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CRITICAL PERIODS OF PLANAR REVERTIBLE VECTOR FIELD WITH THIRD-DEGREE POLYNOMIAL FUNCTIONS

PEI YU* and MAOAN HAN

Department of Applied Mathematics,
The University of Western Ontario,
London, Ontario N6A 5B7, Canada

**Department of Mathematics, Shanghai Normal University,*
Shanghai 200234, P. R. China
**pyu@uwo.ca*

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In this paper, we consider local critical periods of planar vector field. Particular attention is given to revertible systems with polynomial functions up to third degree. It is assumed that the origin of the system is a center. Symbolic and numerical computations are employed to show that the general cubic revertible systems can have six local critical periods, which is the maximal number of local critical periods that cubic revertible systems may have. This new result corrects that in the literature: general cubic revertible systems can at most have four local critical periods.

Keywords: Critical periods; revertible system; center; normal form.

1. Introduction

The study of Hilbert's 16th problem [Hilbert, 1902] has attracted many researchers from the area of nonlinear dynamical systems. The problem seems far away from having a complete solution, since the uniform finiteness problem is not solved even for general quadratic systems. In order to find the upper bound of Hilbert number, many researchers have turned to consider the lower bound of Hilbert number and hoped to get close to the upper bound by raising the lower bound for general planar polynomial systems or for individual degree of systems. Many results have been obtained (e.g. see the review articles [Han, 2002; Li, 2003; Yu, 2006; Han & Zhang, 2006]). One of the research directions is to study small amplitude limit cycles bifurcating from Hopf critical point by computing the focus values (or Lyapunov constants, or normal form of Hopf bifurcation) with the aid of computer algebra systems such as Maple, Mathematica. The earliest result based on focus value computation goes back to Bautin [1954] who proved that a general quadratic system can at most have three small limit cycles bifurcating from an isolated Hopf critical point. Recently, the method of normal forms and efficient computation technique have been used to obtain bifurcation of 12 small limit cycles in cubic polynomial planar systems [Yu & Han, 2004, 2005a, 2005b].

Another interesting problem is bifurcation of limit cycles from equilibria of center type, since the monotonicity of periods of closed orbit surrounding a center is a nondegeneracy condition of subharmonic bifurcation for periodically forced Hamiltonian systems [Chow & Hale, 1982]. Suppose

*Author for correspondence

the planar polynomial vector field is described by the following differential equations:

$$\frac{dx}{dt} = P_n(x, y, \boldsymbol{\mu}), \quad \frac{dy}{dt} = Q_n(x, y, \boldsymbol{\mu}), \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent the nth-degree polynomials of x and y, and $\boldsymbol{\mu} \in R^k$ is a k-dimensional parameter vector. Suppose the origin of system (1) is a fixed point and further it is a nondegenerate center. (If the Jacobian of the system does not have a double zero eigenvalue at the origin, then the origin is called a nondegenerate center.)

Now let $T(h, \boldsymbol{\mu})$ denote the minimum period of closed orbit of system (1) surrounding the origin for $0 < h \ll 1$. Then the origin is said to be a weak center of finite order k of the system for the parameter value $\boldsymbol{\mu} = \boldsymbol{\mu_c}$ if

$$T'(0, \boldsymbol{\mu}_c) = T''(0, \boldsymbol{\mu}_c) = \cdots = T^k(0, \boldsymbol{\mu}_c) = 0, \quad \text{but} \quad T^{k+1}(0, \boldsymbol{\mu}_c) \neq 0. \tag{2}$$

The origin is called an isochronous center if $T^k(0, \boldsymbol{\mu}_c) = 0 \ \forall \ k \geq 1$, or equivalently, $T(h, \boldsymbol{\mu}_c) =$ constant for $0 < h \ll 1$. A local critical period is defined as a period corresponding to a critical point of the period function $T(h, \boldsymbol{\mu})$ which bifurcates from a weak center.

For the quadratic system, given by

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{i+j=2} a_{ij} x^i y^j, \\ \frac{dy}{dt} &= x + \sum_{i+j=2} b_{ij} x^i y^j, \end{aligned} \tag{3}$$

Chicone and Jacobs [1989] discussed weak centers and critical periods which may bifurcate from weak centers. In the same paper, they also studied the following special Hamiltonian system:

$$\ddot{u} + V(u) = 0, \tag{4}$$

where V is a $2n$-degree polynomial of u. Let $u = x$ and $\dot{u} = y$. Then, the Hamiltonian of system (4) can be written as

$$H(x, y) = \frac{1}{2} y^2 + \int_0^x V(s) ds. \tag{5}$$

It has been shown [Chicone & Jacobs, 1989] that system (4) can have at most $n - 2$ critical periods bifurcating from the origin.

In 1993, Rousseau and Toni [1993] studied a special cubic system with third-degree homogeneous polynomials only, as described below:

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{i+j=3} a_{ij} x^i y^j, \\ \frac{dy}{dt} &= x + \sum_{i+j=3} b_{ij} x^i y^j. \end{aligned} \tag{6}$$

They similarly discussed weak centers and bifurcation of critical periods from weak centers.

Recently, Zhang *et al.* [2000] gave a detailed study on cubic revertible polynomial systems — a system is said to be revertible if it is symmetric with respect to a line. Up to translation and rotation of coordinates, any revertible cubic differential systems can be described by (e.g. see [Zhang *et al.*, 2000]):

$$\begin{aligned} \frac{dx}{dt} &= -y + a_{20} x^2 + a_{02} y^2 + a_{21} x^2 y + a_{03} y^3, \\ \frac{dy}{dt} &= x + b_{11} xy + b_{30} x^3 + b_{12} x y^2, \end{aligned} \tag{7}$$

where a_{ij} and b_{ij} are constant parameters. It has been shown [Zhang *et al.*, 2000] that system (7) can have at most four local critical periods.

The work of Maosas and Villadelprat [2006] should be also mentioned. The system considered in [Maosas & Villadelprat, 2006] is a Hamiltonian system with the following Hamiltonian:

$$H(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{a}{4} x^4 + \frac{b}{6} x^6, \tag{8}$$

where a and b are constants, and $b \neq 0$. It is shown that system (8) can at most have one critical period. Note that system (8) is not a special case of system (5), since the term $g(x) = \int_0^x V(s) ds$ in $H(x, y)$ of (5) is a $(2n + 1)$-degree polynomial.

In this paper, we shall consider bifurcation of local critical periods from a weak center in cubic polynomial planar systems. We will show that the revertible system (7) actually can have maximal six local critical periods, rather than four as claimed in [Zhang *et al.*, 2000]. Also, we will give some conditions under which the origin of system (7) is an isochronous center. The method used in this paper is based on normal form theory, with the aid of both symbolic and numerical computations.

In the next section, we outline a perturbation technique for computing the normal form of Hopf bifurcation and discuss how to employ this method to determine local critical periods. The main results for the revertible system (7) are presented in Sec. 3. Some illustrative numerical examples are given in Sec. 4, and finally, conclusion is drawn in Sec. 5.

2. Computation of Critical Periods Using the Method of Normal Forms

In this section, we briefly present an approach for computing local critical periods using normal form theory, associated with Hopf singularity. There are many books and papers in the literature which particularly discuss normal form theory (e.g. see [Marsden & McCracken, 1976; Guckenheimer & Homes, 1992; Ye, 1986; Nayfeh, 1993; Chow *et al.*, 1994]). Here, we shall introduce a perturbation technique based on multiple time scales [Nayfeh, 1993; Yu, 1998]. The technique does not need application of center manifold theory, but instead formulates a unified approach to directly compute the normal forms of Hopf and degenerate Hopf bifurcations for general n-dimensional systems. The technique has been proven to be computationally efficient [Yu & Han, 2004, 2005a, 2005b].

To describe the perturbation technique, consider the following general n-dimensional differential system:

$$\frac{d\mathbf{x}}{dt} = J\mathbf{x} + \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n, \quad \mathbf{f} : R^n \to R^n, \tag{9}$$

where $J\mathbf{x}$ is the linear part of the system, and $\mathbf{f}$ represents the nonlinear part and is assumed analytic. Further, suppose $\mathbf{x} = \mathbf{0}$ is an equilibrium point of the system, i.e. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and that the Jacobian of system (9), evaluated at the equilibrium point $\mathbf{0}$, contains one pair of purely imaginary eigenvalues $\pm i$. Without loss of generality, we may assume that the Jacobian of system (9) is in the Jordan canonical form:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \quad \text{where } A \in R^{(n-2)\times(n-2)}. \tag{10}$$

A is assumed to be stable, i.e. all of its eigenvalues have negative real parts.

The basic idea of the perturbation technique based on multiple scales can be briefly described as follows: Instead of a single time variable t, multiple independent variables or scales, $T_k = \epsilon^k t$, $k = 0, 1, 2, \ldots$, are introduced. Thus, the differentiation with respect to t becomes the summation of partial derivatives with respect to T_k:

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial T_0}{\partial t}\frac{\partial}{\partial T_0} + \frac{\partial T_1}{\partial t}\frac{\partial}{\partial T_1} + \frac{\partial T_2}{\partial t}\frac{\partial}{\partial T_2} + \cdots \\ &= D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots \end{aligned} \tag{11}$$

where the differential operator $D_k = \partial/\partial T_k$.

Next, assume that the solutions of system (9) in the neighborhood of $\mathbf{x} = \mathbf{0}$ are expanded in series as

$$\begin{aligned} \mathbf{x}(t;\epsilon) = \epsilon \mathbf{x_1}(T_0, T_1, \ldots) \\ + \epsilon^2 \mathbf{x_2}(T_0, T_1, \ldots) + \cdots \end{aligned} \tag{12}$$

Note in the above procedure that the same perturbation parameter, ϵ, is used in both time and space scalings, see (11) and (12). This implies that this perturbation approach uses a same scaling to treat time and space.

Now, substituting (11) and (12) into system (9) and solving the resulting ordered nonhomogeneous linear differential equations by eliminating the so-called "secular terms" finally yields the following normal form, given in polar coordinates (a detailed procedure can be found in [Yu, 1998]):

$$\begin{aligned} \frac{dr}{dt} &= \frac{\partial r}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial r}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots \\ &= D_0 r + D_1 r + D_2 r + \cdots \\ \frac{d\phi}{dt} &= \frac{\partial \phi}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial \phi}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial \phi}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots \\ &= D_0 \phi + D_1 \phi + D_2 \phi + \cdots \end{aligned} \tag{13}$$

where $D_i r$ and $D_i\phi$ are uniquely determined. Further, it has been shown [Yu, 1998] that the derivatives $D_i r$ and $D_i \phi$ are functions of r only, and only $D_{2k} r$ and $D_{2k}\phi$ are non-zero, which can be expressed as $D_{2k} r = v_k r^{2k+1}$ and $D_{2k}\phi = b_k r^{2k}$, where both v_k and b_k are expressed in terms of the original system's coefficients. The results are summarized in the following theorem.

Theorem 1. *Suppose the general n-dimensional system (9) has an Hopf-type singular point at the origin, i.e. the linearized system of (9) has one pair of purely imaginary eigenvalues and the remaining eigenvalues have negative real parts. Then the normal form of system (9) for Hopf or generalized Hopf bifurcations up to the $(2k+3)$rd order term is given by*

$$\begin{aligned} \frac{dr}{dt} &= r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k} \\ &\quad + v_{k+1} r^{2k+2}), \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{d\theta}{dt} &= 1 + \frac{d\phi}{dt} \\ &= 1 + b_0 + b_1 r^2 + b_2 r^4 + \cdots + b_k r^{2k} \\ &\quad + b_{k+1} r^{2k+2}, \end{aligned} \tag{15}$$

where the coefficient v_k is usually called the kth-order focus value or Lyapunov constant.

Note here that r and θ represent the amplitude and phase of motion, respectively. v_0 and b_0 correspond to the linear part of system (9) when it contains parameters. For our study in this paper, $v_0 = b_0 = 0$.

Equation (14) (or the focus values) can be used to determine the existence and number of limit cycles that system (9) can have, as what is employed in finding small limit cycles of Hilbert's 16th problem (e.g. [Yu & Han, 2005b]). Equation (15), on the other hand, can be applied to find the period of the periodic solutions and to determine the local critical periods of the solutions.

In the following, we describe how to use Eq. (15) to express the period of periodic motion and how to determine the local critical periods. For convenience, let

$$\begin{aligned} &h = r^2 > 0 \quad \text{and} \\ &p(h) = b_1 h + b_2 h^2 + \cdots + b_{k+1} h^{k+1}. \end{aligned} \tag{16}$$

Then Eq. (15) can be written as

$$d\theta = (1 + p(h))dt \quad (b_0 = 0 \text{ for system (9)})$$

Let the period of motion be $T(h)$. Then integrating the above equation on both sides from 0 to 2π yields

$$2\pi = (1 + p(h))T(h),$$

which gives

$$T(h) = \frac{2\pi}{1 + p(h)} \quad \text{for } 0 < h \ll 1 \quad (\text{and so } 1 + p(h) \approx 1). \tag{17}$$

Now, the local critical periods are determined by $T'(h) = 0$, or

$$T'(h) = \frac{-2\pi p'(h)}{(1 + p(h))^2} = 0. \tag{18}$$

Thus, for $0 < h \ll 1$ (meaning that we consider small limit cycles), the local critical periods are determined by

$$p'(h) = b_1 + 2b_2 h + \cdots + k b_k h^{k-1} + (k+1) b_{k+1} h^k = 0. \tag{19}$$

Similar to the discussion in determining the number of limit cycles using focus values, we can find the sufficient conditions for the polynomial $p'(h)$ to have maximal number of zeros. If $b_1 = b_2 = \cdots = b_k = 0$, but $b_{k+1} \neq 0$, then equation $p'(h) = 0$ can have at most k real roots. Then $b_1, b_2, \ldots, b_k$ (remember that they are expressed in terms of the coefficients of the original system (9)) can be perturbed appropriately to have k real roots. We give a theorem below without proof (see references [Yu & Han, 2004, 2005a, 2005b]), which can be used to determine the maximal number of real roots of $p'(h) = 0$. Assume that b_i depends on k independent system parameters:

$$b_i = b_i(a_1, a_2, \ldots, a_k), \quad i = 1, 2, \ldots, k, \tag{20}$$

where $a_1, a_2, \ldots, a_k$ are the parameters of the original system (9).

Theorem 2. *Suppose that*

$$\begin{gathered} b_i(a_{1c}, a_{2c}, \ldots, a_{kc}) = 0, \quad i = 1, 2, \ldots, k, \\ b_{k+1}(a_{1c}, a_{2c}, \ldots, a_{kc}) \neq 0, \quad \text{and} \\ \det \left[\frac{\partial(b_1, b_2, \ldots, b_k)}{\partial(a_1, a_2, \ldots, a_k)} (a_{1c}, a_{2c}, \ldots, a_{kc}) \right] \neq 0, \end{gathered} \tag{21}$$

where $a_{1c}, a_{2c}, \ldots, a_{kc}$ represent critical values. Then small appropriate perturbations applied to the critical values lead to that equation $p'(h) = 0$ has k real roots.

3. Critical Periods of Cubic Revertible System

Now, we are ready to study local critical periods of the general cubic revertible system, described by system (7). In [Zhang *et al.*, 2000], the authors assumed that the seven parameters $(a_{20}, a_{02}, a_{21}, a_{03}, b_{11}, b_{30}, b_{12})$ are independent. As a matter of fact, we can further reduce the number of parameters by one. In other words, there are in total only six independent parameters. To achieve this, assume that $a_{20} \neq 0$. Then, we use the following scalings:

$$\begin{gathered} x \to \frac{x}{a_{20}}, \quad y \to \frac{-y}{a_{20}}, \\ a_{02} \to m_1 a_{20}, \quad a_{21} \to m_2 a_{20}^2, \\ a_{03} \to m_3 a_{20}^2, \quad b_{11} \to n_1 a_{20}, \\ b_{30} \to n_2 a_{20}^2, \quad b_{12} \to n_3 a_{20}^2, \end{gathered} \tag{22}$$

to obtain a new system (for $a_{20} \neq 0$):

$$\begin{aligned} \frac{dx}{dt} &= y + x^2 + m_1 y^2 - m_2 x^2 y - m_3 y^3, \\ \frac{dy}{dt} &= -x + n_1 xy - n_2 x^3 - n_3 xy^2. \end{aligned} \tag{23}$$

System (23) has only six independent parameters, i.e. $a_{20} \neq 0$ can be chosen arbitrarily if we use the original system (7). This implies that for the cubic revertible polynomial system (23) (or the original system (7)), in general the maximal number of local critical periods that the system can have is six.

Remark 1. The above scaling reduces one more system parameter. The advantage of the reduction makes the computation simpler, particularly for numerical computation. However, it requires to consider more cases (see below), unlike for the analysis based on the original system (7) with seven parameters, only one set of parameters need to be investigated.

When $a_{20} = 0$, there are only six parameters. We may assume $a_{02} \neq 0$, and obtain a similar system like (23) but now $m_1 = 1$ and there is no x^2 term, resulting in a system with only five independent parameters. This clearly shows that such a "degenerate" system has less independent parameters and so in general has less number of critical periods. By doing this, to completely analyze the system there are in total four different cases:

Case 1. $a_{20} = a_{02} = b_{11} = 0$: the corresponding system is given by (no scaling)

$$\frac{dx}{dt} = y - m_2x^2y - m_3y^3, \qquad \frac{dy}{dt} = -x - n_2x^3 - n_3xy^2, \tag{24}$$

where $m_2 = a_{21},\ m_3 = a_{03},\ n_2 = b_{30},\ n_3 = b_{12}$. Note here that the advantage without applying scaling does not necessary assume one of the four parameters being nonzero, and four parameters can be easily handled in computation.

Case 2. $a_{20} = a_{02} = 0,\ b_{11} \neq 0$: the system is described by

$$\frac{dx}{dt} = y - m_2x^2y - m_3y^3, \qquad \frac{dy}{dt} = -x + xy - n_2x^3 - n_3xy^2. \tag{25}$$

Case 3. $a_{20} = 0,\ a_{02} \neq 0$: the system is given by

$$\frac{dx}{dt} = y + y^2 - m_2x^2y - m_3y^3, \qquad \frac{dy}{dt} = -x + n_1xy - n_2x^3 - n_3xy^2. \tag{26}$$

Case 4. $a_{20} \neq 0$: the system is given by Eq. (23).

3.1. *Case 1*: $a_{20} = a_{02} = b_{11} = 0$ (*no scaling*)

The system for this case is described by Eq. (24) which has four parameters. Employing the Maple program [Yu, 1998] we easily obtain the coefficients b_i's. In particular,

$$b_1 = \frac{1}{8}(n_3 - m_2 - 3m_3 + 3n_2). \tag{27}$$

Letting

$$n_3 = m_2 + 3m_3 - 3n_2, \tag{28}$$

we have $b_1 = 0$, and further obtain

$$b_2 = -\frac{1}{16}[2(m_3 + n_2)m_2 + 3(3m_3^2 + n_2^2)]. \tag{29}$$

Thus, by choosing

$$m_2 = -\frac{3(3m_3^2 + n_2^2)}{2(m_3 + n_2)}, \tag{30}$$

we have $b_2 = 0$, and

$$\begin{aligned} b_3 &= \frac{3}{32(m_3+n_2)^2}m_3n_2 \\ &\quad \times (m_3 - n_2)(m_3^2 - 10m_3n_2 + n_2^2), \\ b_4 &= -\frac{3}{128(m_3+n_2)^3}m_3n_2(m_3-n_2)^2 \\ &\quad \times (3m_3^3 - 16m_3^2n_2 + 83m_3n_2^2 - 6n_2^3), \end{aligned} \tag{31}$$

where $m_3 \neq -n_2$, and $(\cdots)$ denotes a homogeneous polynomial of m_3 and n_2.

It is easy to observe from (31) that when $m_3 = 0$, or $n_2 = 0$, or $m_3 = n_2$, in addition to $b_1 = b_2 = 0$, we have $b_3 = b_4 = \cdots = 0$, leading to that the origin is an isochronous center.

Setting $m_3^2 - 10m_3n_2 + n_2^2 = 0$ yields

$$m_3 = (5 \pm 2\sqrt{6})n_2, \tag{32}$$

which, in turn, results in $b_3 = 0$, and $b_4 = -(5/16)(49 \pm 20\sqrt{6})n_2^4$. It is obvious that $n_2 = 0$ leads to a trivial case — a linear system. If $n_2 \neq 0$, then at the critical point,

$$\begin{aligned}(m_{3c}, m_{2c}, n_{3c}) = ((5 \pm 2\sqrt{6})n_2,\ &-(21 \pm 8\sqrt{6})n_2, \\ &-(9 \pm 2\sqrt{6})n_2), \quad (n_2 \neq 0),\end{aligned}$$

system (24) for Case 1 can have at most three local critical periods. Since here we can have perturbation one by one on m_3 for b_3, on m_2 for b_2 and on n_3 for b_1, we know that the system can have three local critical periods after proper small perturbations. Alternatively, it is not difficult to show that

$$\det\left[\frac{\partial(b_1,b_2,b_3)}{\partial(m_3,m_2,n_3)}\right]_{(m_3,m_2,n_3)=(m_{3c},m_{2c},n_{3c})} = \frac{3(2\sqrt{6}\pm 5)}{256}\, n_2^3 \neq 0 \quad \text{when } n_2 \neq 0.$$

So, according to Theorem 2, we know that system (24) for Case (1) can have three local critical periods.

Summarizing the above results, we have the following theorem.

Theorem 3. *For the revertible system* (*24*), *there exist three local critical periods bifurcating from the weak center* (*the origin*) *at the critical point*: $m_3 = (5 \pm 2\sqrt{6})n_2$, $m_2 = -(21 \pm 8\sqrt{6})n_2$, $n_3 = -(9 \pm 2\sqrt{6})n_2$, $(n_2 \neq 0)$. *Moreover, the origin is an isochronous center if one of the following conditions is satisfied*:

(i) $m_3 = 0,\ n_3 = 3m_2 = -(9/2)n_2$;
(ii) $n_2 = 0,\ m_2 = 3n_3 = -(9/2)m_3$;
(iii) $m_3 = n_2,\ m_2 = n_3 = -3n_2$.

Remark 2

(i) The linear isochronous center is included in the above as a special case when $m_2 = m_3 = n_2 = n_3 = 0$.
(ii) System (24) actually has only three independent parameters. One can apply a proper scaling to remove one parameter. For example, if $a_{21} \neq 0$, then substituting the following scalings:

$$x \to \frac{x}{\sqrt{|a_{21}|}}, \quad y \to \frac{y}{\sqrt{|a_{21}|}}, \quad a_{03} \to m_3 a_{21}, \quad b_{30} \to n_2 a_{21}, \quad b_{12} \to n_3 a_{21}, \tag{33}$$

into (7) to obtain

$$\frac{dx}{dt} = y - \mathrm{sign}(a_{21})x^2 y - m_3 y^3, \qquad \frac{dy}{dt} = -x - n_2 x^3 - n_3 x y^2, \tag{34}$$

which has only three independent parameters. So it is not surprising that the maximal number of local critical periods for this case is three. But note that we need to deal with the case $a_{21} = 0$ separately.

3.2. *Case 2*: $a_{20} = a_{02} = 0$, $b_{11} \neq 0$

For this case, the system is given by (25). Again like Case 1, we have only four independent parameters. However, comparing with system (24), we can see that this case has an extra term xy in the second equation. Similarly, applying the Maple program results in

$$b_1 = \frac{1}{8}(n_3 - m_2 - 3m_3 + 3n_2) - \frac{1}{24}. \tag{35}$$

Letting

$$n_3 = m_2 + 3m_3 - 3n_2 + \frac{1}{3} \tag{36}$$

yields $b_1 = 0$, and

$$b_2 = -\frac{1}{8}(m_3 + n_2)m_2 - \frac{3}{16}(3m_3^2 + n_2^2) - \frac{1}{48}(m_3 - n_2). \tag{37}$$

Further, setting $b_2 = 0$ gives

$$m_2 = -\frac{9(3m_3^2 + n_2^2) + m_3 - n_2}{6(m_3 + n_2)}, \tag{38}$$

and then we obtain

```
b3:=-1/622080/(m3+n2)^2*(10*m3*n2-450*m3^2*n2+2430*m3*n2^2+218700*m3^3*n2
    +224370*m3^2*n2^2-21060*m3*n2^3-25*n2^2-m3^2+594*m3^3+450*n2^3+58320*m3*n2^4
    -58320*m3^4*n2+641520*m3^3*n2^2-641520*m3^2*n2^3-2025*n2^4+119151*m3^4):
b4:= 1/14929920/(m3+n2)^3*(15*m3^2*n2-75*m3*n2^2-3924*m3^3*n2+486*m3^2*n2^2
    -12132*m3*n2^3-m3^3+125*n2^3+200853*m3*n2^4-1851741*m3^4*n2-3293622*m3^3*n2^2
    -2475306*m3^2*n2^3-3015*n2^4+729*m3^4+23895*n2^5-20655*m3^5+20111409*m3^2*n2^4
    -7725942*m3^5*n2-14508315*m3^4*n2^2-5360580*m3^3*n2^3-1120230*m3*n2^5+4186647*m3^6
    -61965*n2^6-1049760*m3^6*n2+7698240*m3^5*n2^2-41290560*m3^4*n2^3-33242400*m3^2*n2^5
    +65784960*m3^3*n2^4+2099520*m3*n2^6):
b5:= ...
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Now, we cannot simply solve m_2, or n_2 explicitly from equation $b_3 = 0$. So, eliminating m_3 from the two equations $b_3 = b_4 = 0$ yields the solution:

```
m3:=-5*n2*(56924780201528555200000000*n2^17-10311344144078919300864000000*n2^16
    +46652948781148185480539136000*n2^15+7291113686479713037470819840000*n2^14
    -34401909196256531297478060672000*n2^13+3438895452646143906561357829440*n2^12
    -1378411040396816095486593212352*n2^11+285964218564988124360928779568*n2^10
    -34124043635787788029719058104*n2^9+2432214301711349939660168220*n2^8
    -102825774441224160041266596*n2^7+2435439784546733144583681*n2^6
    -26920684006629130290960*n2^5+28477324117008592587*n2^4-24412806206412462*n2^3
    +227374081948134*n2^2+10829164128*n2-89476)
    /(9323709749208362108620800000000*n2^17-44517267151612466882167296000000*n2^16
    -61412451181308892954545408000000*n2^15+321743370081378172281237529344000*n2^14
    -355843063477576132735248304473600*n2^13+1793705229077538927862679297788880*n2^12
    -52301734187935423342257787875264*n2^11+8886375904760643874661408844576*n2^10
    -868968351982197524387386778472*n2^9+48055547873605778137672378956*n2^8
    -1401016447946025855097185996*n2^7+17115522926246428048567128*n2^6
    -191391090858678055311199*n2^5+541907819900619509 16*n2^4-2055509619790956 93*n2^3
    -232683432511500*n2^2-16584226269*n2-246059):
```
(39)

and the resultant:

```
F1:= n2*(120*n2+1)*(18*n2-1)*(9*n2-1)*(24987557838805401600000000*n2^14
     -17765690890838070067200000*n2^13+132033355862506632806400000*n2^12
     +1169724354873953937666048000*n2^11-3128157134130387297342873600
     *n2^10+3262106484160922583085271040*n2^9-1640735498471830613532864000*n2^8
     +39662487303463530756084240*n2^7-4354024089576423808213776*n2^6
     +167083114350796340077368*n2^5+76365570975002989851 6*n2^4
     -3114647882885854395*n2^3-245131814155149*n2^2-14233762746*n2-559225):
```

A simple numerical scheme can be employed to show that the polynomial equation $F_1(n_2) = 0$ has 14 real solutions for n_2. The first four solutions are: $n_2 = 0, -1/120, 1/18, 1/9$. It is easy to verify that the first three of them are not solutions, while the last one results in $m_3 = m_2 = n_3 = 0$, leading to $b_3 = b_4 = b_5 = \cdots = 0$. This indicates that for this solution, the origin is an isochronous center.

For the remaining ten roots of the equation $F_1(n_2) = 0$, we have used the built-in Maple command *fsolve* to numerically compute these real solutions up to 1000 digit points, guaranteeing the accuracy of computation. (The ten solutions are not listed here for brevity.) Note that for a single-variable polynomial, *fsolve* can be used to find all real roots of the polynomial up to very high accuracy. In fact, the Maple command *solve* can be employed to find all (real and complex) roots of a single-variable polynomial. It can be shown that for all these ten solutions, $b_1 = b_2 = b_3 = b_4 = 0$, but $b_5 \neq 0$. This implies that (n_2, m_3, m_2, n_3) has ten sets of real solutions for which system (25) has four local critical periods bifurcating from the weak center — the origin.

The results obtained above for Case (2) are summarized in the following theorem.

Theorem 4. *For the revertible system* (*25*), *there are ten sets of solutions for the critical point* $(n_{2c}, m_{3c}, m_{2c}, n_{3c})$ *which can be perturbed to generate four local critical periods. Moreover, when* $n_2 = 1/9$, $m_3 = m_2 = n_3 = 0$, *the origin is an isochronous center.*

Remark 3. It should be pointed out that although Theorem 4 states that there are only ten sets of solutions which generate four local critical periods, there are actually infinite number of solutions since b_{11} $(\neq 0)$ can be chosen arbitrarily.

3.3. *Case 3*: $a_{20} = 0$, $a_{02} \neq 0$

Now we consider Case 3 described by Eq. (26), which has five independent parameters. So it is possible to have five local critical periods. Similarly, we apply the Maple program to obtain

$$b_1 = \frac{1}{24}(3n_3 - 3m_2 - 9m_3 + 9n_2 + n_1 - n_1^2) - \frac{5}{12}, \tag{40}$$

which yields

$$n_3 = m_2 + 3m_3 - 3n_2 + \frac{1}{3}n_1(n_1 - 1) + \frac{10}{3} \tag{41}$$

such that $b_1 = 0$. Then,

$$b_2 = -\frac{1}{48}\left[(6m_3 + 6n_2 - n_1 + 11)m_2 + m_3\left(n_1 + \frac{23}{2}\right)^2 + 27\left(m_3 - \frac{17}{24}\right)^2 + (n_1 + 7)^2 + 9\left(n_2 + \frac{1}{18}\right)^2 - n_1^2 n_2 - \frac{13003}{576}\right]. \tag{42}$$

Setting $b_2 = 0$ we obtain

$$m_2 = \frac{n_1^2 n_2 - m_3 n_1(n_1 + 23) - m_3(27m_3 + 94) - n_2(9n_2 + 1) - n_1(n_1 + 14) - 40}{6(m_3 + n_2) - n_1 + 11}. \tag{43}$$

Then b_3, b_4 and b_5 become

$$\begin{aligned} b_3 &= \frac{1}{155520(6m_3 + 6n_2 - n_1 + 11)^2}\, F_1(m_3, n_2, n_1), \\ b_4 &= -\frac{1}{1866240(6m_3 + 6n_2 - n_1 + 11)^2}\, F_2(m_3, n_2, n_1), \\ b_5 &= \frac{1}{1881169920(6m_3 + 6n_2 - n_1 + 11)^2}\, F_3(m_3, n_2, n_1), \end{aligned} \tag{44}$$

where F_1, F_2 and F_3 are polynomials of m_3, n_2 and n_1. Eliminating m_3 from the three polynomials equations $F_1 = F_2 = F_3 = 0$ yields a solution $m_3 = m_3(n_2, n_1)$, and two resultant polynomial equations:

$$P_1 = FF_4(n_2, n_1) \quad \text{and} \quad P_2 = FF_5(n_2, n_1), \tag{45}$$

where

$$F = 432n_2^2 - 24(n_1^2 + 16n_1 - 3)n_2 + 2n_1^3 + 45n_1^2 - 348n_1 - 499, \tag{46}$$

and F_4 and F_5 are respectively 25th and 26th degree polynomials with respect to n_2.

We now want to solve the two equations: $P_1 = P_2 = 0$. It can be shown that the roots of $F = 0$ (e.g. solving n_2 in terms of n_1) are not solutions of the original equations $b_3 = b_4 = b_5 = 0$, since it yields $6(m_3 + n_2) - n_1 + 11 = 0$, giving rise to a zero divisor [see Eq. (43)]. Thus, the only possible solutions come from the two equations: $F_4 = F_5 = 0$. However, it is very difficult to follow the above procedure to eliminate one parameter from these two equations, since their degrees are too high. Therefore, we apply the built-in Maple command *fsolve* here to find the solutions of $F_4 = F_5 = 0$. But Maple has limit on solving multivariate polynomials, which only gives one possible real solution. (For single-variable polynomials, *fsolve* can find all real solutions.) Nevertheless, if the solution is a true solution of the system, it is enough for our purpose since we mainly want to prove the existence of critical periods, rather than finding all their solutions. Certainly, if one can find all solutions, it would be better.

By applying *fsolve* to equations $F_4 = F_5 = 0$, we obtain a solution as follows (up to 1000 digit points):

```
n2 :=-1.037465771140918409177970657729999525446131539720441503141436926337 0 ...
       ... 79336402835255809692665089727046900731772309088198085488679011713048:
n1 := 8.932317071525213581741911335361342609122355717789349800979162279099 5 ...
       ... 83522994180267403875842542364148975622856247830140462101539212923691:
```

Then the values of m_3, m_2 and n_3 directly follow the formulas given in Eqs. (41), (43), and $m_3(n_2, n_1)$ (which is not listed in this paper). By verifying the original equations, we can show that the above solution yields $b_1 = b_2 = b_3 = b_4 = 0$, but $b_5 \neq 0$. Thus, this solution only gives at most four local critical periods, not five as we are expecting.

The problem is caused by numerically solving the roots of the resultant equations, rather than the original equations. We may apply *fsolve* directly to the original equations, with the risk that we may not be able to obtain any solutions at all due to too many equations and variables involved. The following Maple command:

```
with(linalg): Mysolution := fsolve({b1,b2,b3,b4,b5}, {m2,m3,n1,n2,n3}):
```

yields a solution (up to 1000 digit points):

```
m2 :=-21.090600484437152798842381398933514966653977502834642843O7238494 ...
    ... 019234043470591684120459559737824861162984975896990695511170114751:
m3 := 5.125109394169587039145091618355152678331363487291313143329652082 ...
    ... 708996820915283214563810999105222159218993365237562500814877898150:

n1 := .2211444818520424763620979834379313195951475156731323013529805585 ...
    ... 380110623946971134688104347855383954946414228889480850825415792946:
n2 := 4.536766340054913651507181574123480691222115160017318145023788504 ...
    ... 774024708155221034216788445170902581275389079050267890102308702990:
n3 :=-16.04965118875927734665732451579703873331519269547524914368562229 ...
    ... 113889180570162284796205552039147067145683154187478411837014672177:
```

Substituting the above solution (referred to as a critical point C) into b_i's to obtain

$$b_1 = 0, \quad b_2 = -0.128 \times 10^{-997}, \quad b_3 = 0.48 \times 10^{-997}, \quad b_4 = 0.5042 \times 10^{-995}, \quad b_5 = -0.219 \times 10^{-994},$$
$$b_6 = 63.26140030377982283073214398034314178739364579558605952899625352812377 \cdots$$

Further calculating the Jacobian given in Eq. (21) at the above critical point shows that

$$\det \left[\frac{\partial(b_1, b_2, b_3, b_4, b_5)}{\partial(n_1, n_2, m_3, m_2, n_3)} \right]_C = -788.59440734553592526150070851409505290601 04 \cdots \neq 0,$$

implying that for Case 3 there exist five local critical periods bifurcating from the weak center (the origin). The above results are summarized as a theorem below.

Theorem 5. *For the revertible system* (*26*), *there exist values of the parameters* n_1, n_2, m_3, m_2, n_3 *such that five local critical periods are obtained*, *which bifurcate from the weak center* (*the origin*).

3.4. *Case 4*: $a_{20} \neq 0$

Finally, we consider the most general and difficult case $a_{20} \neq 0$. The system, described by Eq. (23), has six independent parameters. So it is expected that the system may have six local critical periods bifurcating from the weak center (the origin). If all the parameters are chosen free, then pure symbolic computation becomes intractable. What we will show below include three cases:

(i) $m_1 = n_1 = 0$: four local critical periods (using symbolic computation only);
(ii) $m_1 = 0$: five local critical periods (using both symbolic and numerical computations);
(iii) No parameter equals zero: six local critical periods (using numerical computation only).

Note here that the four and five local critical periods are different from that presented respectively in Cases 2 and 3, since this case contains the term x^2 in the first equation of (23).

Subcase (*i*) : $m_1 = n_1 = 0$. For this subcase, we have

$$b_1 = \frac{1}{8}(n_3 - m_2 - 3m_3 + 3n_2) - \frac{1}{6}. \tag{47}$$

Thus,

$$n_3 = m_2 + 3m_2 - 3n_2 + \frac{4}{3}, \tag{48}$$

in order to have $b_1 = 0$. Then,

$$b_2 = -\frac{1}{48}\left[(6m_3 + 6n_2 + 1)m_2 + 27m_3^2 + 4m_3 + 9n_2^2 + 17n_2 - \frac{28}{3}\right], \tag{49}$$

which, in turn, gives

$$m_2 = -\frac{3(27m_3^2 + 4m_3 + 9n_2^2 + 17n_2) - 28}{(36m_3 + 6n_2 + 1)}. \tag{50}$$

Having determined n_3, m_2, further calculation on b_i yields

```
b3 := 1/24186470400/(1+6*n2+6*m3)^4
      *(-166336+4299237*m3^2*n2-6138180*m3*n2^3-524880*m3^4*n2-860016*n2+8118774*m3*n2
      +167748*m3+6689538*m3^2+5799564*n2^2-5457861*n2^3-13858047*m3*n2^2+7283439*m3^3
      +790236*m3^4+2055780*m3^3*n2-12629196*m3^2*n2^2-3020976*n2^4-5773680*m3^2*n2^3
      +5773680*m3^3*n2^2+524880*m3*n2^4):
b4 := 1/3482851737600/(1+6*n2+6*m3)^6
      *(23079424+1436655204*m3^2*n2+2741028336*m3^3*n2^3-3587043042*m3^4*n2^2
      -1236483144*m3*n2^3-1899088011*m3^4*n2+45558144*n2+222969024*m3^5*n2
      +111996270*m3^6-1152699903*n2^5-1335699936*m3*n2-96135264*m3-1533692286*m3^2
      -816192288*n2^2+1653973560*n2^3+5023865106*m3*n2^2-1559217870*m3^3+1445706144*m3^4
      +1161666360*m3^3*n2+3218307552*m3^2*n2^2-168315894*m3^2*n2^4-28343520*m3^6*n2
      -133898832*n2^4-897544800*m3^2*n2^5+923597802*m3^2*n2^3-1114845120*m3^4*n2^3
      -1265905584*m3*n2^5-1445873814*m3^3*n2^2-4577964723*m3*n2^4+207852480*m3^5*n2^2
      +1712007657*m3^5+56687040*m3*n2^6-271822230*n2^6+1776193920*m3^3*n2^4):
```

Now eliminating n_2 from the two equations $b_3 = b_4 = 0$ (ignoring the constant facts and the denominator) results in a solution $n_2 = n_2(m_3)$ and the following resultant:

```
F := m3*(432*m3^2-120*m3-143)*(126869487069973102400323268975096320000
     +178488837097252527009160287244355821440 0*m3
     -934865174968558332740089351939778234832 0*m3^2
     -1808000634276696368637345884229962723290 41*m3^3
     -7465817836442890473634741470898947882720 12*m3^4
     -2222378300355895533085580534812469051038 848*m3^5
     -3469925887989885659170689395818689329998 464*m3^6
     -1145234326680884388874832530043049946275 840*m3^7
     +3380460143974513503057245985798526646174 208*m3^8
     +4653823046725235033083004693863807438718 976*m3^9
     +2525328066926045569461981921341628839276 544*m3^10
     +6331023625632464333856975713517214460805 12*m3^11
     +5307146930091595585992414739861136080896 0*m3^12
     -626614375567953738939816181351561297920 0*m3^13
     -149622041236798195415893664068534272000 0*m3^14
     -911255152083551278162503036567552000 00*m3^15
     -126054277488251166385108942848000000 0*m3^16
     +250923216733026906789642240000000 00*m3^17):
```

The solution $m_3 = 0$ gives $n_2 = 4/9$, $m_2 = n_3 = 0$, leading to $b_1 = b_2 = \cdots = 0$, implying that the origin is an isochronous center. The two solutions from the second fact are actually not the solutions of the original equations. So other possible solutions come from the 17th-degree polynomial, which has 13 real solutions. By verifying the original b_i equations: there are only 11 solutions satisfying the original equations. Further, by checking the determinant (nonzero) of the Jacobian in Theorem 2, we know that perturbing each of these 11 solutions results in four local critical periods. Thus, we have

Theorem 6. *For the revertible system* (*23*) *when* $m_1 = n_1 = 0$, *there are* 11 *sets of solutions* (m_3, n_2, m_2, n_3) *leading to four local critical periods. Moreover, when* $n_2 = 4/9$, $m_3 = m_2 = n_3 = m_1 = n_1 = 0$, *the origin is an isochronous center.*

Subcase (*ii*): $m_1 = 0$. For this subcase, if we use elimination procedure, it will lead to very high degree polynomials and it is difficult to obtain the final resultant with only one variable. Thus, we try to use the Maple command, *fsolve*, to find a possible solution, since one solution is enough for proving the existence of certain order critical periods. To do this, let $m_1 = 0$ in b_i, $i = 1, 2, \ldots, 5$. Then use the command

```
with(linalg):
m1 := 0:
Mysolution := fsolve({b1,b2,b3,b4,b5}, {m2,m3,n1,n2,n3}):
```

to obtain a solution (up to 1000 digit points):

```
m1 := 0:
m2 := 6.03540001721923960408836581987637682128304593755918637015083391 7 ...
    ... 521247539246439636893372513430973055913423563721427540938133657870:
m3 :=-2.780809904103370988424442531627905476289418239384590526626481008 ...
    ... 834576020348013746716149424390335117837648597349089950519527455229:
n1 := 0.621849114501545687930704699633765375651291643986255358128944707 ...
    ... 257901313693726483172377467569487002179968213262136823553358496491:
n2 :=-2.047466873502703059425954703630994770898922153574473647213673030 ...
    ... 666859689375467940323918533739308056608575305705993922947509110621:
n3 := 4.261187841650111830405730002765259103184527609789400252719948417 ...
    ... 683595939560330109243994695209408865769445515993036954490477716439:
```

Now, it is very important to verify if this approximate solution indeed implies the existence of a true solution. To do this, we substitute the numerical solution into the explicit expressions of b_i's to obtain

$$b_1 = -0.1 \times 10^{-999}, \quad b_2 = -0.127 \times 10^{-998}, \quad b_3 = 0.836 \times 10^{-998},$$
$$b_4 = 0.121 \times 10^{-996}, \quad b_5 = -0.46863 \times 10^{-995},$$
$$b_6 = -7.421658867638726722085244758030121967219919492961715344089192361574668\cdots$$

Because the symbolic expressions of b_i's are exact before the substitution, the above verification scheme indeed shows that there exists a solution such that $b_i = 0$, $i = 1, 2, \ldots, 5$, but $b_6 \neq 0$.

Moreover, the Jacobian given in Eq. (21) for this case evaluated at the above critical point yields

$$\det \left[\frac{\partial(b_1, b_2, b_3, b_4, b_5)}{\partial(m_2, m_3, n_1, n_2, n_3)} \right]_C = 364.786575577772092246663415903385114393563 7\cdots \neq 0.$$

Thus, based on Theorem 2, we know that Subcase (ii) has five local critical periods bifurcating from the weak center (the origin). A theorem summarizing the above results is given below.

Theorem 7. *For the revertible system (23) when $m_1 = 0$, solution $(m_2, m_3, n_1, n_2, n_3)$ exists such that the system has five local critical periods.*

Subcase (iii) : no parameter equals zero. For this case, computation is more involved than any other cases discussed above. Unless with a very powerful computer system, with purely symbolic computation, it is almost impossible to find the solutions for possible six local critical periods. The Maple command

```
with(linalg):
Mysolution := fsolve({b1,b2,b3,b4,b5,b6}, {m1,m2,m3,n1,n2,n3}):
```

has been used to obtain the following solution (up to 500 digit points):

```
m1 :=-1.911271311412248894318376740313337291799639386556225538177599088 ...
    ... 551695141943450774237846747710516252168805098710405990834877601960:
m2 :=-5.506140892974370699687829878845944063608812076156309266238462197 ...
    ... 453925796588449713220926391370283940499191777465781362693575055392:
m3 :=-0.964624737596251179318221385991593355899241421793789156746030895 ...
    ... 488788745833477563333057595494538855773734927488701053284011587824:
```

```
n1 := 0.010953161557416338490455140243205661271093061527829850636322735 ...
    ... 817276254120361520768053706683747606003767341822911703779322849287:
n2 := 0.000931994068568175728068553100368911145145362148682258162814424 ...
    ... 387802846258161211724575939719403782034564144996916537003441939052:
n3 :=-1.275092497446341268494136712199644529818702522662286744901854359 ...
    ... 410836309921145454838123983140640614491450117156176212929362565911:
```

for which the verification scheme shows that

$$b_1 = -0.9 \times 10^{-499}, \quad b_2 = -0.68 \times 10^{-499}, \quad b_3 = 0.69269 \times 10^{-497},$$
$$b_4 = 0.753 \times 10^{-496}, \quad b_5 = -0.405311585 \times 10^{-494}, \quad b_6 = 0.174948224057419 \times 10^{-492}$$
$$b_7 = 0.000285510949123486875739425029555321579531193374039367118349090525999 \cdots$$

indicating that there exists a solution $(m_1, m_2, m_3, n_1, n_2, n_3)$ such that $b_i = 0,\ i = 1, 2, \ldots, 6$, but $b_7 \neq 0$.

Further, substituting the above critical values into the Jacobian results in

$$\det \left[\frac{\partial(b_1, b_2, b_3, b_4, b_5)}{\partial(m_2, m_3, n_1, n_2, n_3)} \right]_C = 0.0000370807492687558966167886100130198187 49 \cdots \neq 0.$$

Therefore, based on Theorem 2, we can conclude that Subcase (iii) can have six local critical periods bifurcating from the weak center (the origin), as summarized in the following theorem.

Theorem 8. *For the revertible system (23) there exists solution $(m_1, m_2, m_3, n_1, n_2, n_3)$ for the critical point such that six local critical periods bifurcate from the weak center.*

Finally, to end this section, we notice that if we follow the classification given at the beginning of this section, we can have more cases, and combining the case studies with the results obtained in above leads to the following result.

Theorem 9. *For the general revertible system (7), the maximal number of local critical periods bifurcating from the weak center is equal to the number of independent parameters contained in the system.*

4. Numerical Examples

In the previous section, we have established several theorems for the properties of local critical periods and isochronous center of cubic revertible systems. In this section, we present two numerical examples to demonstrate how to perturb parameters from a critical point to obtain the exact number of local critical periods given in the theorems.

Remark 4. We have established Theorem 2 which theoretically guarantees the existence of k local critical periods if the conditions given in the theorem are satisfied. However, in practice it is not easy to find a particular set of perturbations to obtain a numerical realization. If the parameters can be perturbed one by one separately for each of b_i's, the process is straightforward. When the perturbation parameters are coupled, such as those cases considered in Secs. 3.2–3.4, it is very difficult to find such a set of perturbations. In particular, when more parameters are coupled, like the case of six local critical periods (Theorem 8), it is extremely difficult to obtain a numerical set of perturbations.

In the following, we give two examples, one for the three local critical periods considered in Sec. 3.1, and the other for the four local critical periods discussed in Sec. 3.2.

4.1. *Example 1*

Consider the three local critical periods given in Theorem 3. For this example, $T'(h)$ is given by

$$T'(h) = \frac{-2\pi p'(h)}{(1 + p(h))^2},$$

where

$$p_4'(h) = b_1 + 2b_2 h + 3b_3 h^2 + 4b_4 h^3, \tag{51}$$

in which the subscript 4 denotes that $p(h)$ is a fourth-degree polynomial of h.

Taking $n_2 = 0.01$, and applying the following perturbations:

$$m_3 = (5 + 2\sqrt{6})n_2 + 0.1 \times 10^{-3},$$
$$m_2 = -\frac{3(3m_3^2 + n_2^2)}{2(m_3 + n_2)} + 0.1 \times 10^{-8},$$
$$n_3 = m_2 + 3m_3 - 3n_2 + 0.1 \times 10^{-13},$$

yields the third-degree polynomial

$$\begin{aligned} p_4'(h) = 0.125 \times 10^{-14} &- 0.27272820328796 \times 10^{-10} h \\ &+ 0.2046433012965192 \times 10^{-7} h^2 \\ &- 0.1229176145646577605 \times 10^{-5} h^3. \end{aligned}$$

The roots of $p_4'(h) = 0$ are

$$\begin{aligned} h_1 &= 0.47522968629422680728 \times 10^{-4}, \\ h_2 &= 0.14084921112674735809 \times 10^{-2}, \\ h_3 &= 0.15192803075362020703 \times 10^{-1}, \end{aligned} \tag{52}$$

as expected. Therefore, $T'(h_i) = 0,\ i = 1, 2, 3$, and

$$\begin{aligned} &T'(h) > 0 \quad \forall\, h \in (0, h_1) \cup (h_2, h_3) \\ &\text{and} \quad T'(h) < 0 \quad \forall\, h \in (h_1, h_2). \end{aligned}$$

In terms of the amplitude of periodic solution, $r = \sqrt{h}$ [see Eq. (16)], the amplitudes corresponding to the three critical points [see Eq. (52)] are

$$\begin{aligned} r_1 &= 0.0068936904941709328 8, \\ r_2 &= 0.03752988291038853708, \\ r_3 &= 0.12325908922007342952. \end{aligned}$$

In order to show that higher order terms added to $p_4'(h)$ do not affect the number of real roots of $p_4'(h)$ for $0 < h \ll 1$, we expand $p'(h)$ up to b_7 using the above perturbed parameter values to obtain

$$\begin{aligned} p_7'(h) = 0.125 \times 10^{-14} & \\ &- 0.27272820328796 \times 10^{-10} h \\ &+ 0.2046433012965192 \times 10^{-7} h^2 \\ &- 0.122917614564657764 \times 10^{-5} h^3 \\ &+ 0.30977476026963404769 \times 10^{-7} h^4 \\ &- 0.13132778484607037717 \times 10^{-7} h^5 \\ &+ 0.90644113252474127528 \times 10^{-9} h^6, \end{aligned}$$

which has the following four real roots:

$$\begin{aligned} h_1 &= 0.47522968635658762712 \times 10^{-4}, \\ h_2 &= 0.14084868274006266263 \times 10^{-2}, \\ h_3 &= 0.15199198405041823566 \times 10^{-1}, \\ h_4 &= 17.115958097383951681. \end{aligned} \tag{53}$$

Compared with the roots of $p_4'(h)$, the first three roots of $p_7'(h)$ are almost the same as that of $p_4'(h)$ [see Eq. (52)]. The extra root of $p_7'(h)$, 17.115958097383951681, is obviously not in the interval $0 < h \ll 1$. This clearly shows that adding higher-order terms to $p_4'(h)$ does not change the number of local critical periods for small values of h.

4.2. *Example 2*

Consider the case of four local critical periods discussed in Sec. 3.2. For this case,

$$p_5'(h) = b_1 + 2b_2 h + 3b_3 h^2 + 4b_4 h^3 + 5b_5 h^4. \tag{54}$$

Note that for this example, we cannot follow the procedure of Example 1, since for this case the parameters m_3 and n_2 are coupled in the two equations: $b_3(m_3, n_2) = b_4(m_3, n_2) = 0$. Although we obtain the exact expression $m_3 = m_3(n_2)$, given in Eq. (39), we cannot treat these two parameters independently. Thus, we have to find the perturbations simultaneously for b_3 and b_4, by using m_3 and n_2. Having determined perturbations on m_3 and n_2, we can determine the perturbations on m_2 and n_3 one by one since they are separated.

It has been shown in Sec. 3.2 that we have ten sets of real solutions of n_2 for the four local critical periods. The critical values of n_3, m_2 and m_3 are given by Eqs. (36), (38) and (39), respectively. The ten sets of solutions of n_2 are given below (computed with up to 1000 digit points, but here only list the first 30 digits for brevity):

$$\begin{aligned} n_{2c} = -2.98855639079543384724018446571 6, &\quad -0.635382863075051310708014537214 \times 10^{-2}, \\ -0.545466741005699584641353982592 \times 10^{-4}, &\quad 0.270691303457354452445821624251 \times 10^{-2}, \\ 0.105332445282244310486473242133, &\quad 0.146768333158553387163454757140, \\ 0.229725794378667851313343707244, &\quad 1.26245327024229244121803353051 5, \\ 1.82330015790697964733987013697 4, &\quad 4.98613392130385918196636597269 9. \end{aligned}$$

Theoretically, for all the above ten sets of critical solutions, we should be able to find perturbations which yield exactly four local critical periods. However, we have found that except for the ninth solution $n_2 = 1.82330015790697964734$, it is very difficult to use other nine solutions to obtain proper perturbations.

For the ninth set of solutions:

$$n_{2c} = 1.82330015790697964733987 0136974,$$
$$m_{3c} = 0.05474757744936263141191770 8322,$$
$$m_{2c} = -2.50545509802902096030149096 1062,$$
$$n_{3c} = -7.47777950606853867475201491 3683,$$

we have

$$b_1 = b_2 = b_3 = b_4 = 0,$$
$$b_5 = -0.42487704474573231014682262606 8 \times 10^{-2} < 0.$$

Thus, we need perturbations such that

$$b_4 > 0, \quad b_3 < 0, \quad b_2 > 0, \quad b_1 < 0$$
$$\text{and} \quad |b_i| \ll |b_{i+1}| \ll 1 \quad (i = 1, 2, 3, 4).$$

First, consider perturbations simultaneously on n_{2c} and m_{3c} for b_4 and b_3. Following the procedure given in [Yu & Han, 2005b], we obtain

$$\begin{aligned} n_2 &= n_{2c} + \varepsilon_1 \\ &= n_{2c} + 0.001 \\ &= 1.82430015790697964733987013 6974, \\ m_3 &= m_{3c} + \varepsilon_2 \\ &= m_{3c} - 0.000025572 \\ &= 0.05472200544936263141191770 8322, \end{aligned}$$

for which Eq. (54) has two real solutions for h. Then take

$$\varepsilon_3 = -0.1 \times 10^{-15} \quad \text{and} \quad \varepsilon_4 = -0.1 \times 10^{-22},$$

respectively for m_2 and n_3 to obtain

$$\begin{aligned} m_2 &= m_{2c} + \varepsilon_3 \\ &= -2.50697083822942804676399041 4400, \\ n_3 &= n_{3c} + \varepsilon_2 \\ &= -7.48237459293939119955404329 2633. \end{aligned}$$

Under the above perturbed parameter values, we have

$$b_1 = -0.1250000000001 \times 10^{-23},$$
$$b_2 = 0.234877662936404932224 54 \times 10^{-16},$$
$$b_3 = -0.42058141423438673184346818469 1 \times 10^{-10},$$
$$b_4 = 0.33913761594472503735992422835 2 \times 10^{-5},$$
$$b_5 = -0.42714695334058353236697110242 0 \times 10^{-2},$$

for which Eq. (54) has four real roots:

$$\begin{aligned} h_1 &= 0.16981197181642823030092623067 7 \times 10^{-3}, \\ h_2 &= 0.59873098311240285111801902692 5 \times 10^{-3}, \\ h_3 &= 0.30080632281970312839919274946 7 \times 10^{-2}, \\ h_4 &= 0.25033362905761907595465880975 5 \times 10^{-1}, \end{aligned} \tag{55}$$

as expected. If we add two more terms $6b_6h^5$ and $7b_7h^6$ to Eq. (54), it still gives only four real roots, which are almost exactly the same as that given in Eq. (55).

5. Conclusions

It has been shown in this paper that general revertible planar cubic systems can have six local critical periods which bifurcate from a weak center. This new result improves the existing conclusion that such a system can at most have four local critical periods. The methodology used in this paper is based on a perturbation technique for computing normal forms. Also some sufficient conditions are derived under which the center of the system becomes an isochronous center. This approach is proved to be computationally efficient, and can be extended to consider other systems such as Hamiltonian systems.

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