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Bifurcation of limit cycles in quadratic Hamiltonian systems with various degree polynomial perturbations

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ABSTRACT

In this paper, we consider bifurcation of limit cycles in planar quadratic Hamiltonian systems with various degree polynomial perturbations. Attention is focused on the limit cycles which may appear in the vicinity of an isolated center, and up to 20th-degree polynomial perturbations are investigated. Restricted to the first-order Melnikov function, the method of focus value computation is used to determine the maximal number, $H_2(n)$, of small-amplitude limit cycles which may exist in the neighborhood of such a center. Besides the existing results $H_2(2) = 2$ and $H_2(3) = 5$, we shall show that $H_2(n) = \left[\frac{4}{3}(n+1)\right]$ for $n = 3, 4, \ldots, 20$.

1. Introduction

The study of the well-known Hilbert's 16th problem [1] is still very active though it has not been completely solved even for quadratic systems, since it has great impact on the development of modern mathematics. Consider the following planar system:

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \tag{1}$$

where $P_n(x,y)$ and $Q_n(x,y)$ denote nth-degree polynomials of x and y. Roughly speaking, the second part of Hilbert's 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that system (1) can have. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [2].

To help understand and attack the problem the so called weak Hilbert's 16th problem was posed by Arnold [3]. The problem is to ask for the maximal number of isolated zeros of the Abelian integral or Melnikov function:

$$M(h,\delta) = \int_{H(x,y)=h} Q_n\,dx - P_n\,dy, \tag{2}$$

where $H(x,y)$, P_n and Q_n are all real polynomials of x and y with $\deg H = n + 1$, and $\max\{\deg P_n, \deg Q_n\} \leqslant n$. The weak Hilbert's 16th problem itself is a very important and interesting problem, closely related to the following near-Hamiltonian system [4]:

$$\dot{x} = H_y(x,y) + \varepsilon p_n(x,y), \quad \dot{y} = -H_x(x,y) + \varepsilon q_n(x,y), \tag{3}$$

where $H(x,y)$, $p_n(x,y)$ and $q_n(x,y)$ are all polynomial functions of x and y, and $0 < \varepsilon \ll 1$ is a small perturbation. Studying the bifurcation of limit cycles for such a system can be transformed to investigating the zeros of the Melinikov function.

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On the other hand, if the problem is restricted to a neighborhood of isolated fixed points with Hopf singularity, the problem is then reduced to studying degenerate Hopf bifurcations, leading to computation of focus values, and many results have been obtained (e.g., see [5–8]). Alternatively, this is equivalent to computing the normal form of differential equations associated with Hopf or degenerate Hopf bifurcations. Suppose the origin of system (1), $(x,y) = (0,0)$, is an element center. Without loss of generality, we may assume that the eigenvalues of the Jacobian of system (1) evaluated at the origin are a purely imaginary pair, $\pm i$. Further, suppose the normal form associated with this Hopf singularity is given in polar coordinates (obtained by using, say, the method given in [9]):

$$\dot{r} = r\ (v_0 + v_1\ r^2 + v_2\ r^4 + \cdots + v_k\ r^{2k} + \cdots), \tag{4}$$

$$\dot{\theta} = 1 + t_1\ r^2 + t_2\ r^4 + \cdots + t_k\ r^{2k} + \cdots, \tag{5}$$

where r and θ represent, respectively, the amplitude and phase of the limit cycles, and v_i, $i = 0,1,2,\ldots$ are called focus values.

The basic idea of finding k small limit cycles around the origin is as follows: First, find the conditions such that $v_0 = v_1 = v_2 = \cdots = v_{k-1} = 0$, but $v_k \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. For general quadratic system (1) ($n = 2$), in 1952, Bautin [5] proved that there exist 3 small limit cycles around a fine focus point or a center. Until the end of 1970's, concrete examples are given to show that quadratic systems can have 4 limit cycles [10,11], which have (3,1) configuration: three limit cycles enclose a fine focus point, while one limit cycle encloses another element focus point.

In this paper, attention is focused on bifurcation of limit cycles in quadratic Hamiltonian system with various degree polynomial perturbations, and in particular on the limit cycles which bifurcate from a center. Without loss of generality, we may assume that system $(3)_{\varepsilon=0}$ has a center at the origin $(x,y) = (0,0)$. For this problem, we may either use the Melnikov method or the focus value method to determine the number of limit cycles in the vicinity of the origin. In this paper, we will apply the method of focus value computation to study the bifurcation of limit cycles. For $\varepsilon \neq 0$, the focus values of system (3) at the origin can be written in the form of

$$v_i = \bar{v}_i + \varepsilon\ \tilde{v}_i + O(\varepsilon^2), \quad i = 0,1,2,\ldots, \tag{6}$$

where $\bar{v}_i = 0,\ i = 0,1,2,\ldots$ due to the origin being a center. Thus, for sufficiently small ε, we may use $\tilde{v}_i$ to determine the number of small-amplitude limit cycles bifurcating from the origin.

For quadratic Hamiltonian system with 2nd-degree polynomial perturbation, there are many results in the literature, mainly published since the 90's of last century. The conclusion is: quadratic Hamiltonian systems with 2nd-degree polynomial perturbation can have maximal two limit cycles, i.e., $H_2(2) = 2$ [12,13], where the subscript 2 denotes second-order Hamiltonian systems. Recently, it has been shown that quadratic Hamiltonian systems with 3rd-degree polynomial perturbation can have maximal five limit cycles in the vicinity of a center, i.e., $H_2(3) = 5$[14]. In [14], the Melnikov function method is used, on the basis of the following Melnikov function:

$$M(h) = \oint_{L_h} q_n\, dx - p_n\, dy = \int\int_{D(h)} \left(\frac{\partial p_n}{\partial x} + \frac{\partial q_n}{\partial y}\right) dx\, dy, \tag{7}$$

where L_h is a contour around the origin, and $D(h)$ is the region bounded by the contour. Suppose the general polynomial functions $p_n(x,y)$ and $q_n(x,y)$ are given by

$$p_n(x,y) = \sum_{1 \leqslant i+j \leqslant n} a_{ij}\ x^i\ y^j, \quad q_n(x,y) = \sum_{1 \leqslant i+j \leqslant n} b_{ij}\ x^i\ y^j. \tag{8}$$

Then,

$$\frac{\partial p_n}{\partial x} + \frac{\partial q_n}{\partial y} = \sum_{1 \leqslant i+j \leqslant n} i\ a_{ij}\ x^{i-1}\ y^j + \sum_{1 \leqslant i+j \leqslant n} j\ b_{ij}\ x^i\ y^{j-1} = \sum_{0 \leqslant i+j \leqslant n-1} [(i+1)\ a_{(i+1)j} + (j+1)\ b_{i(j+1)}] x^i\ y^j. \tag{9}$$

Thus, without loss of generality, we may assume $a_{ij} = 0$ (i.e., $p_n(x,y) \equiv 0$), and $b_{i0} = 0$.

For a comparison, in this paper we use the method of focus value computation to obtain the same result for $n = 3$. Further, we will consider bifurcation of limit cycles for quadratic Hamiltonian systems with 3rd- to 20th-degree polynomial perturbations. More precisely, we have the following main theorem:

Theorem 1. *A quadratic Hamiltonian system with nth-degree polynomial perturbation can have maximal* $\left[\frac{4}{3}(n+1)\right]$ *small limit-amplitude cycles bifurcating from a center, i.e.,* $H_2(n) = \left[\frac{4}{3}(n+1)\right],\ 3 \leqslant n \leqslant 20$, *where [·] denotes the integer part.*

It should be noted that the case $n = 0$ is meaningless since no perturbation is used, while the cases $n = 1$ and $n = 2$ are known: $H_2(1) = 0$ and $H_2(2) = 2$. Since these two results do not satisfy the formula $H_2(n) = \left[\frac{4}{3}(n+1)\right]$, and are thus excluded from Theorem 1.

The rest of the paper is organized as follows. In the next section, we describe general formula for quadratic Hamiltonian systems and apply the method of focus value computation to re-investigate the cases with 2nd- and 3rd-degree polynomial perturbations, confirming the results $H_2(2) = 2$ and $H_2(3) = 5$. The cases of 4th- to 8th-degree polynomial perturbations are considered in Sections 3, and the results for the cases of 9th- to 20th-degree polynomial perturbations are summarized in Section 4. Conclusion is drawn in Section 5.

2. Quadratic Hamiltonian systems with 2nd- and 3rd-degree polynomial perturbations

In this section, we first derive a simple generic quadratic Hamiltonian, and then re-derive the results of $H_2(2) = 2$ and $H_2(3) = 5$.

2.1. Quadratic Hamiltonian system

To obtain a simple generic quadratic Hamiltonian system, we start from the following general quadratic system:

$$\begin{aligned}\frac{dz_1}{dt} &= c_{100} + c_{110}\, z_1 + c_{101}\, z_2 + c_{120}\, z_1^2 + c_{111}\, z_1\, z_2 + c_{102}\, z_2^2,\\ \frac{dz_2}{dt} &= c_{200} + c_{210}\, z_1 + c_{201}\, z_2 + c_{220}\, z_1^2 + c_{211}\, z_1\, z_2 + c_{202}\, z_2^2,\end{aligned} \tag{10}$$

where c_{ijk}'s are real constant parameters. It is easy to show that this system has at most four singularities, or more precisely, it can have 0, 2 or 4 singularities in real domain. In order for system (10) to have limit cycles, the system must have some singularity. In this paper, we assume that system (10) has at least two singularities. Without loss of generality, we may assume that one singular point is located at the origin (0,0), which implies $c_{100} = c_{200} = 0$, and the second one at (p,q). Further assume the origin is an element center (i.e., the linear part of the system has a center). Then introducing a series of linear transformations, parameter rescaling and time rescaling to system (10) yields the following general quadratic system:

$$\begin{aligned}\tfrac{dx}{dt} &= y + a_1\, x\, y + a_2\, y^2,\\ \tfrac{dy}{dt} &= -x + a_3\, x^2 + a_4\, x\, y + a_5\, y^2,\end{aligned} \tag{11}$$

which has an element center at the origin (0,0) and another singularity at $\left(\frac{1}{a_3}, 0\right)$. System (11) becomes a Hamiltonian system when $a_4 = 0$ and $a_5 = -\frac{1}{2}\, a_1$. Thus, we obtain the following quadratic Hamiltonian system:

$$\begin{aligned}\tfrac{dx}{dt} &= y + a_1\, x\, y + a_2\, y^2,\\ \tfrac{dy}{dt} &= -x + a_3\, x^2 - \tfrac{1}{2}\, a_1\, y^2,\end{aligned} \tag{12}$$

with

$$\left(\frac{1}{a_3}, 0\right) \text{ being a } \begin{cases} \text{center} & \text{if } a_1 < -a_3,\\ \text{saddle point} & \text{if } a_1 > -a_3.\end{cases}$$

The details for deriving (12) from (10) can be found in [15].

Since we are interested in the limit cycles bifurcating from the center (0,0), we will ignore whether the singular point $\left(\frac{1}{a_3}, 0\right)$ is a center or a saddle point. The Hamiltonian of system (12) is given by

$$H(x,y) = \frac{1}{2}\,(x^2 + y^2) - \frac{1}{3}\, a_3\, x^3 + \frac{1}{2}\, a_1\, x\, y^2 + \frac{1}{3}\, a_2\, y^3, \tag{13}$$

In this paper, we concentrate on the generic case: $a_1\, a_2\, a_3 \neq 0$. Note that $a_3 = 0$ implies that the second singular point $\left(\frac{1}{a_3}, 0\right)$ is at the infinity. For $a_3 \neq 0$, we can simply use a scaling to move the singular point $\left(\frac{1}{a_3}, 0\right)$ to (1,0). Alternatively, we may simply set $a_3 = 1$. Thus, in the following analysis, $a_3 = 1$, which does not affect the analysis and results.

The perturbed Hamiltonian system or near-Hamiltonian system of (12) can be generally written as

$$\begin{aligned}\tfrac{dx}{dt} &= y + a_1\, x\, y + a_2\, y^2 + \varepsilon\, p_n(x,y),\\ \tfrac{dy}{dt} &= -x + x^2 - \tfrac{1}{2}\, a_1\, y^2 + \varepsilon\, q_n(x,y),\end{aligned} \tag{14}$$

where, as discussed in the introduction, we have assumed that $p_n(x,y) \equiv 0$ and $q_n(x,y)$ is given in (8) with $b_{i0} = 0$.

2.2. Quadratic Hamiltonian system with 2nd-degree polynomial perturbation

When $n = 2$, $q_2(x,y)$ is given by

$$q_2(x,y) = b_{01}\, y + b_{11}\, x\, y + b_{02}\, y^2. \tag{15}$$

For this case, we have

Theorem 2. *The quadratic near-Hamiltonian system (14) with 2nd-degree polynomial perturbation can have maximal two small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(2) = 2$.*

Proof. First of all, in order to satisfy the condition such that the origin (0,0) is an element center, it requires

$$\tilde{v}_0 = \frac{1}{2}\, b_{01} = 0 \;\Rightarrow\; b_{01} = 0. \tag{16}$$

Then, applying the Maple program [9] to system (14) with $q_2(x,y)$ given in (15), we obtain the focus value $\tilde{v}_1$, given by

$$\tilde{v}_1 = -\frac{1}{16}\,[(a_1-2)\,b_{11} + 4\,a_2\,b_{02}]. \tag{17}$$

We may solve b_{02} from the equation $\tilde{v}_1 = 0$, as

$$b_{02} = -\frac{(a_1-2)}{4\,a_2}\,b_{11}. \tag{18}$$

and then we have the following simplified focus values:

$$\begin{aligned}
\tilde{v}_2 &= \tfrac{5}{192}\,b_{11}\left[(a_1+1)\,(a_1-2)^2 - 4\,a_2^2\right],\\
\tilde{v}_3 &= \tfrac{5}{110592}\,b_{11}\left[(a_1+1)(a_1-2)^2 - 4a_2^2\right](513a_1^2 - 228a_1 + 388 + 868a_2^2),\\
\tilde{v}_4 &= \tfrac{1}{31850496}\,b_{11}\left[(a_1+1)(a_1-2)^2 - 4a_2^2\right] \times \left[651969a_1^4 - 729000a_1^3 + 746424a_1^2 - 324384a_1 + 692368\right.\\
&\quad \left.+ a_2^2(2759400a_1^2 - 871584a_1 + 957728 + 2199568a_2^2)\right],\\
&\;\;\vdots
\end{aligned} \tag{19}$$

It is seen that letting $\tilde{v}_2 = 0$ yields $\tilde{v}_i = 0,\ i = 3,\ 4, \ldots$, leading to a center. In fact, in general, if $b_{11} = 0$ and so $b_{02} = 0$, then system (12) is reduced to unperturbed Hamiltonian system. Hence, for the quadratic perturbation, one may set $\tilde{v}_0 = \tilde{v}_1 = 0$ by choosing $b_{01} = 0$ and b_{02} given in (18), but $\tilde{v}_2 \neq 0$. This indicates that we can have at most two small-amplitude limit cycles around the center (0,0). Further, it is easy to show that by proper perturbations to b_{02} and b_{01}, one can obtain two small-amplitude limit cycles, and hence $H_2(2) = 2$. □

2.3. Quadratic Hamiltonian system with 3rd-degree polynomial perturbation

Now we turn to consider the case $n = 3$, for which $q_3(x,y)$ is given by

$$q_3(x,y) = b_{01}\,y + b_{11}\,x\,y + b_{02}\,y^2 + b_{21}\,x^2\,y + b_{12}\,x\,y^2 + b_{03}\,y^3. \tag{20}$$

For this case, we have the following theorem.

Theorem 3. *The quadratic near-Hamiltonian* system (14)*with 3rd-degree polynomial perturbation can have maximal five small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(3) = 5$.*

Proof. Again set $b_{01} = 0$ in order for the origin (0,0) of the perturbed Hamiltonian system (14) to be an element center, under which $\tilde{v}_0 = 0$. Then, applying the Maple program to system (14) with $q_3(x,y)$ given in (20), we obtain the focus value $\tilde{v}_1$ as

$$\tilde{v}_1 = -\frac{1}{16}\,[(a_1-2)\,b_{11} + 4\,a_2\,b_{02} - 2\,b_{21} - 6\,b_{03}]. \tag{21}$$

Solving (21) for b_{03} results in

$$b_{03} = \frac{1}{6}[(a_1-2)\,b_{11} + 4\,a_2\,b_{02} - 2\,b_{21}]. \tag{22}$$

Then, we obtain $\tilde{v}_2$ as follows:

$$\tilde{v}_2 = -\frac{1}{48}a_2(5a_1-2)\,(a_1b_{02} - b_{12}) - \frac{1}{192}\left[20a_2^2 + (3a_2+10)(a_1-2)\right]\,(b_{11}+b_{21}). \tag{23}$$

First, suppose $a_1 \neq \frac{2}{5}$. We solve $\tilde{v}_2 = 0$ to obtain

$$b_{12} = a_1\,b_{02} + \frac{20\,a_2^2 + (3\,a_2+10)(a_1-2)}{4\,a_2\,(5\,a_1-2)}\,(b_{11}+b_{21}), \tag{24}$$

under which $\tilde{v}_3,\ \tilde{v}_4$, etc. become

$$
\begin{aligned}
\tilde{v}_3 &= \tfrac{35}{3072\ (5\ a_1-2)}\ \left[(a_1+1)(a_1-2)^2-4a_2^2\right](b_{11}+b_{21})\left(3a_1^2+12a_1-4-4a_2^2\right),\\
\tilde{v}_4 &= \tfrac{35}{442368(5a_1-2)}\ \left[(a_1+1)(a_1-2)^2-4a_2^2\right](b_{11}+b_{21})\left[2673a_1^4+7740a_1^3\right.\\
&\quad \left.-9312a_1^2+8592a_1-2704+a_2^2\left(480a_1^2+22992a_1-5024-7504a_2^2\right)\right],\\
\tilde{v}_5 &= \tfrac{7}{1019215872\ (5\ a_1-2)}\ \left[(a_1+1)\ (a_1-2)^2-4a_2^2\right]\ (b_{11}+b_{21})\\
&\quad \times\left\{6673023a_1^6+12848868a_1^5-35889804a_1^4+39108960a_1^3\right.\\
&\quad -28051824a_1^2+26599488a_1-7993664+a_2^2\left[18325764\ a_1^4\right.\\
&\quad +120357792a_1^3-102912288a_1^2+40278144a_1-11246016\\
&\quad \left.\left.-a_2^2\left(26386608a_1^2-141384768a_1+17966016+44573504a_2^2\right)\right]\right\}\\
&\ \ \vdots
\end{aligned}
\tag{25}
$$

There is a common factor $\left[(a_1+1)\ (a_1-2)^2-4a_2^2\right](b_{11}+b_{21})$ in the expressions of $\tilde{v}_i$. Setting this factor to equal zero leads to a center. Thus, in order for it to be non zero, let

$$a_2=\pm\frac{1}{2}\sqrt{3\ (a_1+2)^2-16} \tag{26}$$

for

$$a_1\in\left(-\infty,-\frac{4\sqrt{3}}{3}-2\right)\cup\left(\frac{4\sqrt{3}}{3}-2,\ \infty\right)\approx(-\infty,-4.309401)\cup(0.309401,\ \infty). \tag{27}$$

Under the condition (26), $\tilde{v}_4$ and $\tilde{v}_5$ are reduced to

$$
\begin{aligned}
\tilde{v}_4 &= -\tfrac{7}{4096\ (5\ a_1-2)}\ (a_1+2)^2\ \left(a_1^2-8\ a_1+4\right)\ (b_{11}+b_{21})\times\left(11\ a_1^3+46\ a_1^2-84\ a_1+24\right),\\
\tilde{v}_5 &= -\tfrac{7}{49152\ (5\ a_1-2)}\ (a_1+2)^2\ \left(a_1^2-8\ a_1+4\right)\ (b_{11}+b_{21})\\
&\quad \times\left(638\ a_1^5+3873\ a_1^4+428\ a_1^3-9366\ a_1^2+5168\ a_1-720\right).
\end{aligned}
\tag{28}
$$

Therefore, the only possibility for $\tilde{v}_4=0$ but $\tilde{v}_5\neq 0$ is the roots of the polynomial

$$F_1(a_1)=11\ a_1^3+46\ a_1^2-84\ a_1+24. \tag{29}$$

It is easy to show that the discriminant of the equation $F_1(a_1)=0$ is

$$D=-\frac{86528}{11979}<0,$$

indicating that $F_1(a_1)=0$ has three real roots, given by

$$a_1=-5.6118538340\cdots,\quad 0.3650705869\cdots,\quad 1.0649650652\cdots,$$

which are all located in the interval given in (27).

Further, when a_1 satisfies (29), $\tilde{v}_5$ is reduced to

$$\tilde{v}_5=\frac{91}{743424(5a_1-2)}\ (a_1+2)^2\left(a_1^2-8a_1+4\right)\left(3073a_1^2-5272a_1+1500\right)(b_{11}+b_{21})\neq 0$$

for $b_{11}+b_{21}\neq 0$ and $a_1$taking one of the real roots of $F_1(a_1)=0$.

The above results show that one may choose the perturbation parameters b_{01}, b_{03}, b_{12}, a_2 and a_1 such that $\tilde{v}_i=0,\ i=0,\ 1,\ 2,\ 3,\ 4$, but $\tilde{v}_5\neq 0$. Combining with (26), we obtain a total of six sets of solutions for (a_1,a_2). Thus, for this case, at most 5 small limit cycles can bifurcate from the origin. Further, by proper perturbations on the parameters a_1, a_2, b_{12}, b_{03} and b_{01}, we can obtain five small-amplitude limit cycles. Alternatively, we can show the existence of five limit cycles by verifying the determinant

$$D_3=\det\begin{bmatrix}\frac{\partial\tilde{v}_4}{\partial a_1} & \frac{\partial\tilde{v}_4}{\partial a_2}\\ \frac{\partial\tilde{v}_5}{\partial a_1} & \frac{\partial\tilde{v}_5}{\partial a_2}\end{bmatrix}=\frac{245\,a_2\,(a_1^2-8a_1+4)\,(a_1+2)^2\,(b_{11}+b_{21})^2}{1572864\,(5a_1-2)^3}\times\left(165a_1^6-382a_1^5-4568a_1^4+4896a_1^3+688a_1^2-1696a_1+384\right)\neq 0,$$

if $b_{11}+b_{21}\neq 0$, where $\tilde{v}_3$ and $\tilde{v}_4$ are given in (25), evaluated at the critical point determined by (26) and (29), since the 6th-degree polynomial in the above expression has the real roots:

$$a_1=-4.756694\ \cdots,\quad -0.605725\ \cdots,\quad 0.818912\ \cdots,\quad 6.062837\ \cdots,$$

none of them satisfies (29). This non-zero determinant implies that proper perturbations on a_1 and a_2 can be found such that $|\tilde{v}_4|\ll|\tilde{v}_5|<\epsilon$ and $\tilde{v}_4\ \tilde{v}_5<0$. Other three perturbations on b_{12}, b_{03}, b_{01} can be easily obtained one by one, satisfying $|\tilde{v}_j|\ll|\tilde{v}_{j+1}|$ and $\tilde{v}_j\ \tilde{v}_{j+1}<0$ for $j=0, 1, 2, 3$.

Now, consider the case when $a_1 = \frac{5}{2}$. For this special case, b_{03} and b_{12} become

$$\begin{aligned} b_{03} &= \tfrac{1}{12}\ (b_{11} + 8\ a_2\ b_{02} - 4\ b_{21}), \\ b_{12} &= \tfrac{5}{2}\ b_{02} + \tfrac{5}{168\ a_2}\ \left(16\ a_2^2 + 7\right)\ (b_{11} + b_{21}), \end{aligned} \tag{30}$$

under which $\tilde{v}_1 = \tilde{v}_2 = 0$, and

$$\begin{aligned} \tilde{v}_3 &= \tfrac{5}{147456}\ \left(32\ a_2^2 - 7\right)\ \left(16\ a_2^2 - 179\right)\ (b_{11} + b_{21}), \\ \tilde{v}_4 &= \tfrac{1}{84934656}\ \left(32a_2^2 - 7\right)\ \left(120064a_2^4 - 887296a_2^2 - 2974841\right)(b_{11} + b_{21}), \\ \tilde{v}_5 &= \tfrac{1}{782757787696}\ \left(32\ a_2^2 - 7\right)\ \left(2852704256a_2^6 - 10917094656a_2^4 \right. \\ &\quad \left. -130732044816a_2^2 - 126479453359\right)\ (b_{11} + b_{21}), \\ &\ \ \vdots \end{aligned} \tag{31}$$

Hence,

$$a_2 = \pm\frac{1}{4}\sqrt{179}, \tag{32}$$

and then

$$\tilde{v}_4 = \frac{2302911}{262144}\ (b_{11} + b_{21}), \tag{33}$$

showing that when $a_1 = \frac{5}{2}$, the system can have at most four small-amplitude limit cycles around the origin.

In conclusion, we have shown that $H_2(3) = 5$. □

Remark 1

(i) The method and formulas presented in this section for proving Theorem 3 are different from that given in [14], but lead to the same conclusion.

(ii) The coefficients a_1 and a_2 appeared in the Hamiltonian function do not play any role in determining $H_2(2) = 2$, while they are used to obtain two additional limit cycles in obtaining $H_2(3) = 5$.

(iii) In the case of cubic perturbation, the number of b_{ij} coefficients is more than needed. For example, we may set $b_{11} = b_{02} = 0$. Thus,

$$b_{03} = -\frac{1}{3}\ b_{21}, \quad b_{12} = \frac{20\ a_2^2 + (3\ a_2 + 10)(a_1 - 2)}{4\ a_2\ (5\ a_1 - 2)}\ b_{21}, \tag{34}$$

which will greatly simplify the computation in analysis.

3. Quadratic Hamiltonian systems with 4th- to 8th-degree polynomial perturbations

In this section, we consider the quadratic near-Hamiltonian system (14) with higher-degree polynomial perturbations. In particular, we shall study the cases $n = 4, 5, \ldots, 8$, and leave cases $n = 9, 10, \ldots, 20$ to be discussed in the next section.

3.1. 4th-degree polynomial perturbation – $H_2(4) = 6$

When $n = 4$, we have

$$q_4(x, y) = b_{11}\ x\ y + b_{02}\ y^2 + b_{21}\ x^2\ y + b_{12}\ x\ y^2 + b_{03}\ y^3. + b_{31}\ x^3\ y + b_{22}\ x^2\ y^2 + b_{13}\ x\ y^3 + b_{04}\ y^4, \tag{35}$$

where b_{01} has been set zero in order for the origin $(0, 0)$ to be an element center under perturbation.

Theorem 4. *The quadratic near-Hamiltonian system (14) with 4th-degree polynomial perturbation can have maximal six small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(4) = 6$.*

Proof. The first focus value, $\tilde{v}_1$, is the same as that given in (21), and thus the solution given in (22) also applies to this case. Then, we obtain $\tilde{v}_2$ as follows:

$$\begin{aligned} \tilde{v}_2 = -\frac{1}{192}\ &\{4\ a_2(5a_1 - 2)(a_1 b_{02} - b_{12}) + \left[20a_2^2 + (3a_2 + 10)(a_1 - 2)\right](b_{11} + b_{21}) \\ &+2\ (3\ a_1 - 10)\ b_{31} + 8\ a_2\ b_{22} + 6\ (3\ a_1 - 2)\ b_{13} + 80\ a_2\ b_{04}\}. \end{aligned} \tag{36}$$

To have maximal number of limit cycles, suppose $a_1 \neq \frac{2}{5}$. Then solving b_{12} from the equation $\tilde{v}_2 = 0$ results in

$$b_{12} = a_1 b_{02} + \frac{1}{4a_2(5a_1-2)} \left\{ \left[20\, a_2^2 + (3\, a_2 + 10)(a_1 - 2)\right] (b_{11} + b_{21}) + 2\, (3a_1 - 10) b_{31} + 8a_2 b_{22} + 6\, (3a_1 - 2) b_{13} + 80 a_2 b_{04} \right\}. \tag{37}$$

Next, solving $\tilde{v}_3 = 0$ for b_{21} yields

$$\begin{aligned} b_{21} = & -b_{11} - \frac{1}{7(a_1^3 - 3a_1^2 + 4 - 4a_2^2)(3a_1^2 + 12a_1 - 4 - 4a_2^2)} \times \{ [57a_1^4 - 300a_1^3 + 168a_1^2 + 336a_1 - 112 \\ & + 4(21a_1^2 - 120a_1 + 20)a_2^2]\, b_{31} - 2\, (21a_1^2 - 36a_1 + 20 - 28a_2^2)\, (2a_1 a_2 b_{22} - 3a_1^2 b_{13} + 8a_2 b_{04}) \} \end{aligned} \tag{38}$$

Then we obtain

$$\begin{aligned} \tilde{v}_4 &= -\tfrac{1}{12288}\, Q_4(a_1, a_2)\, F_{41}(a_1, a_2), \\ \tilde{v}_5 &= -\tfrac{1}{442368}\, Q_4(a_1, a_2)\, F_{42}(a_1, a_2), \\ \tilde{v}_6 &= -\tfrac{1}{4756340736}\, Q_4(a_1, a_2)\, F_{43}(a_1, a_2), \end{aligned} \tag{39}$$

where

$$\begin{aligned} Q_4 &= \frac{(a_1^3 - 4a_2^2)\, b_{31} + 4a_1 a_2 b_{22} - 6a_1^2 b_{13} + 16 a_2 b_{04}}{3a_1^2 + 12a_1 - 44a_2^2}, \\ F_{41} &= -880a_2^4 - 8(63a_1^2 - 204a_1 - 212)a_2^2 + 81a_1^4 - 648a_1^3 - 648a_1^2 + 632a_1 - 880, \\ F_{42} &= -114400\, a_2^6 - 48\, (2787\, a_1^2 - 4986\, a_1 - 4088)\, a_2^4 \\ &\quad - 6\, (4203\, a_1^4 - 10968\, a_1^3 + 11112\, a_1^2 - 15840\, a_1 + 5936)\, a_2^2 \\ &\quad + 4617a_1^6 - 39366a_1^5 + 1296a_1^4 + 95184a_1^3 - 131184a_1^2 + 92448a_1 - 35200, \\ F_{43} &= -3933966080\, a_2^8 - 14080\, (495531\, a_1^2 - 640908\, a_1 - 493084)\, a_2^6 \\ &\quad - 96\, (34850991\, a_1^4 - 87072328\, a_1^3 + 7103576\, a_1^2 - 1362400\, a_1 \\ &\quad + 17881520\,)\, a_2^4 - 16\, (13981653\, a_1^6 + 1079028\, a_1^5 + 152452836\, a_1^4 \\ &\quad - 366397536\, a_1^3 + 326596848\, a_1^2 - 124266432\, a_1 + 2321600) a_2^2 \\ &\quad + 63132129a_1^8 - 581959728a_1^7 + 512118288a_1^6 + 1070915904a_1^5 \\ &\quad - 2534554656a_1^4 + 2774148864a_1^3 - 2353103616a_1^2 + 1477155840a_1 - 515905280. \end{aligned}$$

Eliminating a_2 from the equations $F_{41}(a_1,a_2) = F_{42}(a_1,a_2) = 0$ yields the solution for a_2:

$$a_2^2 = G_4(a_1) = \frac{10179a_1^6 - 81864a_1^5 - 179172a_1^4 + 204992a_1^3 - 32496a_1^2 - 124032a_1 + 66880}{4\, (5109\, a_1^4 + 12076\, a_1^3 - 75936\, a_1^2 - 167664\, a_1 + 48944)}, \tag{40}$$

and a resultant equation: $F_{44}(a_1) = (a_1 + 2)\, F_{44}^*(a_1) = 0$, where

$$\begin{aligned} F_{44}^* = & \; 77571a_1^9 - 1561014a_1^8 + 9024720a_1^7 - 9985760a_1^6 - 33089760a_1^5 + 106013376a_1^4 - 124646144a_1^3 \\ & + 66931200a_1^2 - 14081280a_1 + 924160. \end{aligned} \tag{41}$$

Since $G_4(-2) = -4 < 0$, $a_1 = -2$ is not a solution. The equation $F_{44}^* = 0$ has 7 real solutions, among which 6 solutions satisfy $G_4 > 0$, given by

$$\begin{aligned} a_1 = & -2.4319249295\cdots, \quad 0.1214887712\cdots, \quad 0.2396354793\cdots, \\ & 0.8947127237\cdots, \quad 1.6003117430\cdots, \quad 10.4095039074\cdots. \end{aligned}$$

Thus, there are in total 12 solutions such that $\tilde{v}_i,\ i = 0,\ 1,\ \cdots,\ 5$. To show that under these solutions, $\tilde{v}_6 \neq 0$, we simplify F_{43} under the condition (40) to obtain

$$F_{45}(a_1) = -\frac{114048\, (a_1 + 2)^3}{(5109a_1^4 + 12076a_1^3 - 75936a_1^2 + 167664a_1 + 48944)^4}\, F_{45}^*(a_1),$$

where

$$\begin{aligned} F^*_{45} = {} & 1184423008799739489 3\, a_1^{21} - 3057343450524937007 46\, a_1^{20} \\ & + 24185495049939586499 76\, a_1^{19} - 19934607379396402648 80\, a_1^{18} \\ & - 49564195151871383253936\, a_1^{17} + 12444699707356050810864 0\, a_1^{16} \\ & + 33614912025711436961945 6\, a_1^{15} - 11001327599386677158696 96\, a_1^{14} \\ & - 40439089366244183742617 6\, a_1^{13} + 40070674312794241903687 68\, a_1^{12} \\ & - 32820174927585325418045 44\, a_1^{11} - 44461678752196703699353 60\, a_1^{10} \\ & + 10247366647513025385472000\, a_1^{9} - 62637989590148720497950 72\, a_1^{8} \\ & - 34021580541248192214466 56\, a_1^{7} + 99627702184623169656586 24\, a_1^{6} \\ & - 99816023340090264811601 92\, a_1^{5} + 61184563380152236967198 72\, a_1^{4} \\ & - 23645553900266545096949 76\, a_1^{3} + 54444801937626676854784 0\, a_1^{2} \\ & - 66279958873926831964160\, a_1 + 3180778759235116728320. \end{aligned} \tag{42}$$

It can be shown that for the six roots of $F^*_{44}(a_1) = 0$, $F^*_{45}(a_1) \neq 0$. For example, taking the second root and truncated up to 100 decimal points yields the following critical parameter values:

$$a_1 = 0.121488\cdots, \quad a_2 = 0.685579\cdots, \quad b_{21} = -0.918105\cdots b_{31},$$
$$b_{12} = 2.625493\cdots b_{31}, \quad b_{03} = 0.306035\cdots b_{31}, \quad b_{01} = 0,$$

where we have set $b_{11} = b_{02} = b_{22} = b_{13} = b_{04} = 0$, under which the computed focus values, up to 100 decimal points, are

$$\tilde{v}_0 = 0, \quad \tilde{v}_1 = 0, \quad \tilde{v}_2 = 0.6 \times 10^{-100} b_{31}, \quad \tilde{v}_3 = -0.11 \times 10^{-99} b_{31},$$
$$\tilde{v}_4 = 0.16 \times 10^{-99} b_{31}, \quad \tilde{v}_5 = 0.145522 \times 10^{-99} b_{31}, \quad \tilde{v}_6 = 0.004056\cdots b_{31}.$$

Without loss of generality, we may take $b_{31} = 1$. It should be noted that for the exact solutions (in other words, if we obtain the solutions up to infinite decimal points), the focus values $\tilde{v}_i,\ i = 2,\ 3,\ 4,\ 5$ should be exactly equal to zero.

Further, using (39) and (40) it can be show that

$$\begin{aligned} D_4 &= \det \begin{bmatrix} \frac{\partial \tilde{v}_4}{\partial a_1} & \frac{\partial \tilde{v}_4}{\partial a_2} \\ \frac{\partial \tilde{v}_5}{\partial a_1} & \frac{\partial \tilde{v}_5}{\partial a_2} \end{bmatrix} = \frac{1}{5435817984} \left[\frac{\partial\ (Q_4 F_{41})}{\partial a_1}\ \frac{\partial\ (Q_4 F_{42})}{\partial a_2} - \frac{\partial\ (Q_4 F_{41})}{\partial a_2}\ \frac{\partial\ (Q_4 F_{42})}{\partial a_1} \right] \\ &= \frac{1}{5435817984} \left[\left(\frac{\partial Q_4}{\partial a_1}\ F_{41} + Q_4\ \frac{\partial F_{41}}{\partial a_1} \right) \left(\frac{\partial Q_4}{\partial a_2}\ F_{42} + Q_4\ \frac{\partial F_{42}}{\partial a_2} \right) - \left(\frac{\partial Q_4}{\partial a_2}\ F_{41} + Q_4\ \frac{\partial F_{41}}{\partial a_2} \right) \left(\frac{\partial Q_4}{\partial a_1}\ F_{42} + Q_4\ \frac{\partial F_{42}}{\partial a_1} \right) \right] \\ &= \frac{Q_4^2}{5435817984} \left(\frac{\partial F_{41}}{\partial a_1}\ \frac{\partial F_{42}}{\partial a_2} - \frac{\partial F_{41}}{\partial a_2}\ \frac{\partial F_{42}}{\partial a_1} \right) (\text{due to } F_{41} = F_{42} = 0 \text{ at the critical point}) \\ &= -\frac{99\ a_2\ (a_1 + 1)\ (a_1 + 2)^2\ Q_4^2}{262144\ \left(5109\ a_1^4 + 12076\ a_1^3 - 75936\ a_1^2 - 167664\ a_1 + 48944\right)^3} \\ &\quad \times \left(10313314461333\ a_1^{16} - 227551342138848\ a_1^{15} + 1313462632863888\ a_1^{14} + 2295518806514112\ a_1^{13} \right. \\ &\quad - 33581578824741632\ a_1^{12} + 14478127272667136\ a_1^{11} + 261174703841748224\ a_1^{10} - 309215858243984384\ a_1^{9} \\ &\quad - 665789393081826816\ a_1^{8} + 1499325783540473856\ a_1^{7} - 305360348854505472\ a_1^{6} - 1885165145387286528\ a_1^{5} \\ &\quad + 2400549071758000128\ a_1^{4} - 1298010355777404928\ a_1^{3} + 335191187245170688\ a_1^{2} - 37291867563622400\ a_1 \\ &\quad \left. + 1126958853324800 \right. \end{aligned}$$

since $Q_4 \neq 0$, the denominator (which is the denominator of the solution for a_2) is non zero, and the 16th-degree polynomial factor in the above expression has eight real solutions for a_1, but none of them satisfies $F^*_{44}(a_1) = 0$. In fact, for the above chosen parameter values, we obtain $D_4 = 0.003608\ \cdots\ b_{31}^2 \neq 0\ (b_{31} \neq 0)$.

Summarizing the above results shows that one can choose b_{01}, b_{03}, b_{12}, b_{21}, a_2 and a_1 such that $\tilde{v}_i = 0,\ i = 0,\ 1,\ \cdots,\ 5$, but $\tilde{v}_6 \neq 0$. Further, we can perturb these coefficients in backwards to generate

$$|\tilde{v}_j| \ll |\tilde{v}_{j+1}| \quad \text{and} \quad \tilde{v}_j\ \tilde{v}_{j+1} < 0 \quad \text{for } j = 0,\ 1,\ \cdots,\ 5.$$

This finishes the proof. □

Remark 2. Again it is noted that the number of b_{ij} coefficients is more than needed. For example, we may set $b_{11} = b_{02} = b_{22} = b_{13} = b_{04} = 0$, which will greatly simplify the analysis.

3.2. 5th-degree polynomial perturbation –$H_2(5) = 8$

For $n = 5$, we have the following result.

Theorem 5. *The quadratic near-Hamiltonian* system (14)*with 5th-degree polynomial perturbation can have maximal eight small limit cycles bifurcating from the origin, i.e., $H_2(5) = 8$.*

Proof. For this case, $q_5(x,y)$ is given by

$$q_5(x,y) = b_{11}xy + b_{02}y^2 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4 + b_{41}\ x^4\ y + b_{32}\ x^3\ y^2 + b_{23}\ x^2\ y^3 + b_{14}\ x\ y^4 + b_{05}\ y^5, \tag{43}$$

where b_{01} has again been set zero in order for the origin (0,0) to be an element center under perturbation.

Now, based on the focus value computation, solving $\tilde{v}_1 = 0$ for b_{03} gives the same solution as given in (22). Solving $\tilde{v}_2 = 0$ for b_{12}, $\tilde{v}_3 = 0$ for b_{21}, $\tilde{v}_4 = 0$ for b_{05}, and $\tilde{v}_5 = 0$ for b_{14}, yields

$$b_{12} = a_1\ b_{02} + \frac{1}{4a_2(5a_1-2)}\ \{(3a_1^2+4a_1-20+20a_2^2)(b_{11}+b_{21}) + 2(3a_1-10)\ b_{31} + 8\ a_2\ b_{22} + 6\ (3a1-2)\ b_{13} + 80\ a_2\ b_{04} - 12\ (b_{41}+b_{23}) - 60\ b_{05}\},$$

$$b_{21} = -b_{11} - \frac{1}{7(a_1^3-3a_1^2+4-4a_2^2)(3a_1^2+12a_1-4-4a_2^2)} \times \{[57a_1^4 - 300a_1^3 + 168a_1^2 + 336a_1 - 112 + 4\ (21a_1^2 - 120a_1 + 20)a_2^2]\ b_{31} - 2(21a_1^2 - 36a_1 + 20 - 28a_2^2)(2\ a_1a_2b_{22} - 3\ a_1^2\ b_{13} + 8a_2\ b_{04}) - 4[36a_1^3 - 13a_1^2 - 84a_1 + 28 + 12(6a_1-)a_2^2]\ b_{41} + 8a_2(5a_1-2)(3a_1-2)\ b_{32} - 6a_1(9a_1^2+4a_1-12+28a_2^2)\ b_{23} + 16\ a_2(7a_1-2)(5a_1-2)\ b_{14} - 20[3(7a_1^3 - 9a_1^2 + 8a_1 - 4) - 28(a_1-1)\ a_2^2\ ]\ b_{05}$$

and $b_{05} = -\frac{b_{05}^N}{20\ b_{05}^D}$, $b_{14} = \frac{b_{14}^N}{16\ a_2\ b_{14}^D}$, where

$$b_{05}^N = [81a_1^4 - 648a_1^3 - 648a_1^2 + 1632a_1 - 880 - 8\ (63a_1^2 - 204a_1 - 212+110a_2^2)a_2^2][(a_1^3-4a_2^2)\ b_{31} - 6a_1^2\ b_{13} + 4a_1a_2\ b_{22} + 16a_2\ b_{04}] + \{a_1^2\ (387a_1^4 + 1440a_1^3 + 600a_1^2 - 6464a_1 + 5104) - 4a_2^2[363a_1^4 + 728a_1^3 - 1576a_1^2 + 2112a_1 + 304 + 4a_2^2(87a_1^2 - 144a_1 - 512 + 132a_2^2)]\}b_{41} - 4a_2\{27a_1^5 + 102a_1^4 - 1248a_1^3 - 768a_1^2 + 1360a_1 - 352 - [8(21a_1^3 + 93a_1^2 - 212a_1 + 44) - 48(11a_1 - 12)a_2^2]a_2^2\}b_{32} + 6\ a_1\{27a_1^5 + 21a_1^4 - 600a_1^3 - 120a_1^2 - 272a_1 + 528 - 8[21a_1^3 + 30a_1^2 - 8a_1 + 256 - 2(33a_1+19)a_2^2]a_2^2\}b_{23} - 64\ a_2[45a_1^4 - 241a_1^3 - 682a_1^2 + 468a_1 - 72 + a_2^2(77a_1^3 + 119a_1^2 + 132a_1 - 44 - 44a_2^2)]\ b_{14},$$

$$b_{05}^D = 3\ (36a_1^5 - 209a_1^4 - 416a_1^3 + 504a_1^2 - 384a_1 + 176) - 16\ a_2^2[42a_1^3 - 87a_1^2 - 110a_1 + 128 + (22a_1-19)a_2^2],$$

$$b_{14}^N = \Big\{3a_1(a_1-2)^4(27a_1^3 + 378a_1^2 + 468a_1 - 104) - 8a_2^2\ [180a_1^6 + 975a_1^5 - 6258a_1^4 + 6584a_1^3 + 4848a_1^2 - 10704a_1 + 6752 + 4a_2^2(117a_1^5 - 816a_1^4 - 161a_1^3 + 3258a_1^2 + 1812a_1 - 3496 - 4(13a_1^3 + 156a_1^2 + 423a_1 - 130 - 130a_2^2)a_2^2)]\Big\}[(a_1^3-4a_2^2)b_{31} + 4a_1a_2b_{22} - 6a_1^2b_{13} + 16a_2b_{04}] + 4\Big\{3a_1^3(a_1+1)(a_1-2)^4(27a_1^3 + 378a_1^2 + 468a_1 - 104) - a_2^2\Big[(a_1-2)^2 \times (1440a_1^7 + 15081a_1^6 + 12624a_1^5 - 13916a_1^4 + 9824a_1^3 + 23152a_1^2 - 14976a_1 + 2496) + 16a_2^2(192a_1^7 + 1984a_1^6 - 4771a_1^5 + 5862a_1^4 - 272a_1^3 - 10208a_1^2 + 6768a_1 - 1760 - 2a_2^2(a_1(130a_1^4 - 939a_1^3 + 2670a_1^2 - 1372a_1 + 408) - 8\ (65a_1^2 - 3a_1 + 110 - 39a_2^2)a_2^2))\Big]\Big\}\ b_{41} + 4a_2(a_1-1)\Big\{3a_1(a_1-2)^4(27a_1^3 + 378a_1^2 + 468a_1 - 104) - 8a_2^2\Big[(a_1-2)^2 \times (180a_1^4 + 1695a_1^3 - 198a_1^2 - 988a_1 + 1688) + 4a_2^2(117a_1^5 - 816a_1^4 - 161a_1^3 + 3258a_1^2 + 1812a_1 - 3496 - 4a_2^2(13a_1^3 + 156a_1^2 + 423a_1 - 130 - 130a_2^2))\Big]\Big\}\ b_{32} - 6a_1^2\Big\{3a_1(a_1-2)^4(27a_1^3 + 378a_1^2 + 468a_1 - 104) - 8a_2^2\Big[(a_1-2)^2(180a_1^4 + 1695a_1^3 - 198a_1^2 - 988a_1 + 1688) + 4a_2^2(117a_1^5 - 816a_1^4 - 161a_1^3 + 3258a_1^2 + 1812a_1 - 3496 - 4a_2^2(13a_1^3 + 156a_1^2 + 423a_1 - 130 - 130a_2^2))\Big]\Big\}\ b_{23},$$

$$b_{14}^D = 3\ \left(27a_1^7 + 162a_1^6 - 1908a_1^5 + 4360a_1^4 + 400a_1^3 - 11424a_1^2 + 10816a_1 - 1664\right) + 4a_2^2\{468a_1^8 - 3180a_1^7 - 8056a_1^6 + 20955a_1^5 + 14814a_1^4 - 12776a_1^3 + 1584a_1^2 - 2832a_1 - 3232 - 4a_2^2\left[52a_1^6 + 2439a_1^4 + 663a_1^5 - 1659a_1^3 + 4110a_1^2 + 1404a_1 - 3496 - 4\ a_2^2\ \left(143a_1^3 + 156a_1^2 + 429a_1 - 350 - 52a_2^2\right)\right]\}.$$

Then, the 6th focus value becomes

$$\tilde{v}_6 = \frac{143}{294912}\ Q_5(a_1, a_2)\ F_{52}(a_1, a_2) \quad \text{and} \quad \tilde{v}_7 = -\frac{715}{113246208}\ Q_5(a_1, a_2)\ F_{53}(a_1, a_2),$$

where $Q_5(a_1, a_2) = \frac{F_{51}(a_1,a_2)}{F_{50}(a_1,a_2)}$, and

$$\begin{aligned}
F_{50} &= 3(a_1-2)^4\left(27a_1^3 + 378a_1^2 + 468a_1 - 104\right) + 4a_2^2\{(a_1-2)\left(468a_1^7 - 2244a_1^6 - 12544a_1^5 - 4133a_1^4 + 6548a_1^3 + 320a_1^2 + 2224a_1 + 1616\right) - 4\ a_2^2\left[52a_1^6 + 663a_1^5 + 2439a_1^4 - 1659a_1^3 + 4110a_1^2 + 1404a_1 - 3496 - 4a_2^2\left(143a_1^3 + 156a_1^2 + 429a_1 - 350 - 52a_2^2\right)\right]\},\\
F_{51} &= a_2^2\ \left(a_1^3 - 3a_1^2 + 4 - 4a_2^2\right)^3\ \left[(a_1^3 - 4a_2^2)b_{31} + 4a_1b_{22}a_2 - 6a_1^2b_{13} + 16a_2b_{04} + 4\left(a_1^3 + a_1^2 - a_2^2\right)b_{41} + 4a_2(a_1-1)b_{32} - 6a_1^2b_{23}\right],\\
F_{52} &= 405\ a_1^4 + 6264\ a_1^3 + 6264\ a_1^2 - 5664\ a_1 + 1360 - 8\ a_2^2\ \left(99\ a_1^2 + 708\ a_1 + 524 - 170\ a_2^2\right),\\
F_{53} &= 8991a_1^6 + 96228a_1^5 - 890028a_1^4 - 397728a_1^3 + 2094480a_1^2 - 1063872a_1 + 177344 + 4\ a_2^2\left[14553a_1^4 + 269064a_1^3 + 425592a_1^2 - 328032a_1 - 89328 - 4\ a_2^2\left(8307a_1^2 + 66492a_1 + 22332 - 11084a_2^2\right)\right].
\end{aligned}$$

Next, eliminating a_2 from the equations $F_{52}(a_1,a_2) = F_{53}(a_1,a_2) = 0$ yields

$$a_2^2 = G_5(a_1) = \frac{105165a_1^6 + 1945512a_1^5 + 4062996a_1^4 + 620224a_1^3 - 605712a_1^2 + 294528a_1 - 70720}{4\ \left(17499a_1^4 + 78836a_1^3 + 664464a_1^2 + 1162416a_1 - 72176\right)},$$

and a resultant equation: $F_{54}(a_1) = (a_1 + 2)\ F_{54}^*(a_1) = 0$, where

$$F_{54}^*(a_1) = 224181a_1^9 + 7250742a_1^8 + 51634440a_1^7 - 140047504a_1^6 - 106276800a_1^5 + 502223232a_1^4 - 398093440a_1^3 + 109666560a_1^2 - 6539520a_1 + 86528.$$

$a_1 = -2$ is not a solution since $G_5(-2) = -4 < 0$. The polynomial $F_{54}^*(a_1)$ has three real roots, given by

$$a_1 = -2.0772478597\cdots, \quad 0.0187162703\cdots, \quad 0.0563411398\cdots,$$

all of them satisfy $G_5(a_1) > 0$. Thus, there are in total six solutions. It can be shown that for these six solutions, $\tilde{v}_i = 0,\ i = 0,\ 1,\ \cdots,\ 7$, but $\tilde{v}_8 \neq 0$. For example, taking the second root of $F_{53}^*(a_1) = 0$ yields the critical parameter values, up to 100 decimal points:

$$\begin{aligned}
&a_1 = 0.018716\cdots, \quad a_2 = 0.570840\cdots, \quad b_{14} = 0.041161\cdots\ b_{41},\\
&b_{05} = -0.098867\cdots\ b_{41}, \quad b_{21} = -0.998052\cdots\ b_{41},\\
&b_{12} = -1.679929\cdots\ b_{41}, \quad b_{03} = 0.332684\cdots\ b_{41}, \quad b_{01} = 0,
\end{aligned}$$

where we have set $b_{11} = b_{02} = b_{31} = b_{22} = b_{13} = b_{04} = b_{32} = b_{23} = 0$, and the corresponding focus values are

$$\begin{aligned}
&\tilde{v}_0 = 0, \quad \tilde{v}_1 = 0, \quad \tilde{v}_2 = 0.2\times10^{-100}\ b_{41}, \quad \tilde{v}_3 = 0.218\times10^{-100}\ b_{41},\\
&\tilde{v}_4 = -0.1535\times10^{-100}\ b_{41}, \quad \tilde{v}_5 = 0.849035\times10^{-100}\ b_{41},\\
&\tilde{v}_6 = -0.740965\times10^{-100}\ b_{41}, \quad \tilde{v}_7 = 0.8023772269351\times10^{-100}\ b_{41},\\
&\tilde{v}_8 = 0.0001470910\cdots\ b_{41}.
\end{aligned}$$

This shows that there exist at most eight small-amplitude limit cycles around the origin. Moreover, by a similar argument, one can show that

$$D_5 = \det\begin{bmatrix} \frac{\partial \tilde{v}_6}{\partial a_1} & \frac{\partial \tilde{v}_6}{\partial a_2} \\ \frac{\partial \tilde{v}_7}{\partial a_1} & \frac{\partial \tilde{v}_7}{\partial a_2} \end{bmatrix} = -\frac{102245}{33397665693696}\left[\frac{\partial\ (Q_5\ F_{52})}{\partial a_1}\ \frac{\partial\ (Q_5\ F_{53})}{\partial a_2} - \frac{\partial\ (Q_5\ F_{52})}{\partial a_2}\ \frac{\partial\ (Q_5\ F_{53})}{\partial a_1}\right]$$
$$= -\frac{102245\ Q_5^2}{33397665693696}\left(\frac{\partial F_{52}}{\partial a_1}\ \frac{\partial F_{53}}{\partial a_2} - \frac{\partial F_{52}}{\partial a_2}\ \frac{\partial F_{53}}{\partial a_1}\right)$$
$$= \frac{5214495\ a_2\ (a_1+1)\ (a_1+2)^2\ Q_5^2}{33554432\ \left(17499\ a_1^4 + 78836\ a_1^3 + 664464\ a_1^2 + 1162416\ a_1 - 72176\right)^3}$$
$$\times\left(1415952528453\ a_1^{16} - 26280676678068\ a_1^{15} - 1352133619703352\ a_1^{14} - 61067779774350096\ a_1^{13}\right.$$
$$-1302780775819382048\ a_1^{12} - 5579295994527079488\ a_1^{11} + 2382012740290416499 2a_1^{10}$$
$$+ 96183809396670274304 a_1^{9} - 85551175416509872128 a_1^{8} - 33102045729459648000 0a_1^{7}$$
$$+ 226442252205945378816 a_1^{6} + 334597705832994361344 a_1^{5} \quad - 325860493104747700224\ a_1^4$$
$$\left.+ 90087065798674300928\ a_1^3 - 9230619970869952512\ a_1^2 + 392913556957888512\ a_1 - 5852925912743936\right) \neq 0$$

at the critical point. Indeed, for the above chosen parameter values, $D_5 = 0.0000185376\cdots b_{41}^2 \neq 0\ (b_{41} \neq 0)$. This implies that appropriate perturbations can be made to a_1, a_2, b_{14}, b_{05}, b_{21}, b_{12}, b_{03}, b_{01} to obtain eight limit cycles. □

3.3. 6th-degree polynomial perturbation –$H_2(6)$ = 9

In this section, we consider the case n = 6, for which we have the following theorem.

Theorem 6. *The quadratic near-Hamiltonian* system (14) *with 6th-degree polynomial perturbation can have maximal nine small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(6)$ = 9.*

Proof. For this case, $q_6(x,y)$ can take the following form:

$$q_6(x,y) = \sum_{2\leqslant i+j\leqslant 6} b_{ij}x^i y^j,\ \text{with}\ b_{i0} = 0,\quad i = 2,\ 3,\ \cdots,\ 6. \tag{44}$$

First note that the focus values $\tilde{v}_1$ and $\tilde{v}_2$ are identical to that of the case n = 5. Thus, the solutions of b_{03} solved from $\tilde{v}_1 = 0$ and b_{12} from $\tilde{v}_2 = 0$ are identical to that for case n = 5. We then solve $\tilde{v}_3 = 0$ for b_{21}, $\tilde{v}_4 = 0$ for b_{05}, $\tilde{v}_5 = 0$ for b_{14}, and $\tilde{v}_6 = 0$ for b_{23} to obtain

```
b21:=-(35*(a1^3-3*a1^2+4-4*a2^2)*(3*a1^2+12*a1-4-4*a2^2)*b11+5*(57*a1^4-300*a1^3+168*a1^2
    +336*a1-112+4*a2^2*(21*a1^2-120*a1+20))*b31+(21*a1^2-36*a1+20-28*a2^2)*(-20*a1*a2*b22
    +30*a1^2*b13-80*a2*b04)-20*(36*a1^3-13*a1^2-84*a1+28+72*a1*a2^2-12*a2^2)*b41-30*a1
    *(9*a1^2+4*a1-12+28*a2^2)*b23-100*(21*a1^3-27*a1^2+24*a1-12-28*a1*a2^2+28*a2^2)*b05
    +4*(5*a1-2)*(10*a2*(3*a1-2)*b32+20*a2*(7*a1-2)*b14-5*(3*a1-14)*b51-12*a2*b42
    -3*(9*a1-10)*b33-40*a2*b24-15*(5*a1-2)*b15-420*a2*b06))
  /35/(3*a1^5+3*a1^4-40*a1^3+24*a1^2+48*a1-16-4*a1^3*a2^2-48*a1*a2^2+16*a2^4):
b05:=-(5*(81*a1^4-648*a1^3-648*a1^2+1632*a1-880-8*a2^2*(63*a1^2-204*a1-212+110*a2^2))
    *((a1^3-4*a2^2)*b31+4*a1*a2*b22-6*a1^2*b13+16*a2*b04)-5*(a1^2*(387*a1^4+1440*a1^3
    +600*a1^2-6464*a1+5104)-4*a2^2*(363*a1^4-1576*a1^2+728*a1^3+2112*a1+304+4*a2^2*(87
    *a1^2-144*a1-512+132*a2^2)))*b41-20*a2*(27*a1^5+102*a1^4-1248*a1^3-768*a1^2+1360*a1
    -352-8*a2^2*((21*a1^3+93*a1^2-212*a1+44)-6*(11*a1-12)*a2^2))*b32+30*a1*(27*a1^5
    +21*a1^4-600*a1^3-120*a1^2-272*a1+528-8*a2^2*(21*a1^3+30*a1^2-8*a1+256-66*a1*a2^2
    -38*a2^2))*b23-320*a2*(45*a1^4-241*a1^3-682*a1^2+468*a1-72+a2^2*(77*a1^3+119*a1^2
    +132*a1-44-44*a2^2))*b14-80*(a1*(36*a1^4-99*a1^3-750*a1^2+604*a1+88)-a2^2*(-140
    -644*a2^2+1844*a1-247*a1^2+108*a1*a2^2))*b51+16*a2*(171*a1^4+375*a1^3-1914*a1^2+244*a1
    +152-4*a2^2*(105*a1^2+149*a1-38-108*a2^2))*b42-6*(549*a1^5+990*a1^4+4816*a1^2-1416*a1^3
    -6192*a1+1760-8*a2^2*(105*a1^3+114*a1*a2^2-220-360*a2^2+984*a1-167*a1^2))*b33+80*a2
    *(135*a1^4+474*a1^3-884*a1^2-584*a1+288-8*a2^2*(28*a1^2+31*a1-22-22*a2^2))*b24-20
    *(3*(135*a1^5+294*a1^4+80*a1^3+2144*a1^2-1584*a1+288)-4*a2^2*(630*a1^3+1425*a1^2
    -212*a1-76-440*a1*a2^2-76*a2^2))*b15+1680*a2*(45*a1^3-70*a1^2-516*a1+184+44*a1^2*a2^2
    +36*a1*a2^2+40*a2^2)*b06)/100/(3*(36*a1^5-209*a1^4-416*a1^3+504*a1^2-384*a1+176)
    -16*a2^2*(42*a1^3-87*a1^2-110*a1+128+22*a1*a2^2-19*a2^2)):
b14:=...
b23:=...
```

The lengthy expressions for b_{14} and b_{23} are omitted here for brevity. Under the above conditions, computing the 7th and 8th focus values yields

$$\tilde{v}_7 = -\frac{13}{37748736}\ Q_6(a_1,a_2)\ F_{62}(a_1,a_2) \quad \text{and} \quad \tilde{v}_8 = \frac{13}{4076863488}\ Q_6(a_1,a_2)\ F_{63}(a_1,a_2)$$

where $Q_6(a_1,a_2) = \frac{F_{61}(a_1,a_2)}{F_{60}(a_1,a_2)}$, and

$$F_{60} = 405a_1^4 + 6264a_1^3 + 6264a_1^2 - 5664a_1 + 1360 - 8a_2^2\left(99a_1^2 + 708a_1 + 524 - 170a_2^2\right),$$

$$F_{61} = 5a_1^2\left(a_1^3 - 4a_2^2\right)b_{51} + 2\left(a_1^3 + 8a_2^2\right)(2a_2b_{42} - 3a_1b_{33}) + 80a_1^2a_2b_{24} - 20\left(3a_1^3 + 4a_2^2\right)b_{15} + 240a_1a_2b_{06}$$

$$\begin{aligned} F_{62} = {} & 37179a_1^8 - 524880a_1^7 - 4747248a_1^6 - 12436416a_1^5 + 7737120a_1^4 + 13042944a_1^3 \\ & - 17299200a_1^2 + 6945792a_1 - 578816 - 16\ a_2^2\ \{12393\ a_1^6 - 102708\ a_1^5 \\ & - 802548\ a_1^4 - 1317600\ a_1^3 - 40464\ a_1^2 + 232128\ a_1 + 144704 \\ & - 2\ a_2^2\ \left[3\left(11475a_1^4 - 35496a_1^3 - 271896a_1^2 - 38688a_1 + 129712\right)\right. \\ & \left. + 18088\ a_2^2\ \left(3\ a_1^2 + 12\ a_1 - 4 - a_2^2\right)\right]\}, \end{aligned}$$

$$\begin{aligned} F_{63} = {} & 50526261a_1^{10} - 745329600a_1^9 - 5591362932a_1^8 - 10774083456a_1^7 \\ & + 21426635808a_1^6 + 803271168a_1^5 - 30493584000a_1^4 + 34940049408a_1^3 \\ & - 19814582016a_1^2 + 5723332608a_1 - 456107008 - a_2^2\{186043716a_1^8 \\ & - 1187208576a_1^7 - 3220337088a_1^6 + 19132111872a_1^5 + 748891008a_1^4 \\ & - 39153825792a_1^3 + 39396455424a_1^2 - 15435005952a_1 + 2530583552 \\ & - 32a_2^2\left[36018189a_1^6 - 65822544a_1^5 + 13940100a_1^4 + 2598709824a_1^3\right. \\ & + 520859376a_1^2 - 796391424a_1 + 6959296 + 4a_2^2\left(46888227a_1^4\right. \\ & - 2181168a_1^3 - 658959624a_1^2 - 59298816a_1 + 250943344 \\ & \left.\left. + 4522a_2^2\left(6957a_1^2 + 35808a_1 - 10852 - 2948a_2^2\right)\right)\right]\}. \end{aligned}$$

Now, eliminating a_2 from the two equations $F_{62}(a_1,a_2) = F_{63}(a_1,a_2) = 0$ yields $a_2^2 = G_6(a_1) \equiv A_2$, where

```
A2 := (309476589043365 * a1^23 - 5515299346319946 * a1^22 - 65117402421565968 * a1^21
    + 343086542820896832 * a1^20 + 6091504419437183088 * a1^19 + 20302707438672494624 * a1^18
    - 22849915427488200576 * a1^17 - 203037391007744355072 * a1^16 + 9267870908120547840 * a1^15
    + 977666560668401531904 * a1^14 - 163564694668683952128 * a1^13 - 2825180523848282816512 * a1^12
    + 1652248941383106158592 * a1^11 + 4692526746364655321088 * a1^10 - 5725411348687671853056 * a1^9
    - 2190121368214132359168 * a1^8 + 7836770199015990558720 * a1^7 - 4612471098580271235072 * a1^6
    - 604405275220446806016 * a1^5 + 1733007734562386608128 * a1^4 - 602455690407660486656 * a1^3
    - 12215416715527323648 * a1^2 + 37258420417286111232 * a1 - 3839759129400836096)
  /(505036912585500 * a1^21 - 5359814272145448 * a1^20 - 31830654715627344 * a1^19
    + 833586636750573664 * a1^18 + 6440370133222457088 * a1^17 + 8366639248289269248 * a1^16
    - 29454268933673095680 * a1^15 - 28346693116766745600 * a1^14 + 72380694071939567616 * a1^13
    - 241173650638087282688 * a1^12 - 16590897148659154944 * a1^11 + 1650885296665533579264 * a1^10
    - 1242762519649553645568 * a1^9 - 3907821875385169674240 * a1^8 + 5797383741331795476480 * a1^7
    + 1365604425394373787648 * a1^6 - 7928311543044728684544 * a1^5 + 6085713729955457138688 * a1^4
    - 1552106766836555579392 * a1^3 - 139669993254335545344 * a1^2 + 104083794857023242240 * a1
    - 5791825990506250240) :
```

and a resultant equation: $F_{64}(a_1) = (a_1 + 2)\ F_{64}^*(a_1) = 0$, where

$$
\begin{aligned}
F^*_{64} = {} & \left(51\ a_1^3 + 134\ a_1^2 - 208a_1 + 32\right) \\
& \times \left(51779664543825a_1^{27} - 2789137246241070a_1^{26}\right. \\
& + 37823234268486528a_1^{25} + 1330205934696673920a_1^{24} \\
& - 3022610623559552736a_1^{23} - 15840657352752444864a_1^{22} \\
& + 53824460894374680832a_1^{21} + 311838615653778129408a_1^{20} \\
& - 790508745342083391744a_1^{19} - 3090977270839811084800a_1^{18} \\
& + 1089763060184727693312 0a_1^{17} + 11578741470477118267392a_1^{16} \\
& - 11461865328493056283443 2a_1^{15} + 24202505057080572218572 8a_1^{14} \\
& - 16866632342004468115046 4a_1^{13} - 27271442142378781022617 6a_1^{12} \\
& + 88648906977660265031270 4a_1^{11} - 12274732808953181106339 84a_1^{10} \\
& + 10750212937827704037703 68a_1^{9} - 63522495660720083120947 2a_1^{8} \\
& + 24673329993354909528883 2a_1^{7} - 53479864299307757731840a_1^{6} \\
& + 4907666477818262323 20a_1^{5} + 3243995704522992254976a_1^{4} \\
& - 833100146880987267072a_1^{3} + 59909087602453512192a_1^{2} \\
& \left. + 6243138300014493696a_1 - 833543850067755008\right).
\end{aligned}
$$

Again $a_1 = -2$ is not a solution since $G_6(-2) = -4 < 0$. The polynomial $F^*_{64}(a_1)$ has ten real roots, all of them satisfying $G_6(a_1) > 0$, but only seven of them leading to $\tilde{v}_i = 0,\ i = 0,\ 1,\ \cdots,\ 8$, but $\tilde{v}_9 \neq 0$. These solutions are

$$
\begin{aligned}
a_1 = {} & -4.582523\cdots, \quad -1.722947\cdots, \quad -0.218276\cdots, \\
& -0.094202\cdots, 0.148117\cdots, 1.450129\cdots, \qquad 15.521628\cdots.
\end{aligned}
$$

Thus, there are in total twelve solutions. One example for taking the parameter values is given as follows (up to 100 decimal points):

$$
\begin{aligned}
& a_1 = -0.218276\cdots, \quad a_2 = 1.459254\cdots, b_{23} = -26.697001\cdots\ b_{51}, \\
& b_{14} = 1.700641\cdots\ b_{51}, \quad b_{05} = -1.731869\cdots b_{51}, \quad b_{21} = -0.956680\cdots\ b_{51}, \\
& b_{12} = -22.353922\cdots\ b_{51}, \quad b_{03} = 0.318893\cdots\ b_{51}, \quad b_{01} = 0,
\end{aligned}
$$

where b_{11}, b_{02}, b_{31}, b_{22}, b_{13}, b_{04}, b_{41}, b_{32}, b_{42}, b_{33}, b_{24}, b_{15}, b_{06} have been set zero, associated with the following focus values:

$$
\begin{aligned}
& \tilde{v}_0 = \tilde{v}_1 = 0, \quad \tilde{v}_2 = 0.68 \times 10^{-99}\ b_{51}, \quad \tilde{v}_3 = 0.10 \times 10^{-98}\ b_{51}, \\
& \tilde{v}_4 = 0.165 \times 10^{-97}\ b_{51}, \quad \tilde{v}_5 = -0.10 \times 10^{-97}\ b_{51}, \quad \tilde{v}_6 = 0.8245 \times 10^{-96}\ b_{51}, \\
& \tilde{v}_7 = 0.9 \times 10^{-96}\ b_{51}, \quad \tilde{v}_8 = -0.117076701074 \times 10^{-93}\ b_{51}, \\
& \tilde{v}_9 = -0.0058706767\cdots\ b_{51}.
\end{aligned}
$$

Further, we can similarly show that for all the twelve solutions (critical points),

$$
D_6 = \det \begin{bmatrix} \frac{\partial \tilde{v}_7}{\partial a_1} & \frac{\partial \tilde{v}_7}{\partial a_2} \\ \frac{\partial \tilde{v}_8}{\partial a_1} & \frac{\partial \tilde{v}_8}{\partial a_2} \end{bmatrix} \neq 0.
$$

For example, for the above chosen parameter values, $D_6 = -0.000583\cdots\ b_{51}^2 \neq 0\ (b_{51} \neq 0)$. Thus, we can perturb the parameters a_1, a_2, b_{23}, b_{14}, b_{05}, b_{21}, b_{12}, b_{03}, b_{01} to obtain nine limit cycles. □

3.4. 7th-degree polynomial perturbation – $H_2(7) = 10$

For $n = 7$, we have the following result.

Theorem 7. *The quadratic near-Hamiltonian* system (14) *with 7th-degree polynomial perturbation can have maximal eleven small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(7) = 10$.*

Proof. For this case, $q_7(x,y)$ takes the following form

$$
q_7(x,y) = \sum_{2 \leqslant i+j \leqslant 7} b_{ij} x^i y^j, \text{ with } b_{i0} = 0, \quad i = 2,\ 3,\ \cdots,\ 7. \tag{45}
$$

The focus values $\tilde{v}_1$ and $\tilde{v}_2$ for this case are identical to that of cases $n = 5, 6$, and thus the solutions for b_{03} and b_{12} are the same as that given in the previous two cases. Then, similarly solving $\tilde{v}_3 = 0$ for b_{21}, $\tilde{v}_4 = 0$ for b_{05}, $\tilde{v}_5 = 0$ for b_{14}, $\tilde{v}_6 = 0$ for b_{23}, $\tilde{v}_7 = 0$ for b_{07}, and $\tilde{v}_8 = 0$ for b_{16}, yields a family of solutions, for which computing the 9th and 10th focus values yields

$$
\tilde{v}_8 = -\frac{2431}{4076863488}\ Q_7(a_1,a_2)\ F_{72}(a_1,a_2) \quad \tilde{v}_9 = \frac{2431}{652298158080}\ Q_7(a_1,a_2)\ F_{73}(a_1,a_2),
$$

where $Q_7(a_1,a_2) = \frac{F_{71}(a_1,a_2)}{F_{70}(a_1,a_2)}$, and

$$
\begin{aligned}
F_{70} = {} & a_1\big(483327a_1^9 - 5086962a_1^8 + 20097072a_1^7 + 319466592a_1^6 - 161634528a_1^5 \\
& -313108416a_1^4 + 600223488a_1^3 - 477823488a_1^2 + 173065984a_1 - 15049216\big) \\
& - 16a_2^2\{161109a_1^8 - 991602a_1^7 + 1879524a_1^6 + 1579608a_1^5 - 115548624a_1^4 \\
& - 24810336a_1^3 + 121888448a_1^2 - 77237120a_1 + 19723264 \\
& - 2a_2^2[3(149175a_1^6 - 405450a_1^5 + 595416a_1^4 - 11523792a_1^3 - 23979344a_1^2 \\
& -5514784a_1 + 7372288) + 136a_2^2(5187a_1^4 + 3930a_1^3 + 20324a_1^2 \\
& + 283960a_1 + 162624 - a_2^2(1729a_1^2 + 3458a_1 + 72512))]\}, \\
F_{71} = {} & (3a_1^3 + 8a_2^2)[5a_1^2(a_1^3 - 4a_2^2)\ b_{51} + 4a_2(a_1^3 + 8a_2^2)\ b_{42} - 6a_1(a_1^3 + 8a_2^2)\ b_{33} \\
& +80a_1^2a_2\ b_{24} - 20(3a_1^3 + 4a_2^2)\ b_{15} + 240a_1a_2\ b_{06} - 20(3a_1^3 + 4a_2^2)\ b_{25}] \\
& + 15a_1(a_1^7 - 4a_1^6 - 8a_1^4a_2^2 - 8a_1^3a_2^2 + 64a_2^4)\ b_{61} \\
& - (a_1^3 + 8a_2^2)[20a_2(3a_1^2 - 4a_2^2)\ b_{52} - 24(3a_1^3 - 5a_1a_2^2 - 2a_2^2)\ b_{43}] \\
& + 240a_1^2a_2(a_1^3 - a_1^2 + 4a_2^2)\ b_{34} - 240a_2(3a_1^3 - 8a_1a_2^2 - 8a_2^2)\ b_{16}, \\
F_{72} = {} & 3365793a_1^{12} + 60938568a_1^{11} - 774250488a_1^{10} + 1966200480a_1^9 \\
& + 13136171760a_1^8 - 8029124352a_1^7 - 42401159424a_1^6 + 61639418880a_1^5 \\
& + 11348709120a_1^4 - 85053265920a_1^3 + 68653025280a_1^2 - 20425531392a_1 \\
& + 2343047168 - 8a_2^2\{3(1620567a_1^10 + 26340228a_1^9 - 214842132a_1^8 \\
& + 216250560a_1^7 + 2573086176a_1^6 + 131414400a_1^5 - 4093628544a_1^4 \\
& +1881934848a_1^3 + 1137593088a_1^2 - 1275165696a_1 + 718412800) \\
& - 2a_2^2[3(10180485a_1^8 + 153299952a_1^7 - 674144208a_1^6 - 353045952a_1^5 \\
& + 4636649952a_1^4 + 880277760a_1^3 - 3232210176a_1^2 + 170572800a_1 \\
& +1300940032) + 16a_2^2(7853517a_1^6 + 134834868a_1^5 - 120423348a_1^4 \\
& - 748001952a_1^3 + 215457840a_1^2 + 31982400a_1 - 434094272 \\
& - 133a_2^2(3(14175a_1^4 - 72216a_1^3 - 415512a_1^2 - 299616a_1 - 611344) \\
& -8a_2^2(1215a_1^2 - 74988a_1 - 63300 + 8602a_2^2)))]\}, \\
F_{73} = {} & 595745361a_1^{14} + 9456106860a_1^{13} - 180495550692a_1^{12} + 866884039776a_1^{11} \\
& + 1125517505040a_1^{10} - 7989977121984a_1^9 + 3366147119040a_1^8 \\
& + 34380042236928a_1^7 - 59273145771264a_1^6 + 7717979427840a_1^5 \\
& + 76097098183680a_1^4 - 94586831216640a_1^3 + 49990295040000a_1^2 \\
& - 12029752197120a_1 + 1171523584000 - 4a_2^2\{908390133a_1^{12} \\
& + 9845436600a_1^{11} - 161757046008a_1^{10} + 687515327712a_1^9 - 956879159760a_1^8 \\
& - 5927821906176a_1^7 + 11861554007808a_1^6 + 5082805251072a_1^5 \\
& - 21066056398080a_1^4 + 16917933938688a_1^3 - 4446179260416a_1^2 \\
& - 2461820755968a_1 + 1818858868736 - 4a_2^2[2041291125a_1^{10} \\
& + 16484932404a_1^9 - 280533952044a_1^8 + 1027663111872a_1^7 \\
& - 1251444253536a_1^6 - 7805437665408a_1^5 + 10296688491648a_1^4 \\
& + 4987212991488a_1^3 - 6768106036992a_1^2 - 542252141568a_1 \\
& + 1498038592512 + 4a_2^2(6850884663a_1^8 + 97499706480a_1^7 \\
& - 620311388976a_1^6 + 219072133440a_1^5 + 3577077122976a_1^4 \\
& - 2324658546432a_1^3 - 1344157539072a_1^2 + 1227803667456a_1 \\
& - 445085841664 + 4a_2^2(4026591891a_1^6 + 79808495508a_1^5 - 88999950204a_1^4 \\
& - 494926244640a_1^3 + 243954873936a_1^2 - 33890758848a_1 - 111271460416 \\
& - 532a_2^2(5614515a_1^4 - 25396776a_1^3 - 247047768a_1^2 - 72304416a_1 \\
& -175991376 - 4a_2^2(248955a_1^2 - 22082340a_1 - 13355108 + 2150500a_2^2)))))]\}.
\end{aligned}
$$

Then, eliminating a_2 from the two equations $F_{72}(a_1,a_2) = F_{73}(a_1,a_2) = 0$ yields $a_2^2 = G_7(a_1)$, and a resultant equation: $F_{74}(a_1) = (a_1 + 2)\ F_{74}^*(a_1) = 0$, where

```
F74 := (224181 * al^9 + 7250742 * al^8 + 51634440 * al^7 - 140047504 * al^6 - 106276800 * al^5
   + 502223232 * al^4 - 398093440 * al^3 + 109666560 * al^2 - 6539520 * al + 86528)
  * (2319843658348711436300505043425 * al^54 + 168683334913377195606713413919700 * al^53 + 215538
  8002863595533529931128347900 * al^52 - 74072663221288950463925299730960160 * al^51 - 9081938
  03331934965216212380500186960 * al^50 + 26593256218653692191179625239452658240 * al^49 + 595
  90309934254632532125125184435048768 * al^48 - 5768624732920357744210267231054694516736*
  al^47 + 44431384497510110320072710745554850758144 * al^46 + 293884158659712027975922060194
  232945637376 * al^45 - 7908058044715608701919790173283524690180096 * al^44 + 666787884293412
  38249724078588834626032484352 * al^43 - 283432427523733757215332571183936180360175616 * al
  ^42 + 350598541644286746132271912556254260015366144 * al^41 + 2724678536946731479056188139
  870455784846360576 * al^40 - 15652557904665280851995776359745759543055351808 * al^39 + 24552
  467898355009798648226672248370656515391488 * al^38 + 85101943537360971734036388883918354
  553673351168 * al^37 - 520550469399370815728325492400956306030485831680 * al^36 + 9305833133
  96366534043387016394306392653274546176 * al^35 + 106077891869469692195205496772620763132
  1581223936 * al^34 - 9098310548946983636769659974424445102189164101632 * al^33 + 18344409316
  998706643474338776426550233432846237696 * al^32 + 12860952852724352320655291276289482331
  44066113536 * al^31 - 96521917455713372257236632289978434999589965660160 * al^30 + 248094279
  322272697485504727507962022636305902469120 * al^29 - 24092909923996783358548565341497635
  2343434545070080 * al^28 - 326904271795471452249247492666265561486007530946560 * al^27 + 168
  6092611738668486978264339134742554337303173529600 * al^26 - 3320158423697631792101749697
  871193583468283407892480 * al^25 + 3790103642444937840707725267905940807945331838812160*
  al^24 - 1518970667294577288431963056308825973515566796767232 * al^23 - 3809582384717630970
  336159092312824121502481332043776 * al^22 + 10494824674960839460872636857220238630579095
  642046464 * al^21 - 15650734306374777560762171378891627414010838420291584 * al^20 + 17097960
  479181083883475639338440342194670111350063104 * al^19 - 14715249848371228011149119547977
  033947363737126043648 * al^18 + 10196037726987288079370857130712902341167145669361664 * al
  ^17 - 5665172652888856714329418318999902855366202103955456 * al^16 + 243973646067802276152
  2662913159399045806054730366976 * al^15 - 7269304776389307442327493456022446832415256378
  08128 * al^14 + 71071861915935476965731202135144019440064306085888 * al^12 + 724973841764713
  63851974017727963929213163607687168 * al^13 - 550366644255242066600274915915591404191766
  22981120 * al^11 + 23213198872392003098365841495901554285059648782336 * al^10 - 671839725098
  1524185401731669590733429072602333184 * al^9 + 13131483761890419110663494945489943102780
  58344448 * al^8 - 127823673619133472018021432581584894345372237824 * al^7 - 1607111206640139
  7934533970989148212169784950784 * al^6 + 9491832972470796301751972966033478591436554240*
  al^5 - 2066947427209168200427007363391363277337395200 * al^4 + 276012758848568715197597874
  676625159875461120 * al^3 - 23346877759907407965191624301038017582202880 * al^2 + 1137325766
  437048541264886157263118490664960 * al - 23439719787744597763430408974529116241920.
```

Note that $a_1 = -2$ is not a solution since $G_7(-2) = -4 < 0$. The polynomial $F^*_{74}(a_1)$ has eleven real roots, nine of them satisfying $G_6(a_1) > 0$. But only six of them satisfy $\tilde{v}_i = 0,\ i = 0,\ 1,\ \cdots,\ 9$, but $\tilde{v}_{10} \neq 0$. These solutions are

$$
\begin{aligned}
a_1 = {} & -2.3956026741\ \cdots, \quad -1.5368161998\ \cdots, \quad -0.3824986020\ \cdots, \\
& -0.1957571086\ \cdots, \quad 0.5960015015\ \cdots, \quad 0.2940224977\ \cdots.
\end{aligned}
$$

Thus, there are in total twelve solutions. One example for taking the parameter values is given as follows (up to 100 decimal points):

$$
\begin{aligned}
& a_1 = -0.382498\cdots, \quad a_2 = 1.278764\cdots, \quad b_{07} = -0.288840\cdots\ b_{61}, \\
& b_{23} = 0.522636\cdots\ b_{61}, \quad b_{14} = 0.649003\cdots\ b_{61}, \quad b_{05} = 0.381279\cdots\ b_{61}, \\
& b_{21} = -0.311091\cdots\ b_{61}, \quad b_{12} = 1.637032\cdots\ b_{61}, \quad b_{03} = 0.103697\cdots\ b_{61}, \quad b_{01} = 0,
\end{aligned}
$$

where we have set $b_{11} = b_{02} = b_{31} = b_{22} = b_{13} = b_{04} = b_{41} = b_{32} = b_{51} = b_{42} = b_{33} = b_{24} = b_{15} = b_{06} = b_{52} = b_{43} = b_{34} = b_{25} = b_{16} = 0$. The corresponding focus values are:

$$\begin{aligned}
&\tilde{v}_0 = 0, \quad \tilde{v}_1 = 0.2 \times 10^{-100}\ b_{61}, \quad \tilde{v}_2 = 0.25 \times 10^{-100}\ b_{61}, \quad \tilde{v}_3 = -0.12 \times 10^{-99}\ b_{61},\\
&\tilde{v}_4 = 0.2 \times 10^{-99}\ b_{61}, \quad \tilde{v}_5 = -0.248 \times 10^{-98}\ b_{61}, \quad \tilde{v}_6 = -0.116 \times 10^{-97}\ b_{61},\\
&\tilde{v}_7 = 0.2656 \times 10^{-96}\ b_{61}, \quad \tilde{v}_8 = 0.2495 \times 10^{-96}\ b_{61}, \quad \tilde{v}_9 = -0.381 \times 10^{-94}\ b_{61},\\
&\tilde{v}_{10} = -0.0278295512 \cdots\ b_{61}.
\end{aligned}$$

Further, we can similarly show that for all the twelve solutions,

$$D_7 = \det \begin{bmatrix} \frac{\partial \tilde{v}_8}{\partial a_1} & \frac{\partial \tilde{v}_8}{\partial a_2} \\ \frac{\partial \tilde{v}_9}{\partial a_1} & \frac{\partial \tilde{v}_9}{\partial a_2} \end{bmatrix} \neq 0.$$

In particular, for the above chosen parameter values, $D_7 = -0.046634\ \cdots\ b_{61}^2 \neq 0$ ($b_{61} \neq 0$). Thus, we can perturb the parameters a_1, a_2, b_{07}, b_{23}, b_{14}, b_{05}, b_{21}, b_{12}, b_{03}, b_{01} to obtain ten limit cycles. □

Remark 3. Following the pattern seen from solving the cases n = 3, 4, 5, 6, it seems that b_{16} can be used to solve $\tilde{v}_8 = 0$, and thus the case n = 7 may have eleven limit cycles. If this is true, then the number of small limit cycles around the origin would obey the rule $H_2(n) = \left[\frac{1}{2}(3n+1)\right]$, rather than $H_2(n) = \left[\frac{4}{3}(n+1)\right]$. However, it has been observed from the expressions $\tilde{v}_8$ and $\tilde{v}_9$ that at this step all the remaining b_{ij} coefficients, b_{51}, b_{42}, b_{33}, b_{24}, b_{15}, b_{06}, b_{61}, b_{52}, b_{43}, b_{34}, b_{25} and b_{16} appear in a common factor F_{71} (this factor also appears in $\tilde{v}_{10},\ \tilde{v}_{11},\ \cdots$), and so using any of these coefficients to solve $\tilde{v}_8 = 0$ would result in $\tilde{v}_9 = \tilde{v}_{10} = \cdots = 0$. This unusual pattern will also appear in the cases n = 13, 19, to be discussed in the next section.

3.5. 8th-degree polynomial perturbation –$H_2(8)$ = 12

To end this section, we consider the case n = 8, for which we have the following theorem.

Theorem 8. *The quadratic near-Hamiltonian* system (14) *with 8th-degree polynomial perturbation can have maximal twelve small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(8)$ = 12.*

Proof. For this case, $q_8(x,y)$ can be written as

$$q_8(x,y) = \sum_{2 \leqslant i+j \leqslant 8} b_{ij} x^i y^j, \ \text{with}\ b_{i0} = 0, \quad i = 2,\ 3,\ \cdots,\ 8. \tag{46}$$

First it is noted that the focus values $\tilde{v}_1,\ \tilde{v}_2$ and $\tilde{v}_3$ are identical to that of case n = 7, and thus the solutions b_{03}, b_{12} and b_{21} solved from $\tilde{v}_1 = \tilde{v}_2 = \tilde{v}_3 = 0$ are also the same as that obtained in case n = 7. Then, solving $\tilde{v}_4 = 0$ for b_{05}, $\tilde{v}_5 = 0$ for b_{14}, $\tilde{v}_6 = 0$ for b_{23}, $\tilde{v}_7 = 0$ for b_{07}, $\tilde{v}_8 = 0$ for b_{16}, and $\tilde{v}_9 = 0$ for b_{25} we obtain a family of solutions under which computing the 10th and 11th focus values yields

$$\tilde{v}_{10} = -\frac{676039}{1358954496}\ Q_8(a_1,a_2) F_{82}(a_1,a_2), \quad \tilde{v}_{11} = \frac{676039}{260919263232}\ Q_8(a_1,a_2) F_{83}(a_1,a_2),$$

where $Q_8(a_1,a_2) = \frac{F_{81}(a_1,a_2)}{F_{80}(a_1,a_2)}$, and F_{8i}, i = 1, 2, 3 are polynomials of a_1 and a_2. Their lengthy expressions are omitted here for brevity.

Now eliminating a_2 from the two equations $F_{82}(a_1,a_2) = F_{83}(a_1,a_2) = 0$ yields $a_2^2 = G_8(a_1)$ and a resultant equation: $F_{84}(a_1) = (a_1+2)\ F_{84}^*(a_1) = 0$, where

```
F84* := (411075 * a1^9 + 6092010 * a1^8 + 15940512 * a1^7 − 65011456 * a1^6 − 29424288 * a1^5
    + 186530880 * a1^4 − 100326400 * a1^3 + 15025152 * a1^2 + 225024 * a1 + 512)
  * (190919375832007088158733171666452480 − 1440496381199172105087567190529409024 0 * a1 + 4393
    20120939814709233681837672617738240 * a1^2 − 779181449246283569783774200097566359552 0*
    a1^3 + 914339127810620732336668438885683953664 00 * a1^4 − 740886466214196599363340844 1762
    00712585216 * a1^5 + 40125341160385381871645392510281007685959 68 * a1^6 − 11057986974936220
    536665931437395760614735872 * a1^7 + 583439497280164097150243192564497580011028480 * a1^9
   − 3212874324344524293797991978177026882273 2800 * a1^8 − 385787219455862652125436632056955
    3706496819200 * a1^10 + 170218195292388059177367559321690450417082695 68 * a1^11 − 546715468
    1353808480530420536257224974434461286 4 * a1^12 + 1255111082028873703256749110244819249 9
    2347406336 * a1^13 − 16990249819601231505431021043716535275333301043 2 * a1^14 − 72109503455
```

```
8647622110073488372020429494l9925504 * al^15 + 117448019844744553246403962447405717525 0
```

```
86476221100734883720204294941992550 4 * al^15 + 11744801984474455324640396244740571752 50
564546560 * al^16 - 3851879904562185575678563163098643570088361328640 * al^17 + 82448035472
348093150777287938814115608944181248 00 * al^18 - 1296848800424615390699186169718239388 9
383902085120 * al^19 + 150711300927217216321794154585512866379749916672 00 * al^20 - 1190831
4359746895816334193949636485912631684628480 * al^21 + 39625316636007890423311232781834 4
5141645164544000 * al^22 + 4423245894545170451159257484219418436748275875840 * al^23 - 8202
15132851150258122271806845786843238021201920 0 * al^24 + 60569260965867514900370667190 88
474001282328166400 * al^25 - 12929040325392147094249892120466862349273058508 80 * al^26 - 18
178366289401053416910141458554754553152222003 20 * al^27 + 19104913749169672254388482664
844745096941902233 60 * al^28 - 58246883753872570190221906889713683824319135744 0 * al^29 - 3
00144055116251647792073251605824164928050692096 * al^30 + 333750338467325991231107152 78
950833356911987916 8 * al^31 - 7704828798619318080833249577780245462728782643 2 * al^32 - 461
1817278741507079415965422498028770488745984 0 * al^33 + 3271826731837299820706875738966 2
422334402723840 * al^34 - 2303858930899212957977927534883295251293798400 * al^35 - 43469228
4371488988091670961401340345436156723 2 * al^36 + 132246148900649320143863328469252211 19
57467136 * al^37 + 20670776037258678659098795026592464163445145 6 * al^38 - 1467453015714744
2322240369525609705629496115 2 * al^39 + 2343582590897290273699040483926399508054016 * al^
40 + 742175684091767335546705554691882565713920 0 * al^41 - 981359149199676494424472773728
5176152064 0 * al^42 - 27742988652374893825196880515567094620160 0 * al^43 - 1622974452352922
96963141941972877109350 40 * al^44 + 5532901381958046559502773445237064222720 * al^45 + 9956
061443945009497315307451458744448 00 * al^46 + 54398709378856164965098598106635520000 * al
^47 - 11434840916057489198875717153294416 00 * al^48 - 23064440449761866240070828587989920
0 * al^49 - 35205082393677243556327635249324 00 * al^50 + 379675021430515533937951583700000*
al^51 + 63701731397340659156700758625 00 * al^52 - 4470535389543492592015831125 00 * al^53 + 46
57830968006686085650734375 * al^54) :
```

$a_1 = -2$ is not a solution since $G_8(-2) = -4 < 0$. The polynomial $F_{84}^*(a_1)$ has thirteen real roots, eleven of them satisfying $G_8(a_1) > 0$. By verifying the original focus values, only eight of them satisfy $\tilde{v}_i = 0,\ i = 0,\ 1,\ \cdots,\ 11$, but $\tilde{v}_{12} \neq 0$. These solutions are (up to 100 decimal points):

$$
\begin{aligned}
a_1 = {} & -2.2464265320\ \cdots, \quad -1.5081587824\ \cdots, \quad -0.4276404992\ \cdots, \\
& -0.1808215111\ \cdots, \quad 0.3159980682\ \cdots, \quad 0.2005876789\ \cdots, \\
& 0.6971776483\ \cdots, \quad 1.5717242559\ \cdots.
\end{aligned}
$$

Thus, there are in total sixteen solutions. One example for taking the parameter values is given as follows (up to 100 decimal points):

$$
\begin{aligned}
& a_1 = -1.508158\cdots, \quad a_2 = 0.802414\cdots, \quad b_{08} = -0.427430\cdots\ b_{71}, \\
& b_{16} = -1.170520\cdots\ b_{71}, \quad b_{07} = 4.014705\cdots\ b_{71}, \quad b_{23} = -0.544763\cdots\ b_{71}, \\
& b_{14} = 5.561532\cdots\ b_{71}, \quad b_{05} = -0.341210\cdots\ b_{71}, \quad b_{21} = -0.985713\cdots\ b_{71}, \\
& b_{12} = -1.085828\cdots\ b_{71}, \quad b_{03} = 0.328571\cdots\ b_{71}, \quad b_{01} = 0,
\end{aligned}
$$

where we have set $b_{11} = b_{02} = b_{31} = b_{22} = b_{13} = b_{04} = b_{41} = b_{32} = b_{51} = b_{42} = b_{33} = b_{24} = b_{15} = b_{06} = b_{61} = b_{52} = b_{43} = b_{34} = b_{25} = b_{62} = b_{53} = b_{44} = b_{35} = b_{26} = b_{17} = 0$. The corresponding focus values are:

$$
\begin{aligned}
& \tilde{v}_0 = \tilde{v}_1 = \tilde{v}_2 = 0, \quad \tilde{v}_3 = -0.4 \times 10^{-99} b_{71}, \quad \tilde{v}_4 = 0.141 \times 10^{-98} b_{71}, \\
& \tilde{v}_5 = -0.1 \times 10^{-97}\ b_{71}, \quad \tilde{v}_6 = 0.694 \times 10^{-97}\ b_{71}, \quad \tilde{v}_7 = -0.31386 \times 10^{-96}\ b_{71}, \\
& \tilde{v}_8 = -0.356 \times 10^{-95}\ b_{61}, \quad \tilde{v}_9 = -0.15999882 \times 10^{-93}\ b_{71}, \\
& \tilde{v}_{10} = 0.811292 \times 10^{-92}\ b_{71}, \quad \tilde{v}_{11} = 0.295077675 \times 10^{-90}\ b_{71}, \\
& \tilde{v}_{12} = -0.0042354468\cdots\ b_{71}.
\end{aligned}
$$

Further, we can show that for all the sixteen solutions,

$$
D_8 = \det \begin{bmatrix} \frac{\partial \tilde{v}_{10}}{\partial a_1} & \frac{\partial \tilde{v}_{10}}{\partial a_2} \\ \frac{\partial \tilde{v}_{11}}{\partial a_1} & \frac{\partial \tilde{v}_{11}}{\partial a_2} \end{bmatrix} \neq 0.
$$

In particular, for the above chosen parameter values, $D_8 = 0.000489\cdots\ b_{71}^2 \neq 0$ $(b_{71} \neq 0)$. Thus, we can perturb the parameters a_1, a_2, b_{08}, b_{16}, b_{07}, b_{23}, b_{14}, b_{05}, b_{21}, b_{12}, b_{03}, b_{01} to obtain twelve limit cycles. □

Remark 4. Following the pattern in the previous cases, b_{25} might be used to solve $\tilde{v}_9 = 0$. However, it has been found that after solving $\tilde{v}_8 = 0$ for b_{16}, the expression of $\tilde{v}_9$ actually does not contain the coefficient b_{25}. More precisely, it does not contain any remaining of the 7th-order coefficients: b_{61}, b_{52}, b_{43}, b_{34} and b_{25}. So one must use one 8th-order coefficient to solve $\tilde{v}_9 = 0$. Here, we have used b_{08}. This pattern will also appear in the cases n = 14, 20, to be seen in the next section.

4. Quadratic Hamiltonian systems with 9th-to 20th-degree polynomial perturbations

In this section, we shall consider the cases n = 9, 10, $\cdots$, 20. In order to simplify the presentation, first we establish a rule, based on the cases studied in the previous section, and this rule has been verified for $9 \leqslant n \leqslant 20$ by using general computation with all b_{ij}coefficients retained in the system, like what we have done for the cases $3 \leqslant n \leqslant 8$. Then we present three representative cases n = 12, 13, 14, showing the transfer from the regular pattern to the unusual pattern, as we have seen in the previous section. The results for other cases have been obtained, but computations are more involved for larger n.

4.1. The rule of using b_{ij} coefficients in solving focus values

We have the following result.

Theorem 9. *The b_{ij} coefficients used in solving $\tilde{v}_i = 0$ follows the rule shown in Table 1.*

The pattern can be clearly seen from the table.

4.2. 12th-degree polynomial perturbation –$H_2(12)$ = 17

In this section, we consider the case n = 12, for which we have the following theorem.

Theorem 10. *The quadratic near-Hamiltonian system(14) with 12th-degree polynomial perturbation can have maximal seventeen small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(12)$ = 17.*

Table 1
The rule of using coefficients b_{ij} in solving $v_i = 0$.

n	v_i due to b_{ij}	v_i due to (a_1, a_2)	Nonzero v_i due to Nonzero b_{ij}	LC	Note
2	$b_{02} \to v_1$		$b_{11} \neq 0 - v_2 \neq 0$	2	
	$b_{03} \to v_1$				
3	$b_{12} \to v_2$	(v_3, v_4)	$b_{21} \neq 0 \ \to v_5 \neq 0$	5	
4	$b_{21} \to v_2$	(v_4, v_5)	$b_{31} \neq 0 \ \to v_5 \neq 0$	6	
	$b_{05} \to v_4$				
5	$b_{14} \to v_5$	(v_6, v_7)	$b_{41} \neq 0 \to v_8 \neq 0$	8	
6	$b_{23} \to v_6$	(v_7, v_8)	$b_{51} \neq 0 \ \to v_9 \neq 0$	9	
7	$b_{07} \to v_7$	(v_8, v_9)	$b_{61} \neq 0 \ \to v_{10} \neq 0$	10	b_{16}[a]
	$b_{16} \to v_8$				
8	$(b_{08} \to v_9)$	(v_{10}, v_{11})	$b_{71} \ \neq 0 \to v_{12} \neq 0$	12	b_{25}[b]
	$b_{09} \to v_9$				
9	$b_{18} \to v_{10}$	(v_{11}, v_{12})	$b_{81} \ \neq 0 \to v_{13} \neq 0$	13	
10	$b_{27} \to v_{11}$	(v_{12}, v_{13})	$b_{91} \ \neq 0 \to v_{14} \neq 0$	14	
	$b_{011} \to v_{12}$				
11	$b_{110} \to v_{13}$	(v_{14}, v_{15})	$b_{101} \ \neq 0 \to v_{16} \neq 0$	16	
12	$b_{29} \to v_{14}$	(v_{15}, v_{16})	$b_{11,1} \ \neq 0 \to v_{17} \neq 0$	17	
13	$b_{013} \to v_{15}$	(v_{16}, v_{17})	$b_{121} \ \neq 0 \to v_{18} \neq 0$	18	$b_{1,12}$[a]
	$b_{1,12} \to v_{16}$				
14	$(b_{014} \to v_{17})$	(v_{18}, v_{19})	$b_{131} \ \neq 0 \to v_{20} \neq 0$	20	b_{211}[b]
	$b_{015} \to v_{17}$				
15	$b_{1,14} \to v_{18}$	(v_{19}, v_{20})	$b_{141} \ \neq 0 \to v_{21} \neq 0$	21	
16	$b_{213} \to v_{19}$	(v_{20}, v_{21})	$b_{151} \ \neq 0 \to v_{22} \neq 0$	22	
	$b_{017} \to v_{20}$				
17	$b_{1,16} \to v_{21}$	(v_{22}, v_{22})	$b_{161} \ \neq 0 \to v_{23} \neq 0$	23	
18	$b_{215} \to v_{22}$	(v_{23}, v_{24})	$b_{171} \ \neq 0 \to v_{25} \neq 0$	25	
19	$b_{019} \to v_{23}$	(v_{24}, v_{25})	$b_{181} \ \neq 0 \to v_{26} \neq 0$	26	$b_{1,18}$[a]
	$b_{1,18} \to v_{24}$				
20	$(b_{020} \to v_{25})$	(v_{26}, v_{27})	$b_{191} \ \neq 0 \to v_{28} \neq 0$	28	b_{217}[b]

[a] Indicates that the coefficient can not be used.
[b] Denotes that the coefficient does not appear in the process of computation.

Proof. For this case, according to Table 1, $q_{12}(x,y)$ takes the form:

$$q_{12}(x,y) = b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{23}x^2y^3 + b_{14}xy^4 + b_{05}y^5 + b_{16}xy^6 + b_{07}y^7 + b_{27}x^2y^7 + b_{18}xy^8 + b_{09}y^9 + b_{29}x^2y^9 + b_{110}xy^{10} + b_{011}y^{11} + b_{11,1}x^{11}y. \quad (47)$$

We use the coefficients listed in Table 1 for $n = 12$ to solve the focus value equations $\tilde{v}_i = 0$ to obtain

```
b03 := -1/3 * b21 :
b12 := 1/4 * (4 * a1 * b21 + 3 * a1^2 * b21 + 20 * a2^2 * b21 - 20 * b21 - 12 * b23 - 60 * b05)/a2/(-2 + 5 * a1) :
b21 := 2 * (-32 * b14 * a2 + 240 * b05 * a1 - 280 * a1 * b05 * a2^2 - 270 * b05 * a1^2 + 12 * a1^2 * b23 - 36 * a1 * b23 - 420 * b07 * a1
    + 210 * b05 * a1^3 + 280 * b05 * a2^2 + 27 * a1^3 * b23 - 120 * b05 + 168 * b07 + 192 * a1 * b14 * a2 + 84 * a1 * a2^2 * b23
    - 280 * a1^2 * b14 * a2)
  /(48 * a1 + 24 * a1^2 - 48 * a1 * a2^2 + 3 * a1^4 + 16 * a2^4 - 40 * a1^3 - 16 + 3 * a1^5 - 4 * a1^3 * a2^2)/7 :
b05 := -(-6144 * a1 * a2^2 * b23 + 2304 * b14 * a2 + 1584 * a1 * b23 - 816 * a1^2 * b23 - 1344 * b16 * a2 + 192 * a1^2 * a2^2 * b23
    + 44016 * b07 * a1 - 360 * a1^3 * b23 + 840 * b07 * a1^2 + 21824 * a1^2 * b14 * a2 - 15456 * b07 + 672 * b27 + 6048 * b09
    + 7712 * a1^3 * b14 * a2 - 4224 * a1 * b14 * a2^3 + 6496 * b07 * a2^2 + 63 * a1^5 * b23 - 2016 * a1 * b27 + 81 * a1^6 * b23
    + 1584 * a1^2 * a2^4 * b23 + 1408 * b14 * a2^3 - 31920 * a1 * b07 * a2^2 - 14976 * a1 * b14 * a2 - 4536 * a1^3 * b16 * a2
    - 14112 * a1^2 * b07 * a2^2 - 504 * a1^4 * a2^2 * b23 - 18144 * b09 * a1 - 1800 * a1^4 * b23 - 3808 * a1^2 * b14 * a2^3
    - 2464 * a1^3 * b14 * a2^3 - 720 * a1^3 * a2^2 * b23 - 1440 * a1^4 * b14 * a2 + 10080 * a1 * b16 * a2 + 4788 * b07 * a1^3
    + 912 * a1 * a2^4 * b23 + 2268 * b07 * a1^4 - 504 * a1^2 * b27 + 9856 * b07 * a2^4 + 1408 * b14 * a2^5 + 672 * a2^2 * b27
    + 6048 * a1 * b16 * a2^3 - 17136 * a1^2 * b16 * a2 - 1344 * b16 * a2^3 + 6048 * b09 * a2^2 - 4536 * b09 * a1^2)
  /(1392 * a1^2 * a2^2 - 2048 * a2^2 + 108 * a1^5 - 627 * a1^4 - 1248 * a1^3 + 1512 * a1^2 - 672 * a1^3 * a2^2 - 1152 * a1
    + 1760 * a1 * a2^2 - 352 * a2^4 * a1 + 528 + 304 * a2^4)/10 :
b14 := -(4561920 * a1^5 * b09 * a2^2 - 1330560 * a1^6 * b07 * a2^2 + 54912 * a1^5 * a2^6 * b23 + 861696 * a1^6 * a2^4 * b23
    + 329472 * a1^4 * b16 * a2^5 - 290304 * a1^3 * a2^2 * b29 - 2230272 * a1^2 * b18 * a2^5 - 27072 * a1^5 * a2^2 * b27
    - 1913472 * a1^3 * a2^4 * b23 - 8059392 * a1^3 * b09 * a2^4 - 1153152 * a1^5 * b07 * a2^4 + 14212800 * b07 * a1^2
    - 17076240 * a1^5 * b07 * a2^2 - 5381376 * a1^4 * b07 * a2^4 + 1482624 * a1^6 * b16 * a2^3 + 1070784 * a1^4 * b23
    - 1782528 * a1^2 * a2^2 * b23 + 2480896 * a1^3 * b07 * a2^4 + 3691776 * a1^2 * a2^4 * b23 - 24456960 * b07 * a2^2
    - 55782528 * a1^2 * b07 * a2^2 + 44334336 * a1 * b07 * a2^2 - 152064 * a1 * a2^4 * b29 + 18730368 * a1^2 * b16 * a2
    + 2825856 * a1^3 * a2^2 * b23 + 30219264 * a1 * b16 * a2^3 - 4097088 * a2 * a1^5 * b18 + 512512 * a1^3 * b07 * a2^6
    - 1279872 * a1^4 * a2^2 * b23 - 549120 * a1^2 * a2^6 * b23 + 11017728 * a1^3 * b18 * a2 - 1726464 * a1 * a2^2 * b27
    - 3440448 * a1^4 * a2^4 * b23 - 15003648 * a1 * b18 * a2^3 - 11132928 * a1 * b16 * a2^5 + 8797248 * a1^5 * b16 * a2
    - 21223680 * a1^2 * b16 * a2^3 - 19628928 * a1^3 * b16 * a2 - 7871232 * a1^3 * b16 * a2^3 - 14010624 * b07 * a1^3
    - 47264256 * a1 * b09 * a2^2 - 9040896 * a1^2 * b18 * a2 - 1738176 * a1^5 * a2^2 * b23 + 10366560 * a1^4 * b16 * a2
    + 34512384 * a1^3 * b07 * a2^2 - 11328768 * b07 * a1 - 1856640 * a1^2 * a2^2 * b27 + 21676032 * a1^2 * b09 * a2^2
    + 33392016 * a1^4 * b07 * a2^2 + 1652112 * a1^6 * a2^2 * b23 - 9819456 * a1^2 * b07 * a2^4 - 1130976 * a1^5 * b23
    + 170016 * a1^5 * a2^4 * b23 + 2331648 * a1 * b18 * a2^5 - 7185024 * a2 * a1^4 * b18 + 12718464 * a1^2 * b16 * a2^5
    + 658944 * a1^4 * a2^6 * b23 + 6614784 * a1^2 * b011 * a2^2 + 36905472 * a1 * b09 * a2^4 - 149688 * a2 * a1^7 * b16
    - 549120 * a1^2 * a2^8 * b23 + 6918912 * a1^2 * b07 * a2^6 - 257400 * a1^7 * a2^2 * b23 - 5271552 * a1 * b09 * a2^6
    - 439296 * a1^2 * b16 * a2^7 - 3193344 * a1^3 * b011 * a2^2 - 10708992 * a1 * b16 * a2 - 52928256 * a1 * b07 * a2^4
    - 4257792 * a1^4 * b18 * a2^3 - 1317888 * a1 * b16 * a2^7 + 15772416 * a1 * b07 * a2^6 + 601344 * a1^2 * a2^2 * b29
    + 3518592 * a1^3 * b16 * a2^5 - 18718560 * a1^4 * b09 * a2^2 - 18002304 * a1^3 * b09 * a2^2 - 164736 * a1^3 * b23
    + 9547776 * a1^2 * b18 * a2^3 + 1038592 * a1 * a2^4 * b27 - 102912 * a1^3 * a2^2 * b27 + 645696 * a1^2 * a2^4 * b27
    - 5436288 * a1^5 * b16 * a2^3 - 467424 * a1^4 * b16 * a2^3 - 123552 * a1^7 * a2^4 * b23 - 2443608 * a2 * a1^6 * b16
    + 22201344 * a1^2 * b09 * a2^4 + 1786752 * a1^3 * a2^6 * b23 + 84336 * a1^4 * a2^2 * b27 + 760320 * a1 * a2^2 * b29
    + 9593856 * a1^3 * b18 * a2^3 - 1672704 * a1 * b011 * a2^4 - 47520 * a1^8 * a2^2 * b23 + 8363520 * a1 * b011 * a2^2
    - 21120 * a1^3 * a2^4 * b27 + 684288 * a2 * a1^6 * b18 - 28160 * a1 * a2^6 * b27 + 1571328 * b16 * a2 + 5347584 * b07
    - 1181376 * a1^2 * b27 + 48438016 * b07 * a2^4 + 2740992 * a2^2 * b27 + 431640 * a1^7 * b23 - 5474304 * b011 * a1
    + 19782144 * b09 * a1^3 + 4672512 * a1 * b18 * a2 - 4738560 * b16 * a2^3 + 36619776 * b09 * a2^2 + 2509056 * b011
    - 22934016 * b09 * a1^2 + 9481392 * b07 * a1^4 - 1676304 * b07 * a1^5 - 39794688 * b09 * a2^4 - 608256 * b18 * a2
    - 13221120 * b07 * a2^6 - 1104640 * a2^4 * b27 + 39600 * a1^6 * b23 - 1073088 * b09 * a1^5 + 1125576 * b07 * a1^7
    + 658944 * b16 * a2^7 - 5930496 * b011 * a1^3 + 17832960 * b09 * a1 + 2597184 * b09 * a1^4 - 4866156 * b07 * a1^6
    + 7185024 * b011 * a1^2 + 1375488 * a1 * b27 + 1733328 * a1^4 * b27 - 9732096 * b011 * a2^2 - 188892 * a1^8 * b23
    + 513216 * b011 * a1^5 + 1444608 * b011 * a2^4 - 2979504 * b011 * a1^4 + 1586304 * b09 * a1^6 - 497664 * a1 * b29
    + 2359296 * b18 * a2^3 + 3294720 * b09 * a2^6 + 653184 * a1^2 * b29 - 785664 * b27 - 8211456 * b09 + 228096 * b29
    - 2050048 * b07 * a2^8 - 256608 * b09 * a1^7 + 332928 * a1^3 * b27 - 674304 * b16 * a2^5 - 884736 * a2^2 * b29
    - 115456 * a2^6 * b27 + 131328 * a2^4 * b29 - 350208 * b18 * a2^5 - 229716 * a1^6 * b27 - 270864 * a1^4 * b29
```

```
    − 375984 * a1^5 * b27 − 539136 * a1^3 * b29 + 16038 * a1^9 * b23 + 2673 * a1^10 * b23 + 74844 * b07 * a1^8
    + 29808 * a1^7 * b27 + 46656 * a1^5 * b29)
  /(−4992 − 22464 * a2^4 * a1 + 6336 * a1^2 * a2^2 − 12928 * a2^2 − 11328 * a1 * a2^2 − 51104 * a1^3 * a2^2 + 32448 * a1
    + 1200 * a1^3 + 13080 * a1^4 + 55936 * a2^4 − 5724 * a1^5 − 34272 * a1^2 + 486 * a1^6 + 81 * a1^7 − 832 * a1^6 * a2^4
    + 26544 * a1^3 * a2^4 − 22400 * a2^6 − 3328 * a2^8 + 59256 * a1^4 * a2^2 + 9984 * a1^2 * a2^6 − 39024 * a1^4 * a2^4
    + 83820 * a1^5 * a2^2 − 65760 * a1^2 * a2^4 − 32224 * a1^6 * a2^2 − 10608 * a1^5 * a2^4 + 9152 * a1^3 * a2^6
    + 1872 * a1^8 * a2^2 − 12720 * a1^7 * a2^2 + 27456 * a1 * a2^6)/a2/88 :
b23 := ...
b07 := ...
b16 := ...
b09 := ...
b18 := ...
b27 := ...
b011 := ...
b110 := ...
b29 := ...
```

Under the above conditions, computing the 15th and 16th focus values yields

$$\tilde{v}_{15} = -\frac{32395}{51298814505517056}\ Q_{12}(a_1, a_2)\ F_{12a}(a_1, a_2),$$
$$\tilde{v}_{16} = \frac{32395}{10464958159125479424}\ Q_{12}(a_1, a_2)\ F_{12b}(a_1, a_2),$$

where $Q_{12}(a_1,a_2)$ is a common factor (rational function), and F_{12a} and F_{12b} are 16th and 17th-degree polynomials of a_2^2, respectively. Eliminating a_2 from the two polynomial equations $F_{12a} = F_{12b} = 0$ results in $a_2^2 = G_{12}(a_1)$, and a resultant equation: $F_{12c} = (a_1 + 2)\ F^*_{12c1}(a_1)\ F^*_{12c2}(a_1) = 0$, where F^*_{12c1} and F^*_{12c2} are 300th-and 108th-degree polynomials of a_1, respectively. It has been found that F^*_{12c2} has 16 real roots, but none of them satisfies $\tilde{v}_{15} = \tilde{v}_{16} = 0$, though these roots generate $G_{12} > 0$. On the other hand, F^*_{12c1} has 24 real roots, 21 of them satisfy $G_{12} > 0$ and $\tilde{v}_i, i = 1, 2, \ldots, 16$, but $\tilde{v}_{17} \neq 0$. Therefore, there are in total 42 solutions for this case. One example is given by choosing the following parameter values (up to 100 decimal points):

$$\begin{aligned}
&a_1 = -0.3332384216\cdots, \quad a_2 = 0.7285122463\cdots,\\
&b_{29} = -335.2660464\cdots\ b_{11,1}, \quad b_{110} = 7.050743261\cdots\ b_{11,1},\\
&b_{011} = -32.21729250\cdots\ b_{11,1}, \quad b_{27} = -2393.202281\cdots\ b_{11,1},\\
&b_{18} = -1591.226517\cdots\ b_{11,1}, \quad b_{09} = -399.8174247\cdots\ b_{11,1},\\
&b_{16} = -1257.562384\cdots\ b_{11,1}, \quad b_{07} = 235.6307204\cdots\ b_{11,1},\\
&b_{23} = 11707.09117\cdots\ b_{11,1}, \quad b_{14} = 242.2101585\cdots\ b_{11,1},\\
&b_{05} = 517.9980964\cdots\ b_{11,1}, \quad b_{21} = -1.0014595077\cdots b_{11,1},\\
&b_{12} = 16057.95547\cdots b_{11,1}, \quad b_{03} = 0.3338198359\cdots b_{11,1}, \quad b_{01} = 0,
\end{aligned}$$

for which the corresponding focus values are:

$$\begin{aligned}
&\tilde{v}_0 = \tilde{v}_1 = 0, \quad \tilde{v}_2 = 0.575 \times 10^{-98}\ b_{11,1}, \quad \tilde{v}_3 = -0.1 \times 10^{-96}\ b_{11,1},\\
&\tilde{v}_4 = -0.31 \times 10^{-95}\ b_{11,1}, \quad \tilde{v}_5 = 0.231 \times 10^{-95}\ b_{11,1},\\
&\tilde{v}_6 = -0.1209 \times 10^{-94}\ b_{11,1}, \quad \tilde{v}_7 = 0.160888898 \times 10^{-93}\ b_{11,1},\\
&\tilde{v}_8 = -0.78577 \times 10^{-93}\ b_{11,1}, \quad \tilde{v}_9 = -0.102844007 \times 10^{-92}\ b_{11,1},\\
&\tilde{v}_{10} = -0.2878923898 \times 10^{-91}\ b_{11,1}, \quad \tilde{v}_{11} = 0.5454574658 \times 10^{-90}\ b_{11,1},\\
&\tilde{v}_{12} = -0.3944240092 \times 10^{-88}\ b_{11,1}, \quad \tilde{v}_{13} = 0.2832152908 \times 10^{-87}\ b_{11,1},\\
&\tilde{v}_{14} = 0.6290073553 \times 10^{-86}\ b_{11,1}, \quad \tilde{v}_{15} = 0.7136148552 \times 10^{-85}\ b_{11,1},\\
&\tilde{v}_{16} = -0.4395054496 \times 10^{-83}\ b_{11,1}, \quad \tilde{v}_{17} = -0.0000369260\cdots\ b_{11,1}.
\end{aligned}$$

Further, we can show that for the above critical values,

$$D_{12} = \det \begin{bmatrix} \frac{\partial \tilde{v}_{15}}{\partial a_1} & \frac{\partial \tilde{v}_{15}}{\partial a_2} \\ \frac{\partial \tilde{v}_{16}}{\partial a_1} & \frac{\partial \tilde{v}_{16}}{\partial a_2} \end{bmatrix} = 0.0000125138\ \cdots\ b_{11,1}^2 \neq 0, \quad (b_{11,1} \neq 0),$$

which indicates that one can perturb a_1, a_2 and the b_{ij} coefficients to obtain seventeen limit cycles. □

4.3. 13th-degree polynomial perturbation –$H_2(13) = 18$

For case $n = 13$, we have the following theorem.

Theorem 11. *The quadratic near-Hamiltonian* system (14) *with 13th-degree polynomial perturbation can have maximal eighteen small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(13) = 18$.*

Proof. According to Table 1, for this case, $q_{13}(x,y)$ takes the following form:

$$q_{13}(x,y) = b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{23}x^2y^3 + b_{14}xy^4 + b_{05}y^5 + b_{16}xy^6 + b_{07}y^7 + b_{27}x^2y^7 + b_{18}xy^8 + b_{09}y^9 + b_{29}x^2y^9 + b_{110}xy^{10} + b_{011}y^{11} + b_{121}x^{12}y + b_{013}y^{13}. \quad (48)$$

We use the coefficients listed in the Table 1 to solve the focus value equations $\tilde{v}_i = 0$ to obtain b_{03}, b_{12}, b_{21}, b_{05} and b_{14}, which are the same as that given for the case $n = 12$. Other lengthy expressions for b_{23}, b_{07}, b_{16}, b_{09}, b_{18}, b_{27}, b_{011}, b_{110}, b_{29} and b_{013} are omitted here for brevity. Having found the above coefficients, computing the 16th and 17th focus values yields

$$\tilde{v}_{16} = \frac{21607465}{205195258022068224}\ Q_{13}(a_1,a_2)\ F_{13a}(a_1,a_2),$$
$$\tilde{v}_{17} = -\frac{21607465}{1772887029310669455\,36}\ Q_{13}(a_1,a_2)\ F_{13b}(a_1,a_2),$$

where $Q_{13}(a_1,a_2)$ is a common factor (rational function), and F_{13a} and F_{13b} are 20th and 21th-degree polynomials of a_2^2, respectively. We use the built-in solver "fsolve" in Maple to solve the two polynomial equations: $F_{13a} = F_{13b} = 0$, to find a solution up to 1000 decimal points:

$$a_1 = 0.0257481687\ \cdots, \qquad a_2 = 0.7260217686\ \cdots$$

Other corresponding b_{ij} coefficients are given by (up to 1000 decimal points):

$$\begin{aligned}
&b_{013} = -0.2063119737\cdots b_{121}, && b_{29} = -3317.631354\cdots\ b_{121},\\
&b_{110} = 1.6405785173\cdots\ b_{121}, && b_{011} = 35.65167073\cdots\ b_{121},\\
&b_{27} = -44835.84962\cdots\ b_{121}, && b_{18} = -21136.38636\cdots\ b_{121},\\
&b_{09} = 443.3249551\cdots\ b_{121}, && b_{16} = -31711.19244\cdots\ b_{121},\\
&b_{07} = 15.98673120\cdots\ b_{121}, && b_{23} = 65849.79281\cdots\ b_{121},\\
&b_{14} = 58.47934625\cdots\ b_{121}, && b_{05} = -533.8629596\cdots\ b_{121},\\
&b_{21} = -0.9999950065\cdots b_{121}, && b_{12} = 139513.1753\cdots b_{121},\\
&b_{03} = 0.3333316688\cdots b_{121}, && b_{01} = 0,
\end{aligned}$$

for which the corresponding focus values are:

$$\begin{aligned}
&\tilde{v}_0 = \tilde{v}_1 = 0, \quad \tilde{v}_2 = -0.3641792\times 10^{-995}\ b_{121}, \quad \tilde{v}_3 = 0.9407780\times 10^{-995}\ b_{121},\\
&\tilde{v}_4 = -0.107\times 10^{-994}\ b_{121}, \qquad \tilde{v}_5 = 0.2063478\times 10^{-994}\ b_{121},\\
&\tilde{v}_6 = -0.3023\times 10^{-994}\ b_{121}, \qquad \tilde{v}_7 = 0.2176313347\times 10^{-993}\ b_{121},\\
&\tilde{v}_8 = -0.8321231688\times 10^{-993}\ b_{121}, \qquad \tilde{v}_9 = 0.8378132516\times 10^{-992}\ b_{121},\\
&\tilde{v}_{10} = -0.1045852\times 10^{-990}\ b_{121}, \qquad \tilde{v}_{11} = 0.2959753605\times 10^{-989}\ b_{121},\\
&\tilde{v}_{12} = -0.8352600340\times 10^{-989}\ b_{121}, \qquad \tilde{v}_{13} = 0.5771797060\times 10^{-988}\ b_{121},\\
&\tilde{v}_{14} = 0.7579489695\times 10^{-986}\ b_{121}, \qquad \tilde{v}_{15} = -0.1040540821\times 10^{-984}\ b_{121},\\
&\tilde{v}_{16} = 0.2101032732\times 10^{-983}\ b_{121}, \qquad \tilde{v}_{17} = 0.8098677917\times 10^{-982}\ b_{121},\\
&\tilde{v}_{18} = 0.13113103624\cdots\times 10^{-5}\ b_{121}.
\end{aligned}$$

Further, we can show that for the above critical values,

$$D_{13} = \det\begin{bmatrix} \frac{\partial\tilde{v}_{16}}{\partial a_1} & \frac{\partial\tilde{v}_{16}}{\partial a_2} \\ \frac{\partial\tilde{v}_{17}}{\partial a_1} & \frac{\partial\tilde{v}_{17}}{\partial a_2} \end{bmatrix} = 0.28521385848\ \cdots\ \times 10^{-8}\ b_{121}^2 \neq 0, \quad (b_{121}\neq 0),$$

implying that one can perturb a_1, a_2 and the b_{ij} coefficients to obtain eighteen limit cycles. □

4.4. 14th-degree polynomial perturbation –$H_2(14) = 20$

For the last case, $n = 14$, we have the following result.

Theorem 12. *The quadratic near-Hamiltonian system*(14) *with 14th-degree polynomial perturbation can have maximal twenty small-amplitude limit cycles bifurcating from the origin, i.e., $H_2(14) = 20$.*

Proof. For this case, $q_{14}(x,y)$ can be taken as

$$q_{14}(x,y) = b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{23}x^2y^3 + b_{14}xy^4 + b_{05}y^5 + b_{16}xy^6 + b_{07}y^7 + b_{27}x^2y^7 + b_{18}xy^8 + b_{09}y^9 + b_{29}x^2y^9 + b_{110}xy^{10} + b_{011}y^{11} + b_{1,12}xy^{12} + b_{013}y^{13} + b_{131}x^{13}y + b_{014}y^{14}. \quad (49)$$

We use the coefficients listed in Table 1 to solve the focus value equations $\tilde{v}_i = 0$ to obtain b_{03}, b_{12}, b_{21}, b_{05} and b_{14}, which are the same as that given for the case $n = 12$. Other lengthy expressions for b_{23}, b_{07}, b_{16}, b_{09}, b_{18}, b_{27}, b_{011}, b_{110}, b_{29}, b_{013}, $b_{1,12}$, b_{014} and b_{131} are omitted for brevity. Having determined the above coefficients, computing the 18th and 19th-order focus values yields

$$\tilde{v}_{18} = -\frac{112137057175}{28855583159353344}\ Q_{14}(a_1,a_2)\ F_{14a}(a_1,a_2),$$
$$\tilde{v}_{19} = \frac{156991880045}{615585774066204672}\ Q_{14}(a_1,a_2)\ F_{14b}(a_1,a_2),$$

where $Q_{14}(a_1,a_2)$ is a common factor (rational function), and F_{14a} and F_{14b} are 20th and 21th-degree polynomials of a_2^2, respectively. We use the built-in solver ''fsolve'' in Maple to solve the two polynomial equations: $F_{14a} = F_{14b} = 0$, to find a solution up to 1000 decimal points:

$$a_1 = 0.1393454226\ \cdots, \qquad a_2 = 0.4716609811\ \cdots$$

Other corresponding b_{ij} coefficients are given by (up to 1000 decimal points):

$$\begin{aligned}
&b_{014} = 0.0021865068\cdots b_{131}, \qquad b_{1,12} = -0.0528713061\cdots b_{131},\\
&b_{013} = -0.0008911582\cdots b_{131}, \qquad b_{29} = -127.0526814\cdots\ b_{131},\\
&b_{110} = 4.4143060020\cdots\ b_{131}, \qquad b_{011} = 8.2775044971\cdots\ b_{131},\\
&b_{27} = -5872.921032\cdots\ b_{131}, \qquad b_{18} = -1585.602652\cdots\ b_{131},\\
&b_{09} = 294.6646844\cdots\ b_{131}, \qquad b_{16} = -7133.416532\cdots\ b_{131},\\
&b_{07} = 212.2641879\cdots\ b_{131}, \qquad b_{23} = 142578.7773\cdots\ b_{131},\\
&b_{14} = -136.7209413\cdots\ b_{131}, \qquad b_{05} = -7347.932109\cdots\ b_{131},\\
&b_{21} = -0.9999999952\cdots b_{131}, \qquad b_{12} = 516531.7769\cdots b_{131},\\
&b_{03} = 0.3333333317\cdots b_{121}, \qquad b_{01} = 0,
\end{aligned}$$

for which the associated focus values are:

$$\begin{aligned}
&\tilde{v}_0 = \tilde{v}_1 = 0, \qquad \tilde{v}_2 = 0.1946572 \times 10^{-995}\ b_{131}, \qquad \tilde{v}_3 = -0.2420 \times 10^{-994}\ b_{131},\\
&\tilde{v}_4 = 0.64 \times 10^{-995}\ b_{131}, \qquad \tilde{v}_5 = 0.10778 \times 10^{-994}\ b_{131},\\
&\tilde{v}_6 = 0.54055 \times 10^{-994}\ b_{131}, \qquad \tilde{v}_7 = 0.7293242778 \times 10^{-994}\ b_{131},\\
&\tilde{v}_8 = 0.86612349 \times 10^{-993}\ b_{131}, \qquad \tilde{v}_9 = 0.4677922416 \times 10^{-992}\ b_{131},\\
&\tilde{v}_{10} = -0.9548521108 \times 10^{-992}\ b_{131}, \qquad \tilde{v}_{11} = -0.1447200857 \times 10^{-989}\ b_{131},\\
&\tilde{v}_{12} = -0.7146995602 \times 10^{-989}\ b_{131}, \qquad \tilde{v}_{13} = -0.1008519561 \times 10^{-988}\ b_{131},\\
&\tilde{v}_{14} = 0.3157956249 \times 10^{-987}\ b_{131}, \qquad \tilde{v}_{15} = 0.7850157892 \times 10^{-986}\ b_{131},\\
&\tilde{v}_{16} = 0.1526212083 \times 10^{-984}\ b_{131}, \qquad \tilde{v}_{17} = -0.1126378381 \times 10^{-983}\ b_{131},\\
&\tilde{v}_{18} = -0.2323564724 \times 10^{-981}\ b_{131}, \qquad \tilde{v}_{19} = -0.7445896836 \times 10^{-980}\ b_{131},\\
&\tilde{v}_{20} = 0.1288587954\cdots \times 10^{-7}\ b_{131}.
\end{aligned}$$

Further, we can show that for the above critical values,

$$D_{14} = \det\begin{bmatrix} \frac{\partial \tilde{v}_{18}}{\partial a_1} & \frac{\partial \tilde{v}_{18}}{\partial a_2} \\ \frac{\partial \tilde{v}_{19}}{\partial a_1} & \frac{\partial \tilde{v}_{19}}{\partial a_2} \end{bmatrix} = 0.799045764\ \cdots\ \times 10^{-12}\ b_{131}^2 \neq 0, \quad (b_{131} \neq 0),$$

which shows that one can perturb a_1, a_2 and the b_{ij} coefficients to obtain twenty limit cycles. □

Finally, summarizing the results obtained in this section as well as in the previous section shows that the proof for the main Theorem 1 is complete.

5. Conclusion

In this paper, we have studied bifurcation of limit cycles from perturbing quadratic Hamiltonian systems with nth-degree polynomial perturbations. It has been shown at least up to $n = 20$ the number of small-amplitude limit cycles obeys a simple rule, $H_2(n) = \left[\frac{4}{3}(n+1)\right]$. For general n, we have the following conjecture.

Conjecture 1. *The quadratic near-Hamiltonian* system (14) *with nth-degree polynomial perturbation can have maximal* $\left[\frac{4}{3}(n+1)\right]$ $(n \geqslant 3)$ *small-amplitude limit cycles bifurcating from the origin, i.e.,* $H_2(n) = \left[\frac{4}{3}(n+1)\right]$ *for* $n \geqslant 3$.

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