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Ten limit cycles around a center-type singular point in a 3-d quadratic system with quadratic perturbation[☆]

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ABSTRACT

In this paper, we show that perturbing a simple 3-d quadratic system with a center-type singular point can yield at least 10 small-amplitude limit cycles around a singular point. This result improves the 7 limit cycles obtained recently in a simple 3-d quadratic system around a Hopf singular point. Compared with Bautin's result for quadratic planar vector fields, which can only have 3 small-amplitude limit cycles around an elementary center or focus, this result of 10 limit cycles is surprisingly high. The theory and methodology developed in this paper can be used to consider bifurcation of limit cycles in higher-dimensional systems.

1. Introduction

The phenomenon of limit cycles can arise in almost all areas of science and engineering, from physics, biology, ecology, economy, mechanics, electronics, and even from financial and social systems. Limit cycle theory has been playing a very important role in the study of dynamical behavior of nonlinear systems. The study of limit cycles was initiated by Poincaré [1], and its further development was perhaps motivated by the well-known Hilbert's 16th problem [2], which was posed by Hilbert in 1900. The second part of this problem is to find the upper bound, called the Hilbert number $H(n)$, on the number of limit cycles that planar polynomial systems of degree n can have. Many results have been obtained in study of the Hilbert number; see for example [3–7]. However, the finiteness problem remains unsolved even for quadratic polynomial systems, that is, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the lower bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [8,9].

When we consider the limit cycles bifurcating from isolated fixed points, Hilbert's 16th problem becomes studying degenerate Hopf bifurcations. This local problem has been completely solved only for generic quadratic systems [10], showing existence of 3 limit cycles in the vicinity of a singular point which may be a focus or center. For cubic systems, regarding the case of focus, the best result obtained so far is 9 limit cycles; see for example [11] and references therein. In the case

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of center, on the other hand, it has been shown that more limit cycles can bifurcate from a singular point. The best result obtained recently is 12 limit cycles; see more details in [12] where a literate review is given.

Higher-dimensional vector fields may not only more likely exhibit limit cycles, but also may co-exist complex dynamical behaviors such as chaos. But not many results have been obtained on limit cycles for higher-dimensional vector fields since the analysis for such systems is much more complex than planar systems. A very natural extension from the study of limit cycles in 2-d systems is to study 3-d systems with a Hopf singular point. A similar problem is to investigate the maximal number of limit cycles which may exist in the vicinity of such a singular point under proper perturbations. This is a very challenging problem. There are very few results in the literature. However, unlike 2-d systems, an interesting finding has revealed that even a quadratic 3-d system can exhibit infinitely many small limit cycles, which appear on an infinite family of algebraic invariant surfaces [13,14]. In this paper, we want to ask: What is an upper bound for the cyclicity of a singular point of a quadratic 3-d system restricted to a single center manifold? Although we cannot answer this open problem, we will try to provide a better lower bound in this paper and hope that this will help promote research in this direction.

Restricted to a single center manifold, a well-known 3-d competitive Lotka–Volterra model has been studied extensively over the last twenty years. The model is described by a 3-d differential system:

$$\dot{x}_i = x_i\Big(b_i - \sum_{j=1}^{3} a_{ij}x_j\Big), \quad i = 1, 2, 3, \tag{1}$$

where the dot indicates differentiation with respect to time, t, x_i represents the population of the ith species, and the coefficients take positive real values, $b_i > 0, a_{ij} > 0, i, j = 1, 2, 3$. This is a special case of general 3-d quadratic systems. This system has 8 equilibria and only one of them is a positive equilibrium (i.e., all components of the equilibrium take positive values). An interesting problem is to consider the limit cycles around the positive equilibrium. In the past two decades, several researchers have paid attention to this model and particularly studied bifurcation of limit cycles around the positive equilibrium (e.g., see [15–19]). So far, the best result is 4 limit cycles, obtained by Gyllenberg & Yan [19], using appropriate parameter values. These 4 limit cycles include 3 small-amplitude limit cycles, proved by using focus value computation, and one large limit cycle, shown by constructing a heteroclinic loop. Recently, Tian and Yu revisited this problem [20] and showed that this system, which may not be a competitive system, might have maximal 8 limit cycles, but it is very difficult to prove this using the existing methodology.

For general 3-d quadratic systems with a Hopf singular point, the best result obtained so far is 7 small-amplitude limit cycles bifurcating from the origin of a simple 3-d system [21]. In this paper, we shall consider a quadratic system similar to the one in [21] but with a center-type critical point at the origin. Then, by adding quadratic polynomial perturbations in the form of $\varepsilon\, p_i(x_1, x_2, x_3)$ to each equation of the system, we obtain 10 small-amplitude limit cycles around the origin. In the next section, we formulate the 3-d quadratic system, and then in Section 3 we prove existence of 10 limit cycles.

2. Formulation of 3-d system

In this section, we present a 3-d quadratic system which may yield 10 limit cycles. To achieve this, we start from the following generic 3-d quadratic polynomial system:

$$\begin{aligned}
\dot{x}_1 &= \alpha\, x_1 + x_2 + a_{11}\, x_1^2 + (2b_{11} + a_{12})x_1x_2 + a_{22}\, x_2^2 + a_{33}\, x_3^2 + a_{13}\, x_1x_3 + a_{23}\, x_2x_3,\\
\dot{x}_2 &= -x_1 + \alpha\, x_2 + b_{11}\, x_1^2 + (2a_{11} + b_{12})x_1x_2 + b_{22}\, x_2^2 + b_{33}\, x_3^2 + b_{13}\, x_1x_3 + b_{23}\, x_2x_3,\\
\dot{x}_3 &= -\,\beta\, x_3 + c_{11}\, x_1^2 + c_{12}\, x_1x_2 + c_{22}\, x_2^2 + c_{33}\, x_3^2 + c_{13}\, x_1x_3 + c_{23}\, x_2x_3,
\end{aligned} \tag{2}$$

where $\alpha, \beta > 0$, a_{ij}, b_{ij} and c_{ij} are parameters. Note that the first two equations have taken Bautin's system format, which can be obtained by a rotating transformation. System (2) has a Hopf singularity at the origin when $\alpha = 0$. To simplify the generic system (2), first note that without loss of generality, suppose $b_{11} \neq 0,\ c_{12} \neq 0$, and then we can introduce proper scaling into system (2) so that $b_{11} = c_{12} = 1$. Then, we want that the focus values of the resulting system around the origin are all zero, i.e. the origin becomes a center-type point. There are many possibilities, but a complete classification is difficult. Here, we set $\alpha = a_{33} = a_{13} = a_{23} = b_{33} = b_{13} = b_{23} = c_{11} = c_{22} = c_{13} = c_{23} = 0$, and $\beta = c_{33} = 1$. According to Bautin's theory, setting $a_{22} = -a_{11}$ in the resulting system yields all focus values to vanish, i.e. the origin becomes a center restricted to a center manifold under this condition. Finally, we add quadratic perturbations to system (2) under above conditions to obtain the following perturbed system:

$$\begin{aligned}
\dot{x}_1 \ &= \ x_2 + a_{11}\, x_1^2 + (2 + a_{12})x_1x_2 - a_{11}\, x_2^2 + \varepsilon(a_{100}\, x_1 + a_{020}\, x_2\\
&\qquad + a_{110}\, x_1^2 + a_{120}\, x_1x_2 + a_{220}\, x_2^2 + a_{330}\, x_3^2 + a_{130}\, x_1x_3 + a_{230}\, x_2x_3),\\
\dot{x}_2 \ &= \ -x_1 + x_1^2 + 2\, a_{11}\, x_1x_2 - x_2^2 + \varepsilon(b_{100}\, x_1 + b_{020}\, x_2 + b_{110}\, x_1^2\\
&\qquad + b_{120}\, x_1x_2 + b_{220}\, x_2^2 + b_{330}\, x_3^2 + b_{130}\, x_1x_3 + b_{230}\, x_2x_3),\\
\dot{x}_3 \ &= \ -x_3 + x_1x_2 + x_3^2 + \varepsilon(c_{100}\, x_1 + c_{020}\, x_2 + c_{110}\, x_1^2 + c_{120}\, x_1x_2\\
&\qquad + c_{220}\, x_2^2 + c_{330}\, x_3^2 + c_{130}\, x_1x_3 + c_{230}\, x_2x_3),
\end{aligned} \tag{3}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter.

Recently, a sub-family of system (3) has been studied for bifurcation of small limit cycles and a method different from that of this paper is given to show that restricted to a single center manifold, at most 3 small limit cycles can bifurcate from the origin [22].

3. Main result

Now, based on the perturbed system (3), we prove the following theorem.

Theorem 1. *The perturbed system (3) can have at least 10 small-amplitude limit cycles bifurcating from the origin by properly choosing the perturbation coefficients, a_{ij0}, b_{ij0} and c_{ij0}, as well as the system coefficients a_{11} and a_{12}.*

In order to prove Theorem 1, we need the following lemma which is based on Theorem 1 in [23] and Theorem 1 in [24].

Lemma 1. *Suppose the ε-order focus values, obtained from the general n –d dynamical system, $\dot{\mathbf{x}} = \mathbf{f_1}(\mathbf{x}, \mathbf{p_{k_1}}) + \varepsilon\, \mathbf{f_2}(\mathbf{x}, \mathbf{p_{k_2}})$ (which is an integrable system when $\varepsilon = 0$) associated with a Hopf bifurcation, are functions of $k = k_1 + k_2$ parameters $p_1, p_2, \ldots, p_{k_1}, p_{k_1+1}, \ldots, p_k$, in which $p_1, p_2, \ldots, p_{k_1}$ are system parameters, while $p_{k_1+1}, p_{k_1+2}, p_k$ are perturbation parameters. Further, assume that at a critical point, p_c defined by $(p_1, p_2, \ldots, p_k) = (p_{1c}, p_{2c}, \ldots, p_{kc})$, the focus values satisfy $v_j(p_c) = 0$, $j = 0, 1, \ldots, k-1$, $v_k(p_c) \neq 0$, and $\det\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(p_1, p_2, \ldots, p_k)}\right]_{p_c} \neq 0$. Then, proper perturbations can be made to the parameters $p_1, p_2, \ldots, p_k$ around the critical point p_c to generate k small-amplitude limit cycles in the vicinity of the Hopf singular point.*

Proof. The approach to prove Theorem 1 is to compute the focus values and show that they satisfy the conditions in Lemma 1. First note that when $\varepsilon = 0$, the origin of system (3) is a center restricted to a center manifold. When $\varepsilon \neq 0$, the origin becomes a Hopf singular point. Therefore, we can use the method of normal forms to find the ε-order focus values v_i, $i = 0, 1, 2, \ldots$. Since $v_0 = \frac{1}{2}(a_{100} + b_{020})$, in order to compute higher-order focus values, we let $v_0 = 0$, and for convenience we further let the third eigenvalue of the perturbed system be unchanged, i.e. make it equal 1 after perturbation. Thus, $b_{020} = -a_{100}$, $c_{030} = \frac{1}{2}(b_{100} - a_{020})$, under which the linearized system of (3) has eigenvalues $\pm i$ and -1.

Now, we apply the Maple program developed in [20] for computing the normal form of general n-d systems associated with the Hopf bifurcation to find the focus values v_i, $i = 1, 2, \ldots$. It is shown from the computations that the following perturbation parameters can be set zero: $a_{110} = a_{100} = a_{020} = a_{120} = b_{100} = b_{110} = b_{120} = c_{110} = c_{120} = c_{220} = c_{330} = c_{130} = c_{230} = 0$. Now system (3) only contains 8 perturbation parameters and two system parameters. Further, we notice that one perturbation parameter, say, $b_{230} \neq 0$ can be treated as a free parameter. Hence, the focus values of system (3) are obtained as follows:

$$v_1 = \frac{1}{10}(20a_{11}b_{220} - 5a_{12}a_{220} + a_{230} + b_{130} + 2a_{130} - 2b_{230}), \qquad v_2 = \cdots.$$

Then, solve $v_1 = v_2 = \cdots = v_7 = 0$ for the seven parameters, a_{220}, b_{220}, a_{330}, b_{330}, a_{130}, b_{130} and a_{230} to obtain the solutions, given in the form of $a_{ij0} = A_{ij0}(a_{11}, a_{12})\, b_{230}$ and $b_{ij0} = B_{ij0}(a_{11}, a_{12})\, b_{230}$, where A_{ij0} and B_{ij0} are polynomial functions in a_{11} and a_{12}. Then, the focus values v_8, v_9 and v_{10} become

$$v_8 = -\frac{b_{230}}{131281402809375000\, F_0(a_{11}, a_{12})}\, F_8(a_{11}, a_{12}),$$
$$v_9 = -\frac{b_{230}}{200039127094256673234375000\, F_0(a_{11}, a_{12})}\, F_9(a_{11}, a_{12}),$$
$$v_{10} = -\frac{b_{230}}{1636210592803614048093122938813125000000000\, F_0(a_{11}, a_{12})}\, F_{10}(a_{11}, a_{12}),$$

where F_0, F_8, F_9 and F_{10} are all polynomial functions in the two system parameters, a_{11} and a_{12}, with respect to a_{11} in 37th, 49th, 51th and 53th degree, respectively. Therefore, the best choice for obtaining maximal number of limit cycles is to find the solutions of a_{11} and a_{12} such that $F_8 = F_9 = 0$, but $F_0\, F_{10} \neq 0$, which results in at most 10 small-amplitude limit cycles.

To find the solutions of $F_8 = F_9 = 0$, we use the Maple built-in command *resultant* to eliminate a_{12}, yielding

$$F_{89} = \text{resultant}(F_8, F_9, a_{12}) = C_{89}(3a_{11} + 1)\, F_1(a_{11})\, F_{89a}(a_{11}),$$
$$F_{810} = \text{resultant}(F_8, F_{10}, a_{12}) = C_{810}\, (3a_{11} + 1)\, F_1(a_{11})\, F_{810a}(a_{11}),$$

where C_{89} and C_{810} are nonzero integers, and F_1, F_{89a} and F_{810a} are respectively 825th, 1762th and 1862th degree polynomials in a_{11}. Further, we can use the same command to find

$$F_{8910} = \text{resultant}(F_{89a}, F_{810a}, a_{11}) = -0.7644995585 \cdots \times 10^{14824452} \neq 0,$$

which indicates that if $F_{89a} = 0$ has solutions, then these solutions guarantee $F_{810a} \neq 0$.

Finally, we need to solve $F_{89a} = 0$ to find the solutions of a_{11}. It has been shown that this 1762th-degree polynomial equation has 104 real solutions, which in turn yield corresponding 104 solutions for a_{12} (the solution a_{12} is obtained from the computation of F_{89}). However, by checking the original equations $F_8 = F_9 = 0$, we found that only 87 sets of them satisfy the original functions. We take one set of the solutions:

$$a_{11} = -0.1668823226\cdots \qquad a_{12} = 0.5173132565\cdots.$$

In addition, we choose $b_{230} = 0.0000001$. Then, the other seven perturbation parameters are equal to

$$\begin{aligned} a_{220} &= -0.4254505775\cdots \times 10^{-6}, \qquad b_{220} = -0.2249632036\cdots \times 10^{-8},\\ a_{330} &= 0.2794082093\cdots \times 10^{-6}, \qquad b_{330} = 0.1151307669\cdots \times 10^{-7},\\ a_{130} &= -0.1783412978\cdots \times 10^{-6}, \qquad b_{130} = -0.1855085690\cdots \times 10^{-6},\\ a_{230} &= -0.3657734304\cdots \times 10^{-6}. \end{aligned}$$

The above critical values can be used to define a critical point, called p_c, for which the ε-order focus values become

$$v_i = 0, \qquad i = 1, 2, \ldots, 9, \qquad v_{10} = -0.5978563575\cdots \times 10^{-5} \neq 0.$$

Moreover, a direct calculation shows that the Jacobin evaluated at the critical point p_c is given by

$$\det\left[\frac{\partial(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)}{\partial(a_{220}, b_{220}, a_{330}, b_{330}, a_{130}, b_{130}, a_{230}, a_{11}, a_{12})}\right]_{p_c} = 0.8938528113\cdots \times 10^{-6} \neq 0,$$

implying that system (3) can have 10 small-amplitude limit cycles bifurcating from the center-type singular point (the origin).

The proof of Theorem 1 is complete. □

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