) are indeed centers, indicating that the condition (4) is sufficient for the two singular points $(\pm 1, 0)$ of system (1.9) to be bi-center.

This completes the proof of Theorem 4.1. □

Remark 4.1. The two bi-center conditions given in (4.1) are obtained under the restrictions listed in (1.7) and (1.8). These restrictions are made for simplifying the Lyapunov constant computation in order to find 18 limit cycles. If these restrictions are removed, the Lyapunov constant computation becomes much more difficult and 18 limit cycles cannot be obtained, but more center conditions might be found. But in this general case, the linear transformation yielding the Jordan canonical form will introduce coordinates rotations. In order to keep the upper half vector field and the lower half vector field to rotate in the same direction with a same angle, the following conditions must be satisfied:

$$\begin{aligned} & b_{11}(a_{01} + a_{21}) + 2\,b_{02}(b_{01} + b_{21}) = 0, \\ & \big[(b_{01} + b_{21} - b_{11})^2 + 2(b_{00} + b_{10})(a_{01} + a_{21} - a_{11})\big] \\ \times & \big[(b_{01} + b_{21} + b_{11})^2 - 2(b_{00} + b_{10})(a_{01} + a_{21} + a_{11})\big] > 0. \end{aligned} \tag{4.9}$$

However, it can be shown that for the general case, the Lyapunov computation becomes extremely involved and even L_5 is hard to be obtained, which makes it almost impossible to determine more center conditions.

5. Conclusion

In this paper, we have shown that cubic planar switching polynomial systems can have at least 18 limit cycles around two symmetric singular points of focus type. This is a new lower bound on the number of limit cycles bifurcating in such systems. In addition, we have identified four necessary and sufficient conditions for the two symmetric singular points to be centers.

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