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Bifurcation of limit cycles in 3rd-order Z_2 Hamiltonian planar vector fields with 3rd-order perturbations ☆

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ABSTRACT

In this paper, we show that a Z_2-equivariant 3rd-order Hamiltonian planar vector fields with 3rd-order symmetric perturbations can have at least 10 limit cycles. The method combines the general perturbation to the vector field and the perturbation to the Hamiltonian function. The Melnikov function is evaluated near the center of vector field, as well as near homoclinic and heteroclinic orbits.

1. Introduction

Limit cycle theory has been extensively studied for many years and many methodologies have been developed. In particular, a well-known problem related to limit cycles is Hilbert's 16th problem, which has not been solved since Hilbert proposed the 23 mathematical problems in 1990 [5]. For a review of the problem, the reader is refereed to, for example, the survey article [7]. To more precisely describe Hilbert's 16th problem, consider the following planar system:

$$\frac{dx}{dt} = P_n(x,y), \quad \frac{dy}{dt} = Q_n(x,y), \tag{1}$$

where $P_n(x,y)$, and $Q_n(x,y)$, represent nth-degree polynomials of x, and y. The second part of Hilbert's 16th problem is to find the upper bound on the number of limit cycles that the system can have, called Hilbert number $H(n)$, which only depends on the polynomial degree n.

In general, determining $H(n)$, is a very difficult problem. In the past several decades, researchers have paid attention to finding the lower bound of the limit cycles that system (1) with particular n and hope to gain information for finding $H(n)$. If the problem is restricted to the neighborhood of isolated singular points, then the question is reduced to studying degenerate Hopf bifurcations, which gives rise to fine focus points, and many results have been obtained (e.g., see [1,6,8,12]). Moreover, in the last two decades, much progress on finite cyclicity near a fine focus point or a homoclinic loop has been achieved.

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Another important and interesting problem, called weak Hilbert's 16th problem, is to find the maximal number of isolated zeros of the Abelian integral or Melnikov function

$$M(h,\delta) = \int_{H(x,y)=h} Q_n dx - P_n dy, \tag{2}$$

where $H(x,y), P_n$, and Q_n, are all real polynomials of x, and y, with $\deg H = n+1$, and $\max\{\deg P_n, \deg Q_n\} \leqslant n$. The weak Hilbert's 16th problem is closely related to the maximal number of limit cycles of the following so-called near-Hamiltonian system [2]:

$$\frac{dx}{dt} = H_y(x,y) + \varepsilon p(x,y), \quad \frac{dy}{dt} = -H_x(x,y) + \varepsilon q(x,y), \tag{3}$$

where $H(x,y)$, is a Hamiltonian function with $\deg H = n+1$, and the perturbation functions $p_n(x,y)$, and $q_n(x,y)$, are nth-degree polynomials.

It has been shown that a cubic system (1) with Z_2 symmetry can have at least 12 limit cycles [13,14]. These 12 limit cycles are distributed in the neighborhood of two symmetric fine focus points. Recently, an additional limit cycle has been found at infinity of this Z_2 symmetric vector field [9]. Another example of 13 limit cycles has been obtained by perturbing a cubic Hamiltonian system with five centers [10]. In this paper, we prove that a Z_2-equivariant third-order Hamiltonian planar vector field with symmetric third-order perturbations can have at least 10 limit cycles, which surround two symmetric centers. The Melnikov function method is used to prove the existence of the 10 limit cycles. In particular, in addition to the parameters in the perturbation functions p and q, the parameters involved in the Hamiltonian H are also used in finding the limit cycles.

The rest of the paper is organized as follows. In the next section, we present a recently developed method for computing Melnikov function near a center. In Section 3, a general Hamiltonian function for cubic-order Z_2-equivalent vector fields is derived, which is used in Section 4, together with the calculation of Melnikov function to prove the existence of 10 small limit cycles. Section 5 is devoted to study existence of possible large limit cycles that such a Hamiltonian system may have. Finally, conclusion is drawn in Section 6.

2. General mathematical formulas

In this section, we briefly describe the method of computing the Melnikov function near a center of vector fields. More details can be found in [3]. We study a C^∞, system of the form:

$$\frac{dx}{dt} = H_y(x,y,h_{ij}) + \varepsilon p(x,y,a_{ij}), \quad \frac{dy}{dt} = -H_x(x,y,h_{ij}) + \varepsilon q(x,y,b_{ij}), \tag{4}$$

where $H(x,y,h_{ij})$, $p(x,y,a_{ij})$, and $q(x,y,b_{ij})$ are C^∞ system functions, $\epsilon \geqslant 0$, is small, and $h_{ij}, a_{ij}, b_{ij} \in \mathbf{R}$, are parameters. For $\varepsilon = 0$, (4) becomes

$$\frac{dx}{dt} = H_y, \quad \frac{dy}{dt} = -H_x, \tag{5}$$

which is a Hamiltonian system. Hence, (4) is called a near-Hamiltonian system.

Suppose that (4) has an elementary center at the origin, namely, the Hamiltonian function H, satisfies $H_x(0,0,h_{ij}) = H_y(0,0,h_{ij}) = 0$, and

$$\det \frac{\partial(H_y, -H_x)}{\partial(x,y)}(0,0) > 0.$$

Therefore, without loss of generality, we may assume that

$$H_{yy}(0,0,h_{ij}) = 1, \quad H_{xx}(0,0,h_{ij}) > 0 \ \text{ and } \ H_{xy}(0,0,h_{ij}) = 0.$$

It then follows that in the vicinity of the origin, H can be expanded as

$$H(x,y,h_{ij}) = \frac{1}{2}y^2 + h_{20}x^2 + \sum_{i+j\geqslant 3} h_{ij}x^i y^j \quad (h_{20} > 0). \tag{6}$$

Here, note that the coefficients h_{ij}, are usually fixed real values. If they allow to be perturbed, then the perturbations on the coefficients can be considered the parameter vector λ. For example, $h_{ij} = h_{ij}^* + \lambda_{ij}$, where h_{ij}^*, is a fixed value while λ_{ij}, is a perturbation on h_{ij}. For our purpose, it is required that

$$0 < \varepsilon \ll \|\lambda\| \ll 1. \tag{7}$$

In the following we present a result which is a generalization of [3] and will be used in Section 4 to prove the existence of 10 small limit cycles. Without loss of generality, we may assume that the parameters h_{ij} in the Hamiltonian function H depend on a parameter vector $\xi \in \mathbf{R}^n$, and the perturbation parameters a_{ij}, and b_{ij}, linearly depend on a parameter vector $\zeta \in \mathbf{R}^m$. Then by formulas in [3] we have the expansion of the Melnikov function around the origin in the form of

$$M(h,\zeta,\xi) = h\sum_{j\geqslant 0}\mu_j(\zeta,\xi)h^j. \tag{8}$$

For simplicity, suppose the functions p and q in system (4) are linear in ζ, and thus the coefficients $\mu_j(\zeta,\xi)$, are linear in ζ. Further, suppose there exist an integer $k>0$, and vectors $\zeta_0 \in \mathbf{R}^m$, $\xi_0 \in \mathbf{R}^n$, such that

$$\mu_j(\zeta_0,\xi_0) = 0,\ j=0,1,\ldots,k-1, \quad \text{but} \ \frac{\partial(\mu_0,\mu_1,\ldots,\mu_{k-1})}{\partial(\zeta_1,\zeta_2,\ldots,\zeta_k)}(\xi_0) \neq 0. \tag{9}$$

Then, the linear equations $\mu_j = 0, j=0,1,\ldots,k-1$, of ζ, have a unique solution in the form of

$$(\zeta_1,\zeta_2,\ldots,\zeta_k) = \kappa(\zeta_{k+1},\ldots,\zeta_m,\xi)$$

for ξ, near ξ_0. It is obvious that κ, is linear in $\zeta_{k+1},\ldots,\zeta_m$.

Further, let

$$\mu_{k+j}|_{(\zeta_1,\ldots,\zeta_k)=\kappa(\zeta_{k+1},\ldots,\zeta_m,\xi)} = L_j(\zeta_{k+1},\ldots,\zeta_m)R_j(\xi), \quad j=0,1,\ldots,l. \tag{10}$$

Then, we have the following result [4].

Theorem 1. *Consider the near-Hamiltonian system (4), where $H(x,y,h_{ij})$ satisfies (6) and the coefficients h_{ij}, linearly depend on $\xi \in \mathbf{R}^n$ and the coefficients a_{ij}, in p and b_{ij}, in q are linear in $\zeta \in \mathbf{R}^m$. Suppose there exist integer $k>0$ and $\zeta_0 = (\zeta_{10},\ldots,\zeta_{m0}) \in \mathbf{R}^m, \xi_0 \in \mathbf{R}^n$ such that (9) and (10) hold with*

$$\begin{aligned} &L_j(\zeta_{(k+1)0},\ldots,\zeta_{m0}) \neq 0,\ j=0,\ldots,l, \\ &R_j(\xi_0) = 0,\ j=0,\ldots,l-1,\ R_l(\xi_0)\neq 0, \end{aligned} \tag{11}$$

and

$$\det\frac{\partial(R_0,\ldots,R_{l-1})}{\partial(\xi_1,\ldots,\xi_l)}(\xi_0) \neq 0. \tag{12}$$

Then, for all (ε,ζ,ξ) near $(0,\zeta_0,\xi_0)$, system (4) has at most $k+l$ limit cycles near the origin, and for some (ε,ζ,ξ) near $(0,\zeta_0,\xi_0)$ system (4) can have $k+l$ limit cycles near the origin.

The proof can be found in [4], and thus omitted here for brevity.

3. A 3rd-order Z_2-equivariant Hamiltonian vector field with symmetric 3rd-order perturbation

Now consider the following cubic Hamiltonian system with cubic perturbations:

$$\begin{aligned} \frac{dx}{dt} &= \widetilde{H}_y + \varepsilon\tilde{p}(x,y), \\ \frac{dy}{dt} &= -\widetilde{H}_x + \varepsilon\tilde{q}(x,y), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \widetilde{H}(x,y) &= \frac{1}{2}y^2 - \tilde{h}_{20}x^2 + \tilde{h}_{30}x^3 + \tilde{h}_{21}x^2y + \tilde{h}_{12}xy^2 + \tilde{h}_{03}y^3 + \tilde{h}_{40}x^4 + \tilde{h}_{31}x^3y + \tilde{h}_{22}x^2y^2 + \tilde{h}_{13}xy^3 + \tilde{h}_{04}y^4, \\ \tilde{p}(x,y) &= \tilde{a}_{10}x + \tilde{a}_{01}y + \tilde{a}_{20}x^2 + \tilde{a}_{11}xy + \tilde{a}_{02}y^2 + \tilde{a}_{30}x^3 + \tilde{a}_{21}x^2y + \tilde{a}_{12}xy^2 + \tilde{a}_{03}y^3, \\ \tilde{q}(x,y) &= \tilde{b}_{10}x + \tilde{b}_{01}y + \tilde{b}_{20}x^2 + \tilde{b}_{11}xy + \tilde{b}_{02}y^2 + \tilde{b}_{30}x^3 + \tilde{b}_{21}x^2y + \tilde{b}_{12}xy^2 + \tilde{b}_{03}y^3, \end{aligned} \tag{14}$$

with $\tilde{h}_{20} \neq 0$, and $\varepsilon > 0$, is a small perturbation parameter.

It is obvious that the non-perturbed system

$$\frac{dx}{dt} = \widetilde{H}_y, \quad \frac{dy}{dt} = -\widetilde{H}_x,$$

is a Hamiltonian system, and the origin $(x,y)=(0,0)$, is a saddle point when $\tilde{h}_{20}>0$, but a center when $\tilde{h}_{20}<0$. Suppose system (13) is Z_2 equivariant, then

$$\tilde{h}_{30} = \tilde{h}_{21} = \tilde{h}_{12} = \tilde{h}_{03} = 0, \quad \tilde{a}_{20} = \tilde{a}_{11} = \tilde{a}_{02} = 0 \ \text{and} \ \tilde{b}_{20} = \tilde{b}_{11} = \tilde{b}_{02} = 0. \tag{15}$$

Further, assume that besides the origin, the unperturbed Z_2 equivalent vector filed has two fixed points at $(\pm 1,0)$ (otherwise, using a rotation around the origin to achieve this), which, in turn, requires that

$$\tilde{h}_{40} = \frac{1}{2}\tilde{h}_{20}, \quad \tilde{h}_{31} = 0. \tag{16}$$

Hence, system (13) becomes

$$\begin{aligned} \frac{dx}{dt} &= y + 2\tilde{h}_{22}x^2y + 3\tilde{h}_{13}xy^2 + 4\tilde{h}_{04}y^3 + \varepsilon\left[\tilde{a}_{10}x + \tilde{a}_{01}y + \tilde{a}_{30}x^3 + \tilde{a}_{21}x^2y + \tilde{a}_{12}xy^2 + \tilde{a}_{03}y^3\right] \\ \frac{dy}{dt} &= 2\tilde{h}_{20}x - 2\tilde{h}_{20}x^3 - 2\tilde{h}_{22}xy^2 - \tilde{h}_{13}y^3 + \varepsilon\left[\tilde{b}_{10}x + \tilde{b}_{01}y + \tilde{b}_{30}x^3 + \tilde{b}_{21}x^2y + \tilde{b}_{12}xy^2 + \tilde{b}_{03}y^3\right] \end{aligned} \tag{17}$$

Since $\tilde{h}_{20} \neq 0$, we apply the following scaling

$$y \to \sqrt{2|\tilde{h}_{20}|}\,y, \;\; t \to \frac{t}{\sqrt{2|\tilde{h}_{20}|}}, \;\; \tilde{h}_{13} \to \frac{h_{13}}{\sqrt{2|\tilde{h}_{20}|}}, \;\; \tilde{h}_{04} \to \frac{h_{04}}{2|\tilde{h}_{20}|}, \;\; \text{and} \;\; \tilde{h}_{22} \to h_{22}$$

into Eq. (17) to obtain

$$\begin{aligned} \sqrt{2|\tilde{h}_{20}|}\,\frac{dx}{dt} &= \sqrt{2|\tilde{h}_{20}|}\,y + 2h_{22}x^2\sqrt{2|\tilde{h}_{20}|}\,y + 3\frac{h_{13}}{\sqrt{2|\tilde{h}_{20}|}}x(2|\tilde{h}_{20}|)y^2 + 4\frac{h_{04}}{2|\tilde{h}_{20}|}(2|\tilde{h}_{20}|)\sqrt{2|\tilde{h}_{20}|}\,y^3 \\ &\quad + \varepsilon\left[\tilde{a}_{10}x + \tilde{a}_{01}\sqrt{2|\tilde{h}_{20}|}\,y + \tilde{a}_{30}x^3 + \tilde{a}_{21}x^2\sqrt{2|\tilde{h}_{20}|}\,y + \tilde{a}_{12}x(2|\tilde{h}_{20}|)y^2 + \tilde{a}_{03}(2|\tilde{h}_{20}|)\sqrt{2|\tilde{h}_{20}|}\,y^3\right] \\ 2|\tilde{h}_{20}|\frac{dy}{dt} &= 2\tilde{h}_{20}x - 2\tilde{h}_{20}x^3 - 2h_{22}x(2|\tilde{h}_{20}|)y^2 - \frac{h_{13}}{\sqrt{2|\tilde{h}_{20}|}}(2|\tilde{h}_{20}|)\sqrt{2|\tilde{h}_{20}|}\,y^3 \\ &\quad + \varepsilon\left[\tilde{b}_{10}x + \tilde{b}_{01}\sqrt{2|\tilde{h}_{20}|}\,y + \tilde{b}_{30}x^3 + \tilde{b}_{21}x^2\sqrt{2|\tilde{h}_{20}|}\,y + \tilde{b}_{12}x(2|\tilde{h}_{20}|)y^2 + \tilde{b}_{03}(2|\tilde{h}_{20}|)\sqrt{2|\tilde{h}_{20}|}\,y^3\right] \end{aligned} \tag{18}$$

which, after renaming the coefficients, can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= y + 2h_{22}x^2y + 3h_{13}xy^2 + 4h_{04}y^3 + \varepsilon\left[a_{10}x + a_{01}y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3\right] \\ &\equiv H_y + \varepsilon p(x, y, a_{ij}), \\ \frac{dy}{dt} &= \pm(x - x^3) - 2h_{22}xy^2 - h_{13}y^3 + \varepsilon\left[b_{10}x + b_{01}y + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3\right] \\ &\equiv -H_x + \varepsilon q(x, y, b_{ij}), \end{aligned} \tag{19}$$

where

$$H(x, y, h_{ij}) = \frac{1}{2}y^2 \pm \left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right) + h_{22}x^2y^2 + h_{13}xy^3 + h_{04}y^4. \tag{20}$$

Thus,

$$h_0 = H(0, 0, h_{ij}) = 0, \;\; h_1 = H(\pm 1, 0, h_{ij}) = \pm\frac{1}{4}, \;\; \text{and} \;\; h_2^* = H(x^*, y^*, h_{ij}), \tag{21}$$

where (x^*, y^*), is any of other fixed points of system (19). Here, in (20) and (21), the "+" and "−" signs correspond to the cases of the origin being a center and a saddle point, respectively.

Further, we define the Melnikov function of (19) as

$$M(h, h_{ij}, a_{ij}, b_{ij}) = \oint_{H(x,y,h_{ij})=h} q(x, y, b_{ij})dx - p(x, y, a_{ij})dy. \tag{22}$$

where $p(x, y, a_{ij})$, and $q(x, y, b_{ij})$, are given in (19) and $H(x, y, h_{ij})$, is given in (20).

It is easy to show that the two fixed points $(\pm 1, 0)$, of the unperturbed Hamiltonian system $(19)_{\varepsilon=0}$ are centers when

$$1 + 2h_{22} > 0. \tag{23}$$

It should be noted that the unperturbed system $(19)_{\varepsilon=0}$, with three free coefficients h_{22}, h_{13} and h_{04}, can be considered as a "normal form" of the 3rd-order Hamiltonian system with two centers at $(\pm 1, 0)$. A different "normal form" of such Hamiltonian system is given in [11], where several phase portraits of the system for different values of the free coefficients are presented. These phase portraits given in [11] show, besides the origin (which is either a center or a saddle point), additional four saddle points. In this paper, we will consider the cases which give maximal number of limit cycles around the two centers $(\pm 1, 0)$, with the origin being a saddle point, and then investigate possible existence of large limit cycles.

4. Ten Small limit cycles bifurcating from the centers $(\pm 1, 0)$

We can apply the method of computing the focus values or the method of computing the Melnikov function, developed by Han [2].

First, we shift the system to one of the center, say, $(1, 0)$, by introducing

$$x = 1 + \bar{x}$$

to obtain

$$\begin{aligned}\frac{d\bar{x}}{dt} &= (1+2h_{22})y + 4h_{22}\bar{x}y + 3h_{13}y^2 + 2h_{22}\bar{x}^2y + 3h_{13}\bar{x}y^2 + 4h_{04}y^3 \\ &\quad + \varepsilon\big[(a_{10}+a_{30}) + (a_{10}+3a_{30})\bar{x} + (a_{01}+a_{21})y + 3a_{30}\bar{x}^2 + 2a_{21}\bar{x}y + a_{12}y^2 + a_{30}\bar{x}^3 + a_{21}\bar{x}^2y + a_{12}\bar{x}y^2 + a_{03}y^3\big] \\ \frac{dy}{dt} &= -2\bar{x} - 3\bar{x}^2 - 2h_{22}y^2 - \bar{x}^3 - 2h_{22}\bar{x}y^2 - h_{13}y^3 \\ &\quad + \varepsilon\big[(b_{10}+b_{30}) + (b_{10}+3b_{30})\bar{x} + (b_{01}+b_{21})y + 3b_{30}\bar{x}^2 + 2b_{21}\bar{x}y + b_{12}y^2 + b_{30}\bar{x}^3 + b_{21}\bar{x}^2y + b_{12}\bar{x}y^2 + b_{03}y^3\big].\end{aligned} \tag{24}$$

By a time scaling

$$\tau = \rho t, \quad \text{where } \rho = 1 + 2h_{22} > 0, \tag{25}$$

the above system (24) can be written as

$$\begin{aligned}\frac{d\bar{x}}{d\tau} &= \overline{H}_y + \varepsilon\bar{p}(x,y), \\ \frac{dy}{d\tau} &= -\overline{H}_{\bar{x}} + \varepsilon\bar{q}(x,y),\end{aligned} \tag{26}$$

where

$$\begin{aligned}\overline{H}(\bar{x},y) &= \frac{1}{2}y^2 + \bar{h}_{20}\bar{x}^2 + \bar{h}_{30}\bar{x}^3 + \bar{h}_{12}\bar{x}y^2 + \bar{h}_{03}y^3 + \bar{h}_{40}\bar{x}^4 + \bar{h}_{22}\bar{x}^2y^2 + \bar{h}_{13}\bar{x}y^3 + \bar{h}_{04}y^4, \\ \bar{p}(\bar{x},y) &= \bar{a}_{10}\bar{x} + \bar{a}_{01}y\bar{a}_{20}\bar{x}^2 + \bar{a}_{11}\bar{x}y + \bar{a}_{02}y^2\bar{a}_{30}\bar{x}^3 + \bar{a}_{21}\bar{x}^2y + \bar{a}_{12}\bar{x}y^2 + \bar{a}_{03}y^3, \\ \bar{q}(\bar{x},y) &= \bar{b}_{10}\bar{x} + \bar{b}_{01}y\bar{b}_{20}\bar{x}^2 + \bar{b}_{11}\bar{x}y + \bar{b}_{02}y^2\bar{b}_{30}\bar{x}^3 + \bar{b}_{21}\bar{x}^2y + \bar{b}_{12}\bar{x}y^2 + \bar{b}_{03}y^3,\end{aligned} \tag{27}$$

with

$$\begin{aligned}&\bar{h}_{20} = \bar{h}_{30} = 4\bar{h}_{40} = \frac{1}{\rho},\ \ \frac{1}{2}\bar{h}_{12} = \bar{h}_{22} = \frac{1}{\rho}h_{22},\ \ \bar{h}_{03} = \bar{h}_{13} = \frac{1}{\rho}h_{13},\ \ \bar{h}_{04} = \frac{h_{04}}{\rho}, \\ &\bar{a}_{00} = \frac{a_{10}+a_{30}}{\rho}\ \ \bar{a}_{10} = \frac{a_{10}+3a_{30}}{\rho},\ \ \bar{a}_{01} = \frac{a_{01}+a_{21}}{\rho},\ \ \frac{1}{3}\bar{a}_{20} = \bar{a}_{30} = \frac{a_{30}}{\rho}, \\ &\frac{1}{2}\bar{a}_{11} = \bar{a}_{21} = \frac{a_{21}}{\rho},\ \ \bar{a}_{02} = \bar{a}_{12} = \frac{a_{12}}{\rho},\ \ \bar{a}_{03} = \frac{1}{\rho}a_{03}, \\ &\bar{b}_{00} = \frac{b_{10}+b_{30}}{\rho}\ \ \bar{b}_{10} = \frac{b_{10}+3b_{30}}{\rho},\ \ \bar{b}_{01} = \frac{b_{01}+b_{21}}{\rho},\ \ \frac{1}{3}\bar{b}_{20} = \bar{b}_{30} = \frac{b_{30}}{\rho}, \\ &\frac{1}{2}\bar{b}_{11} = \bar{b}_{21} = \frac{b_{21}}{\rho},\ \ \bar{b}_{02} = \bar{b}_{12} = \frac{b_{12}}{\rho},\ \ \bar{b}_{03} = \frac{1}{\rho}b_{03}.\end{aligned}$$

Now, let $\overline{H}(\bar{x},y) = \bar{h}$ (or $H(x,y) = h - h_1$), then by (22) we have

$$M_1(\bar{h},h_{ij},a_{ij},b_{ij}) = \bar{h}\sum_{k\geqslant 0}\mu_k(h_{ij},a_{ij},b_{ij})\bar{h}^k, \quad \text{for } 0 < \bar{h} \ll 1, \tag{28}$$

where $\bar{h} = h - h_1$, and the coefficient μ_0, can be found as

$$\mu_0 = \frac{\sqrt{2}\pi}{\bar{h}_{20}^{1/2}}(\bar{a}_{10} + \bar{b}_{01}) = \frac{2\pi}{(2\rho)^{1/2}}(3a_{30} + a_{10} + b_{01} + b_{21}). \tag{29}$$

Letting $\mu_0 = 0$, yields

$$a_{30} = -\frac{1}{3}(a_{10} + b_{01} + b_{21}), \tag{30}$$

under which we have

$$\begin{aligned}\mu_1 &= \frac{\sqrt{2}\pi}{24\bar{h}_{20}^{3/2}}\Big[-6(1+\bar{h}_{12})(\bar{b}_{11} + 2\bar{a}_{20}) - 36\bar{h}_{03}\bar{h}_{20}(\bar{a}_{11} + 2\bar{b}_{02})\Big] \\ &= \frac{2\pi}{(2\rho)^{3/2}}[3\rho b_{03} - 6h_{13}a_{21} + \rho a_{12} - 6h_{13}b_{12} + \rho(2\rho-1)(a_{10} + b_{01})].\end{aligned} \tag{31}$$

Thus, we set

$$b_{03} = -\frac{1}{3}a_{12} + \frac{2h_{13}}{\rho}(a_{21} + b_{12}) - \frac{1}{3}(2\rho - 1)(a_{10} + b_{01}), \tag{32}$$

under which $\mu_1 = 0$, and then letting $\mu_2 = 0$, results in

$$a_{21} = -b_{12} - \frac{\rho\left[\rho(2\rho-1) + 4\rho h_{04} - 20h_{13}^2\right]}{2h_{13}(6\rho - 5 + 20h_{04})}\,(a_{10}+b_{01}), \quad (h_{13}(6\rho-5+20h_{04}) \neq 0). \tag{33}$$

Further calculating μ_3, yields

$$\mu_3 = \frac{5\pi(a_{10}+b_{01})}{(2\rho)^{5/2}(6\rho-5+20h_{04})}\left[28\rho h_{04} + 42h_{13}^2 + \rho(8\rho-7)\right](4h_{04}+4h_{13}+2\rho-1)(4h_{04}-4h_{13}+2\rho-1). \tag{34}$$

Letting $\mu_3 = 0$, results in

$$\begin{aligned} h_{04}^{(1)} &= -\tfrac{1}{28\rho}\left[42h_{13}^2 + \rho(8\rho-7)\right],\\ h_{04}^{(2)} &= -h_{13} - \tfrac{1}{4}(2\rho-1),\\ h_{04}^{(3)} &= h_{13} - \tfrac{1}{4}(2\rho-1). \end{aligned} \tag{35}$$

Taking $(a_{10}+b_{01}) = 0$, or $h_{04} = h_{04}^{(2)}$, or $h_{04} = h_{04}^{(3)}$, leads to $\mu_4 = \mu_5 = \mu_6 = \cdots = 0$. In order to obtain maximal number of small limit cycles bifurcating from the two centers, we choose $h_{04} = h_{04}^{(1)}$. Then, we obtain

$$\begin{aligned} \mu_4 &= \frac{3\sqrt{2}\pi}{49\rho^{11/2}f_1}\,(a_{10}+b_{01})f_2f_4,\\ \mu_5 &= \frac{12\sqrt{2}\pi}{343\rho^{15/2}f_1}\,(a_{10}+b_{01})f_2f_5,\\ &\vdots \end{aligned} \tag{36}$$

where

$$\begin{aligned} f_1 &= \rho^2 - 105h_{13}^2,\\ f_2 &= (21h_{13}^2 + 14\rho h_{13} - 3\rho^2)(21h_{13}^2 - 14\rho h_{13} - 3\rho^2),\\ f_4 &= 1617h_{13}^4 - 210\rho^2h_{13}^2 + \rho^4,\\ f_5 &= 294294h_{13}^6 + 539\rho^2(63\rho-104)h_{13}^4 - 14\rho^4(315\rho-209)h_{13}^2 + 3\rho^6(7\rho-4). \end{aligned} \tag{37}$$

Choosing $f_1 \neq 0$ (i.e., $\rho \neq \pm\sqrt{105}h_{13}$), then when $(a_{10}+b_{01})f_2 = 0$, we have $\mu_4 = \mu_5 = \cdots = 0$, leading to two centers at $(\pm1, 0)$. Thus, letting $f_4(h_{13}^2) = 0$ (and so $\mu_4 = 0$), we have

$$h_{13}^{2+} = \frac{(15+8\sqrt{3})\rho^2}{231} > 0, \quad h_{13}^{2-} = \frac{(15-8\sqrt{3})\rho^2}{231} = \frac{\rho^2}{7(15+8\sqrt{3})} > 0, \tag{38}$$

both of which are positive. For these two values, we obtain the following two sets of solutions:

$$S^{\pm} = \begin{pmatrix} a_{30} = -\frac{1}{3}(a_{10}+b_{01}+b_{21}) & \Rightarrow & \mu_0 = 0\\ b_{03}^{\pm} = -\frac{1}{3}a_{12} + \frac{1+(\pm4\sqrt{3}-11)\rho}{3}(a_{10}+b_{01}) & \Rightarrow & \mu_1 = 0\\ a_{21}^{\pm} = -b_{12} + \frac{(\pm4\sqrt{3}-9)\sqrt{231}\rho}{6\sqrt{15\pm8\sqrt{3}}}(a_{10}+b_{01}) & \Rightarrow & \mu_2 = 0\\ h_{04}^{\pm} = \frac{1}{4} - \frac{1}{154}(59\pm8\sqrt{3})\rho & \Rightarrow & \mu_3 = 0\\ h_{13}^{2\pm} = \frac{1}{231}(15\pm8\sqrt{3})\rho^2 & \Rightarrow & \mu_4 = 0 \end{pmatrix} \tag{39}$$

for which μ_5, becomes

$$\mu_5^{\pm} = \mu_5(S^{\pm}) = \frac{32768(90\pm37\sqrt{3})\sqrt{2\rho}\pi}{1369599}(a_{10}+b_{01}). \tag{40}$$

Noticing $90 - 37\sqrt{3} > 0$ we know that both μ_5^+, and μ_5^-, have the same sign as that of $(a_{10}+b_{01})$. It should be pointed out that the above two sets of solutions are the critical values of the parameters for the focus values. However, for system (19), there are actually four sets of solutions, since the four different values of $\pm h_{13}^+$, $\pm h_{13}^-$, give four different Hamiltonian functions (20). The above results show that one can choose the five parameters a_{30}, a_{21}, a_{12}, h_{04}, and h_{20}, such that $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$, but $\mu_5 \neq 0$. Thus, there exist at most 5 small limit cycles bifurcating from the center $(1, 0)$. Furthermore, by backwards small perturbations, λ_1, on h_{20}, for μ_4, λ_2, on h_{04}, for μ_3, ε_3, on a_{12}, for μ_2, ε_4, on a_{21}, for μ_1, and ε_5, on a_{30}, for μ_0, we can obtain 5 small limit cycles in the vicinity of $(1, 0)$. Here,

$$0 < \varepsilon \ll |\varepsilon_5| \ll |\varepsilon_4| \ll |\varepsilon_3| \ll |\lambda_2| \ll |\lambda_1| \ll 1.$$

By symmetry, system (17) can thus have 10 small limit cycles near the two symmetric points $(\pm1, 0)$.

Summarizing the above results gives the following theorem.

Theorem 2. *The Z_2-equivariant 3rd-order Hamiltonian system (17) can have maximal 10 small limit cycles, with 5 bifurcating from each of the two symmetric centers.*

In the next section, we consider possible limit cycles bifurcating from homoclinic or heteroclinic orbits.

5. Bifurcation of large limit cycles inside homoclinic and heteroclinic orbits

In the previous section, we have shown that under the conditions given in (39) system (19) can have 10 small limit cycles bifurcating from the two symmetric centers $(\pm 1, 0)$. Now suppose the conditions given in (39) still hold, we want to investigate the limit cycles which may bifurcate from homoclinic or heteroclinic loops.

First note that we have four sets of solutions:

$$\begin{aligned}(h_{04}, h_{13})^+ &= \left(\tfrac{1}{4} - \tfrac{1}{154}\,(59+8\sqrt{3})(1+2h_{22}),\ \pm\sqrt{\tfrac{15+8\sqrt{3}}{231}}\,(1+2h_{22})\right),\\ (h_{04}, h_{13})^- &= \left(\tfrac{1}{4} - \tfrac{1}{154}\,(59-8\sqrt{3})(1+2h_{22}),\ \pm\sqrt{\tfrac{15-8\sqrt{3}}{231}}\,(1+2h_{22})\right).\end{aligned} \tag{41}$$

Except for the three fixed points $(0,0)$, $(\pm 1, 0)$, other possible fixed points can be found from Eq. (19) in which $\varepsilon = 0$, yielding

$$y_e = \frac{\left[2(8h_{04}^2 - 8h_{04}h_{22}^2 + 3h_{13}^2h_{22})x_e^2 + (3h_{13}^2 - 16h_{04}^2 - 8h_{04}h_{22})\right]x_e}{h_{13}\left[(32h_{04}h_{22} - 9h_{13}^2)x_e^2 + 4h_{04}\right]},$$

where x_e, is determined from the following equation:

$$\begin{aligned}0 = &\left[64h_{04}(h_{04} - h_{22}^2)^2 - h_{13}^2(27h_{13}^2 + 16h_{22}^2)\right]x_e^6\\ &- \left[64h_{04}(2h_{04} + h_{22})(h_{04} - h_{22}^2)^2 + 3h_{13}^2(4h_{22}^2 - 9h_{13}^2 + 48h_{04}h_{22} - 12h_{04}\right]x_e^4 + 4h_{04}(3h_{13} + 4h_{04} + 2h_{22})(4h_{04}\\ &+ 2h_{22} - 3h_{13})x_e^2 + h_{13}^2.\end{aligned}$$

This equation is a cubic polynomial of x_e^2, and has at least one real solution. For the first two sets of parameter values $(h_{04}, h_{13})^+$, this solution is always positive for $h_{22} \in (-\frac{1}{2}, \infty)$, while for the second two sets of parameter values $(h_{04}, h_{13})^-$, this solution is positive only for $h_{22} \in (h_{22}^*, \infty)$, where

$$\begin{aligned}h_{22}^* = &-\tfrac{61}{154} + \tfrac{40\sqrt{3}}{231} - \tfrac{1}{231}\left[793071 - 457410\sqrt{3} + 396\sqrt{164406 - 94919\sqrt{3}}\right]^{1/3}\\ &- \tfrac{(2263-1292\sqrt{3})}{77}\left[793071 - 457410\sqrt{3} + 396\sqrt{164406 - 94919\sqrt{3}}\right]^{-1/3}\\ &\approx -0.17337533355610033812\end{aligned}$$

Thus, choosing a value of h_{22}, from these two intervals, we obtain two additional fixed points which are symmetric about the origin. It can be shown that these two fixed points are saddle points.

In order to compute the Melnikov function near a homoclinic loop, we choose a fixed value of h_{22}, to determine $(h_{04}, h_{13})^+$, and $(h_{04}, h_{13})^-$. For convenience, let $h_{22} = 1$, and so $\rho = 3$. Then we obtain two sets of parameter values and the associated fixed points, listed as follows (approximated to 10 decimal points):

$$\begin{aligned}h_{22} = 1,\quad h_{13} = \pm 1.0603186202,\quad h_{04} = -1.1692806453,\quad a_{30} = -\frac{1}{3}(a_{10} + b_{01} + b_{21}), b_{03}\\ = -\frac{1}{3}a_{12} - 3.7384634364(a_{10} + b_{01}),\quad a_{21} = -b_{12} - 2.9309068947(a_{10} + b_{01}), (\pm x_1, \mp y_1)\\ = (\pm 0.8236724666,\ \mp 0.4830363744),\quad \mu_5^+ = 28.3690534323(a_{10} + b_{01}),\end{aligned} \tag{42}$$

and

$$\begin{aligned}h_{22} = 1,\quad h_{13} = \pm 0.2110819567,\quad h_{04} = -0.6294206534,\quad a_{30} = -\frac{1}{3}(a_{10} + b_{01} + b_{21}), b_{03}\\ = -\frac{1}{3}a_{12} - 17.5948698969(a_{10} + b_{01}),\quad a_{21} = -b_{12} - 113.1897070620(a_{10} + b_{01}), (\pm x_2, \mp y_2)\\ = (\pm 0.4469955744,\ \mp 0.6914971979),\quad \mu_5^- = 4.7710994603(a_{10} + b_{01}).\end{aligned} \tag{43}$$

The unperturbed Hamiltonian systems for the two sets of parameter values with positive h_{13}, are shown in Figs. 1 and 2, respectively.

5.1. Inside the homoclinic orbits passing through $(\pm x_1, \mp y_1)$

First, we consider the two symmetric saddle points $(\pm x_1, \mp y_1)$. Note that at the two fixed points $(\pm x_1, \mp y_1)$,

$$h_2 = H(\pm x_1, \mp y_1) = -0.1112780483 > h_1 = H(\pm 1, 0) = -0.25 \quad \text{(see Eq. (21))}.$$

By introducing

$$x = x_1 + \tilde{x}, \quad y = y_1 + \tilde{y} \tag{44}$$

into system (19), we obtain

$$\begin{aligned}
\frac{d\tilde{x}}{dt} = & -0.8492612604\tilde{x} - 3.4481582636\tilde{y} - 0.9660727498\tilde{x}^2 + 0.2216550944\tilde{x}\tilde{y} + 9.3977267624\tilde{y}^2 + 2\tilde{x}^2\tilde{y} \\
& + 3.1809558607\tilde{x}\tilde{y}^2 - 4.6771225813\tilde{y}^3 + \varepsilon\{1.5978885074a_{10} - 0.4830363744a_{01} + 0.1921826691a_{12} \\
& - 0.1127040461a_{03} + 0.7742160408b_{01} - 0.1862697757b_{21} + 0.3277094262b_{12} + [2.6537669538(a_{10} + b_{01}) \\
& + 0.2333241390a_{12} + 1.6537669538b_{01} - 0.6784363322b_{21} + 0.7957275239b_{12}]\tilde{x} + [a_{01} - 1.9884337237(a_{10} + b_{01}) \\
& - 0.7957275239a_{12} + 0.6999724169a_{03} - 0.6784363322b_{12}]\tilde{y} + [0.5920621735(a_{10} + b_{01}) - 0.8236724666b_{21} \\
& + 0.4830363744b_{12}]\tilde{x}^2 - 4.8282146226(a_{10} + b_{01}) - 0.9660727488a_{12} - 1.6473449332b_{12}]\tilde{x}\tilde{y} \\
& + (0.8236724666a_{12} - 1.4491091231a_{03})\tilde{y}^2 - \frac{1}{3}(a_{10} + b_{01} + b_{21})\tilde{x}^3 + [b_{12} - 2.9309068947(a_{10} + b_{01})]\tilde{x}^2\tilde{y} \\
& + a_{12}\tilde{x}\tilde{y}^2 + a_{03}\tilde{y}^3 \equiv \widetilde{H}_{\tilde{y}}(\tilde{x}, \tilde{y}) + \varepsilon\tilde{p}(\tilde{x}, \tilde{y}, a_{ij}, b_{ij}),
\end{aligned}$$

$$\begin{aligned}
\frac{d\tilde{y}}{dt} = & -1.5019572745473863348 9\tilde{x} + 0.84926126042998520430\tilde{y} - 2.47101739974075548014\tilde{x}^2 \\
& + 1.93214549750312157554\tilde{x}\tilde{y} - 0.11082754717800281508\tilde{y} - \tilde{x}^3 - 2\tilde{x}\tilde{y}^2 - 1.06031862021993399065\tilde{y}^3 \\
& + \varepsilon\{0.4213399556a_{10} + 0.3756801538a_{12} + 0.8236724666b_{10} - 0.0616964187b_{01} + 0.5588093272b_{30} \\
& - 0.3277094262b_{21} + 0.1921826691b_{12} + (b_{10} - 0.7957275239b_{21} + 0.2333241390b_{12} + 2.0353089966b_{30})\tilde{x} \\
& - (2.6168212871a_{10} + 1.6168212871b_{01} + 0.2333241390a_{12} + 0.7957275239b_{12} - 0.6784363322b_{21})\tilde{y} \\
& - (0.4830363744b_{21} - 2.4710173997b_{30})\tilde{x}^2 + (1.6473449332b_{21} - 0.9660727488b_{12})\tilde{x}\tilde{y} \\
& + [+5.4174414722(a_{10} + b_{01}) + 0.4830363744a_{12} + 0.8236724666b_{12}]\tilde{y}^2 + b_{30}\tilde{x}^3 + b_{21}\tilde{x}^2\tilde{y} + b_{12}\tilde{x}\tilde{y}^2 \\
& - \left[\frac{1}{3}a_{12} + 3.7384634364(a_{10} + b_{01})\right]\tilde{y}y^3\Big\} \equiv -\widetilde{H}_{\tilde{x}}(\tilde{x}, \tilde{y}) + \varepsilon\tilde{q}(\tilde{x}, \tilde{y}, a_{ij}, b_{ij}),
\end{aligned} \tag{45}$$

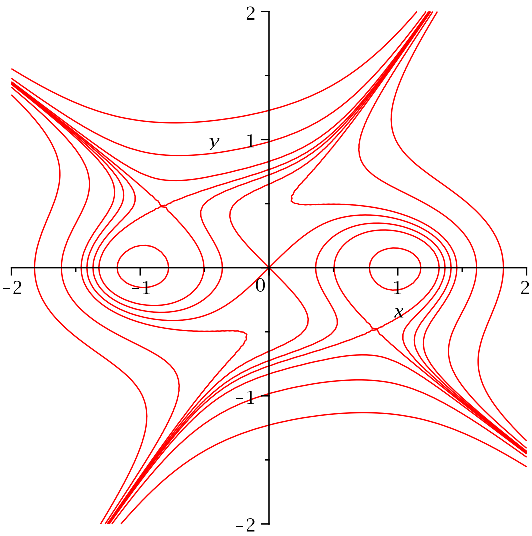

Fig. 1. Phase portrait of unperturbed system $(19)_{\varepsilon=0}$ for $h_{22} = 1$, $h_{13} = 1.0603186202$, $h_{04} = -1.1692806453$.

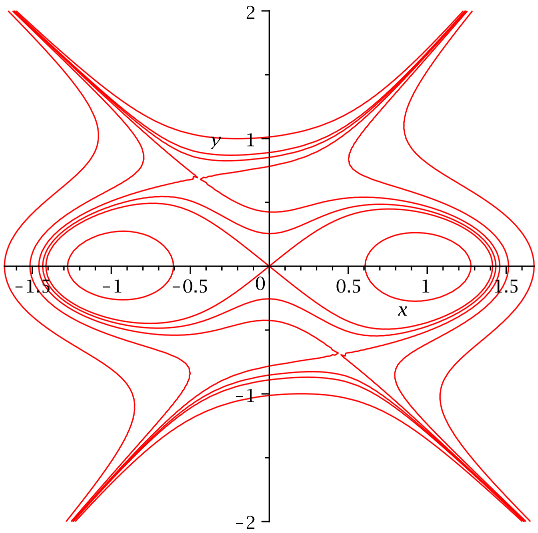

Fig. 2. Phase portrait of unperturbed system (19)$_{\varepsilon=0}$ for $h_{22}=1$, $h_{13}=0.2110819567$, $h_{04}=-0.6294206534$.

where

$$\begin{aligned}\widetilde{H}(\tilde{x},\tilde{y}) = {} & 0.75097863727369316744\tilde{x}^2 - 0.84926126042998520430\tilde{x}\tilde{y} - 1.72407913177696846553\tilde{y}^2\\ & + 0.82367246658025182671\tilde{x}^3 - 0.96607274875156078777\tilde{x}^2\tilde{y} + 0.11082754717800281508\tilde{x}\tilde{y}^2\\ & + 3.13257558747987804952\tilde{y}^3 + 0.25\tilde{x}^4 + \tilde{x}^2\tilde{y}^2 + 1.06031862021993399065\tilde{x}\tilde{y}^3\\ & - 1.16928064533540944834\tilde{y}^4 \end{aligned} \tag{46}$$

Now, based on system (45) with $\widetilde{H}(\tilde{x},\tilde{y}) = \tilde{h}$ (or $H(x,y) = h - h_2$), using the formulas given by Han [3] and (22), we obtain the expansion of the Melnikov function near the homoclinic orbit:

$$M_2(h,h_{ij},a_{ij},b_{ij}) = \sum_{k\geqslant 0} v_{2k}\tilde{h}^k + \sum_{k\geqslant 0} v_{2k+1}\tilde{h}^{k+1}\ln|\tilde{h}|, \quad 0 < -\tilde{h} \ll 1, \tag{47}$$

where $\tilde{h} = h_2 - h$, and

$$v_0 = \oint_{L_2} \tilde{q}(\tilde{x},\tilde{y},a_{ij},b_{ij})d\tilde{x} - \tilde{p}(\tilde{x},\tilde{y},a_{ij},b_{ij})d\tilde{y} = -\oint_{L_2} p^*(\tilde{x},\tilde{y},a_{ij},b_{ij})d\tilde{y}, \tag{48}$$

where

$$p^*(\tilde{x},\tilde{y},a_{ij},b_{ij}) = \tilde{p}(\tilde{x},\tilde{y},a_{ij},b_{ij}) - \tilde{p}(0,\tilde{y},a_{ij},b_{ij}) + \int_0^{\tilde{x}} \tilde{q}_{\tilde{y}}(u,\tilde{y},a_{ij},b_{ij})du,$$

and the closed curve L_2, is the homoclinic loop passing through the saddle point $(x_1,-y_1)$ which is now the origin in the $\tilde{x}$-$\tilde{y}$ coordinate system, defined by $\widetilde{H}(\tilde{x},\tilde{y}) = 0$.

With the above formulas and parameter values given in (42), we obtain

$$v_0 = 0.0001477405(a_{10}+b_{01}). \tag{49}$$

It can be shown that when $a_{10}+b_{01}=0$, $\mu_5=\mu_6=\mu_7=\cdots=0$, and $v_0=v_1=v_2=v_3=0$, indicating that the perturbed system (19) is an integral system with the two centers at $(\pm 1,0)$. When $a_{10}+b_{01}\neq 0$, without loss of generality, we may assume $a_{10}+b_{01}>0$. Thus, under the conditions $\mu_0=\mu_1=\cdots=\mu_4=0$, we have

$$M(h,h_{ij},a_{ij},b_{ij}) = \begin{cases} M_1(\bar{h},h_{ij},a_{ij},b_{ij}) & = \mu_5^+(h-h_1)^6 + h.o.t.\\ & = 28.3690534323(a_{10}+b_{01})(h-h_1)^6\\ & +h.o.t. > 0, \quad \text{for } 0 < h-h_1 \ll 1,\\ M_2(\tilde{h},h_{ij},a_{ij},b_{ij}) & = v_0 + h.o.t.\\ & = 0.0001477405(a_{10}+b_{01}) + h.o.t. > 0,\\ & \text{for } 0 < h_2 - h \ll 1. \end{cases} \tag{50}$$

Since $M_1M_2>0$, it does not lead to conclusion on the existence of large limit cycles near the homoclinic loops passing through the two saddle points $(\pm x_1, \mp y_1)$.

In order to investigate the possibility of large limit cycles which may exist inside the two homoclinic loops passing through the two saddle points $(\pm x_1, \mp y_1)$, we have used (22) to numerically compute the Melnikov function as h gradually decreases from $h_2 = -0.1112780483$, to $h_1 = -0.25$ and found that it remains positive when $a_{10} + b_{01} > 0$. This clearly indicates that no large limit cycles exist between the homoclinic loop and the 5 small limit cycles. By symmetry, we have shown that there do not exist additional (large) limit cycles around the two elementary centers at $(\pm 1, 0)$.

5.2. Inside the homoclinic orbits passing through the origin $(0,0)$

Now, we consider possible (large) limit cycles bifurcating from the symmetric double homoclinic loops passing through the origin (see Fig. 2). First, note that

$$h_0 = H(0,0) = 0 > h_1 = H(\pm 1, 0) = -0.25 \quad (\text{see Eq. (21)}).$$

For this case, due to symmetry, we use (22) to simply calculate v_0, along one loop of the symmetric double homoclinic loops (say, the right loop), yielding

$$\begin{aligned} v_{0,R}^{h_0} &= M_R(h_0, h_{ij}, a_{ij}, b_{ij}) \qquad (\text{the R denotes the right loop}) = \oint_{H_R(x,y,h_{ij})=0} q(x,y,b_{ij})dx - p(x,y,a_{ij})dy \\ &= \oint_{H_R(x,y,h_{ij})=0} q^*(x,y,b_{ij})dx \qquad (\text{the R denotes the right loop}) \\ &= \oint_{H_R(x,y,h_{ij})=0} \left[q(x,y,b_{ij}) - q(x,0,b_{ij}) + \int_0^y p_x(x,v,a_{ij})dv \right] dx \approx 0.00952505(a_{10} + b_{01}). \end{aligned} \tag{51}$$

Thus, under the conditions $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4$, we obtain

$$M(h, h_{ij}, a_{ij}, b_{ij}) = \begin{cases} M_1(\bar{h}, h_{ij}, a_{ij}, b_{ij}) & = \mu_5^-(h-h_1)^6 + h.o.t. \\ & = 4.7710994603(a_{10}+b_{01})(h-h_1)^6 \\ & +h.o.t. > 0, \quad \text{for } 0 < h - h_1 \ll 1, \\ M_2(\tilde{h}, h_{ij}, a_{ij}, b_{ij}) & = v_{0,R}^{h_0} + h.o.t. \\ & = 0.00952505(a_{10}+b_{01}) + h.o.t. > 0, \\ & \text{for } 0 < -h \ll 1. \end{cases} \tag{52}$$

Hence, due to $M_1 M_2 > 0$, for this case, we can not make any conclusion on the existence of large limit cycles bifurcating from the homoclinic orbits passing through the origin. Moreover, numerically computing the Melnikov function for $h \in (h_1, h_2)$, shows that the sign of the Melnikov function does not change. This implies that no additional (large) limit cycles exist between the right homoclinic loop and the 5 small limit cycles. Due to symmetry, we have proved that except for the 10 small limit cycles, no more limit cycles exist inside the symmetric double homoclinic loops passing through the origin.

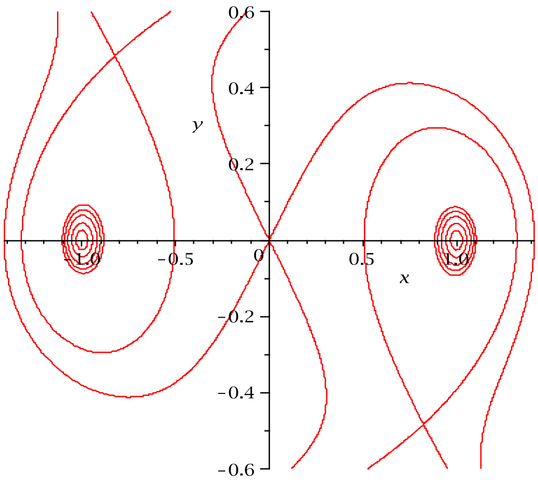

Fig. 3. Phase portrait of the perturbed system (19) with the critical parameter values given in (42).

5.3. Between the homoclinic orbits passing through the origin (0, 0), and the heteroclinic orbit passing through the saddle points $(\pm x_2, \mp y_2)$

Finally, we investigate possible existence of large limit cycles between the double homoclinic orbits passing through the origin (0, 0), and the double heteroclinic orbit passing through the saddle points $(\pm x_2, \mp y_2)$ (see Fig. 2). For this case, we have

$$h_2 = H(\pm x_2, \mp y_2) = 0.0695908328 > h_0 = H(0,0) = 0.$$

In the previous section we have obtained $v_{0,R}^{h_0}$. Thus, due to symmetry, the v_0, along the symmetric double homoclinic loops is equal to $v_0^{h_0} = 2\,v_{0,R}^{h_0} = 0.0190501(a_{10} + b_{01})$, and so here we only need to compute $v_0^{h_2}$. Using formula (51) yields

$$v_0^{h_2} = M(h_2, h_{ij}, a_{ij}, b_{ij}) = \oint_{H(x,y,h_{ij})=h_2} \left[q(x,y,b_{ij}) - q(x,0,b_{ij}) + \int_0^y p_x(x, v, a_{ij})dv \right] dx \approx 0.718807(a_{10} + b_{01}). \tag{53}$$

Since $v_0^{h_0} v_0^{h_2} > 0$, we are not able to claim the existence of large limit cycles bifurcating from the annuluses between the symmetric double homoclinic loops and the symmetric double hetroclinic loops. In fact, numerical computation of the Melnikov function for $h \in (0, h_2)$, shows that it remains positive for $a_{10} + b_{01} > 0$, implying that no limit cycles exist between the symmetric double homoclinic loops and the symmetric double hetroclinic loops.

Summarizing the above results gives the following theorem.

Theorem 3. *Z_2-equivariant third-order Hamiltonian vector fields with symmetric third-order perturbations can have at least 10 limit cycles, all of which are small limit cycles bifurcating from two symmetric centers.*

One example showing the 10 small limit cycles is depicted in Fig. 3.

6. Conclusion

In this paper, we have shown that Z_2-equivariant 3rd-order Hamiltonian planar vector fields with 3rd-order symmetric perturbations can have at least 10 limit cycles, all of which are small limit cycles bifurcating from two symmetric centers. The methods presented in this paper can be extended to consider bifurcation of limit cycles in integral systems.

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