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On limit cycles of the Liénard equation with Z_2 symmetry

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Abstract

This paper considers the limit cycles in the Liénard equation, described by $\ddot{x} + f(x)\dot{x} + g(x) = 0$, with Z_2 symmetry (i.e., the vector filed is symmetric with the y-axis). Particular attention is given to the existence of small-amplitude (local) limit cycles around fine focus points when $g(x)$ is a third-degree, odd polynomial function and $f(x)$ is an even function. Such a system has three fixed points on the x-axis, with one saddle point at the origin and two linear centres which are symmetric with the origin. Based on normal form computation, it is shown that such a system can generate more limit cycles than the existing results for which only the origin is considered. In general, such a Liénard equation can have $2m$ small limit cycles, i.e., $H(2m, 3) \geqslant 2m$, where H denotes the Hilbert number of the system, $2m$ and 3 are the degrees of f and g, respectively.

1. Introduction

Hilbert's 16th problem is still an open problem since Hilbert presented the well-known 23 mathematical problems to the Second International Congress of Mathematicians in Paris in 1900 [1]. This problem contains two parts. The second part of the problem is, roughly speaking, to find a uniform upper bound $H(n)$ for the number of limit cycles of planar polynomial systems, where n is the degree of polynomial vector field [2]. This problem is not only important in theoretical studies, but also important in applications. In fact, limit cycles are common solutions for all types of dynamical systems. They model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing. For example, the Holling–Tanner predator–prey model [3] shows that all trajectories lying in the first quadrant of the phase plane are drawn to a closed periodic cycle. Therefore, no matter what the initial values are, the populations of predator and prey event usually rise and fall periodically. The isolated periodic trajectory is a stable limit cycle. This model appears to match very well with what happens for many predator–prey species in the natural world. Other examples of self-excited oscillation are: beating of a heart; rhythms in body temperature; hormone secretion; chemical reactions that oscillate spontaneously, vibrations in bridges and airplane wings, etc.

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Due to the wide occurrence of limit cycles in science and technology, limit cycle theory has also been extensively studied by physicists, and more recently by chemists, biologists and economists. Although it has not been possible to obtain the uniform upper bound for the Hilbert number $H(n)$, various efforts have been made in finding the maximal number of limit cycles and raising the lower bound of Hilbert number $H(n)$ for general planar polynomial systems or for individual degree of systems. This way people hope to get a close estimation of the upper bound of $H(n)$. Nevertheless, even estimating the lower bound of $H(n)$ is generally very difficult, in particular, for determining large (global) limit cycles. For recent progress on the research of Hilbert's 16th problem, readers are referred to [4,5] and references therein.

If the study on the second part of Hilbert's 16th problem is restricted to the neighborhood of isolated fixed points, the problem is reduced to considering degenerate Hopf bifurcations. Many research results have been obtained in the past 50 years on the local problem (e.g., see [6–12]). In the last two decades, much progress on finite cyclicity near a fine focus point or a homoclinic loop has been achieved. For a quadratic system, Bautin [6] proved that the maximal number of small limit cycles was three. For cubic order systems, recently ten [13], eleven [5,14] and twelve [15–19] limit cycles were obtained. The main idea in the study of local (small) limit cycles is to compute the focus values of the system associated with Hopf-type singular points, and then to take appropriate perturbations to prove the existence of the limit cycles. Calculating focus values is equivalent to computing the normal form of Hopf bifurcation. Thus, symbolic computations with the aid of Maple or Mathematica can be efficiently applied.

Another type of Hilbert's 16th problem is called weakened problem, or infinitesimal, or tangential problem. For the weakened problem, the general idea is to perturb a Hamiltonian vector field or integral system so that the problem can be transferred to considering the zeros of Abelian integrals. A Hamiltonian system or an integral system has only saddle points and centres. Thus, under perturbations, the centres may become fine focus points and local small limit cycles may bifurcate. On the other hand, under perturbations, homoclinic orbits near saddle points may bifurcate into global limit cycles, and the number of the limit cycles is determined by the zeros of the corresponding Abelian integral, which is the first order Melnikov integral [20]. A recently developed technique called detection function [5] has been successfully applied to investigate higher-order Hamiltonian systems and new results have been obtained [21,22].

A simplified version of Hilbert's 16th problem—the Liénard equation has attracted many researchers. The Liénard equation is described by [23]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where the dot denotes differentiation with respect to time t, and $f(x)$ and $g(x)$ are polynomial functions of x. Smale [2] chose the Liénard equation with $g(x) = x$ as a simplified version of Hilbert's 16th problem in his book ‘‘Mathematical Problems for the Next Century’’. Hilbert's 16th problem for the Liénerd equation is to find an upper bound on the number of limit cycles through the degree of polynomial $f(x)$. It should be clarified that until now, no upper bound has ever been found even for the simplified Liénard equation.

Small limit cycles in the Liénard system have been extensively studied by many researchers and many results have been obtained (e.g., see [24–26]). In this paper, we will consider a class of Liénard equations with $g(x)$ being a third-degree, odd polynomial function and $f(x)$ an even function of x. Such a system has three fixed points with one saddle point at the origin, and two linear centres on the x-axis which are symmetric with the origin. We shall investigate the small limit cycles bifurcating from these two linear centres by appropriately choosing focus values. Suppose that the degree of $f(x)$ is $2m$. Then we have found that $\bar{H}(2m, 3) = 2m$ for $m = 1, 2, \ldots, 10$, where $\bar{H}$ denotes the number of small limit cycles.

In the next section, we shall briefly discuss the Liénard equation, and present our main results in Section 3. Conclusion is given in Section 4.

2. The Liénard equation

In this section, for completeness we briefly discuss the Liénard equation (1). Most of the early history in the theory of limit cycles was stimulated by practical problems displaying periodic behaviour. For example, in 1877 Rayleigh derived a differential equation to describe the oscillation of a violin string [27] in the phase space

$$\dot{x} = y, \quad \dot{y} = -x - \epsilon\left(\frac{1}{3}y^2 - 1\right)y, \tag{2}$$

where ϵ is a small perturbation parameter. Following the invention of the triode vacuum tube, which was able to produce stable self-excited oscillations of constant amplitude, van der Pol [28] used the following differential equation to describe this phenomenon

$$\dot{x} = y, \quad \dot{y} = -x - \epsilon(x^2 - 1)y. \tag{3}$$

A generalization of Eq. (3) represents a famous class of differential equations given by Eq. (1), which was first investigated by Liénard [23] in 1928, can be rewritten in the phase space as

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{4}$$

Further, let $y = \tilde{y} - F(x)$, where $F(x) = \int_0^x f(s)\,\mathrm{d}s$. Then we have the following equivalent system:

$$\begin{aligned} &\dot{x} = y = \tilde{y} - F(x), \\ &\dot{\tilde{y}} = \dot{y} + \frac{\mathrm{d}F}{\mathrm{d}x}\dot{x} = -g(x) - f(x)y + f(x)y = -g(x). \end{aligned} \tag{5}$$

The system considered by Smale [2] is a special case of system (5) when $g(x) = x$.

For definiteness, let

$$\begin{aligned} &F(x) = a_1x + a_2x^2 + a_3x^3 + \cdots, \\ &g(x) = b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \cdots, \end{aligned} \tag{6}$$

where $b_1 > 0$. If the attention is focused on the dynamic behaviour of the system in the vicinity of the origin, then one may introduce a local coordinate transformation [24] into Eq. (5) to obtain

$$\begin{cases} \dot{u} = y - (A_1u + A_2u^2 + A_3u^3 + A_4u^4 + A_5u^5 + \cdots), \\ \dot{y} = -u, \end{cases} \tag{7}$$

where $A_1 = a_1$, $A_2 = a_2$, and other A_i's are given explicitly in terms of a_i's and b_i's.

It is easy to see from Eq. (7) that the origin $(u, y) = (0, 0)$ is a unique fixed point—a linear centre. Further, one can apply the Maple program developed in [29] to system (7) to obtain the following focus values:

$$\begin{aligned} v_0 &= -A_1, \\ v_1 &= -\frac{3}{8}A_3, \\ v_2 &= -\frac{5}{16}A_5 - \frac{5}{24}A_2^2A_3, \\ v_3 &= -\frac{35}{128}A_7 - \frac{205}{1152}A_4^2A_5 - \left(\frac{1885}{13824}A_4^2 + \frac{2}{3}A_2A_4 + \frac{999}{8192}A_3^2\right)A_3, \\ v_4 &= -\frac{63}{256}A_9 - \frac{413}{2304}A_2^2A_7 - \left(\frac{47}{96}A_2A_4 + \frac{2115}{4096}A_3^2 + \frac{4297}{41472}A_2^4\right)A_5 \\ &\quad - \left(\frac{141}{160}A_2A_6 + \frac{149}{240}A_4^2 + \frac{1093}{1152}A_2^3A_4 + \frac{20599}{49152}A_2^2A_3^2 + \frac{109483}{1244160}A_2^6\right)A_3, \\ &\ \vdots \end{aligned} \tag{8}$$

It follows from Eq. (8) that

$$v_0 = v_1 = v_2 = v_3 = 0, \quad v_4 \neq 0 \iff A_1 = A_3 = A_5 = A_7 = 0, \quad A_9 \neq 0.$$

In general, one can show that

$$v_0 = v_1 = v_2 = \cdots = v_{m-1} = 0, \quad v_m \neq 0 \iff A_1 = A_3 = A_5 = \cdots = A_{2m-1} = 0, \quad A_{2m+1} \neq 0.$$

Therefore, in order for system (7) or (1) to have m small limit cycles around the origin, it requires that $v_0 = v_1 = v_2 = \cdots = v_{m-1} = 0$, but $v_m \neq 0$, or $A_1 = A_3 = A_5 = \cdots = A_{2m-1} = 0$, but $A_{2m+1} \neq 0$. For example, when $A_1 = 0$, but $A_3 \neq 0$, then system (7) or (1) has at most one limit cycle around the origin; when $A_1 = A_3 = 0$, $A_5 \neq 0$, the system (7) or (1) has at most two limit cycles in the neighborhood of the origin; etc. Since the coefficients A_i's are given in terms of a_i's and b_i's, one needs to determine the values of a_i's and b_i's to satisfy the necessary conditions. Further, based on the sufficient conditions for the existence of small-amplitude limit cycles [13,17,18], we can apply appropriate perturbations to obtain exactly m limit cycles.

Let $\hat{H}(i,j)$ be the maximal number of small-amplitude limit cycles of system (1) in the vicinity of the origin, where i and j are the degrees of f and g, respectively. Then the existing results for the Liénard system (1) are summarized in Table 1 [25]. Note that the numbers given in this table are symmetric with respect to f and g [26], i.e., $\hat{H}(i,j) = \hat{H}(j,i)$. Thus, one only needs to prove the cases $i \geqslant j$. It should be pointed out that the notation $\hat{H}(i,j)$ denotes the maximal number of small limit cycles which may exist in the vicinity of the origin. It does not include global (large)

Table 1
The values of $\hat{H}(m,n)$ for the generalized Liénard systems associated with the origin when f and g are of varying degrees

	50	↑	↑	38														
	49	24	33	38														
	48	24	32	36														
	⋮	⋮	⋮	⋮														
	13	6	9	10														
	12	6	8	10														
	11	5	7	8														
	10	5	7	8														
deg(f)	9	4	6	8	9													
	8	4	5	6	9													
	7	3	5	6	8													
	6	3	4	6	7													
	5	2	3	4	6	6												
	4	2	3	4	4	6	7	8	9	9								
	3	1	2	2	4	4	6	6	6	8	8	8	10	10	⋯	36	38	38
	2	1	1	2	3	3	4	5	5	6	7	7	8	9	⋯	32	33	→
	1	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
		1	2	3	4	5	6	7	8	9	10	11	12	13	⋯	48	49	50
		deg(g)																

limit cycles, nor contain other local (small) limit cycles which may appear in the neighborhood of other non-zero focus points.

3. Limit cycles of the Liénard equation when deg(*g*) = 3

In this section, we will consider a particular class of Liénard equations in which $g(x)$ is a third-degree, odd polynomial while $f(x)$ is an even function of x. To be more specific, consider the following system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{1}{2}b^2 x(x^2-1) - y\sum_{i=0}^{m} a_i x^{2i}, \end{aligned} \tag{9}$$

where $b \neq 0$ and a_i's are real coefficients. Eq. (9) has three fixed points: $(0,0)$ and $(\pm 1, 0)$. It is easy to use a linear analysis to show that the origin $(0,0)$ is a saddle point (with eigenvalues $\frac{1}{2}(-a_0 \pm \sqrt{a_0^2 + 2b^2})$). In order to have the two fixed points $(\pm 1, 0)$ being linear centres, the following condition:

$$\sum_{i=0}^{m} a_i = 0, \quad \text{or} \quad a_0^* = -\sum_{i=1}^{m} a_i, \tag{10}$$

must be satisfied, where * denotes the critical value of the coefficient. Then the eigenvalues of the Jacobian of system (9) evaluated at $(\pm 1, 0)$ are $\pm|b|i$. What we want to do is, for a given positive integer m, to choose appropriate values of a_i's such that system (9) has maximal limit cycles in the neighborhood of the two fixed points $(\pm 1, 0)$. This local analysis is based on the calculation of focus values or the normal forms associated with Hopf singularity.

Since we do not intend to discuss normal form computation in this paper, we assume that the normal form for the general system (9) has been obtained in the polar coordinates as follows (interested readers can find the details of normal form computation in [29]):

$$\dot{r} = r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2m}), \tag{11}$$

$$\dot{\theta} = \omega + t_1 r^2 + t_2 r^4 + \cdots + t_k r^{2m}, \tag{12}$$

up to the $(2m+1)$th-order term, where both v_i and t_i are expressed in terms of the original system's coefficients. v_i is called the ith-order focus value of the Hopf-type critical point.

The basic idea of finding k small limit cycles around a Hopf-type critical point is as follows: First, find the conditions such that $v_1 = v_2 = \cdots = v_{m-1} = 0$ ($v_0 = 0$ is automatically satisfied at the critical point), but $v_m \neq 0$, and then perform appropriate small perturbations to prove the existence of m limit cycles. This indicates that the procedure for finding multiple small limit cycles involves two steps: computing the focus values (i.e., computing the normal form) and solving the coupled non-linear polynomial equations: $v_1 = v_2 = \cdots = v_{m-1} = 0$. The sufficient conditions for the existence of small-amplitude limit cycles can be found in [13,17–19].

It should be noted that the coefficient b does not affect the results. In other words, different values of b (as long as they are not equal to zero) do not change the number of limit cycles. To show this, first introduce the following scalings:

$$a_i \Rightarrow b a_i, \quad i = 0, 1, \ldots, m. \tag{13}$$

Then, apply the transformation, given by

$$x = \pm(1 + u), \quad y = \pm b v, \tag{14}$$

and, in addition, the time scaling

$$\tau = bt, \tag{15}$$

into system (9) to obtain

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\tau} &= v, \\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= -u - \frac{3}{2}u^2 - \frac{1}{2}u^3 - v\sum_{i=0}^{m} a_i(1+u)^{2i}. \end{aligned} \tag{16}$$

The Jacobian matrix of system (16) evaluated at the origin $(u, v) = (0, 0)$ (i.e., at $(x, y) = (\pm 1, 0)$) is now in Jordan canonical form. The above procedure shows that the coefficient b can be chosen as any non-zero real values, which does not affect the qualitative behaviour of the system. In particular, it does not change the number of limit cycles. Thus, without loss of generality, one may assume $b = 1$, and so $\tau = t$. Thus, Eq. (16) can be written as

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u - \frac{3}{2}u^2 - \frac{1}{2}u^3 \\ &\quad - \left\{ a_0 + a_1\left[\binom{2}{0} + \binom{2}{1}u + \binom{2}{2}u^2\right] + a_2\left[\binom{4}{0} + \binom{4}{1}u + \binom{4}{2}u^2 + \binom{4}{3}u^3 + \binom{4}{4}u^4\right] \right. \\ &\qquad + \cdots \\ &\qquad + a_{m-1}\left[\binom{2m-2}{0} + \binom{2m-2}{1}u + \cdots \binom{2m-2}{2m-2}u^{2m-2}\right] \\ &\qquad \left. + a_m\left[\binom{2m}{0} + \binom{2m}{1}u + \cdots \binom{2m}{2m}u^{2m}\right]\right\} v \\ &= -u - \frac{3}{2}u^2 - \frac{1}{2}u^3 - \left\{ \left(\sum_{i=1}^{m}\binom{2i}{1}a_i\right)u + \left(\sum_{i=1}^{m}\binom{2i}{2}a_i\right)u^2 \right. \\ &\qquad + \left(\sum_{i=2}^{m}\binom{2i}{3}a_i\right)u^3 + \left(\sum_{i=2}^{m}\binom{2i}{4}a_i\right)u^4 \\ &\qquad + \cdots \\ &\qquad + \left(\sum_{i=m-1}^{m}\binom{2i}{2m-3}a_i\right)u^{2m-3} + \left(\sum_{i=m-1}^{m}\binom{2i}{2m-2}a_i\right)u^{2m-2} \\ &\qquad \left. + \left(\sum_{i=m}^{m}\binom{2i}{2m-1}a_i\right)u^{2m-1} + \left(\sum_{i=m}^{m}\binom{2i}{2m}a_i\right)u^{2m}\right\} v, \end{aligned} \tag{17}$$

where condition (10) has been used. Based on Eq. (17) we have the following observations.

(i) All the coefficients in Eq. (17) are linear combinations of the original coefficients $a_i, i = 1, 2, \ldots, m$.
(ii) If all the coefficients a_i's are equal to zero, then the original system (9) is integrable. Thus, without a_i's present, the two terms $-\frac{3}{2}u^2 - \frac{1}{2}u^3$ have no contribution to focus values.
(iii) Starting from the 4th-order term, each order has exactly one term in the form of $u^k v$. The highest term is $-a_m u^{2m} v$.

In the following, we use the early developed Maple program [29] to compute the focus values v_i ($i \geqslant 1$) of system (17), starting from $m = 1$. Note that the zeroth-order focus value of system (9) associated with the fixed points $(\pm 1, 0)$ is given by

$$v_0 = -\sum_{i=0}^{m} a_i, \tag{18}$$

and $v_0 = 0$ at the critical point $a_0 = a_0^*$, which is given in Eq. (10).

3.1. m = 1

For this case, only $a_1 \neq 0$. When $a_0 = a_0^*$, the first order focus value is given by

$$v_1 = \frac{1}{4} a_1 \neq 0. \tag{19}$$

Hence, system (17) can have at most one small limit cycle in the vicinity of the origin $(u, v) = (0, 0)$. In this case, $a_0^* = -a_1$. Thus, we may properly perturb a_0 from the critical value a_0^* to obtain exactly one small limit cycle. For example, suppose $0 < -a_1 \ll 1$. We then let $a_0 = a_0^* - \epsilon_1$, where $0 < \epsilon_1 \ll |a_1|$. Therefore, $v_0 = \epsilon_1$ and $0 < v_0 \ll -v_1 \ll 1$ which satisfy the sufficient conditions for the existence of small limit cycles [13,17,18]. The origin $(u, v) = (0, 0)$ is unstable and the bifurcating limit cycle is stable since $v_1 < 0$. Hence, the original system (9) has two small limit cycles, one around each of the two focus points, that is, $\bar{H}(2, 3) = 2$.

3.2. m = 2

For $m = 2$, both a_1 and a_2 are non-zero, and $a_0^* = -(a_1 + a_2)$. It is easy to show that v_1 is again given by Eq. (19). Thus, in order to have $v_1 = 0$, it is required that $a_1 = 0$. Then executing the Maple program [29] yields v_2, given by

$$v_2 = \frac{1}{4} a_2. \tag{20}$$

Therefore, system (17) can have at most two limit cycles near the origin. Similarly, we can perturb the coefficients a_1 and a_0 to obtain exactly two limit cycles. For example, let $0 < -a_2 \ll 1$. Then we choose $a_1 = \epsilon_1$ and $a_0 = -a_2 - \epsilon_1 + \epsilon_2$ in which $0 < \epsilon_2 \ll \epsilon_1 \ll |a_2| \ll 1$, implying that

$$0 < -v_0 \ll v_1 \ll -v_2 \ll 1.$$

Hence, the origin $(u, v) = (0, 0)$ is stable, the inner limit cycle is unstable while the outer limit cycle is stable. This shows that $\bar{H}(4, 3) = 4$ for system (9).

3.3. m = 3

Now, $a_i \neq 0$, $i = 1, 2, 3$, and $a_0^* = -(a_1 + a_2 + a_3)$. Executing the Maple program results in

$$v_1 = \frac{1}{4}(a_1 - 3a_3). \tag{21}$$

Setting $v_1 = 0$ yields

$$a_1^* = 3a_3. \tag{22}$$

Then v_2 can be found as

$$v_2 = \frac{1}{4}(a_2 + 5a_3). \tag{23}$$

In order to have $v_2 = 0$, we need to choose

$$a_2^* = -5a_3. \tag{24}$$

With the above chosen parameters, executing the Maple program gives

$$v_3 = -\frac{5}{16} a_3, \tag{25}$$

which implies that system (17) can have at most three limit cycles around the origin. Similarly, suppose $0 < a_3 \ll 1$. Then we find the following perturbations:

$$a_2 = -5a_3 + \epsilon_1, \quad a_1 = 3a_3 - \epsilon_2, \quad a_0 = a_3 - \epsilon_1 + \epsilon_2 - \epsilon_3,$$

where $0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_3$. Thus, the following inequalities hold:

$$0 < v_0 = \epsilon_3 \ll -v_1 = \frac{1}{4}\epsilon_2 \ll v_2 = \frac{1}{4}\epsilon_1 \ll -v_3 = \frac{5}{16}a_3 \ll 1,$$

which indicate that the sufficient conditions for the existence of small limit cycles are satisfied. Hence, system (17) has exactly three small limit cycles in the neighborhood of the origin, i.e., the original system (9) has six small limit cycles, $\bar{H}(6,3) = 6$. The stability of the bifurcating limit cycles can be easily obtained from the signs of the focus values. For the above chosen parameters, we know that the focus points are unstable, then the smallest limit is stable, the second one is unstable, and the largest one (the third one) is stable.

The above procedure can be processed for integers $m \geqslant 4$ to obtain the conditions under which the focus values equal zero and appropriate perturbations which show the existence of exact number of limit cycles. The procedure is similar to that used in this subsection and previous subsection. We shall omit the details but list the results in tables.

3.4. $m = 4$

For this case, $a_i \neq 0$, $i = 1, 2, \ldots, 4$, and $a_0 = -(a_1 + a_2 + a_3 + a_4)$. It can be shown that $\bar{H}(8,3) = 8$. The focus values, critical parameter values and the perturbed parameter values are listed in Table 2.

For the perturbed parameter values given in Table 2, we have

$$0 < -v_0 = \epsilon_4 \ll v_1 = \frac{1}{4}\epsilon_3 \ll -v_2 = \frac{1}{4}\epsilon_2 \ll v_3 = \frac{5}{16}\epsilon_3 \ll -v_4 = \frac{7}{16}a_4 \ll 1,$$

indicating that the sufficient conditions for the existence of small limit cycles are satisfied.

3.5. $m = 5, 6, 7$

In this subsection, we list the results for the cases $m = 5, 6, 7$ in Tables 3–5, respectively. The results show that $\bar{H}(2m,3) = 2m$ for $m = 5, 6, 7$.

When $m = 5$, the results given in Table 3 show that

$$0 < -v_0 = \epsilon_5 \ll v_1 = \frac{1}{4}\epsilon_4 \ll -v_2 = \frac{1}{4}\epsilon_3 \ll v_3 = \frac{5}{16}\epsilon_2 \ll -v_4 = \frac{7}{16}\epsilon_1 \ll v_5 = \frac{21}{32}a_{10} \ll 1,$$

where $0 < \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_5 \ll 1$. Therefore, the sufficient conditions for the existence of small limit cycles are satisfied and so system (9) can have exactly ten small limit cycles, five around each of the two symmetric focus points.

Table 2
The conditions for the limit cycles of system (9) for $m = 4$

Focus values	Critical parameter values	Perturbed parameter values
$v_0 = -(a_0 + a_1 + a_2 + a_3 + a_4)$	$a_0 = -(a_1 + a_2 + a_3 + a_4)$	$a_0 = a_4 - \epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4$
$v_1 = \frac{1}{4}(a_1 - 3a_3 - 8a_4)$	$a_1 = 3a_3 + 8a_4$	$a_1 = 8a_4 - 3\epsilon_1 + \epsilon_3$
$v_2 = \frac{1}{4}(a_2 + 5a_3 + 10a_4)$	$a_2 = -5a_3 - 10\ a_4$	$a_2 = -10a_4 + 5\epsilon_1 - \epsilon_2$
$v_3 = -\frac{5}{16}a_3$	$a_3 = 0$	$a_3 = -\epsilon_1$
$v_4 = -\frac{7}{16}a_4$	$0 < a_4 \ll 1$	$0 < \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_4$

Table 3
The conditions for the limit cycles of system (9) for $m = 5$

Focus values	Critical parameter values	Perturbed parameter values
$v_0 = -\sum_{i=0}^{5} a_i$	$a_0 = -\sum_{i=1}^{5} a_i$	$a_0 = -a_5 + \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5$
$v_1 = \frac{1}{4}(a_1 - 3a_3 - 8a_4 - 15a_5)$	$a_1 = 3a_3 + 8a_4 + 15a_5$	$a_1 = -15a_5 + 8\epsilon_1 - 3\epsilon_2 + \epsilon_4$
$v_2 = \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5)$	$a_2 = -(5a_3 + 10a_4 + 10a_5)$	$a_2 = 10a_5 - 10\epsilon_1 + 5\,\epsilon_2 - \epsilon_3$
$v_3 = -\frac{5}{16}(a_3 - 14a_5)$	$a_3 = 14a_5$	$a_3 = 10a_5 - \epsilon_2$
$v_4 = -\frac{7}{16}(a_4 + 9a_5)$	$a_4 = -9a_5$	$a_4 = -9a_5 + \epsilon_1$
$v_5 = \frac{21}{32}a_5$	$0 < a_5 \ll 1$	$0 < \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_5$

Table 4
The conditions for the limit cycles of system (9) for $m = 6$

Focus values	Critical parameter values	Perturbed parameter values
$v_0 = -\sum_{i=0}^{6} a_i$	$a_0 = -\sum_{i=1}^{6} a_i$	$a_0 = -a_6 + \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5 - \epsilon_6$
$v_1 = \frac{1}{4}(a_1 - 3a_3 - 8a_4 - 15a_5 - 24a_6)$	$a_1 = 3a_3 + 8a_4 + 15a_5 + 24a_6$	$a_1 = -24a_6 + 15\epsilon_1 - 8\epsilon_2 + 3\epsilon_3 - \epsilon_5$
$v_2 = \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5 - 5a_6)$	$a_2 = -5a_3 - 10a_4 - 10a_5 + 5a_6$	$a_2 = -5a_6 - 10\epsilon_1 + 10\epsilon_2 - 5\epsilon_3 + \epsilon_4$
$v_3 = -\frac{5}{16}(a_3 - 14a_5 - 56a_6)$	$a_3 = 14a_5 + 56a_6$	$a_3 = 56a_6 - 14\epsilon_1 + \epsilon_3$
$v_4 = -\frac{7}{16}(a_4 + 9a_5 + 27a_6)$	$a_4 = -9a_5 – 27a_6$	$a_4 = -27a_6 + 9\epsilon_1 - \epsilon_2$
$v_5 = \frac{21}{32}a_5$	$a_5 = 0$	$a_5 = -\epsilon_1$
$v_6 = \frac{33}{32}a_6$	$0 < a_6 \ll 1$	

Similarly, for $m = 6$ (see Table 4), we have

$$v_0 = \epsilon_6, \quad v_1 = -\frac{1}{4}\epsilon_5, \quad v_2 = \frac{1}{4}\epsilon_4, \quad v_3 = -\frac{5}{16}\epsilon_3, \quad v_4 = \frac{7}{16}\epsilon_2, \quad v_5 = -\frac{21}{32}\epsilon_1, \quad v_6 = \frac{33}{32}a_6,$$

where $0 < \epsilon_6 \ll< \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_6 \ll 1$. Hence, the sufficient conditions are again satisfied and the conclusion is true.

It follows from Table 5 that for $m = 7$,

$$v_0 = \epsilon_7, \quad v_1 = -\frac{1}{4}\epsilon_6, \quad v_2 = \frac{1}{4}\epsilon_5, \quad v_3 = -\frac{5}{16}\epsilon_4, \quad v_4 = \frac{7}{16}\epsilon_3, \quad v_5 = -\frac{21}{32}\epsilon_2, \quad v_6 = \frac{33}{32}\epsilon_1, \quad v_7 = -\frac{429}{256}a_7,$$

Table 5
The conditions for the limit cycles of system (9) for $m = 7$

Focus values	Critical parameter values	Perturbed parameter values
$v_0 = -\sum_{i=0}^{7} a_i$	$a_0 = -\sum_{i=1}^{7} a_i$	$a_0 = a_7 - \epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 - \epsilon_5 + \epsilon_6 - \epsilon_7$
$v_1 = \frac{1}{4}(a_1 - 3a_3 - 8a_4 - 15a_5 - 24a_6 - 35a_7)$	$a_1 = 3a_3 + 8a_4 + 15a_5 + 24a_6 + 35a_7$	$a_1 = 35a_7 - 24\epsilon_1 + 15\epsilon_2 - 8\epsilon_3 + 3\epsilon_4 - \epsilon_6$
$v_2 = \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5 - 5a_6 - 49a_7)$	$a_2 = -5a_3 - 10a_4 - 10a_5 + 5a_6 + 49a_7$	$a_2 = 49a_7 - 5\epsilon_1 - 10\epsilon_2 + 10\epsilon_3 - 5\epsilon_4 + \epsilon_5$
$v_3 = -\frac{5}{16}(a_3 - 14a_5 - 56a_6 - 133a_7)$	$a_3 = 14a_5 + 56a_6 + 133a_7$	$a_3 = -133a_7 + 56\epsilon_1 - 14 \epsilon_2 + \epsilon_4$
$v_4 = -\frac{7}{16}(a_4 + 9a_5 + 27a_6 + 27a_7)$	$a_4 = -9a_5 - 27a_6 - 27a_7$	$a_4 = 27a_7 - 27\epsilon_1 + 9\ \epsilon_2 - \epsilon_3$
$v_5 = \frac{21}{32}(a_5 - 33a_7)$	$a_5 = 33a_7$	$a_5 = 33a_7 - \epsilon_2$
$v_6 = \frac{33}{32}(a_6 + 13a_7)$	$a_6 = -13a_7$	$a_6 = -13a_7 + \epsilon_1$
$v_7 = -\frac{429}{256}a_7$	$0 < a_7 \ll 1$	

where $0 < \epsilon_7 \lll< \epsilon_6 \lll< \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_7 \ll 1$. This implies that the sufficient conditions for the existence of small limit cycles are satisfied, which proves that $\bar{H}(14, 3) = 14$.

It is noted from the above results that although the perturbations given for different cases are different (see Tables 2–5), the expressions for the perturbed focus values with a smaller integer m are identical to that in the focus values with a larger integer m. In other words, having obtained the perturbed focus values for $m = k$, one can directly use these values for the case $m = k + 1$, and only needs to find one more higher focus value.

3.6. $m = 8$

In this subsection, we consider the case $m = 8$, and will present a numerical example for this case to show that exact sixteen limit cycles can be obtained using appropriate perturbations. For this case, $a_0^* = -\sum_{i=1}^{8} a_i$ under which $v_0 = 0$. Then executing the Maple program [29] yields

$$v_1 = \frac{1}{4}(a_1 - 3a_3 - 8a_4 - 15a_5 - 24a_6 - 35a_7 - 48a_8). \tag{26}$$

Setting $v_1 = 0$ results in

$$a_1^* = 3a_3 + 8a_4 + 15a_5 + 24a_6 + 35a_7 + 48a_8. \tag{27}$$

Then, v_2 can be found as

$$v_2 = \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5 - 5a_6 - 49a_7 - 140a_8). \tag{28}$$

Hence, letting $v_2 = 0$ leads to

$$a_2^* = -(5a_3 + 10a_4 + 10a_5 - 5a_6 - 49a_7 - 140a_8). \tag{29}$$

Then, under the conditions (10), (27) and (29), one similarly obtains

$$v_3 = -\frac{5}{16}(a_3 - 14a_5 - 56a_6 - 133a_7 - 224a_8), \tag{30}$$

which, in turn, yields

$$a_3^* = 14a_5 + 56a_6 + 133a_7 + 224a_8 \tag{31}$$

in order to have $v_3 = 0$. Similarly, one may find

$$v_4 = -\frac{7}{16}(a_4 + 9a_5 + 27a_6 + 27a_7 - 90a_8), \tag{32}$$

and thus

$$a_4^* = -(9a_5 + 27a_6 + 27a_7 - 90a_8) \tag{33}$$

under which $v_4 = 0$.

Processing the above procedure further yields the following results:

$$\begin{aligned} v_5 &= \frac{21}{32}(a_5 - 33a_7 - 176a_8) \quad \text{with} \quad a_5^* = 33a_7 + 176a_8; \\ v_6 &= \frac{33}{32}(a_6 + 13a_7 + 52a_8) \quad \text{with} \quad a_6^* = -(13a_7 + 52a_8); \\ v_7 &= -\frac{429}{256}a_7 \quad \text{with} \quad a_7^* = 0; \end{aligned}$$

and

$$v_8 = -\frac{715}{256}a_8.$$

For convenience, we rewrite the critical values of the coefficients in a reverse order as follows:

$$\begin{aligned} a_7^* &= 0, \\ a_6^* &= -(13a_7 + 52a_8), \\ a_5^* &= 33a_7 + 176a_8, \\ a_4^* &= -(9a_5 + 27a_6 + 27a_7 - 90a_8), \\ a_3^* &= 14a_5 + 56a_6 + 133a_7 + 224a_8, \\ a_2^* &= -(5a_3 + 10a_4 + 10a_5 - 5a_6 - 49a_7 - 140a_8), \\ a_1^* &= 3a_3 + 8a_4 + 15a_5 + 24a_6 + 35a_7 + 48a_8, \\ a_0^* &= -(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8), \end{aligned} \tag{34}$$

under which $v_i = 0$, $i = 0, 1, \ldots, 7$.

Next, we want to perform appropriate perturbations to the critical parameter values to obtain exactly eight limit cycles around each of the two fine focus points $(\pm 1, 0)$. Without loss of generality, assume $0 < a_8 \ll 1$. Then we need to find perturbations to a_7, a_6, a_5, a_4, a_3, a_2, a_1 and a_0 such that

$$0 < -v_0 \ll v_1 \ll -v_2 \ll v_3 \ll -v_4 \ll v_5 \ll -v_6 \ll v_7 \ll -v_8 \ll 1.$$

Note that all the focus values v_i, $i = 0, 1, \ldots, 8$ are given in linear forms of the coefficients a_i. Further, consider back order perturbations one by one: First on a_7 for v_7, then on a_6 for v_6, and so on until on a_0 for v_0. Therefore, the perturbation procedure is straightforward. Since $a_7^* = 0$, one may choose $a_7 = a_7^* - \epsilon_1 = -\epsilon_1$ $(0 < \epsilon_1 \ll a_8)$ so that $0 < v_7 = \frac{429}{256}\epsilon_1 \ll -v_8 = \frac{715}{256}a_8$.

Next consider perturbing v_6. Noticing that $\frac{\partial v_6}{\partial a_6} = \frac{33}{32} > 0$, one may perturb $a_6 = -(13a_7 + 52a_8)$ to $a_6 = -(13a_7 + 52a_8) - \epsilon_2$ $(0 < \epsilon_2 \ll \epsilon_1)$, and thus $v_6 = -\frac{33}{32}\epsilon_2 < 0$, and $0 < -v_6 \ll v_7$. This procedure can be systematically carried out until reaching the last parameter a_0 for v_0. The results for this case are summarized in the following theorem.

Theorem 1. *Given the Liénard equation (9) for* $m = 8$, *which has a saddle point at the origin and a pair of symmetric fine focus points at* $(x, y) = (\pm 1, 0)$, *under the condition* $b \neq 0$ *and* $a_0 = -\sum_{i=1}^{8} a_i$. *Further, without loss of generality, take* $b = 1$, *and perturb* a_7, a_6, a_5, a_4, a_3, a_2, a_1 *and* a_0 *as*

$$\begin{aligned} a_7 &= a_7^* - \epsilon_1 = -\epsilon_1, \\ a_6 &= a_6^* - \epsilon_2 = -52a_8 + 13\epsilon_1 - \epsilon_2, \\ a_5 &= a_5^* + \epsilon_3 = 176a_8 - 33\epsilon_1 + \epsilon_3, \\ a_4 &= a_4^* + \epsilon_4 = -90a_8 - 27\epsilon_1 + 27\epsilon_2 - 9\epsilon_3 + \epsilon_4, \\ a_3 &= a_4^* - \epsilon_5 = -224a_8 + 133\epsilon_1 - 56\epsilon_2 + 14\epsilon_3 - \epsilon_5, \\ a_2 &= a_2^* - \epsilon_6 = 140a_8 - 49\epsilon_1 + 5\epsilon_2 + 10\epsilon_3 - 10\epsilon_4 + \epsilon_5 - \epsilon_6, \\ a_1 &= a_1^* + \epsilon_7 = 48a_8 - 35\epsilon_1 + 24\epsilon_2 - 15\epsilon_3 + 8\epsilon_4 - 3\epsilon_5 + \epsilon_7, \\ a_0 &= a_0^* + \epsilon_8 = a_8 - \epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 - \epsilon_5 + \epsilon_6 - \epsilon_7 + \epsilon_8, \end{aligned} \tag{35}$$

where $0 < \epsilon_8 \ll \epsilon_7 \ll \epsilon_6 \ll \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_8 \ll 1$, *then system* (9) *has exactly sixteen small limit cycles.*

Proof. First note that

$$0 < -v_8 = \frac{715}{256} a_8 \ll 1$$

since $0 < a_8 \ll 1$. Then for the given perturbation $a_7 = -\epsilon_1$, we have

$$0 < v_7 = \frac{429}{256}\epsilon_1 \ll -v_8 = \frac{715}{256} a_8,$$

where ϵ_1 is chosen such that $0 < \epsilon_1 \ll a_8$. Similarly, for $a_6 = -52a_8 + 13\epsilon_1 - \epsilon_2$, we obtain

$$v_6 = -\frac{33}{32}\epsilon_2,$$

where $0 < \epsilon_2 \ll \epsilon_1 \ll 1$, and so $0 < -v_6 \ll v_7$.

Next, for the perturbed parameter values given in Eq. (35), we have

$$v_5 = \frac{21}{32}(a_5 - 33a_7 - 176a_8) = \frac{21}{32}\epsilon_3.$$

Hence, by choosing $0 < \epsilon_3 \ll \epsilon_2$, one obtains $0 < v_5 \ll -v_6$.

For v_4, we find

$$v_4 = -\frac{7}{16}(a_4 + 9a_5 + 27a_6 + 27a_7 - 90a_8) = -\frac{7}{16}\epsilon_4.$$

Then one may select $0 < \epsilon_4 \ll \epsilon_3$ so that $0 < -v_4 \ll v_5$. Next, similarly, we obtain

$$v_3 = -\frac{5}{16}(a_3 - 14a_5 - 56a_6 - 133a_7 - 224a_8) = \frac{5}{16}\epsilon_5 \ll -v_4,$$

by choosing $0 < \epsilon_5 \ll \epsilon_4$. And

$$v_2 = \frac{1}{4}(a_2 + 5a_3 + 10a_4 + 10a_5 - 5a_6 - 49a_7 - 140a_8) = -\frac{1}{4}\epsilon_6,$$

which gives $0 < -v_2 \ll v_3$ if $\epsilon_6 \ll \epsilon_5$. Further, for v_1, one obtains

$$v_1 = \frac{1}{4}(a_1 - 3a_3 - 8a_4 - 15a_5 - 24a_6 - 35a_7 - 48a_8) = \frac{1}{4}\epsilon_7 \ll -v_2,$$

provided $0 < \epsilon_7 \ll \epsilon_6$.

Finally, substituting the parameter values given in Eq. (35) into v_0 yields

$$v_0 = -(a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8) = -\epsilon_8.$$

Thus, $0 < -v_0 \ll v_1$ when $0 < \epsilon_8 \ll \epsilon_7$.

Summarizing the above perturbation results gives

$$\begin{aligned} 0 < -v_0 = \epsilon_8 \ll v_1 = \frac{1}{4}\epsilon_7 \ll -v_2 = \frac{1}{4}\epsilon_6 \ll v_3 = \frac{5}{16}\epsilon_5 \ll -v_4 = \frac{7}{16}\epsilon_4 \ll v_5 = \frac{21}{32}\epsilon_3 \ll -v_6 = \frac{33}{32}\epsilon_2 \\ \ll v_7 = \frac{429}{256}\epsilon_1 \ll -v_8 = \frac{715}{256}a_8 \ll 1, \end{aligned}$$

where $0 < \epsilon_8 \ll \epsilon_7 \ll \epsilon_6 \ll \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll a_8 \ll 1$. Therefore, the sufficient conditions for the existence of small limit cycles [13,17,18] are satisfied and so system (9) can have eight small-amplitude limit cycles near each of the two fine focus points $(\pm 1, 0)$. $\square$

To end this section, we present a numerical example of choosing proper perturbations to have sixteen small limit cycles. Let $b = 1$ and

$$a_8 = 0.02 \quad \Rightarrow \quad v_8 = -0.055859375, \tag{36}$$

and further choose the following perturbations:

$$\begin{aligned}
\epsilon_1 &= 0.1 \times 10^{-2} \Rightarrow v_7 = 0.167578125 \times 10^{-2},\\
\epsilon_2 &= 0.1 \times 10^{-4} \Rightarrow v_6 = -0.103125 \times 10^{-4},\\
\epsilon_3 &= 0.1 \times 10^{-7} \Rightarrow v_5 = 0.65625 \times 10^{-8},\\
\epsilon_4 &= 0.2 \times 10^{-11} \Rightarrow v_4 = -0.875 \times 10^{-12},\\
\epsilon_5 &= 0.1 \times 10^{-15} \Rightarrow v_3 = 0.3125 \times 10^{-16},\\
\epsilon_6 &= 0.1 \times 10^{-20} \Rightarrow v_2 = -0.25 \times 10^{-21},\\
\epsilon_7 &= 0.1 \times 10^{-26} \Rightarrow v_1 = 0.25 \times 10^{-27},\\
\epsilon_8 &= 0.1 \times 10^{-34} \Rightarrow v_0 = -0.1 \times 10^{-34}.
\end{aligned} \tag{37}$$

Then, the normal form (11) associated with the fine focus points $(\pm 1, 0)$ up to term r^{17} becomes

$$\begin{aligned}
\dot{r} = r(&-0.1 \times 10^{-34} + 0.25 \times 10^{-27} r^2 - 0.25 \times 10^{-21} r^4 + 0.3125 \times 10^{-16} r^6 - 0.875 \times 10^{-12} r^8 + 0.65625\\
&\times 10^{-8} r^{10} - 0.103125 \times 10^{-4} r^{12} + 0.167578125 \times 10^{-2} r^{14} - 0.055859375 r^{16}).
\end{aligned} \tag{38}$$

Numerically solving the polynomial equation $\dot{r} = 0$ yields the following eight positive roots for r:

$$\begin{aligned}
r_1 &= 0.20428541169921635282582009364207759574193326249016 \times 10^{-3},\\
r_2 &= 0.10557450404492965285350949490871952069267135033321 \times 10^{-2},\\
r_3 &= 0.30863297664155462969625931143584766431552752527455 \times 10^{-2},\\
r_4 &= 0.63246745671122582379747204119182923242615141366736 \times 10^{-2},\\
r_5 &= 0.10748050029906136301169392775166208250728875477932 \times 10^{-1},\\
r_6 &= 0.23087718506626126654947264909323427140614083498976 \times 10^{-1},\\
r_7 &= 0.86815345019225265714510542942016780037491378781084 \times 10^{-1},\\
r_8 &= 0.14752575800171135311165842026036596278036792799907,
\end{aligned} \tag{39}$$

which are the approximate solutions for the amplitudes of periodic solutions (limit cycles) including both stable and unstable motions. The above solutions are solved using Maple up to 50 decimal places.

Under the perturbations given in Eq. (37), the perturbed parameter values are

$$\begin{aligned}
a_0 &= 0.019009990001999900000999999900000001,\\
a_1 &= 0.92523985001599970000000000001,\\
a_2 &= 2.75105009998000004999999,\\
a_3 &= -4.3475598600000001,\\
a_4 &= -1.826730089998,\\
a_5 &= 3.48700001,\\
a_6 &= -1.02701,\\
a_7 &= -0.001,\\
a_8 &= 0.02.
\end{aligned} \tag{40}$$

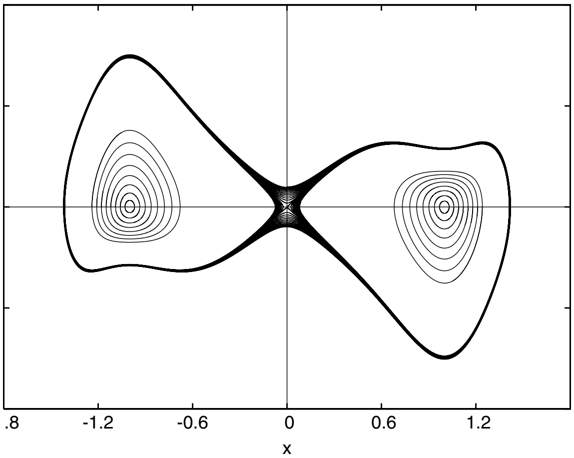

Fig. 1. The phase portrait of system (9) showing sixteen small limit cycles around the fine focus points $(\pm 1, 0)$ under the perturbed parameter values: $b = 1$, $a_0 = 0.019009990001999900000999999900000001$, $a_1 = 0.92523985001599970000000000001$, $a_2 = 2.751050099980000499999$, $a_3 = -4.3475598600000001$, $a_4 = -1.826730089998$, $a_5 = 3.48700001$, $a_6 = -1.02701$, $a_7 = -0.001$, $a_8 = 0.02$.

Table 6
Comparison between the number of limit cycles in the vicinity of one focus point and two focus points

deg(*f*)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Focus point $(0,0)$	1	2	2	4	4	6	6	6	8	8	8	10	10	12	12	12	14	14	14	16
Focus points $(\pm 1\ 0)$		2		4		6		8		10		12		14		16		18		20

For the parameter values given in Eq. (40), the phase portrait for system (9) obtained from computer simulation is shown in Fig. 1, where sixteen small limit cycles are depicted near the two fine focus points $(\pm 1, 0)$. It should be pointed out that the trajectories which are not near the two points $(\pm 1, 0)$ can be obtained quite accurately using numerical simulations. However, one cannot obtain the computer simulation results for the small limit cycles since the accuracy of some parameters is even higher than the machine precision. That is why one must employ certain theoretical approach (like the one presented in this paper) to prove the existence of small limit cycles. In fact, in the neighborhood of a highly degenerate focus point, trajectories behave like centres, as shown in Fig. 1. The stabilities of these small limit cycles can be easily determined by the signs of the focus values. For convenience, let these small limit cycles be named, from the smallest to the largest, as $l_1, l_2, \ldots, l_8$. Since $v_0 < 0$, the focus points $(\pm 1, 0)$ are stable. Then the smallest limit cycle l_1 is unstable, and thus l_2 is table, and so on. The largest one is stable.

Remark. We have also considered the cases $m = 9$ and $m = 10$, and showed that both the two cases have the same results. That is, $\bar{H}(2m, 3) = 2m$. A comparison between the results presented in this paper with the existing results obtained with the restriction to the origin is given in Table 6.

Finally, we present a conjecture for this problem as follows.

Conjecture. *For any integer* $m \geqslant 1$, *the maximal number of small limit cycles that system* (9) *can have is* $2m$, *i.e.*, $\bar{H}(2m, 3) = 2m$, *and therefore*, $H(2m, 3) \geqslant 2m$.

4. Conclusion

In this paper, we have investigated a class of Liénard equations with Z_2 symmetry. The system has a saddle point at the origin and two linear centres on the x-axis which are symmetric with the origin. Ten cases are considered and exact limit cycles are obtained using appropriate perturbations. It has been shown that $\bar{H}(2m, 3) = 2m$ for $m = 1, 2, \ldots, 10$. It is conjectured that $\bar{H}(2m, 3) = 2m$ for all integers $m \geqslant 1$.

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