# FOUR LIMIT CYCLES FROM PERTURBING QUADRATIC INTEGRABLE SYSTEMS BY QUADRATIC POLYNOMIALS* 

PEI YU ${ }^{\dagger, \ddagger, \S}$ and MAOAN HAN ${ }^{\ddagger}$<br>${ }^{\dagger}$ Department of Mathematics, Shanghai Normal University, Shanghai 200234, P. R. China<br>${ }^{\ddagger}$ Department of Applied Mathematics, The University of Western Ontario, London, Ontario, Canada N6A 5B7<br>§pyu@uwo.ca

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#### Abstract

In this paper, we present four limit cycles in quadratic near-integrable polynomial systems. It is shown that when a quadratic integrable system has two centers and is perturbed by quadratic polynomials, it can generate at least four limit cycles with $(3,1)$-distribution. This result provides a positive answer to an open question in this research area.


Keywords: Hilbert's 16th problem; quadratic near-integrable system; limit cycle; reversible system; Hopf bifurcation; Poincaré bifurcation; Melnikov function.

## 1. Introduction

The well-known Hilbert's 16th problem has remained unsolved since Hilbert proposed the 23 mathematical problems at the Second International Congress of Mathematics in 1900 [Hilbert, 1902]. Recently, a modern version of the second part of the 16th problem was formulated by Smale [1998], chosen as one of the 18 challenging mathematical problems for the 21st century. To be more specific, consider the following planar system:

$$
\begin{equation*}
\frac{d x}{d t}=P_{n}(x, y), \quad \frac{d y}{d t}=Q_{n}(x, y), \tag{1}
\end{equation*}
$$

where $P_{n}(x, y)$ and $Q_{n}(x, y)$ represent $n$th degree polynomials of $x$ and $y$. The second part of Hilbert's 16th problem is to find the upper bound $H(n) \leq n^{q}$ on the number of limit cycles that the system can have, where $q$ is a universal constant, and $H(n)$ is called Hilbert number. In the early 1990's,

Ilyashenko and Yakovenko [1991], Écalle [1992] proved the finiteness theorem pioneered by Dulac, for given planar polynomial vector fields. In general, the finiteness problem has not been solved even for quadratic systems. Recent survey articles (e.g. see [Li, 2003; Yu, 2006] and more references therein) have comprehensively discussed this problem and reported the recent progress.

If the problem is restricted to the neighborhood of isolated fixed points, then the question on studying degenerate Hopf bifurcations gives rise to weak (fine) focus points. In the past six decades, many researchers have considered the local problem and obtained many results (e.g. see [Kukles, 1944; Bautin, 1952; Malkin, 1964; Liu \& Li, 1989; Li \& Liu, 1991; Yu \& Han, 2005a, 2005b]). In the last 20 years, much progress on finite cyclicity near a weak focus point or a homoclinic loop has been achieved. Roughly speaking, the so-called finite

[^0]cyclicity means that at most a finite number of limit cycles can exist in some neighborhood of focus points or homoclinic loop under small perturbations on the system's parameters.

In this paper, we particularly consider bifurcation of limit cycles in quadratic systems. Early results can be found in a survey article by Ye [1982]. Some recent progress has been reported in a number of papers (e.g. see [Roussarie, 1998; Roussarie \& Schlomiuk, 2002]). For general quadratic system (1) ( $n=2$ ), in 1952, Bautin proved that there exist 3 small limit cycles around a weak focus point or a center [Bautin, 1952]. After 30 years, until the end of 1970's, concrete examples were given to show that general quadratic systems can have 4 limit cycles [Shi, 1979; Chen \& Wang, 1979], around two foci with $(3,1)$-configuration. Since then, many researchers have paid attention to integrable quadratic systems, and a number of results have been obtained. A question was naturally raised: Can near-integrable quadratic systems have 4 limit cycles? A quadratic system is called near-integrable if it is a perturbation of a quadratic integrable system by quadratic polynomials. On one hand, it is reasonable to believe that the answer should be positive since general quadratic systems have at least 4 limit cycles; while on the other hand, near-integrable quadratic systems have restrictions on their system parameters and thus it is more difficult to find 4 limit cycles in such systems. In fact, this is still an open problem after another 30 years since the finding of 4 limit cycles in general quadratic systems, and many researchers are working on this problem. It should be mentioned that 4 limit cycles have been discussed by Llibre and Schlomiuk [2004], Artés et al. [2006] using general polynomial perturbations applied to integral quadratic systems, which are not near-integral quadratic systems defined in this paper.

The study of bifurcation of limit cycles in near-integrable systems is related to the so-called weak Hilbert's 16th problem [Arnold, 1977], which is transformed to finding the maximal number of isolated zeros of the Abelian integral or Melnikov function:

$$
\begin{equation*}
M(h, \delta)=\oint_{H(x, y)=h} Q_{n} d x-P_{n} d y \tag{2}
\end{equation*}
$$

where $H(x, y), P_{n}$ and $Q_{n}$ are all real polynomials of $x$ and $y$ with $\operatorname{deg} H=n+1$, and $\max \left\{\operatorname{deg} P_{n}, \operatorname{deg} Q_{n}\right\} \leq n$. The weak Hilbert's 16th
problem is a very important problem, closely related to the maximal number of limit cycles of the following near-Hamiltonian system [Han, 2006]:

$$
\begin{align*}
& \frac{d x}{d t}=\frac{\partial H(x, y)}{\partial y}+\varepsilon p_{n}(x, y)  \tag{3}\\
& \frac{d y}{d t}=-\frac{\partial H(x, y)}{\partial x}+\varepsilon q_{n}(x, y)
\end{align*}
$$

where $H(x, y), p_{n}(x, y)$ and $q_{n}(x, y)$ are polynomials of $x$ and $y$, and $0<\varepsilon \ll 1$ is a small perturbation.

General quadratic systems with one center have been classified, for example, by Żolạdek [1994] using a complex analysis on the condition of the center, as four systems: $Q_{3}^{L V}$ - the Lotka-Volterra system; $Q_{3}^{H}$ - Hamiltonian system; $Q_{3}^{R}$ - reversible system; and $Q_{4}$ - codimension-4 system. In 1994, Horozov and Iliev [1994] proved that in quadratic perturbation of generic quadratic Hamiltonian vector fields with one center and three saddle points there can appear at most two limit cycles, and this bound is exact. Later, Gavrilov [2001] extended Horozov and Iliev's method to give a fairly complete analysis on quadratic Hamiltonian systems with quadratic perturbations. Quadratic Hamiltonian systems, with at most four singularities, can be classified as three cases [Gavrilov, 2001]: (i) one center and three saddle points; (ii) one center and one saddle point; and (iii) two centers and two saddle points. Gavrilov [2001] showed that like case (i), cases (ii) and (iii) can also have at most two limit cycles. Therefore, generic quadratic Hamiltonian systems with quadratic perturbations can have maximal two limit cycles, and this case has been completely solved.

For the $Q_{3}^{R}$ reversible system, there have been many results published. For example, Dumortier et al. [1997] studied a case of $Q_{3}^{R}$ system with two centers and two unbounded heteroclinic loops, and presented a complete analysis of quadratic 3parameter unfolding. It was proved that 3 is the maximal number of limit cycles surrounding a single focus, and only the ( 1,1 )-configuration can occur in case of simultaneous nests of limit cycles. That is, 3 is the maximal number of limit cycles for the system they studied [Dumortier et al., 1997]. Later, Peng [2002] considered a similar case with a homoclinic loop and showed that 2 is the maximal number of limit cycles which can bifurcate from the system. Around the same time, Yu and Li [2002] investigated a similar case as Peng considered
but with a varied parameter in a certain interval, and obtained the same conclusion as Peng's. Later, Iliev et al. [2005] reinvestigated the same case but for the parameter values varied at a different interval (which yields two centers) and got the same conclusion as that of [Dumortier et al., 1997], i.e. 3 is the maximal number of limit cycles which can be obtained from this case. Recently, Li and Llibre [2010] considered a different case of $Q_{3}^{R}$ system which can exhibit the configurations of limit cycles: $(0,0),(1,0),(1,1)$ and $(1,2)$. Again, no 4 limit cycles were found. In order to explain why the above authors did not find 4 limit cycles from the $Q_{3}^{R}$ reversible system, consider the $Q_{3}^{R}$ system with quadratic perturbations, which can be described by [Dumortier et al., 1997]

$$
\begin{align*}
& \dot{x}=-y+a x^{2}+b y^{2}+\varepsilon\left(\mu_{1} x+\mu_{2} x y\right) \\
& \dot{y}=x(1+c y)+\varepsilon \mu_{3} x^{2} \tag{4}
\end{align*}
$$

where $a, b, c$ are real parameters, $\mu_{i}, i=1,2,3$ are real perturbation parameters, and $0<\varepsilon \ll 1$. When $\varepsilon=0$, system $(4)_{\varepsilon=0}$ is a reversible integrable system. It has been noted that in all the cases considered in [Dumortier et al., 1997; Peng, 2002; Yu \& Li, 2002; Iliev et al., 2005], the parameters $a$ and $c$ were chosen $a=-3, c=-2$, but with $b=1$ in [Dumortier et al., 1997]; $b=-1$ in [Peng, 2002], $b \in(-\infty,-1) \cup(-1,0)$ in [Yu \& Li, 2002], and $b \in(0,2)$ in [Iliev et al., 2005]. In these papers, complete analysis on the perturbation parameters was carried out with the aid of Poincaré transformation and the Picard-Fuchs equation, but it needed to fix all (or most of) the parameters $a, b$ and $c$. This way it may miss the opportunity to find more limit cycles, such as possible existence of 4 limit cycles. As a matter of fact, for the cases considered in [Yu \& Li, 2002; Iliev et al., 2005], a simple scaling on the parameter $b(b \neq 0)$ can be used to eliminate $b$. So, suppose the nonperturbed system $(4)_{\varepsilon=0}$ has two free parameters and let us consider the twodimensional parameter plane. Then, all the cases studied in the above mentioned articles are special cases, represented by just a point or a line segment in the two-dimensional parameter plane (see more details in Sec. 2). It has been noted that a different method was used in [Li \& Llibre, 2010] with Melnikov function up to second order, but no more limit cycles were found.

It should be mentioned that Zhang [2002] has proved that the possible cycle distributions in general quadratic systems with two foci must
be $(0,1)$-distribution or $(1, i)$-distribution, $i=$ $0,1,2,3, \ldots$ So far, no results have been obtained for $i \geq 4$. This result also rules out the possibility of $(2,2)$-distribution. It is conjectured that at most 3 limit cycles can exist around one focus point. The problem of bifurcation of 3 limit cycles near an isolated homoclinic loop is still open.

In this paper, we turn to a different angle and consider bifurcation of limit cycles in quadratic near-integrable systems with two centers. We shall leave more free parameters in the integrable systems, so that we will have chances to find more limit cycles. The basic idea is as follows: we first consider bifurcation of multiple limit cycles from Hopf singularity, which does not need to fix any parameters, and use expansion of Melnikov function near centers to get as many as possible such limit cycles. This leads to the determination of a maximal number of parameters. Then, for the remaining undetermined parameters, we compute the global Melnikov function to look for possible large limit cycles. Indeed, although, due to the complex integrating factor in the analysis, we are not able to give a complete analysis for classifying the perturbation unfolding, we do get a positive answer to the open question of existence of 4 limit cycles in quadratic near-integrable systems. In particular, we will show that perturbing a reversible, integrable quadratic system with two centers can have at least 4 limit cycles, with $(3,1)$-distribution, bifurcating from the two centers under quadratic perturbations.

The rest of paper is organized as follows. In Sec. 2, we give a different classification in real domain for quadratic systems with one center, and compare it with that given by Żolądek [1994]. Also, we use our classification to present a simple summary on some of the existing results for the reversible near-integrable system. Section 3 is devoted to the analysis on bifurcation of small limit cycles from Hopf singularity. In Sec. 4, we show how to find large limit cycles bifurcating from closed orbits to obtain a total of 4 limit cycles. Finally, conclusion is drawn in Sec. 5. The main results of this manuscript has been posted on arXiv.org since February 2010 [Yu \& Han, 2010].

## 2. Classification of Generic Quadratic Systems with at Least One Center

In this section, we give a different classification in real domain for quadratic systems with a center,
which is consistent with the Hamiltonian systems considered in [Horozov \& Iliev, 1994; Gavrilov, 2001]. We start from the following general quadratic system:

$$
\begin{align*}
\frac{d z_{1}}{d t}= & c_{100}+c_{110} z_{1}+c_{101} z_{2}+c_{120} z_{1}^{2} \\
& +c_{111} z_{1} z_{2}+c_{102} z_{2}^{2} \\
\frac{d z_{2}}{d t}= & c_{200}+c_{210} z_{1}+c_{201} z_{2}+c_{220} z_{1}^{2}  \tag{5}\\
& +c_{211} z_{1} z_{2}+c_{202} z_{2}^{2}
\end{align*}
$$

where $c_{i j k}$ 's are real constant parameters. It is easy to show that this system has at most four singularities, or more precisely, it can have 0,2 or 4 singularities in real domain. In order for system (5) to have limit cycles, the system must have some singularity. In this paper, we assume that system (5) has at least two singularities. Without loss of generality, we may assume that one singular point is located at the origin $(0,0)$, which implies $c_{100}=c_{200}=0$, and the other at $(p, q)\left(p^{2}+q^{2} \neq 0\right)$. Further assume the origin is an elementary center. Then introducing a series of linear transformations, parameter rescaling and time rescaling to system (5) yields the following general quadratic system:

$$
\begin{align*}
& \frac{d x}{d t}=y+a_{1} x y+a_{2} y^{2} \\
& \frac{d y}{d t}=-x+x^{2}+a_{3} x y+a_{4} y^{2} \tag{6}
\end{align*}
$$

which has an elementary center at the origin $(0,0)$ and another singularity at $(1,0)$.

In order to have the origin of system (6) being a center, we may calculate the focus values of system (6) and find four cases under which $(0,0)$ is a center, listed in the following theorem (here we use Żola̧dek's notation in our classification).

Theorem 1. The origin of $(6)$ is a center if and only if one of the following conditions is satisfied:
$Q_{3}^{R} —$ Reversible system: $a_{3}=a_{2}=0$, under which system (6) becomes

$$
\begin{aligned}
& \frac{d x}{d t}=y+a_{1} x y \\
& \frac{d y}{d t}=-x+x^{2}+a_{4} y^{2}
\end{aligned}
$$

with
$(1,0)$ being a $\begin{cases}\text { center } & \text { if } a_{1}<-1, \\ \text { saddle point } & \text { if } a_{1}>-1 .\end{cases}$
$Q_{3}^{H}-$ Hamiltonian system: $a_{3}=a_{1}+2 a_{4}=0$, under which system ( 6 ) is reduced to

$$
\begin{align*}
& \frac{d x}{d t}=y+a_{1} x y+a_{2} y^{2}  \tag{8}\\
& \frac{d y}{d t}=-x+x^{2}-\frac{1}{2} a_{1} y^{2}
\end{align*}
$$

with
$(1,0)$ being a $\begin{cases}\text { center } & \text { if } a_{1}<-1, \\ \text { saddle point } & \text { if } a_{1}>-1 .\end{cases}$
$Q_{3}^{L V}-$ Lokta-Volterra system: $a_{2}=1+a_{4}=0$, under which system (6) becomes

$$
\begin{align*}
& \frac{d x}{d t}=y+a_{1} x y  \tag{9}\\
& \frac{d y}{d t}=-x+x^{2}+a_{3} x y-y^{2}
\end{align*}
$$

with
$(1,0)$ being a

$$
\begin{cases}\text { focus } & \text { if } a_{1}<-\left(1+\frac{1}{4} a_{3}^{2}\right) \\ \text { node } & \text { if }-\left(1+\frac{1}{4} a_{3}^{2}\right)<a_{1}<-1 \\ \text { saddle point } & \text { if } a_{1}>-1\end{cases}
$$

$Q_{4}$ - Codimension- 4 system:

$$
\begin{align*}
a_{3}-5 a_{2} & =a_{1}-\left(5+3 a_{4}\right) \\
& =a_{4}+2\left(1+a_{2}^{2}\right)=0 \tag{10}
\end{align*}
$$

under which system (6) can be rewritten as

$$
\begin{align*}
& \frac{d x}{d t}=y-\left(1+6 a_{2}^{2}\right) x y+a_{2} y^{2} \\
& \frac{d y}{d t}=-x+x^{2}+5 a_{2} x y-2\left(1+a_{2}^{2}\right) y^{2} \tag{11}
\end{align*}
$$

with $(1,0)$ being a node for $a_{2} \neq 0$.

Remark 2.1. There is one more case found from the above process, defined by the following conditions:

$$
\begin{aligned}
a_{3}-5 a_{2} & =a_{1}-\left(5+3 a_{4}\right) \\
& =3\left(a_{4}+2\right)\left(a_{4}+1\right)^{2}-\left(5 a_{4}+6\right) a_{2}^{2} \\
& =0
\end{aligned}
$$

We will show later in this section, when we compare our above real classification with the complex classification given by Żola̧dek [1994], that the case defined by (12) actually belongs to the $Q_{3}^{R}$ — reversible system.

Proof. Necessity is easy to be verified by computing the focus values of system (6) associated with the origin. Some focus values will not equal zero if the condition is not satisfied.

For sufficiency, we find an integrating factor for each case when the condition holds. For the $Q_{3}^{H}-$ Hamiltonian system (8), we know that the integrating factor is 1 , and the Hamiltonian is given by

$$
\begin{align*}
H(x, y)= & \frac{1}{2}\left(x^{2}+y^{2}\right)-\frac{1}{3} x^{3}+\frac{1}{2} a_{1} x y^{2}  \tag{12}\\
& +\frac{1}{3} a_{2} y^{3} \tag{13}
\end{align*}
$$

which is exactly the same as that given in [Horozov \& Iliev, 1994; Gavrilov, 2001].

For the $Q_{3}^{R}$ - reversible system (7), the integrating factor is

$$
\begin{equation*}
\gamma=\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}} \tag{14}
\end{equation*}
$$

and the first integral of the system is given by

$$
\begin{equation*}
F(x, y)=\frac{1}{2} \operatorname{sign}\left(1+a_{1} x\right)\left|1+a_{1} x\right|^{-\frac{2 a_{4}}{a_{1}}}\left[y^{2}+\frac{\left(1+a_{1}-a_{4}\right)\left(1+2 a_{4} x\right)}{a_{4}\left(a_{1}-a_{4}\right)\left(a_{1}-2 a_{4}\right)}-\frac{x^{2}}{a_{1}-a_{4}}\right] \tag{15}
\end{equation*}
$$

For the $Q_{3}^{L V}$ - Lokta-Volterra system (9), we find the integrating factor as

$$
\begin{equation*}
\gamma=|g(x, y)|^{-1}, \quad \text { where } g(x, y)=\left(1+a_{1} x\right)\left[(x-1)^{2}+a_{3}(x-1) y-\left(1+a_{1}\right) y^{2}\right] \tag{16}
\end{equation*}
$$

and the first integral of the system is

$$
F(x, y)=\left\{\begin{array}{r}
-\frac{\operatorname{sign}(g(x, y))}{2 a_{1}\left(1+a_{1}\right)}\left\{2 \ln \left|1+a_{1} x\right|+a_{1} \ln \left|\left(1+a_{1}\right) y^{2}-a_{3} y(x-1)-(x-1)^{2}\right|\right. \\
\quad+\frac{2 a_{1} a_{3}(x-1)}{\sqrt{\left[a_{3}^{2}+4\left(1+a_{1}\right)\right](x-1)^{2}}} \tanh ^{-1}\left[\frac{a_{3}(x-1)-2\left(1+a_{1}\right) y}{\left.\left.\sqrt{\left[a_{3}^{2}+4\left(1+a_{1}\right)\right](x-1)^{2}}\right]\right\},} \begin{array}{r}
\text { when } a_{3}^{2}+4\left(1+a_{1}\right)>0, \\
-\frac{\operatorname{sign}\left(1+a_{1} x\right)}{2 a_{1}\left(1+a_{1}\right)}\left\{2 \ln \left|1+a_{1} x\right|+a_{1} \ln \left[\left(1+a_{1}\right) y^{2}-a_{3} y(x-1)-(x-1)^{2}\right]\right. \\
\left.-\frac{2 a_{1} a_{3}(x-1)}{\sqrt{\left[-a_{3}^{2}-4\left(1+a_{1}\right)\right](x-1)^{2}}} \tan ^{-1}\left[\frac{a_{3}(x-1)-2\left(1+a_{1}\right) y}{\sqrt{\left[-a_{3}^{2}-4\left(1+a_{1}\right)\right](x-1)^{2}}}\right]\right\},
\end{array}\right\},
\end{array}\right.
$$

Finally, for the $Q_{4}$ - codimension- 4 system (11), we have

$$
\begin{equation*}
\gamma=|g(x, y)|^{-5 / 2}, \quad \text { where } g(x, y)=1-2\left(1+2 a_{2}^{2}\right) x-2 a_{2} y+\left(1+4 a_{2}^{2}\right)\left(x+a_{2} y\right)^{2} \tag{18}
\end{equation*}
$$

and the first integral of the system is equal to

$$
F(x, y)=\frac{\operatorname{sign}(g(x, y))}{12 a_{2}^{6}}|g(x, y)|^{-3 / 2} f(x, y)
$$

where

$$
\begin{align*}
f(x, y)= & -\left(1+a_{2}^{2}\right)+3\left(x+a_{2} y+2 a_{2}^{2} x\right) \\
& \times\left[1+a_{2}^{2}-\left(1+3 a_{2}^{2}\right)\left(x+a_{2} y\right)\right]  \tag{20}\\
& +\left(1+3 a_{2}^{2}\right)\left(1+4 a_{2}^{2}\right)\left(x+a_{2} y\right)^{3} \tag{19}
\end{align*}
$$

The proof is complete.

Note that among the four classifications of the integrable system (6), the first three classified systems (7)-(9) have two free parameters, while the last system (11) has only one free parameter.

Remark 2.2. We now show that our classification in Theorem 1 is equivalent to that given by Żolądek [1994]. The general quadratic system considered by Żoladdek is given in the complex form:

$$
\begin{equation*}
\frac{d z}{d t}=(i+\lambda) z+A z^{2}+B z \bar{z}+C \bar{z}^{2} \tag{21}
\end{equation*}
$$

where $z=x+i y$, and $A, B$ and $C$ are complex coefficients. It has been shown by Żolądek [1994] that the point $z=0$ is a center if and only if one of the following conditions is fulfilled:

$$
\begin{align*}
Q_{3}^{L V}: \lambda & =B=0, \\
Q_{3}^{H}: \lambda & =2 A+\bar{B}=0, \\
Q_{3}^{R}: \lambda & =\operatorname{Im}(A B)=\operatorname{Im}\left(\bar{B}^{3} C\right)  \tag{22}\\
& =\operatorname{Im}\left(A^{3} C\right)=0, \\
Q_{4}: \lambda & =A-2 \bar{B}=|C|-|B|=0 .
\end{align*}
$$

In the following, we first use real differential equation to give a brief proof (different from Żolądek's [1994]), and then show that our classification is equivalent to Żolądek's when system (21) is assumed to have a nonzero singularity. To prove this, let

$$
\begin{gathered}
A=A_{1}+i A_{2}, \quad B=B_{1}+i B_{2}, \\
C=C_{1}+i C_{2}, \quad\left(i^{2}=-1\right),
\end{gathered}
$$

and then rewrite the complex equation (21) in the real form:

$$
\begin{align*}
\frac{d x}{d t}= & \lambda x+y+\left(A_{1}+B_{1}+C_{1}\right) x^{2} \\
& +2\left(A_{2}-C_{2}\right) x y-\left(A_{1}-B_{1}+C_{1}\right) y^{2}  \tag{23}\\
\frac{d y}{d t}= & -x+\lambda y-\left(A_{2}+B_{2}+C_{2}\right) x^{2} \\
& +2\left(A_{1}-C_{1}\right) x y+\left(A_{2}-B_{2}+C_{2}\right) y^{2}
\end{align*}
$$

where $y \rightarrow-y$ has been used. Letting $\lambda=0$ yields the focus value $v_{0}=0$. Then, it is easy to find the first focus value (or the first Lyapunov constant) as

$$
\begin{equation*}
v_{1}=-A_{1} B_{2}-B_{1} A_{2}=-\operatorname{Im}(A B) \tag{24}
\end{equation*}
$$

Letting $v_{1}=0$ results in $\operatorname{Im}(A B)=0$, which gives $B_{2}=-\frac{B_{1} A_{2}}{A_{1}}, \quad$ under the assumption of $A_{1} \neq 0$.
(The degenerate case $A_{1}=0$ can be similarly analyzed and the details are omitted here.) Then, we apply our Maple program (e.g. see [Yu, 1998]) to system (23), with the conditions $\lambda=0$ and (25), to obtain

$$
\begin{aligned}
& v_{2}=\frac{-f\left(A_{1}-2 B_{1}\right)}{3 A_{1}^{3}}, \quad v_{3}=\frac{-f f_{3}}{216 A_{1}^{5}}, \\
& v_{4}=\frac{-f f_{4}}{9720 A_{1}^{7}}, \quad v_{5}=\frac{-f f_{5}}{466560 A_{1}^{9}}, \ldots
\end{aligned}
$$

where

$$
\begin{aligned}
f= & B_{1}\left(2 A_{1}+B_{1}\right)\left(C_{2} A_{1}^{3}+3 C_{1} A_{2} A_{1}^{2}\right. \\
& \left.-3 A_{2}^{2} C_{2} A_{1}-C_{1} A_{2}^{3}\right),
\end{aligned}
$$

and $f_{3}, f_{4}$, etc. are polynomials of $A_{1}, A_{2}, C_{1}, C_{2}$ and $B_{1}$. Letting $f=0$, i.e.

$$
\begin{aligned}
& B_{1}=0 \quad \text { or } \quad 2 A_{1}+B_{1}=0 \quad \text { or } \\
& C_{2} A_{1}^{3}+3 C_{1} A_{2} A_{1}^{2}-3 A_{2}^{2} C_{2} A_{1}-C_{1} A_{2}^{3} \\
& \quad=\operatorname{Im}\left(A^{3} C\right)=0
\end{aligned}
$$

yields $v_{2}=v_{3}=\cdots=0$.
Indeed, $B_{1}=0$ implies $B_{2}=0$ due to the condition (25), and so $B=0$. Thus, we obtain $\lambda=B=0$, corresponding to the $Q_{3}^{L V}$ case.

For the condition $2 A_{1}+B_{1}=0$, it follows from (25) that $2 A_{2}-B_{2}=0$, i.e. $2 A+\bar{B}=0$, which plus the condition $\lambda=0$ gives the $Q_{3}^{H}$ case.

The third condition $\operatorname{Im}\left(A^{3} C\right)=0$, with $\lambda=0$ and $\operatorname{Im}(A B)=0$, corresponds to the $Q_{3}^{R}$ case. Further, it is easy to show that under the condition $\operatorname{Im}(A B)=0, \operatorname{Im}\left(A^{3} C\right)=0$ and $\operatorname{Im}\left(\bar{B}^{3} C\right)=0$ are equivalent. Thus, the conditions $\lambda=\operatorname{Im}(A B)=$ $\operatorname{Im}\left(\bar{B}^{3} C\right)=0$ are also applicable for this case. So for this case, either $\operatorname{Im}\left(A^{3} C\right)=0$ or $\operatorname{Im}\left(\bar{B}^{3} C\right)=0$ is needed, but not both of them. In the following, we show one more case to join this case, leading to both the two conditions being needed.

Note that there is one more condition $A_{1}=2 B_{1}$ which renders $v_{2}=0$. Letting $A_{1}=2 B_{1}$, and so $A_{2}=-2 B_{2}$ [see (25)], implying that $A-2 \bar{B}=0$. Under the condition $A=2 \bar{B}, v_{1}=v_{2}=0$, and the
other focus values become

$$
\begin{aligned}
v_{3}= & \frac{25}{8}\left(C_{1}^{2}+C_{2}^{2}-B_{1}^{2}-B_{2}^{2}\right)\left(C_{2} B_{1}^{3}-3 C_{1} B_{1}^{2} B_{2}-3 C_{2} B_{1} B_{2}^{2}+C_{1} B_{2}^{3}\right), \\
v_{4}= & \frac{v_{3}}{45}\left[45 B_{1}^{2}+585 B_{2}^{2}+60\left(B_{1} C_{1}+B_{2} C_{2}\right)-196\left(C_{1}^{2}+C_{2}^{2}\right)\right], \\
v_{5}= & \frac{v_{3}}{6480}\left[648\left(7 B_{1}^{4}+124 B_{1}^{2} B_{2}^{2}+1557 B_{2}^{4}\right)-3\left(961 B_{1}^{2} C_{1}^{2}-7680 B_{1} B_{2} C_{1} C_{2}+202345 B_{2}^{2} C_{2}^{2}\right)\right. \\
& +576 B_{1} C_{1}\left(106 B_{1}^{2}+307 B_{2}^{2}\right)+288 B_{2} C_{2}\left(371 B_{1}^{2}+773 B_{2}^{2}\right)-3\left(4801 B_{1}^{2} C_{2}^{2}+206185 B_{2}^{2} C_{1}^{2}\right) \\
& \left.-80688\left(C 1^{2}+C 2^{2}\right)\left(B_{1} C_{1}+B_{2} C_{2}\right)+86144\left(C 1^{2}+C 2^{2}\right)^{2}\right],
\end{aligned}
$$

Hence, under the conditions $\lambda=A-2 \bar{B}=0$, there are two possibilities such that $v_{3}=v_{4}=\cdots=0$. The first possibility is

$$
C_{1}^{2}+C_{2}^{2}-B_{1}^{2}-B_{2}^{2}=0, \quad \text { i.e. }|C|-|B|=0,
$$

which is one of the conditions given for the $Q_{4}$ case [see (22)].

The second possibility is given by the condition:

$$
\begin{aligned}
& C_{2} B_{1}^{3}-3 C_{1} B_{1}^{2} B_{2}-3 C_{2} B_{1} B_{2}^{2}+C_{1} B_{2}^{3} \\
& \quad=\operatorname{Im}\left(\bar{B}^{3} C\right)
\end{aligned}
$$

$$
\begin{equation*}
=\frac{1}{8} \operatorname{Im}\left(A^{3} C\right)=0, \tag{26}
\end{equation*}
$$

due to $A=2 \bar{B}$. Since these conditions can be included in the conditions $\lambda=\operatorname{Im}(A B)=$ $\operatorname{Im}\left(\bar{B}^{3} C\right)=\operatorname{Im}\left(A^{3} C\right)=0$, this possibility belongs to the $Q_{3}^{R}$ case.

The remaining task is to show that the conditions classified in (22) are sufficient. This can be done by finding an integrating factor for each case. For brevity, we only list these integrating factors below (while the lengthy expressions of the first integrals are omitted):

$$
\gamma=\left\{\begin{array}{rlrl}
\mid 1+ & 4\left(A_{2} x-A_{1} y\right)+4\left(A_{1} C_{2}+A_{2} C_{1}-2 A_{1} A_{2}\right) x y+\left[\left(A_{1}+C_{1}\right)\left(A_{1}-3 C_{1}\right)\right. &  \tag{27}\\
& \left.+\left(A_{2}+C_{2}\right)\left(5 A_{2}-3 C_{2}\right)\right] x^{2}+\left[\left(A_{2}+C_{2}\right)\left(A_{2}-3 C_{2}\right)\right. \\
& \left.+\left(A_{1}+C_{1}\right)\left(5 A_{1}-3 C_{1}\right)\right] y^{2}+2\left(A_{1}^{2}+A_{2}^{2}-C_{1}^{2}-C_{2}^{2}\right)\left[\left(A_{2}+C_{2}\right) x^{3}\right. & & \\
& \left.-\left(A_{1}+C_{1}\right) y^{3}-\left(A_{1}-3 C_{1}\right) x^{2} y+\left(A_{2}-3 C_{2}\right) x y^{2}\right]\left.\right|^{-1}, & & \text { for } Q_{3}^{L V}, \\
1, & & \text { for } Q_{3}^{H}, \\
\left|1-2\left(A_{1}-C_{1}\right) y\right|^{-\frac{2 A_{1}+B_{1}}{A_{1}-C_{1}},} & & \text { for } Q_{3}^{R}, \\
\mid 1-4\left(B_{2} x+B_{1} y\right)+2\left(B_{1}^{2}+B_{2}^{2}\right)\left(x^{2}+y^{2}\right) & & \text { for } Q_{4} .
\end{array}\right.
$$

For the integrating factors of degenerate cases (e.g. $A_{1}-C_{1}=0$ ), one can easily find them.
Next, compare the classification listed in (22) with ours given in Theorem 1. First, consider the $Q_{3}^{L V}$ case. Letting $\lambda=B_{1}=B_{2}=0$ in (23) yields

$$
\begin{align*}
& \frac{d x}{d t}=y+\left(A_{1}+C_{1}\right) x^{2}+2\left(A_{2}-C_{2}\right) x y-\left(A_{1}+C_{1}\right) y^{2}  \tag{28}\\
& \frac{d y}{d t}=-x-\left(A_{2}+C_{2}\right) x^{2}+2\left(A_{1}-C_{1}\right) x y+\left(A_{2}+C_{2}\right) y^{2} .
\end{align*}
$$

Then, let

$$
\begin{gather*}
k=\tan (\theta), \quad \text { and so } \\
\sin (\theta)=\frac{k}{\sqrt{1+k^{2}}}, \quad \cos (\theta)=\frac{1}{\sqrt{1+k^{2}}} \tag{29}
\end{gather*}
$$

where $k$ is solved from the following cubic polynomial:

$$
\begin{align*}
P_{1}(k)= & \left(A_{2}+C_{2}\right) k^{3}+\left(A_{1}-3 C_{1}\right) k^{2} \\
& +\left(A_{2}-3 C_{2}\right) k+A_{1}+C_{1}=0 \tag{30}
\end{align*}
$$

This cubic polynomial at least has one real solution for $k$, which gives the slope of the line on which a second fixed point is located. $k=0$ if $A_{1}+C_{1}=0$, otherwise, $k \neq 0$. Let $\bar{k}$ be a real root of $P_{1}(k)$, i.e. $P_{1}(\bar{k})=0$.

Further, introducing the linear transformation (rotation):

$$
\begin{align*}
& x=\cos (\theta) u-\sin (\theta) v \\
& y=\sin (\theta) u+\cos (\theta) v \tag{31}
\end{align*}
$$

into (28) yields

$$
\begin{align*}
& \frac{d x}{d t}=y+m_{120} x^{2}+m_{111} x y+m_{102} y^{2}  \tag{32}\\
& \frac{d y}{d t}=-x+m_{220} x^{2}+m_{211} x y+m_{202} y^{2}
\end{align*}
$$

where

$$
\begin{aligned}
m_{120}= & -m_{102} \\
= & \left(1+\bar{k}^{2}\right)^{-3 / 2} P_{1}(\bar{k})=0 \\
m_{220}= & -m_{202} \\
= & \left(1+\bar{k}^{2}\right)^{-3 / 2}\left[\left(A_{1}+C_{1}\right) \bar{k}^{3}-\left(A_{2}-3 C_{2}\right) \bar{k}^{2}\right. \\
& \left.+\left(A_{1}-3 C_{1}\right) \bar{k}-A_{2}-C_{2}\right] \\
m_{111}= & -2\left(1+\bar{k}^{2}\right)^{-3 / 2}\left[\left(A_{1}-C_{1}\right) \bar{k}^{3}\right. \\
& -\left(A_{2}+3 C_{2}\right) \bar{k}^{2}+\left(A_{1}+3 C_{1}\right) \bar{k} \\
& \left.-A_{2}+C_{2}\right] \\
m_{211}= & 2\left(1+\bar{k}^{2}\right)^{-3 / 2}\left[\left(A_{2}-C_{2}\right) \bar{k}^{3}+\left(A_{1}+3 C_{1}\right) \bar{k}^{2}\right. \\
& \left.+\left(A_{2}+3 C_{2}\right) \bar{k}+A_{1}-C_{1}\right]
\end{aligned}
$$

Suppose $m_{220} \neq 0$. Then, introducing $\bar{x}=m_{220} x$, $\bar{y}=m_{220} y$ into (32) results in

$$
\frac{d \bar{x}}{d t}=\bar{y}+\frac{m_{111}}{m_{220}} \bar{x} \bar{y}
$$

$$
\begin{equation*}
\frac{d \bar{y}}{d t}=-\bar{x}+\bar{x}^{2}+\frac{m_{211}}{m_{220}} x y-\bar{y}^{2} \tag{33}
\end{equation*}
$$

which is identical to (9) as long as letting $a_{1}=\frac{m_{111}}{m_{220}}$ and $a_{3}=\frac{m_{211}}{m_{220}}$. This shows that the four parameters $A_{1}, A_{2}, C_{1}$ and $C_{2}$ are not independent. Thus, alternatively, we may simply take $\bar{k}=0$ (which renders the second singularity of (28) on the $x$-axis), yielding $C_{1}=-A_{1}$. Thus, (28) becomes

$$
\begin{aligned}
& \frac{d x}{d t}=y+2\left(A_{2}-C_{2}\right) x y \\
& \frac{d y}{d t}=-x-\left(A_{2}+C_{2}\right) x^{2}+4 A_{1} x y+\left(A_{2}+C_{2}\right) y^{2}
\end{aligned}
$$

Suppose $A_{2}+C_{2} \neq 0$. Introducing $\bar{x}=-\left(A_{2}+\right.$ $\left.C_{2}\right) x, \bar{y}=-\left(A_{2}+C_{2}\right) y$ into the above equations we obtain

$$
\begin{align*}
& \frac{d \bar{x}}{d t}=\bar{y}-\frac{2\left(A_{2}-C_{2}\right)}{A_{2}+C_{2}} \bar{x} \bar{y} \\
& \frac{d \bar{y}}{d t}=-\bar{x}+\bar{x}^{2}-\frac{4 A_{1}}{A_{2}+C_{2}} \bar{x} \bar{y}-\bar{y}^{2} \tag{34}
\end{align*}
$$

which is identical to (9) if letting $a_{1}=\frac{-2\left(A_{2}-C_{2}\right)}{A_{2}+C_{2}}$ and $a_{3}=\frac{-4 A_{1}}{A_{2}+C_{2}}$. In the following, we will use this simple approach for other cases.

For the $Q_{3}^{H}$ case, substituting $\lambda=0, B_{1}=$ $-2 A_{1}$ and $B_{2}=2 A_{2}$ into system (23) results in

$$
\begin{aligned}
\frac{d x}{d t}= & y-\left(A_{1}-C_{1}\right) x^{2}+2\left(A_{2}-C_{2}\right) x y \\
& -\left(3 A_{1}+C_{1}\right) y^{2} \\
\frac{d y}{d t}= & -x-\left(3 A_{2}+C_{2}\right) x^{2}+2\left(A_{1}-C_{1}\right) x y \\
& -\left(A_{2}-C_{2}\right) y^{2}
\end{aligned}
$$

Further, taking $C_{1}=A_{1}$ in the above equations gives another singularity on the $x$-axis, and introducing $\bar{x}=-\left(3 A_{2}+C_{2}\right) x, \bar{y}=-\left(3 A_{2}+C_{2}\right) y$ into the resulting equations yields

$$
\begin{align*}
& \frac{d \bar{x}}{d t}=\bar{y}-\frac{2\left(A_{2}-C_{2}\right)}{3 A_{2}+C_{2}} \bar{x} \bar{y}+\frac{4 A_{1}}{3 A_{2}+C_{2}} \bar{y}^{2}  \tag{35}\\
& \frac{d \bar{y}}{d t}=-\bar{x}+\bar{x}^{2}+\frac{A_{2}-C_{2}}{3 A_{2}+C_{2}} \bar{y}^{2}
\end{align*}
$$

which is identical to (8) if we set $a_{1}=\frac{-2\left(A_{2}-C_{2}\right)}{3 A_{2}+C_{2}}$ and $a_{2}=\frac{4 A_{1}}{3 A_{2}+C_{2}}$.

For the $Q_{3}^{R}$ reversible case, it follows from [Żolądek, 1994] that all the coefficients $A, B$ and $C$ are real, and thus we obtain the following real form from the complex system (21)

$$
\begin{align*}
& \frac{d x}{d t}=-y+a x^{2}+b y^{2} \\
& \frac{d y}{d t}=x+c x y \tag{36}
\end{align*}
$$

where

$$
\begin{gathered}
a=A_{1}+B_{1}+C_{1}, \quad b=B_{1}-A_{1}-C_{1}, \\
c=2 A_{1}-2 C_{1} .
\end{gathered}
$$

Suppose $b \neq 0$. Then, introducing $\bar{x}=b y, \bar{y}=b x$ into (36) results in

$$
\begin{align*}
& \frac{d \bar{x}}{d t}=\bar{y}+\frac{c}{b} \bar{x} \bar{y}, \\
& \frac{d \bar{y}}{d t}=-\bar{x}+\bar{x}^{2}+\frac{a}{b} \bar{y}^{2}, \tag{37}
\end{align*}
$$

which is identical to (7) if

$$
\begin{aligned}
& a_{1}=\frac{c}{b}=\frac{2\left(A_{1}-C_{1}\right)}{B_{1}-A_{1}-C_{1}} \quad \text { and } \\
& a_{4}=\frac{a}{b}=\frac{A_{1}+B_{1}+C_{1}}{B_{1}-A_{1}-C_{1}}
\end{aligned}
$$

For the last $Q_{4}$ case, under the condition $\lambda=$ $A-2 \bar{B}=0$, by setting $C_{1}=-3 B_{1}$ (which renders a nonzero singularity on the $x$-axis) in (23) we obtain

$$
\begin{aligned}
\frac{d x}{d t}= & y-2\left(2 B_{2}+C_{2}\right) x y+2 B_{1} y^{2} \\
\frac{d y}{d t}= & -x+\left(B_{2}-C_{2}\right) x^{2}+10 B_{1} x y \\
& -\left(3 B_{2}-C_{2}\right) y^{2} .
\end{aligned}
$$

Suppose $B_{2}-C_{2} \neq 0$. Then, introducing $\bar{x}=$ $\left(B_{2}-C_{2}\right) x, \bar{y}=\left(B_{2}-C_{2}\right) y$ into the above equations yields

$$
\begin{align*}
& \frac{d \bar{x}}{d t}=\bar{y}-\frac{2\left(2 B_{2}+C_{2}\right)}{B_{2}-C_{2}} \bar{x} \bar{y}+\frac{2 B_{1}}{B_{2}-C_{2}} \bar{y}^{2}  \tag{38}\\
& \frac{d \bar{y}}{d t}=-\bar{x}+\bar{x}^{2}+\frac{10 B_{1}}{B_{2}-C_{2}} \bar{x} \bar{y}-\frac{3 B_{2}-C_{2}}{B_{2}-C_{2}} \bar{y}^{2} .
\end{align*}
$$

Comparing the coefficients of the above system (38) with our system (6) results in

$$
\begin{gather*}
a_{1}=-\frac{2\left(2 B_{2}+C_{2}\right)}{B_{2}-C_{2}}, \quad a_{2}=\frac{2 B_{1}}{B_{2}-C_{2}},  \tag{39}\\
a_{3}=\frac{10 B_{1}}{B_{2}-C_{2}}, \quad a_{4}=-\frac{3 B_{2}-C_{2}}{B_{2}-C_{2}},
\end{gather*}
$$

which in turn implies that $a_{3}-5 a_{2}=a_{1}-(5+$ $\left.3 a_{4}\right)=0$, and

$$
\begin{aligned}
a_{4}+2\left(1+a_{2}^{2}\right) & =\frac{8 B_{1}^{2}+C_{2}^{2}-B_{2}^{2}}{\left(B_{2}-C_{2}\right)^{2}} \\
& =\frac{C_{1}^{2}+C_{2}^{2}-B_{1}^{2}-B_{2}^{2}}{\left(B_{2}-C_{2}\right)^{2}} \\
& =0, \quad \text { for }|C|-|B|=0
\end{aligned}
$$

The above conditions are the exact conditions given in (10) for the $Q_{4}$ case.

Finally, we turn to the conditions given in (12). It follows from (39) that

$$
\begin{align*}
& 3\left(a_{4}+2\right)\left(a_{4}+1\right)^{2}-\left(5 a_{4}+6\right) a_{2}^{2} \\
& \quad=-\frac{4}{\left(B_{2}-C_{2}\right)^{3}}\left(3 B_{2}^{3}+3 B_{2}^{2} C_{2}-C_{1}^{2} B_{2}-B_{1}^{2} C_{2}\right) . \tag{40}
\end{align*}
$$

On the other hand, under the condition $C_{1}=$ $-3 B_{1}$, the condition (26) for the second possibility becomes

$$
\begin{aligned}
& C_{2} B_{1}^{3}-3 C_{1} B_{1}^{2} B_{2}-3 C_{2} B_{1} B_{2}^{2}+C_{1} B_{2}^{3} \\
& \quad=C_{2} B_{1}^{3}+C_{1}^{2} B_{1} B_{2}-3 C_{2} B_{1} B_{2}^{2}-3 B_{1} B_{2}^{3} \\
& \quad=-B_{1}\left(3 B_{2}^{3}+3 B_{2}^{2} C_{2}-C_{1}^{2} B_{2}-B_{1}^{2} C_{2}\right)=0
\end{aligned}
$$

which implies, by Eq. (40), $3\left(a_{4}+2\right)\left(a_{4}+1\right)^{2}-\left(5 a_{4}+\right.$ 6) $a_{2}^{2}=0$ for $B_{1} \neq 0$. Hence, according to Żolądek's classification [see (22)], this case should be included in the $Q_{3}^{R}$ case. However, one cannot prove this by directly using the conditions in (12) as well as that for the $Q_{3}^{R}$ case (see Theorem 1). One must trace back to the original system coefficients.

In the paper [Żoladek, 1994], the author used Bautin's system to verify his classification. Bautin's system is described by [Bautin, 1952]

$$
\begin{align*}
& \frac{d x}{d t}=\lambda_{1} x-y+\lambda_{3} x^{2}+\left(2 \lambda_{2}+\lambda_{5}\right) x y+\lambda_{6} y^{2}, \\
& \frac{d y}{d t}=x+\lambda_{1} y+\lambda_{2} x^{2}+\left(2 \lambda_{3}+\lambda_{4}\right) x y-\lambda_{2} y^{2} . \tag{41}
\end{align*}
$$

It is seen from (23) and (41) that Bautin's system has only six parameters, while Żolądek's system has
seven (in real domain) parameters. This indicates that Żoladek's system has one redundant parameter. In fact, putting Bautin's system in Żolądek's complex form gives the following expressions:

$$
\begin{aligned}
\lambda=\lambda_{1}, \quad A & =\frac{1}{4}\left(\lambda_{3}+\lambda_{4}-\lambda_{6}-i \lambda_{5}\right), \\
B & =-\frac{1}{2}\left(\lambda_{3}-\lambda_{6}\right),
\end{aligned}
$$

$$
C=\frac{1}{4}\left[-\left(3 \lambda_{3}+\lambda_{4}+\lambda_{6}\right)+i\left(4 \lambda_{2}+\lambda_{5}\right)\right] .
$$

Then, applying the formulas given in (23) will immediately generate the center conditions obtained by Bautin [1952]. The above expressions clearly show that $B_{2}=0$. As a matter of factor, the integrating factor for the system, corresponding to the second possibility, i.e. when $\lambda=A-2 \bar{B}=\operatorname{Im}\left(\bar{B}^{3} C\right)=0$, is given by

$$
\left|1+2\left[\frac{C_{1}\left(B_{1}^{2}+B_{2}^{2}\right)}{B_{1}\left(B_{1}^{2}-3 B_{2}^{2}\right)}-2\right]\left(B_{2} x+B_{1} y\right)\right|^{\frac{5 B_{1}\left(B_{1}^{2}-3 B_{2}^{2}\right)}{C_{1}\left(B_{1}^{2}+B_{2}^{2}\right)-2 B_{1}\left(B_{1}^{2}-3 B_{2}^{2}\right)}} .
$$

For $B_{2}=0$, the above expression is reduced to

$$
\left|1-2\left(2 B_{1}-C_{1}\right) y\right|^{\frac{5 B_{1}}{C_{1}-2 B_{2}}}=\left|1-2\left(A_{1}-C_{1}\right) y\right|^{-\frac{2 A_{1}+B_{1}}{A_{1}-C_{1}}} \quad\left(\text { due to } A_{1}=2 B_{1}\right)
$$

which is the integrating factor for the $Q_{3}^{R}$ system, as shown in (27).

Now we return to system (6). Among the four classifications, the Hamiltonian system $\left(Q_{3}^{H}\right)$ has been completely studied in [Horozov \& Iliev, 1994; Gavrilov, 2001]: the system can have maximal two limit cycles. In this paper, we will concentrate on the $Q_{3}^{R}$ - reversible case. Special cases for the reversible system have been investigated by a number of authors (e.g. see [Dumortier et al., 1997; Peng, 2002; Yu \& Li, 2002; Iliev et al., 2005; Li \& Llibre, 2010]). It is easy to see that system (7) is invariant under the mapping $(t, y) \rightarrow(-t,-y)$, where $a_{1}$ and $a_{4}$ can be considered as perturbation parameters. The singular point $(1,0)$ of $(7)$ is a center when $a_{1}<-1$; but a saddle point when $a_{1}>-1 . a_{1}=-1$ gives a degenerate singular point at $(1,0)$. Further, it is easy to verify that when $\left(a_{1}+1\right) a_{4}>0$, there are no more singularity; while when $\left(a_{1}+1\right) a_{4}<0$, there exist additional two saddle points, given by

$$
\left(x^{*}, y^{*}\right)=\left(-\frac{1}{a_{1}}, \pm \frac{\sqrt{-a_{4}\left(a_{1}+1\right)}}{a_{1} a_{4}}\right)
$$

$a_{4}=0$ is a critical value, yielding the two additional saddle points at infinity: $\left(x^{*}, y^{*}\right)=\left(-\frac{1}{a_{1}}, \pm \infty\right)$. In summary, the distribution of singularity of the reversible system (7) has the following possibility (see Fig. 1, where $1 C+1 S$ stands for one center and one saddle point, similar meaning applies to $2 C, 2 C+2 S$ and $1 C+3 S)$ :

Two centers when $a_{1}<-1$ and $a_{4}<0$;

Two centers and two saddle points

$$
\text { when } a_{1}<-1 \text { and } a_{4}>0 \text {; }
$$

One center and one saddle point

$$
\text { when } a_{1}>-1 \text { and } a_{4}>0 ;
$$

One center and three saddle points

$$
\begin{equation*}
\text { when } a_{1}>-1 \text { and } a_{4}<0 . \tag{42}
\end{equation*}
$$

In this paper, we pay particular attention to $a_{1}<-1, a_{4}<0$, for which system (7) has only two singularities at $(0,0)$ and $(1,0)$, both of them are centers.

By adding quadratic perturbations to system (7) we obtain the following perturbed quadratic system:

$$
\begin{align*}
\frac{d x}{d t}= & y\left(1+a_{1} x\right)+\varepsilon P(x, y) \\
= & y\left(1+a_{1} x\right)+\varepsilon\left(a_{10} x+a_{01} y\right. \\
& \left.+a_{20} x^{2}+a_{11} x y+a_{02} y^{2}\right),  \tag{43}\\
\frac{d y}{d t}= & -x+x^{2}+a_{4} y^{2}+\varepsilon Q(x, y) \\
= & -x+x^{2}+a_{4} y^{2}+\varepsilon\left(b_{10} x+b_{01} y\right. \\
& \left.+b_{20} x^{2}+b_{11} x y+b_{02} y^{2}\right),
\end{align*}
$$

where $0<\varepsilon \ll 1, a_{i j}$ 's and $b_{i j}$ 's are perturbation parameters.


Fig. 1. Case studies for the $Q_{3}^{R}$ reversible system.
Remark 2.3. The special system considered by Dumortier et al. [1997] is system (4) with

$$
a=-3, \quad c=-2, \quad b=1 .
$$

This is equivalent to our system when $a_{1}=-2$ and $a_{4}=-3$ for which the system has only two elementary centers at $(0,0)$ and $(1,0)$. Consider the $a_{1}-a_{4}$ parameter plane, as shown in Fig. 1. It can be seen that this case is just a point, $\left(a_{1}, a_{4}\right)=(-2,-3)$, in the parameter plane, marked by a blank circle in the third quadrant on the line $a_{4}=\frac{3}{2} a_{1}$ (see Fig. 1).

The special system studied by Peng [2002] is system (4) with

$$
a=-3, \quad c=-2, \quad b=-1 .
$$

This is equivalent to our system when $a_{1}=2$ and $a_{4}=3$, for which the system has one center at $(0,0)$ and one saddle point at $(1,0)$. Thus, this case is again a point, $\left(a_{1}, a_{4}\right)=(2,3)$, in the $a_{1}-a_{4}$ parameter plane, marked by another blank circle in the first quadrant on the line $a_{4}=\frac{3}{2} a_{1}$ (see Fig. 1).

The cases considered in [Yu \& Li, 2002; Iliev et al., 2005] correspond to system (4) with $a=-3$, $c=-2$, and $b \in(-\infty,-1) \cup(-1,0)$ in $[Y u \& L i$, 2002], and $b \in(0,2)$ in [Iliev et al., 2005].

When $\varepsilon=0$ in system (4), one can use the following transformation:

$$
x=\frac{\tilde{y}}{b}, \quad y=\frac{\tilde{x}}{b},
$$

to transform system (4) $)_{\varepsilon=0}$ to

$$
\begin{aligned}
& \frac{d \tilde{x}}{d t}=\tilde{y}\left(1+\frac{c}{b} \tilde{x}\right), \\
& \frac{d \tilde{y}}{d t}=-\tilde{x}+\tilde{x}^{2}+\frac{a}{b} \tilde{y}^{2},
\end{aligned}
$$

which is our system (7) with

$$
\begin{equation*}
a_{1}=\frac{c}{b}, \quad a_{4}=\frac{a}{b} . \tag{45}
\end{equation*}
$$

Equation (45) yields

$$
\begin{equation*}
a_{4}=\frac{a}{c} a_{1} \quad(b \neq 0) \tag{46}
\end{equation*}
$$

which represents a line in the $a_{1}-a_{4}$ parameter plane, passing through the origin with the slope $\frac{a}{c}$. In particular, the parameter values: $a=-3, c=-2$, $b \in(-\infty,-1) \cup(-1,0) \cup(0,2)$, yielding $a_{1}=-\frac{2}{b}$ and $a_{4}=-\frac{3}{b}$, correspond to a part of the line, described by

$$
\begin{equation*}
a_{4}=\frac{3}{2} a_{1} \quad \forall a_{1} \in(-\infty,-1) \cup(0, \infty), \tag{47}
\end{equation*}
$$

as shown in Fig. 1, where the dotted line for $a_{1} \in$ $[-1,0]$ is excluded from the study in [Yu \& Li, 2002; Iliev et al., 2005].

It should be noted that when $a=-3, c=-2$, the point $\left(0, \frac{1}{b}\right)$ is a saddle point if and only if

$$
1+\frac{c}{b}=1-\frac{2}{b}>0 \Rightarrow b \in(-\infty, 0) \cup(2,+\infty)
$$

Thus, the case considered in [Yu \& Li, 2002] has one center and one saddle point; while the case studied in [Iliev et al., 2005] has two elementary centers. But even these two studies together do not cover the whole line $a_{4}=\frac{3}{2} a_{1}$ (the missing part is denoted by a dotted line segment in Fig. 1).

Another alternative form for a special case of our system (7) considered by Han [1997] is described by

$$
\begin{align*}
& \frac{d \bar{x}}{d t}=\bar{y}\left[1+2(1-e)\left(\bar{x}+\frac{1}{d}\right)\right]  \tag{48}\\
& \frac{d \bar{y}}{d t}=\bar{x}+d \bar{x}^{2}+e \bar{y}^{2}
\end{align*}
$$

where $e$ and $d(\neq 0)$ are parameters. This system has a saddle point at the origin and a center at $(\bar{x}, \bar{y})=\left(-\frac{1}{d}, 0\right)$. Based on the two parameters, seven cases are classified [Han, 1997]. We can apply the following transformation:

$$
\bar{x}=\frac{1}{d}(x-1), \quad \bar{y}=\frac{1}{d} y,
$$

to system (48), yielding

$$
\frac{d x}{d t}=y\left[1+\frac{2(1-e)}{d} x\right]
$$

$$
\begin{equation*}
\frac{d y}{d t}=-x+x^{2}+\frac{e}{d} y^{2} \tag{49}
\end{equation*}
$$

which has a center at the origin and a saddle point at $(1,0)$. Then, setting

$$
\begin{equation*}
a_{1}=\frac{2(1-e)}{d}, \quad a_{4}=\frac{e}{d}, \tag{50}
\end{equation*}
$$

in system (49) leads to our system (7). Equation (50) denotes a line, given by

$$
\begin{equation*}
a_{4}=\frac{e}{2(1-e)} a_{1} \tag{51}
\end{equation*}
$$

in the $a_{1}-a_{4}$ parameter plane, passing through the origin with the slope $\frac{e}{2(1-e)}$. However, it is easy to see that using our system (7) in analysis is simpler to using system (48). In fact, all the seven cases classified in [Han, 1997] together denote a region in Fig. 1, see the shaded area in this figure. This area covers most of the region, defined by $a_{1}>-1$. But the study given in [Han, 1997] for the seven cases is restricted to local analysis on the bifurcation of limit cycles near a homoclinic loop, except the two lines (see Fig. 1):

$$
\begin{equation*}
a_{4}=a_{1} \quad \forall a_{1} \in(-1,0) \cup(0, \infty) \tag{52}
\end{equation*}
$$

which corresponds to the parameter value $e=\frac{2}{3}$, and

$$
\begin{equation*}
a_{4}=-\frac{1}{2} a_{1} \quad \forall a_{1} \in(0, \infty), \tag{53}
\end{equation*}
$$

which corresponds to $e \rightarrow \pm \infty$. It has been shown in [Han, 1997] that except the above two lines, for the parameter values in the shaded area, system (48) can have at most 2 limit cycles near a homoclinic loop under quadratic perturbation.

Figure 1 shows the $a_{1}-a_{4}$ parameter plane associated with the reversible system (7), where the above mentioned case studies are indicated on the line $a_{4}=\frac{3}{2} a_{1}$ as well as in the shaded area. More precisely, a complete global analysis given in [Yu \& Li, 2002], which includes the result in [Dumortier et al., 1997] as a special case, shows that corresponding to each point on the line segment $a_{4}=\frac{3}{2} a_{1}$ ( $a_{1}>0$ ), the system has one center and one saddle point, and has maximal 2 limit cycles. In [Han, 1997] it is shown for each point in the shaded area [except the two line segments $a_{4}=a_{1}\left(a_{1}>-1\right)$ and $a_{4}=-\frac{1}{2} a_{1}\left(a_{1}>0\right)$ ], which contains the above line segment, the system has one center and
one (or three) saddle(s), and has maximal 2 limit cycles, but restricted to local analysis near one homoclinic loop. Similarly, a global analysis given in [Iliev et al., 2005], which contains the result in [Dumortier et al., 1997] as a special case, proves that corresponding to each point on the line segment $a_{4}=\frac{3}{2} a_{1}\left(a_{1}<-1\right)$, the system has two centers, and exhibits maximal 3 limit cycles around one center. The technique of Poincaré transformation and Picar-Puchs equation, used for the above mentioned global analysis on parameter unfolding, seems not possible to be generalized to consider the general situation for arbitrary points in the $a_{1}-a_{4}$ parameter plane. The two particular dash-dotted lines: $a_{4}=\frac{1}{3}\left(a_{1}-5\right) \forall a_{1} \in(-\infty,-1) \cup(-1, \infty)$, and $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right) \forall a_{1} \in(-\infty,-1)$, as well as the five dark circles correspond to our results, presented in the next two sections. In particular, we will show that there exist 3 small limit cycles on the two dash-dotted lines, and at least 4 limit cycles for the parameter values marked by the five dark circles.

In the following, we will use the perturbed quadratic system (43) for our study on bifurcation of limit cycles. First, we need the following lemma, which will greatly simplify the analysis.

Lemma 1. The perturbed quadratic reversible system (43) can have three independent perturbation parameters.

Proof. First, note that the integrating factor for the unperturbed reversible system (7) is $\gamma=\mid 1+$ $\left.a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}$. Thus, let $t=\gamma \tau$. Then system (7) can be transformed to

$$
\begin{align*}
& \frac{d x}{d \tau}=\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left(y+a_{1} x y\right)  \tag{54}\\
& \frac{d y}{d \tau}=\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left(-x+x^{2}+a_{4} y^{2}\right)
\end{align*}
$$

which has the Hamiltonian function:

$$
\begin{align*}
H(x, y)= & \frac{1}{2} \operatorname{sign}\left(1+a_{1} x\right)\left|1+a_{1} x\right|^{-\frac{2 a_{4}}{a_{1}}} \\
& \times\left[y^{2}+\frac{\left(1+a_{1}-a_{4}\right)\left(1+2 a_{4} x\right)}{a_{4}\left(a_{1}-a_{4}\right)\left(a_{1}-2 a_{4}\right)}\right. \\
& \left.-\frac{x^{2}}{a_{1}-a_{4}}\right] . \tag{55}
\end{align*}
$$

Then, the Melnikov function of system (43) along a loop defined by $L_{h}: H(x, y)=h$, can be expressed as

$$
\begin{align*}
M(h, & \left.a_{1}, a_{4}, a_{i j}, b_{i j}\right) \\
= & \oint_{L_{h}} \gamma Q\left(x, y, b_{i j}\right) d x-\gamma P\left(x, y, a_{i j}\right) d y \\
= & \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left(b_{01}+b_{11} x\right) y d x-\oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left(a_{10} x+a_{20} x^{2}+a_{02} y^{2}\right) d y \\
= & \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left[\left(b_{01}+b_{11} x\right)+a_{10}+2 a_{20} x-\frac{a_{1}+2 a_{4}}{\left(1+a_{1} x\right)}\left(a_{10} x+a_{20} x^{2}\right)\right] y d x \\
& -a_{02} \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}} y^{2} d y \\
= & \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left[\left(a_{10}+b_{01}\right)+\left(b_{11}+2 a_{20}\right) x-\frac{a_{1}+2 a_{4}}{\left(1+a_{1} x\right)}\left(a_{10} x+a_{20} x^{2}\right)\right] y d x \\
& -\frac{1}{3}\left(a_{1}+2 a_{4}\right) a_{02} \oint_{L_{h}} \frac{\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}}{1+a_{1} x} y^{3} d x . \tag{56}
\end{align*}
$$

Note that

$$
\begin{equation*}
a_{4} \oint_{L_{h}} \frac{\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}}{1+a_{1} x} y^{3} d x=\frac{3 a_{4}}{a_{1}+2 a_{4}} \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}} y^{2} d y . \tag{57}
\end{equation*}
$$

Further, it follows from Eq. (7) that

$$
\begin{equation*}
\left(-x+x^{2}+a_{4} y^{2}\right) d x=\left(1+a_{1} x\right) y d y \tag{58}
\end{equation*}
$$

which is multiplied by $\frac{\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}}{1+a_{1} x} y$ on both sides and then the resulting equation is integrated along $L_{h}$ to yield

$$
\oint_{L_{h}} \frac{\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}}{1+a_{1} x}\left(-x+x^{2}+a_{4} y^{2}\right) y d x=\oint_{L_{h}} \frac{\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}}{1+a_{1} x}\left(1+a_{1} x\right) y^{2} d y .
$$

Combining the above equation with (57) we obtain

$$
\begin{equation*}
\oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}} y^{2} d y=\frac{a_{1}+2 a_{4}}{a_{1}-a_{4}} \oint_{L_{h}} \frac{\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}}{1+a_{1} x}\left(-x+x^{2}\right) y d x . \tag{59}
\end{equation*}
$$

Substituting the above result into (56) yields

$$
\begin{aligned}
& M\left(h, a_{1}, a_{4}, a_{10}, b_{01}, b_{11}, a_{20}, a_{02}\right) \\
& \quad=\oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left[\left(a_{10}+b_{01}\right)+\left(b_{11}+2 a_{20}\right) x-\frac{a_{1}+2 a_{4}}{\left(1+a_{1} x\right)}\left(a_{10} x+a_{20} x^{2}\right)\right] y d x \\
& \quad-\frac{a_{1}+2 a_{4}}{a_{1}-a_{4}} a_{02} \oint_{L_{h}} \frac{\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}}{1+a_{1} x}\left(-x+x^{2}\right) y d x .
\end{aligned}
$$

$$
\begin{align*}
= & \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left\{\left(a_{10}+b_{01}\right)+\left(b_{11}+2 a_{20}\right) x\right. \\
& \left.-\frac{a_{1}+2 a_{4}}{\left(a_{1}-a_{4}\right)\left(1+a_{1} x\right)}\left[\left(a_{1}-a_{4}\right)\left(a_{10} x+a_{20} x^{2}\right)+a_{02}\left(-x+x^{2}\right)\right]\right\} y d x \tag{60}
\end{align*}
$$

Next, rewriting the term in the square bracket of (60) gives

$$
\begin{align*}
M(h, & \left.a_{1}, a_{4}, a_{10}, a_{20}, a_{02}, b_{01}, b_{11}\right) \\
= & \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left\{\left(a_{10}+b_{01}\right)+\left(b_{11}+2 a_{20}\right) x-\frac{a_{1}+2 a_{4}}{\left(a_{1}-a_{4}\right)\left(1+a_{1} x\right)}\right. \\
& \times\left[\frac{1}{a_{1}}\left[\left(a_{1}-a_{4}\right) a_{20}+a_{02}\right]\left(1+a_{1} x\right) x+\frac{1}{a_{1}^{2}}\left[\left(a_{1}-a_{4}\right)\left(a_{1} a_{10}-a_{20}\right)\right.\right. \\
& \left.\left.\left.-\left(1+a_{1}\right) a_{02}\right]\left(1+a_{1} x\right)-\frac{1}{a_{1}^{2}}\left[\left(a_{1}-a_{4}\right)\left(a_{1} a_{10}-a_{20}\right)-\left(1+a_{1}\right) a_{02}\right]\right]\right\} \\
= & \oint_{L_{h}}\left|1+a_{1} x\right|^{-\frac{a_{1}+2 a_{4}}{a_{1}}}\left\{\frac{a_{1}+2 a_{4}}{a_{1}^{2}\left(a_{1}-a_{4}\right)}\left[\left(a_{1}-a_{4}\right)\left(a_{1} a_{10}-a_{20}\right)-\left(1+a_{1}\right) a_{02}\right] \frac{1}{1+a_{1} x}\right. \\
& +\left(b_{01}+a_{10}-\frac{a_{1}+2 a_{4}}{a_{1}^{2}\left(a_{1}-a_{4}\right)}\left[\left(a_{1}-a_{4}\right)\left(a_{1} a_{10}-a_{20}\right)-\left(1+a_{1}\right) a_{02}\right]\right. \\
& \left.-\frac{1}{a_{1}}\left[b_{11}+2 a_{20}-\frac{a_{1}+2 a_{4}}{a_{1}\left(a_{1}-a_{4}\right)}\left[\left(a_{1}-a_{4}\right) a_{20}+a_{02}\right]\right]\right) \\
& \left.+\frac{1}{a_{1}}\left(b_{11}+2 a_{20}-\frac{a_{1}+2 a_{4}}{a_{1}\left(a_{1}-a_{4}\right)}\left[\left(a_{1}-a_{4}\right) a_{20}+a_{02}\right]\right)\left(1+a_{1} x\right)\right\} \\
= & c_{0} I_{0}+c_{1} I_{1}+c_{2} I_{2}, \tag{61}
\end{align*}
$$

where

$$
\begin{align*}
& I_{0}=\oint_{L_{h}} \operatorname{sign}\left(1+a_{1} x\right)\left(1+a_{1} x\right)^{-\frac{2\left(a_{1}+a_{4}\right)}{a_{1}}} y d x \\
& I_{1}=\oint_{L_{h}} \operatorname{sign}\left(1+a_{1} x\right)\left(1+a_{1} x\right)^{-\frac{2\left(a_{1}+a_{4}\right)}{a_{1}}} y\left(1+a_{1} x\right) d x  \tag{62}\\
& I_{2}=\oint_{L_{h}} \operatorname{sign}\left(1+a_{1} x\right)\left(1+a_{1} x\right)^{-\frac{2\left(a_{1}+a_{4}\right)}{a_{1}}} y\left(1+a_{1} x\right)^{2} d x
\end{align*}
$$

and

$$
\begin{align*}
& c_{0}=\frac{a_{1}+2 a_{4}}{a_{1}} a_{10}-\frac{a_{1}+2 a_{4}}{a_{1}^{2}} a_{20}-\frac{\left(1+a_{1}\right)\left(a_{1}+2 a_{4}\right)}{a_{1}^{2}\left(a_{1}-a_{4}\right)} a_{02}  \tag{63}\\
& c_{1}=b_{01}-\frac{2 a_{4}}{a_{1}} a_{10}-\frac{1}{a_{1}} b_{11}+\frac{4 a_{4}}{a_{1}^{2}} a_{20}+\frac{\left(2+a_{1}\right)\left(a_{1}+2 a_{4}\right)}{a_{1}^{2}\left(a_{1}-a_{4}\right)} a_{02}  \tag{64}\\
& c_{2}=\frac{1}{a_{1}} b_{11}+\frac{a_{1}-2 a_{4}}{a_{1}^{2}} a_{20}+\frac{a_{1}+2 a_{4}}{a_{1}^{2}\left(a_{1}-a_{4}\right)} a_{02} \tag{65}
\end{align*}
$$

It is obvious that the expression in (61) contains only three independent perturbation parameters, though the parameters $a_{1}$ and $a_{4}$ are involved in $I_{i}, i=0,1,2$. Thus, we may let two of them equal to zero. For example, letting $a_{20}=a_{02}=0$ yields

$$
\begin{align*}
& c_{0}=\left(1+\frac{2 a_{4}}{a_{1}}\right) a_{10} \\
& c_{1}=b_{01}-\frac{2 a_{4} a_{10}+b_{11}}{a_{1}}  \tag{66}\\
& c_{2}=\frac{1}{a_{1}} b_{11}
\end{align*}
$$

which indeed shows that $a_{10}, b_{01}$ and $b_{11}$ can be used as the three independent perturbation parameters.

Thus, without loss of generality, we may assume that $a_{01}=a_{20}=a_{11}=a_{02}=b_{10}=b_{20}=b_{02}=0$, under which system (43) is reduced to

$$
\begin{align*}
& \frac{d x}{d t}=y\left(1+a_{1} x\right)+\varepsilon a_{10} x  \tag{67}\\
& \frac{d y}{d t}=-x+x^{2}+a_{4} y^{2}+\varepsilon\left(b_{01} y+b_{11} x y\right)
\end{align*}
$$

where $a_{1}<-1$ and $0<\varepsilon \ll 1$.

## 3. Hopf Bifurcation Associated with the Two Centers

In this section, we study Hopf bifurcation of system (67) from two centers $(0,0)$ and $(1,0)$, leading to the bifurcation of multiple limit cycles. The result is summarized in the following theorem.

Theorem 2. When $a_{1}<-1$, the quadratic near-integrable system (67) can have small limit cycles bifurcating from the two centers $(0,0)$ and $(1,0)$ with distributions: $(3,0),(0,3),(2,0),(0,2)$ and $(1,1)$. (2,1)- or (1,2)-distribution does not exist.

Proof. Consider system (67) for $a_{1}<-1$. The system $(67)_{\varepsilon=0}$ is a reversible integrable system. In order to compute the Melnikov function near the two centers $(0,0)$ and $(1,0)$, we multiply ( 67 ) by the integrating factor $\gamma$ [given in (14)] to obtain the following perturbed Hamiltonian system:

$$
\frac{d x}{d \tau}=\gamma\left(y+a_{1} x y\right)+\varepsilon \gamma a_{10} x
$$

$$
\begin{equation*}
\frac{d y}{d \tau}=\gamma\left(-x+x^{2}+a_{4} y^{2}\right)+\varepsilon \gamma\left(b_{01} y+b_{11} x y\right) \tag{68}
\end{equation*}
$$

with the Hamiltonian of the unperturbed system (7) (i.e. $(68)_{\varepsilon=0}$ ), given by (55), with $a_{4} \neq 0, a_{1} \neq a_{4}$, $a_{1} \neq 2 a_{4}$. The cases $a_{4}=0, a_{1}=a_{4}$ or $a_{1}=2 a_{4}$ will not be considered in this paper.

Note that

$$
\begin{align*}
& h_{00}=H(0,0) \\
&=\frac{1+a_{1}-a_{4}}{2 a_{4}\left(a_{1}-a_{4}\right)\left(a_{1}-2 a_{4}\right)}, \\
& \quad \text { for } 1+a_{1} x>0, \\
& h_{10}=H(1,0)  \tag{69}\\
&=-\frac{\left(a_{1}+1\right)\left(a_{4}+1\right)}{2 a_{4}\left(a_{1}-a_{4}\right)\left(a_{1}-2 a_{4}\right)}\left(-1-a_{1}\right)^{-\frac{2 a_{4}}{a_{1}}}, \\
& \quad \text { for } 1+a_{1} x<0 .
\end{align*}
$$

Since in this paper, we concentrate on the case that system $(67)_{\varepsilon=0}$ has only two centers, we assume $a_{1}<-1, a_{4}<0$. Thus,

$$
\begin{aligned}
& \lim _{x \rightarrow-{\frac{1}{a_{1}}}^{-}} H(x, y)=+\infty \quad \text { and } \\
& \lim _{x \rightarrow-{\frac{1}{a_{1}}}^{+}} H(x, y)=-\infty
\end{aligned}
$$

It is easy to see from system (67) that the trajectories of $(67)_{\varepsilon=0}$ rotate around the center $(0,0)$ in the clockwise direction, while rotating around the center $(1,0)$ in the counter clockwise direction, as shown in Fig. 2. Thus, the values of $h$ in $H(x, y)=h$


Fig. 2. A phase portrait of the reversible system (7) with two centers for $a_{1}=-3, a_{4}=-\frac{8}{3}$.
are taken from the two intervals: $h \in\left(h_{00}, \infty\right)$ for $1+a_{1} x>0$, and $h \in\left(-\infty, h_{10}\right)$ for $1+a_{1} x<0$. It should be noted that $h_{00}$ is not necessarily larger than $h_{10}$. The analyses on the two half-plane in the $x-y$ plane (see Fig. 2), divided by the singular line $1+a_{1} x=0$, are independent.

Next, introduce

$$
\begin{align*}
L_{h}: & H(x, y) \\
& =h \begin{cases}h \in\left(h_{00}, \infty\right), & \text { for } 1+a_{1} x>0, \\
h \in\left(-\infty, h_{10}\right), & \text { for } 1+a_{1} x<0,\end{cases} \tag{70}
\end{align*}
$$

and define the Melnikov function:

$$
\begin{align*}
& M\left(h, a_{i j}, b_{i j}\right) \\
& \quad=\oint_{L_{h}} q\left(x, y, b_{i j}\right) d x-p\left(x, y, a_{i j}\right) d y \tag{71}
\end{align*}
$$

where $p\left(x, y, a_{i j}\right)=\gamma a_{10} x$ and $q\left(x, y, b_{i j}\right)=\gamma\left(b_{01}+\right.$ $\left.b_{11} x\right) y$. Using the results in [Han, 2000, 2006; Han \& Chen, 2000], we can expand $M$ near $h=h_{00}$ and $h=h_{10}$ as

$$
\begin{aligned}
& M_{0}\left(h, a_{i j}, b_{i j}\right) \\
& \quad=\mu_{00}\left(h-h_{00}\right)+\mu_{01}\left(h-h_{00}\right)^{2}+\mu_{02}\left(h-h_{00}\right)^{3} \\
& \quad+\mu_{03}\left(h-h_{00}\right)^{4}+O\left(\left(h-h_{00}\right)^{5}\right), \\
& \quad \text { for } 0<h-h_{00} \ll 1, \\
& M_{1}\left(h, a_{i j}, b_{i j}\right) \\
& =\mu_{10}\left(h_{10}-h\right)+\mu_{11}\left(h_{10}-h\right)^{2}+\mu_{12}\left(h_{10}-h\right)^{3} \\
& \quad+\mu_{13}\left(h_{10}-h\right)^{4}+O\left(\left(h_{10}-h\right)^{5}\right),
\end{aligned}
$$

$$
\begin{equation*}
\text { for } 0<h_{10}-h \ll 1 \tag{72}
\end{equation*}
$$

where the coefficients $\mu_{i j}, i=0,1 ; j=0,1,2, \ldots$ can be obtained by using the Maple programs developed in [Han et al., 2009] as follows:

$$
\begin{aligned}
\mu_{00}= & 2 \pi\left(a_{10}+b_{01}\right), \\
\mu_{01}= & \frac{\pi}{12}\left[\left(10-13 a_{1}-14 a_{4}+13 a_{1}^{2}+7 a_{1} a_{4}-20 a_{4}^{2}\right) a_{10}+\left(10-a_{1}+10 a_{4}+a_{1}^{2}\right.\right. \\
& \left.\left.-5 a_{1} a_{4}+4 a_{4}^{2}\right) b_{01}+12\left(1+a_{4}\right) b_{11}\right], \\
\mu_{02}= & \frac{\pi}{864}\left[\left(1540-980 a_{1}-280 a_{4}+861 a_{1}^{2}-1512 a_{1} a_{4}-3948 a_{4}^{2}-626 a_{1}^{3}+1566 a_{1}^{2} a_{4}+1620 a_{1} a_{4}^{2}\right.\right. \\
& \left.-4432 a_{4}^{3}+313 a_{1}^{4}-1018 a_{1}^{3} a_{4}-279 a_{1}^{2} a_{4}^{2}+3080 a_{1} a_{4}^{3}-2096 a_{4}^{4}\right) a_{10}+\left(1540+700 a_{1}+3080 a_{4}\right. \\
& +21 a_{1}^{2}+168 a_{1} a_{4}+2772 a_{4}^{2}-2 a_{1}^{3}+126 a_{1}^{2} a_{4}-828 a_{1} a_{4}^{2}+1424 a_{4}^{3}+a_{1}^{4}-58 a_{1}^{3} a_{4}+369 a_{1}^{2} a_{4}^{2} \\
& \left.\left.-712 a_{1} a_{4}^{3}+400 a_{4}^{4}\right) b_{01}+24 b_{11}\left(1+a_{4}\right)\left(70+35 a_{1}+70 a_{4}+a_{1}^{2}-17 a_{1} a_{4}+52 a_{4}^{2}\right) b_{11}\right], \\
\mu_{03}= & \frac{\pi}{622080}\left[\left(3403400-300300 a_{1}+3003000 a_{4}+690690 a_{1}^{2}-4984980 a_{1} a_{4}-7327320 a_{4}^{2}\right.\right. \\
& -500885 a_{1}^{3}+3314850 a_{1}^{2} a_{4}-4430580 a_{1} a_{4}^{2}-17811640 a_{4}^{3}+323121 a_{1}^{4}-2444439 a_{1}^{3} a_{4} \\
& +4201218 a_{1}^{2} a_{4}^{2}+5794692 a_{1} a_{4}^{3}-18033936 a_{4}^{4}-168603 a_{1}^{5}+1420500 a_{1}^{4} a_{4}-3253551 a_{1}^{3} a_{4}^{2} \\
& -1296282 a_{1}^{2} a_{4}^{3}+12107904 a_{1} a_{4}^{4}-10462368 a_{4}^{5}+56201 a_{1}^{6}-520311 a_{1}^{5} a_{4}+1471287 a_{1}^{4} a_{4}^{2} \\
& \left.-407053 a_{1}^{3} a_{4}^{3}-4589772 a_{1}^{2} a_{4}^{4}+7149264 a_{1} a_{4}^{5}-3159616 a_{4}^{6}\right) a_{10}+\left(3403400+3303300 a_{1}\right. \\
& +10210200 a_{4}+690690 a_{1}^{2}+5825820 a_{1} a_{4}+14294280 a_{4}^{2}+11935 a_{1}^{3}+404250 a_{1}^{2} a_{4} \\
& +2721180 a_{1} a_{4}^{2}+12236840 a_{4}^{3}-699 a_{1}^{4}-11379 a_{1}^{3} a_{4}+262458 a_{1}^{2} a_{4}^{2}-1891308 a_{1} a_{4}^{3} \\
& +6994704 a_{4}^{4}+417 a_{1}^{5}+1380 a_{1}^{4} a_{4}-149091 a_{1}^{3} a_{4}^{2}+1121838 a_{1}^{2} a_{4}^{3}-2964576 a_{1} a_{4}^{4} \\
& +2670432 a_{4}^{5}-139 a_{1}^{6}-291 a_{1}^{5} a_{4}+46227 a_{1}^{4} a_{4}^{2}-366193 a_{1}^{3} a_{4}^{3}+1076988 a_{1}^{2} a_{4}^{4} \\
& \left.-1335216 a_{1} a_{4}^{5}+578624 a_{4}^{6}\right) b_{01}+\left(3603600+3603600 a_{1}+10810800 a_{4}+790020 a_{1}^{2}\right.
\end{aligned}
$$

$$
\begin{aligned}
& +6597360 a_{1} a_{4}+15024240 a_{4}^{2}+12600 a_{1}^{3}+480060 a_{1}^{2} a_{4}+3764880 a_{1} a_{4}^{2}+12514320 a_{4}^{3} \\
& +180 a_{1}^{4}-10800 a_{1}^{3} a_{4}+11340 a_{1}^{2} a_{4}^{2}-618480 a_{1} a_{4}^{3}+6566400 a_{4}^{4}+180 a_{1}^{4} a_{4}-23400 a_{1}^{3} a_{4}^{2} \\
& \left.\left.+321300 a_{1}^{2} a_{4}^{3}-1389600 a_{1} a_{4}^{4}+1869120 a_{4}^{5}\right) b_{11}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
& \mu_{10}=2 \pi\left(-1-a_{1}\right)^{3 / 2}\left[\left(1-2 a_{4}\right) a_{10}+\left(1+a_{1}\right)\left(b_{01}+b_{11}\right)\right], \\
& \mu_{11}=\frac{\pi}{12}\left(-1-a_{1}\right)^{-\frac{2\left(a_{1}-a_{4}\right)}{a_{1}}}\left[\left(10+33 a_{1}-6 a_{4}+36 a_{1}^{2}-21 a_{1} a_{4}-24 a_{1}^{2} a_{4}+30 a_{1} a_{4}^{2}-8 a_{4}^{3}\right) a_{10}\right. \\
& \left.+\left(1+a_{1}\right)\left(10+21 a_{1}-10 a_{4}+12 a_{1}^{2}-15 a_{1} a_{4}+4 a_{4}^{2}\right) b_{01}-\left(1+a_{1}\right)\left(1+a_{4}\right)\left(2+3 a_{1}-4 a_{4}\right) b_{11}\right], \\
& \mu_{12}=\frac{\pi}{864}\left(-1-a_{1}\right)^{-\frac{\left(5 a_{1}-8 a_{4}\right)}{2 a_{1}}}\left[\left(1540+7140 a_{1}-2800 a_{4}+13041 a_{1}^{2}-11592 a_{1} a_{4}+2212 a_{4}^{2}+11448 a_{1}^{3}\right.\right. \\
& -18072 a_{1}^{2} a_{4}+8628 a_{1} a_{4}^{2}-1112 a_{4}^{3}+752 a_{4}^{4}-12024 a_{1}^{3} a_{4}+12213 a_{1}^{2} a_{4}^{2}-5232 a_{1} a_{4}^{3} \\
& \left.+4320 a_{1}^{4}-1728 a_{1}^{4} a_{4}+6192 a_{1}^{3} a_{4}^{2}-7938 a_{1}^{2} a_{4}^{3}+4272 a_{1} a_{4}^{4}-800 a_{4}^{5}\right) a_{10} \\
& +\left(1+a_{1}\right)\left(1540+5460 a_{1}-3080 a_{4}+7161 a_{1}^{2}-9072 a_{1} a_{4}+2772 a_{4}^{2}+4104 a_{1}^{3}\right. \\
& \left.-9030 a_{1}^{2} a_{4}+6372 a_{1} a_{4}^{2}-1424 a_{4}^{3}+864 a_{1}^{4}-3096 a_{1}^{3} a_{4}+3969 a_{1}^{2} a_{4}^{2}-2136 a_{1} a_{4}^{3}+400 a_{4}^{4}\right) b_{01} \\
& -\left(1+a_{1}\right)\left(1+a_{4}\right)\left(140+420 a_{1}-420 a_{4}+423 a_{1}^{2}-996 a_{1} a_{4}+576 a_{4}^{2}\right. \\
& \left.\left.+144 a_{1}^{3}-633 a_{1}^{2} a_{4}+888 a_{1} a_{4}^{2}-400 a_{4}^{3}\right) b_{11}\right], \\
& \mu_{13}=\frac{\pi}{1244160}\left(-1-a_{1}\right)^{-\frac{2\left(a_{1}-3 a_{4}\right)}{a_{1}}}\left[\left(3403400+20720700 a_{1}-9809800 a_{4}+53243190 a_{1}^{2}-54234180 a_{1} a_{4}\right.\right. \\
& +13093080 a_{4}^{2}+74334645 a_{1}^{3}-123735150 a_{1}^{2} a_{4}+65571660 a_{1} a_{4}^{2}-10776920 a_{4}^{3}+60023916 a_{1}^{4} \\
& -147900519 a_{1}^{3} a_{4}+131934978 a_{1}^{2} a_{4}^{2}-49682268 a_{1} a_{4}^{3}+6439744 a_{4}^{4}+27002160 a_{1}^{5} \\
& -95460120 a_{1}^{4} a_{4}+132380865 a_{1}^{3} a_{4}^{2}-89408610 a_{1}^{2} a_{4}^{3}+29027880 a_{1} a_{4}^{4}-3527040 a_{4}^{5} \\
& +5443200 a_{1}^{6}-28946160 a_{1}^{5} a_{4}+63998532 a_{1}^{4} a_{4}^{2}-74879613 a_{1}^{3} a_{4}^{3}+48498336 a_{1}^{2} a_{4}^{4} \\
& -16296336 a_{1} a_{4}^{5}+2181248 a_{4}^{6}-1555200 a_{1}^{6} a_{4}+9603360 a_{1}^{5} a_{4}^{2}-24061752 a_{1}^{4} a_{4}^{3} \\
& \left.+31232358 a_{1}^{3} a_{4}^{4}-22072536 a_{1}^{2} a_{4}^{5}+8011296 a_{1} a_{4}^{6}-1157248 a_{4}^{7}\right) a_{10} \\
& +\left(1+a_{1}\right)\left(3403400+17117100 a_{1}-10210200 a_{4}+35225190 a_{1}^{2}-45225180 a_{1} a_{4}\right. \\
& +14294280 a_{4}^{2}+37785825 a_{1}^{3}-79202970 a_{1}^{2} a_{4}+54455940 a_{1} a_{4}^{2}-12236840 a_{4}^{3} \\
& +22125636 a_{1}^{4}-68371209 a_{1}^{3} a_{4}+77864598 a_{1}^{2} a_{4}^{2}-38601828 a_{1} a_{4}^{3}+6994704 a_{4}^{4} \\
& +6629040 a_{1}^{5}-28984608 a_{1}^{4} a_{4}+49687587 a_{1}^{3} a_{4}^{2}-41614974 a_{1}^{2} a_{4}^{3}+16953984 a_{1} a_{4}^{4} \\
& -2670432 a_{4}^{5}+777600 a_{1}^{6}-4801680 a_{1}^{5} a_{4}+12030876 a_{1}^{4} a_{4}^{2}-15616179 a_{1}^{3} a_{4}^{3} \\
& \left.+11036268 a_{1}^{2} a_{4}^{4}-4005648 a_{1} a_{4}^{5}+578624 a_{4}^{6}\right) b_{01}
\end{aligned}
$$

$$
\begin{aligned}
& -\left(1+a_{1}\right)\left(1+a_{4}\right)\left(200200+900900 a_{1}-800800 a_{4}+1600830 a_{1}^{2}-3132360 a_{1} a_{4}\right. \\
& +1530760 a_{4}^{2}+1397655 a_{1}^{3}-4596480 a_{1}^{2} a_{4}+5008500 a_{1} a_{4}^{2}-1808240 a_{4}^{3}+594864 a_{1}^{4} \\
& -3001266 a_{1}^{3} a_{4}+5594022 a_{1}^{2} a_{4}^{2}-4568112 a_{1} a_{4}^{3}+1379936 a_{4}^{4}+97200 a_{1}^{5}-736776 a_{1}^{4} a_{4} \\
& \left.\left.+2162079 a_{1}^{3} a_{4}^{2}-3080268 a_{1}^{2} a_{4}^{3}+2136528 a_{1} a_{4}^{4}-578624 a_{4}^{5}\right) b_{11}\right]
\end{aligned}
$$

Remark 3.1. The coefficients $\mu_{0 j}$ listed above are applicable as long as $(0,0)$ is a center, and the coefficients $\mu_{1 j}$ are applicable as long as $(1,0)$ is a center, regardless of the number and distribution of the system's singularities. Therefore, for each point on the whole line $a_{4}=\frac{1}{3}\left(a_{1}-5\right)$ (see Fig. 1), there always exist 3 small limit cycles bifurcating from the center $(0,0)$, no matter whether the system has two centers, or one center and three saddle points, or one center and one saddle point. For each point on the line segment $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)\left(a_{1}<-1\right)$, the system can have 3 limit cycles bifurcating from the center $(1,0)$. This indicates that the results given in [Dumortier et al., 1997; Peng, 2002; Han, 1997] showing that the reversible near-integrable systems with one center and one saddle point can have maximal 2 limit cycles is conservative, since on the part of the line $a_{4}=\frac{1}{3}\left(a_{1}-5\right)$ in the first quadrant ( $a_{1}>5$ ) such a system can have at least 3 limit cycles.

First, we consider the maximal number of limit cycles which can bifurcate from the center $(0,0)$. Setting $\mu_{00}=0$ yields

$$
\begin{equation*}
b_{01}=-a_{10}, \tag{73}
\end{equation*}
$$

and then we have
$\mu_{01}=\pi\left[\left(a_{1}-1-a_{4}\right)\left(a_{1}+2 a_{4}\right) a_{10}+\left(1+a_{4}\right) b_{11}\right]$.

In order to have $\mu_{01}=0$, we suppose $a_{4} \neq-1$ and choose

$$
\begin{equation*}
b_{11}=-\frac{\left(a_{1}-1-a_{4}\right)\left(a_{1}+2 a_{4}\right)}{1+a_{4}} a_{10} \tag{75}
\end{equation*}
$$

Then, $\mu_{02}$ and $\mu_{03}$ are simplified to

$$
\begin{aligned}
& \mu_{02}=\frac{\pi}{3} a_{1}\left(a_{1}-a_{4}\right)\left(a_{1}+2 a_{4}\right)\left(a_{1}-3 a_{4}-5\right) a_{10}, \\
& \mu_{03}=-\frac{\pi}{144} a_{1}\left(a_{1}-a_{4}\right)\left(a_{1}+2 a_{4}\right)\left(770+105 a_{1}\right.
\end{aligned}
$$

$$
\begin{align*}
& +1400 a_{4}+42 a_{1}^{2}-434 a_{1} a_{4}+1274 a_{4}^{2} \\
& \left.-13 a_{1}^{3}+128 a_{1}^{2} a_{4}-415 a_{1} a_{4}^{2}+444 a_{4}^{3}\right) a_{10} \tag{76}
\end{align*}
$$

There are five choices for $\mu_{02}=0$. Except the choice $a_{1}-3 a_{4}-5=0$, all other choices lead to $\mu_{0 i}=0, i=3,4, \ldots$ Thus, letting

$$
\begin{equation*}
a_{4}=\frac{1}{3}\left(a_{1}-5\right), \tag{77}
\end{equation*}
$$

which implies $a_{1} \neq 2$ when $a_{4} \neq-1$. Since we assume $a_{1}<-1$, for this case (i.e. when the condition (77) holds), $a_{4} \neq-1$ is guaranteed. Then, we have

$$
\begin{aligned}
& \mu_{03}=-\frac{25 \pi}{162} a_{1}\left(a_{1}+1\right)\left(a_{1}-2\right)^{2}\left(2 a_{1}+5\right) a_{10}, \\
& \mu_{04}=-\frac{5 \pi}{8748} a_{1}\left(a_{1}+1\right)\left(a_{1}-2\right)^{2}\left(2 a_{1}+5\right) \\
& \times\left(a_{1}+4\right)\left(17 a_{1}+518\right) a_{10} \\
& \vdots \\
& \mu_{10}=-\frac{10 \pi}{3}\left(-1-a_{1}\right)^{-3 / 2} a_{1}\left(2 a_{1}+5\right) a_{10} \\
& \mu_{11}= \frac{25 \pi}{324}\left(-1-a_{1}\right)^{-\frac{2\left(2 a_{1}+5\right)}{3 a_{1}}} \\
& \times a_{1}\left(a_{1}-2\right)^{2}\left(2 a_{1}+5\right) a_{10}, \\
& \vdots
\end{aligned}
$$

implying that in addition we need

$$
\begin{equation*}
\left(2 a_{1}+5\right) a_{10} \neq 0 \tag{78}
\end{equation*}
$$

Under the above conditions (73), (75), (77) and (78), we obtain $\mu_{00}=\mu_{01}=\mu_{02}=0$, but $\mu_{03} \neq 0$, $\mu_{10} \neq 0$. Hence, at most 3 small limit cycles can bifurcate from the center $(0,0)$ with no limit cycles bifurcating from the center $(1,0)$. Further, giving proper perturbations to the parameters $a_{4}$ (or $a_{1}$ ),
$b_{11}$ and $b_{01}$, we can obtain 3 small limit cycles bifurcating from the origin. This shows that the conclusion is true for the case of $(3,0)$-distribution.

Next, consider the ( 0,3 )-distribution. Similarly, letting $\mu_{10}=0$ yields

$$
\begin{equation*}
b_{01}=-b_{11}+\frac{2 a_{4}-1}{1+a_{1}} a_{10} . \tag{79}
\end{equation*}
$$

Then, $\mu_{11}$ becomes

$$
\begin{align*}
\mu_{11}= & \pi\left(-1-a_{1}\right)^{-\frac{2\left(a_{1}-a_{4}\right)}{a_{1}}} \\
& \times\left[\left(a_{1}+2 a_{4}\right)\left(2 a_{1}-a_{4}+1\right) a_{10}\right. \\
& \left.-\left(1+a_{1}\right)^{2}\left(a_{1}-a_{4}+1\right) b_{11}\right] . \tag{80}
\end{align*}
$$

Hence, we set

$$
\begin{array}{r}
b_{11}=\frac{\left(a_{1}+2 a_{4}\right)\left(2 a_{1}-a_{4}+1\right)}{\left(1+a_{1}\right)^{2}\left(a_{1}-a_{4}+1\right)} a_{10}, \\
\left(a_{1}-a_{4}+1 \neq 0\right), \tag{81}
\end{array}
$$

to yield $\mu_{11}=0$, and

$$
\begin{align*}
\mu_{12}= & \frac{\pi}{3}\left(-1-a_{1}\right)^{-\frac{5 a_{1}-8 a_{4}}{2 a_{1}}} a_{1}\left(a_{1}-a_{4}\right) \\
& \times\left(a_{1}+2 a_{4}\right)\left(6 a_{1}-3 a_{4}+5\right) a_{10}, \\
\mu_{13}= & \frac{\pi}{288}\left(-1-a_{1}\right)^{-\frac{2\left(a_{1}-3 a_{4}\right)}{a_{1}}} a_{1}\left(a_{1}-a_{4}\right)  \tag{82}\\
& \times\left(a_{1}+2 a_{4}\right)\left(770+2205 a_{1}-1400 a_{4}\right. \\
& +2142 a_{1}^{2}-3234 a_{1} a_{4}+1274 a_{4}^{2}-720 a_{1}^{3} \\
& \left.-1962 a_{1}^{2} a_{4}+1689 a_{1} a_{4}^{2}-444 a_{4}^{3}\right) a_{10} .
\end{align*}
$$

The only choice for $\mu_{12}=0$ is $6 a_{1}-3 a_{4}+5=0$, from which we have

$$
\begin{equation*}
a_{4}=\frac{1}{3}\left(6 a_{1}+5\right) . \tag{83}
\end{equation*}
$$

This implies that $a_{1}-a_{4}+1=-\left(a_{1}+\frac{2}{3}\right)>0$ for $a_{1}<-1$. Further, we obtain

$$
\begin{aligned}
\mu_{13}= & -\frac{25 \pi}{324}\left(-1-a_{1}\right)^{\frac{10+11 a_{1}}{a_{1}}} a_{1}\left(3 a_{1}+2\right)^{2} \\
& \times\left(3 a_{1}+5\right) a_{10}, \\
\mu_{14}= & -\frac{5 \pi}{17496}\left(-1-a_{1}\right)^{\frac{80+87 a_{1}}{6 a_{1}}} a_{1}\left(3 a_{1}+2\right)^{2} \\
& \times\left(3 a_{1}+5\right)\left(3 a_{1}+4\right)\left(501 a_{1}+518\right) a_{10} \\
\vdots &
\end{aligned}
$$

$$
\begin{aligned}
\mu_{00} & =\frac{10 \pi}{3\left(1+a_{1}\right)^{2}} a_{1}\left(3 a_{1}+5\right) a_{10}, \\
\mu_{01} & =-\frac{25 \pi}{324\left(1+a_{1}\right)^{2}} a_{1}\left(3 a_{1}+5\right)\left(3 a_{1}+2\right)^{2} a_{10}, \\
& \vdots
\end{aligned}
$$

implying that in addition we require

$$
\begin{equation*}
\left(3 a_{1}+5\right) a_{10} \neq 0 . \tag{84}
\end{equation*}
$$

Under the above conditions (79), (81), (83) and (84), we have $\mu_{10}=\mu_{11}=\mu_{12}=0$, but $\mu_{13} \neq 0, \mu_{00} \neq 0$. Further, by properly perturbing the parameters $a_{4}$ (or $a_{1}$ ), $b_{11}$ and $b_{01}$, we can obtain 3 small limit cycles bifurcating from the center $(1,0)$, but no limit cycles from the origin. This proves the case of $(0,3)$-distribution.

For the case of ( 2,0 )-distribution, it follows from the conditions (73) and (75), and $a_{4} \neq-1$ that $\mu_{00}=\mu_{01}=0$, and

$$
\begin{aligned}
\mu_{02}= & \frac{\pi}{3} a_{1}\left(a_{1}-a_{4}\right)\left(a_{1}+2 a_{4}\right)\left(a_{1}-3 a_{4}-5\right) a_{10}, \\
\mu_{10}= & -\frac{2 \pi}{\left(1+a_{4}\right)\left(-1-a_{1}\right)^{3 / 2}} a_{1}\left(a_{1}-a_{4}\right) \\
& \times\left(a_{1}+2 a_{4}\right) a_{10} .
\end{aligned}
$$

Thus, $\mu_{02} \neq 0$ implies $\mu_{10} \neq 0$, indicating that the conclusion holds for the case of $(2,0)$-distribution, if $a_{4} \neq-1$,

When $a_{4}=-1,(74)$ becomes

$$
\mu_{01}=\pi a_{1}\left(a_{1}-2\right) a_{10}
$$

$$
\neq 0 \text { for } a_{1}<-1 \text { and } a_{10} \neq 0 .
$$

Under the conditions $b_{01}=-a_{10}$ and $a_{4}=-1, \mu_{10}$ and $\mu_{11}$ becomes
$\mu_{10}=-2 \pi\left(-1-a_{1}\right)^{-3 / 2}\left[\left(a_{1}-2\right) a_{10}-\left(1+a_{1}\right) b_{11}\right]$,
$\mu_{11}=\pi\left(-1-a_{1}\right)^{-\frac{2+a_{1}}{a_{1}}} a_{1}\left(a_{1}-2\right) a_{10}$,
which shows that $\mu_{11} \neq 0$ for $a_{1}<-1$ and $a_{10} \neq 0$. But we can choose

$$
b_{11}=\frac{a_{1}-2}{1+a_{1}} a_{10}
$$

to obtain $\mu_{10}=0$. Thus, for this case we have a ( 1,1 )-distribution.

Similarly, for the ( 0,2 )-distribution, we use the conditions (79) and (81) to obtain

$$
\begin{aligned}
\mu_{12}= & \frac{\pi}{3}\left(-1-a_{1}\right)^{-\frac{5 a_{1}-8 a_{4}}{2 a_{1}}} a_{1}\left(a_{1}-a_{4}\right)\left(a_{1}+2 a_{4}\right) \\
& \times\left(6 a_{1}-3 a_{4}+5\right) a_{10} \\
\mu_{00}= & \frac{2 \pi}{\left(1+a_{1}\right)^{2}\left(a_{1}-a_{4}+1\right)} a_{1}\left(a_{1}-a_{4}\right) \\
& \times\left(a_{1}+2 a_{4}\right) a_{10}
\end{aligned}
$$

This indicates that $\mu_{12} \neq 0$ implies $\mu_{00} \neq 0$, and so the conclusion for the case of $(0,2)$-distribution is also true if $a_{1}-a_{4}+1 \neq 0$.

When $a_{1}-a_{4}+1=0$, i.e. $a_{4}=a_{1}+1<0$, (80) is reduced to

$$
\begin{array}{r}
\mu_{11}=\pi\left(-1-a_{1}\right)^{-\frac{2\left(a_{1}-a_{4}\right)}{a_{1}}} a_{1}\left(3 a_{1}+2\right) a_{10} \neq 0 \\
\text { for } a_{1}<-1 \text { and } a_{10} \neq 0
\end{array}
$$

and $\mu_{00}$ and $\mu_{01}$ become

$$
\begin{align*}
& \mu_{00}=\frac{2 \pi}{1+a_{1}}\left[\left(a_{1}+2\right) a_{10}-\left(1+a_{1}\right) b_{11}\right],  \tag{86}\\
& \mu_{01}=-\frac{\pi}{1+a_{1}} a_{1}\left(3 a_{1}+2\right) a_{10},
\end{align*}
$$

which clearly shows that $\mu_{01} \neq 0$ for $a_{1}<-1$ and $a_{10} \neq 0$. However, we may choose

$$
b_{11}=\frac{a_{1}+2}{1+a_{1}} a_{10}
$$

to obtain $\mu_{00}=0$. Thus, for $a_{1}-a_{4}+1=0$, we have a (1, 1)-distribution.

Finally, suppose the condition given in (73) is satisfied, i.e. $b_{01}=-a_{10}$, then substitute this into $\mu_{10}$ to solve $b_{11}$ to obtain

$$
\begin{equation*}
b_{11}=\frac{a_{1}+2 a_{4}}{1+a_{1}} \tag{87}
\end{equation*}
$$

Then, under the conditions (73) and (87), we obtain

$$
\begin{align*}
\mu_{01}= & \frac{\pi}{1+a_{1}} a_{1}\left(a_{1}-a_{4}\right)\left(a_{1}+2 a_{4}\right) a_{10} \\
\mu_{11}= & -\pi\left(-1-a_{1}\right)^{-\frac{2\left(a_{1}-a_{4}\right)}{a_{1}}} a_{1}\left(a_{1}-a_{4}\right)  \tag{88}\\
& \times\left(a_{1}+2 a_{4}\right) a_{10}
\end{align*}
$$

which shows that $\mu_{01} \neq 0$ implies $\mu_{11} \neq 0$, and thus in general the conclusion is true for the case of ( 1,1 )-distribution.

As we have seen in the above analysis, if the condition (77), $a_{4}=\frac{1}{3}\left(a_{1}-5\right)$, is not used, then we can only have 2 limit cycles bifurcating from the origin, but no limit cycles can occur from the center $(1,0)$. In other words, we can obtain one more limit cycle, by using the condition $a_{4}=\frac{1}{3}\left(a_{1}-5\right)$, only bifurcating from the center $(0,0)$. Similarly, if the condition (83), $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)$, is not used, then we can have only 2 limit cycles bifurcating from the center $(1,0)$, but no limit cycles can bifurcate from the origin. Then, condition $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)$ can be only used to get one more limit cycle around the center $(1,0)$, rather than the origin. Therefore, $(2,1)$ - or $(1,2)$-distribution is not possible.

This completes the proof of Theorem 2.

## 4. Limit Cycles Bifurcating from Closed Orbits

In this section, based on the results of the small limit cycles obtained in the previous section, we wish to investigate the possibility of existence of large limit cycles by applying the Melnikov function, defined in (71). We have the following result.

Theorem 3. For the case of bifurcation of small limit cycles from the two centers $(0,0)$ and $(1,0)$ with (3, 0)-distribution (resp., ( 0,3 )-distribution) there exists at least one large limit cycle near $L_{h}$ for some $h \in\left(-\infty, h_{10}\right)$ (resp. for some $h \in$ $\left.\left(h_{00}, \infty\right)\right)$. For the case of limit cycles with $(2,0)$ distribution (resp., (0,2)-distribution) there exist at least two large limit cycles, one near $L_{h_{1}}$ for some $h_{1} \in\left(-\infty, h_{10}\right)$ and one near $L_{h_{2}}$ for some $h_{2} \in$ $\left(h_{00}, \infty\right)$. The corresponding values of the parameters $a_{1}$ and $a_{4}$ for the existence of 4 limit cycles can appear at least in some regions in the $a_{1}-a_{4}$ parameter plane.

Remark 4.1. Theorem 3 gives a positive answer to the open question of existence of limit cycles in near-integrable quadratic systems: at least 4 limit cycles can exist. For the case of $(1,1)$-distribution, so far no more large limit cycles have been found.

Proof. We use the formulas given in (61) and (66) in the following calculations. Since one cannot find the closed form of the integrals $I_{i}\left(h, a_{1}, a_{4}\right), i=$ $0,1,2$, for general $a_{1}$ and $a_{4}$, nor the technique of Picard-Fuchs equation can be applied here, we shall choose some values for $a_{1}$ and $a_{4}$ and then find numerical values of the integral. We first use the
results given in the previous section to determine $b_{01}, b_{11}$, and $a_{4}$, and then choose proper values for $a_{1}$ to find more limit cycles. We consider four distributions: $(3,0),(0,3),(2,0)$ and $(0,2)$, and each case can have 4 limit cycles.
(A) First, consider the (3, 0)-distribution. For this case, we have

$$
\begin{gathered}
a_{4}=\frac{1}{3}\left(a_{1}-5\right), \quad b_{01}=-a_{10} \\
b_{11}=-10\left(1+a_{1}\right) a_{10}
\end{gathered}
$$

Taking $a_{1}=-\frac{30}{7}$ yields $a_{4}=-\frac{65}{21}$, which denotes a point (a blank circle) on the line $a_{4}=\frac{1}{3}\left(a_{1}-5\right)$ in the $a_{1}-a_{4}$ parameter plane (see Fig. 1). Further, we have $b_{11}=\frac{230}{7} a_{10}$, and

$$
\gamma=\left(1-\frac{30}{7} x\right)^{-\frac{22}{9}} \quad\left(x \neq \frac{7}{30}\right)
$$

Then, the Hamiltonian (55) becomes

$$
\begin{array}{r}
H(x, y)=\frac{16250 y^{2}+13650 x^{2}+2730 x-441}{32500\left(1-\frac{30}{7} x\right)^{13 / 9}} \\
\text { for } x \neq \frac{7}{30}
\end{array}
$$

with

$$
h_{00}=-\frac{441}{32500}>h_{10}=-\frac{15939}{32500}\left(\frac{7}{23}\right)^{13 / 9}
$$


(a)

The Melnikov functions $M_{i}\left(h, a_{10}\right)$ can be expressed as

$$
\begin{equation*}
M_{i}\left(h, a_{10}\right)=M_{i 0}(h) a_{10}, \quad i=0,1 \tag{89}
\end{equation*}
$$

Without loss of generality, we may assume

$$
\begin{equation*}
a_{10}>0 \tag{90}
\end{equation*}
$$

and thus $M_{i}\left(h, a_{10}\right)$ and $M_{i 0}(h)$ have the same sign. It is noted that for the above chosen parameter values, we have

$$
\begin{aligned}
& \mu_{03}=\frac{139150000 \pi}{453789} a_{10}>0 \quad \text { and } \\
& \mu_{10}=-\frac{2500 \sqrt{161} \pi}{3703} a_{10}<0
\end{aligned}
$$

The computation results of $M_{00}(h)$ for $h \in$ $\left(h_{00}, \infty\right)$ and $M_{10}(h)$ for $h \in\left(-\infty, h_{10}\right)$ are shown, respectively, in Figs. 3(a) and 3(b). Figure 3(a) shows that $M_{00}(h)>0$ for $h \in\left(h_{00}, \infty\right)$, and its sign agrees with that of $\mu_{03}>0$ for $0<h-h_{00} \ll 1$, as expected. It is also noted, as shown in Fig. 3(b), that the sign of $M_{10}(h)$ agrees with that of $\mu_{10}<0$ for $0<h_{10}-h \ll 1$. However, unlike the interval $h \in\left(h_{00}, \infty\right)$, this interval contains a critical value $h=h_{1}^{*} \in(-0.9250363254,-0.9250363253)$ at which $M_{10}\left(h_{1}^{*}\right)=0$ and the function $M_{10}(h)$ changes its sign as $h$ crosses this critical point. Thus, for this case, besides the 3 small limit cycles, there exists at least one large limit cycle bifurcating from the closed orbit $L_{h_{1}^{*}}$ of (70). This large limit cycle is shown in Fig. 4(a), which encloses the center $(1,0)$;

(b)

Fig. 3. Functions $M_{00}(h)$ and $M_{10}(h)$ under the conditions $\mu_{00}=\mu_{01}=\mu_{02}=0, \mu_{03} \neq 0$ and $\mu_{10} \neq 0$, for $a_{1}=-\frac{30}{7}$ and $a_{4}=\frac{1}{3}\left(a_{1}-5\right)=-\frac{65}{21}: ~(a) M_{00}(h)>0$ for $h \in\left[h_{00},+\infty\right)$, with $h_{0}=-\frac{441}{32500} \approx-0.01357$; and (b) $M_{10}(h)$ for $h \in\left(-\infty, h_{1}\right]$, with $h_{10}=-\frac{15939}{32500}\left(\frac{7}{23}\right)^{13 / 9} \approx-0.08797$, crossing the $h$-axis at $h=h_{1}^{*} \in(-0.9250363254,-0.9250363253)$.


Fig. 4. Illustration of the existence of 4 limit cycles when $a_{1}=-\frac{30}{7}, a_{4}=\frac{1}{3}\left(a_{1}-5\right)=-\frac{65}{21}-\varepsilon_{1}$, and $a_{10}=\frac{1}{2000}$, $b_{11}=\frac{230}{21} a_{10}-\varepsilon_{2}, b_{01}=-a_{10}-\varepsilon_{3}$, where $0<\varepsilon_{3} \ll \varepsilon_{2} \ll \varepsilon_{1} \ll \varepsilon$ : (a) An unstable large limit cycle enclosing the center $(1,0)$; and (b) zoomed area around the center ( 0,0 ) showing the existence of 3 small limit cycles.
and Fig. 4(b) illustrates the existence of 3 small limit cycles around the center $(0,0)$.
(B) For the case of the $(0,3)$-distribution, we have

$$
\begin{gathered}
a_{4}=\frac{1}{3}\left(6 a_{1}+5\right), \quad b_{01}=-b_{11}+\frac{2 a_{4}-1}{1+a_{1}} a_{10}, \\
b_{11}=\frac{\left(a_{1}+2 a_{4}\right)\left(2 a_{1}-a_{4}+1\right)}{\left(1+a_{1}\right)^{2}\left(a_{1}-a_{4}+1\right)} a_{10} .
\end{gathered}
$$

By choosing $a_{1}=-\frac{70}{51}$, we have $a_{4}=-\frac{55}{51}, b_{01}=$ $-\frac{5611}{361} a_{10}$ and $b_{11}=\frac{8670}{361} a_{10}$. The point $\left(a_{1}, a_{4}\right)=$ $\left(-\frac{70}{51},-\frac{55}{51}\right)$ is marked by a blank circle on the line $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)$ in the $a_{1}-a_{4}$ parameter plane (see Fig. 1). Moreover,

$$
\gamma=\left(1-\frac{70}{51} x\right)^{-\frac{18}{7}} \quad\left(x \neq \frac{51}{70}\right)
$$

and the Hamiltonian is

$$
\begin{array}{r}
H(x, y)=\frac{2750 y^{2}+9350 x^{2}-16830 x+7803}{5500\left(1-\frac{70}{51} x\right)^{11 / 7}} \\
\quad \text { for } x \neq \frac{51}{70}
\end{array}
$$

with

$$
h_{00}=\frac{7803}{5500}>h_{10}=-\frac{323}{5500}\left(\frac{51}{19}\right)^{11 / 7}
$$

For this case, $\mu_{00}$ and $\mu_{13}$ become

$$
\begin{aligned}
& \mu_{00}=-\frac{10500 \pi}{361} a_{10}<0 \text { and } \\
& \mu_{13}=\frac{4561235000}{565036352721}\left(\frac{51}{19}\right)^{2 / 7} \pi a_{10}>0 .
\end{aligned}
$$

The computation results of $M_{00}(h)$ for $h \in$ $\left(h_{00}, \infty\right)$ and $M_{10}(h)$ for $h \in\left(-\infty, h_{10}\right)$ are shown in Figs. 5(a) and 5(b), respectively. As shown in Fig. 5(a), the sign of $M_{00}(h)$ agrees with that of $\mu_{00}<0$ for $0<h-h_{00} \ll 1$, and in addition the function $M_{00}(h)$ crosses a critical value at $h=h_{2}^{*} \in(13.3847179116,13.3847179117)$, at which it changes sign. Figure 5(b) shows that $M_{10}(h)>0$ for $h \in\left(-\infty, h_{10}\right)$, and its sign agrees with that of $\mu_{13}>0$ for $0<h_{10}-h \ll 1$. Hence, for this case, in addition to the 3 small limit cycles, there also exists at least one large limit cycle bifurcating from the closed orbit $L_{h_{2}^{*}}$ of (70). This large limit cycle is depicted in Fig. 6(a), which encloses the center $(0,0)$; and Fig. 6(b) illustrates the existence of 3 small limit cycles around the center $(1,0)$.
(C) Now consider the ( 2,0 )-distribution. For this case, the condition $a_{4}=\frac{1}{3}\left(a_{1}-5\right)$ is not used. We need to determine the values for both $a_{1}$ and $a_{4}$. We choose

$$
a_{1}=-4, \quad a_{4}=-\frac{18}{5}
$$



Fig. 5. Functions $M_{00}(h)$ and $M_{10}(h)$ under the conditions $\mu_{10}=\mu_{11}=\mu_{12}=0, \mu_{13} \neq 0$ and $\mu_{00} \neq 0$, for $a_{1}=-\frac{70}{51}$ and $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)=-\frac{55}{51}:$ (a) $M_{00}(h)$ for $h \in\left[h_{0},+\infty\right)$, with $h_{00}=\frac{7803}{5500} \approx 1.41873$, crossing the $h$-axis at $h=h_{2}^{*} \in(13.3847179116,13.3847179117)$; and (b) $M_{10}(h)>0$ for $h \in\left(-\infty, h_{1}\right]$, with $h_{10}=-\frac{323}{5500}\left(\frac{51}{19}\right)^{11 / 7} \approx-0.27714$.
which represents a point in the third quadrant of the $a_{1}-a_{4}$ parameter plane (see the dark circle in Fig. 1 near the line $\left.a_{4}=\frac{1}{3}\left(a_{1}-5\right)\right)$. Thus,

$$
\gamma=(1-4 x)^{-\frac{14}{5}} \quad\left(x \neq \frac{1}{4}\right) .
$$

In addition, we have $b_{01}=-a_{10}, b_{11}=\frac{392}{65} a_{10}$, and

$$
\begin{array}{r}
H(x, y)=\frac{192 y^{2}+480 x^{2}-180 x+25}{384(1-4 x)^{9 / 5}} \\
\text { for } x \neq \frac{1}{4}
\end{array}
$$


with

$$
h_{00}=\frac{25}{384}>h_{10}=-\frac{325}{384}\left(\frac{1}{3}\right)^{9 / 5} .
$$

For this case, $\mu_{02}$ and $\mu_{10}$ are reduced to

$$
\begin{aligned}
& \mu_{02}=-\frac{1344}{125} \pi a_{10}<0 \quad \text { and } \\
& \mu_{10}=-\frac{40 \sqrt{3}}{9} \pi a_{10}<0 .
\end{aligned}
$$

The computation results of $M_{00}(h)$ for $h \in$ $\left(h_{00}, \infty\right)$ and $M_{10}(h)$ for $h \in\left(-\infty, h_{10}\right)$ are shown,

Fig. 6. Illustration of the existence of 4 limit cycles when $a_{1}=-\frac{70}{51}, a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)=-\frac{55}{51}-\varepsilon_{1}$, and $a_{10}=10$, $b_{11}=\frac{8670}{361} a_{10}-\varepsilon_{2}, b_{01}=-\frac{5611}{361} a_{10}+\varepsilon_{3}$, where $0<\varepsilon_{3} \ll \varepsilon_{2} \ll \varepsilon_{1} \ll \varepsilon$ : (a) An unstable large limit cycle enclosing the center $(0,0)$; and (b) zoomed area around the center ( 1,0 ) showing the existence of 3 small limit cycles.


Fig. 7. Functions $M_{00}(h)$ and $M_{10}(h)$ under the conditions $\mu_{00}=\mu_{01}=0, \mu_{02} \neq 0$ and $\mu_{10} \neq 0$, for $a_{1}=-4$ and $a_{4}=-\frac{18}{5}:\left(\right.$ a) $M_{00}(h)$ for $h \in\left[h_{00},+\infty\right)$, with $h_{0}=\frac{25}{384} \approx 0.06510$, crossing the $h$-axis at $h=h_{3}^{*} \in$ (0.1448192224, 0.1448192225 ); and (b) $M_{10}(h)$ for $h \in\left(-\infty, h_{1}\right]$, with $h_{10}=-\frac{325}{384}\left(\frac{1}{3}\right)^{9 / 5} \approx-0.11715$, crossing the $h$-axis at $h=h_{4}^{*} \in$ ( $-0.5822537644,-0.5822537643$ ).
respectively, in Figs. 7(a) and 7(b). As shown in Fig. 7(a), the sign of $M_{00}(h)$ agrees with that of $\mu_{02}<0$ for $0<h-h_{00} \ll 1$. Moreover, the function $M_{00}(h)$ crosses a critical value at $h=h_{3}^{*} \in$ ( $0.1448192224,0.1448192225$ ) at which it changes sign. Figure $7(\mathrm{~b})$ shows $M_{10}(h)$ for $h \in\left(-\infty, h_{10}\right)$, whose sign agrees with that of $\mu_{10}<0$ for $0<h_{10}-$ $h \ll 1$. Also, $M_{10}(h)$ crosses a critical value at $h=$ $h_{4}^{*} \in(-0.5822537644,-0.5822537643)$ at which it changes sign. Therefore, for this case, besides the 2 small limit cycles, there exist at least 2 large limit cycles bifurcating from the two different closed

(a)
orbits $L_{h_{3}^{*}}$ and $L_{h_{4}^{*}}$ of (70). One large limit cycle surrounding the center $(1,0)$ is shown in Fig. 8(a), while another large limit cycle enclosing the center $(0,0)$ with 2 small limit cycles is depicted in Fig. 8(b).
(D) Finally, consider the ( 0,2 )-distribution. For this case, the condition $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)$ is not used. Taking

$$
a_{1}=-\frac{4}{3}, \quad a_{4}=-\frac{6}{5}
$$


(b)

Fig. 8. Illustration of the existence of 4 limit cycles when $a_{1}=-4, a_{4}=-\frac{18}{5}$, and $a_{10}=\frac{1}{100}, b_{11}=\frac{392}{65} a_{10}-\varepsilon_{1}$, and $b_{01}=-a_{10}-\varepsilon_{2}$, where $0<\varepsilon_{2} \ll \varepsilon_{1} \ll \varepsilon$ : (a) An unstable large limit cycle enclosing the center ( 1,0 ); and (b) Zoomed area around the center $(0,0)$ showing the existence of 1 large limit cycle and 2 small limit cycles.


Fig. 9. Functions $M_{00}(h)$ and $M_{10}(h)$ under the conditions $\mu_{10}=\mu_{11}=0, \mu_{12} \neq 0$ and $\mu_{00} \neq 0$, for $a_{1}=-\frac{4}{3}$ and $a_{4}=-\frac{6}{5}$ : (a) $M_{00}(h)$ for $h \in\left[h_{00},+\infty\right)$, with $h_{0}=\frac{325}{128} \approx 2.53096$, crossing the $h$-axis at $h=h_{5}^{*} \in$ (12.6197809949, 12.6197809950); and (b) $M_{10}(h)$ for $h \in\left(-\infty, h_{1}\right]$, with $h_{10}=-\frac{25}{128} 3^{9 / 5} \approx-1.41107$, crossing the $h$-axis at $h=h_{6}^{*} \in(-3.1388150376,-3.1388150375)$.
yields

$$
\gamma=\left(1-\frac{4}{3} x\right)^{-\frac{14}{5}} \quad\left(x \neq \frac{3}{4}\right)
$$

The point $\left(-\frac{4}{3},-\frac{6}{5}\right)$ is marked by a dark circle near the line $a_{4}=\frac{1}{3}\left(6 a_{1}+5\right)$ in the $a_{1}-a_{4}$ parameter plane (see Fig. 1). Further, we have $b_{01}=-\frac{513}{65} a_{10}$, $b_{11}=\frac{1176}{65} a_{10}$, and
$H(x, y)=\frac{64 y^{2}+480 x^{2}-780 x+325}{128\left(1-\frac{4}{3} x\right)^{9 / 5}}$ for $x \neq \frac{3}{4}$,
with

$$
h_{00}=\frac{325}{128}>h_{10}=-\frac{25}{128} 3^{9 / 5} .
$$

For this case, $\mu_{00}$ and $\mu_{12}$ are simplified as

$$
\begin{aligned}
& \mu_{00}=-\frac{896}{65} \pi a_{10}<0 \quad \text { and } \\
& \mu_{12}=-\frac{448}{30375} 3^{9 / 10} \pi a_{10}<0 .
\end{aligned}
$$

The computation results of $M_{00}(h)$ for $h \in$ $\left(h_{00}, \infty\right)$ and $M_{10}(h)$ for $h \in\left(-\infty, h_{10}\right)$ are shown in Figs. 9(a) and 9(b), respectively. As shown in Fig. 9(a), the sign of $M_{00}(h)$ agrees with that of $\mu_{00}<0$ for $0<h-h_{00} \ll 1$, and the function $M_{00}(h)$ crosses a critical value at $h=h_{5}^{*} \in$ $(12.6197809949,12.6197809950)$ at which it changes sign. Figure 9(b) shows $M_{10}(h)$ for $h \in\left(-\infty, h_{10}\right)$, whose sign agrees with that of $\mu_{12}<0$ for
$0<h_{10}-h \ll 1$. Moreover, $M_{10}(h)$ crosses a critical value at $h=h_{6}^{*} \in(-3.1388150376,-3.1388150375)$ at which it changes sign. Therefore, for this case, in addition to the 2 small limit cycles, there also exist at least 2 large limit cycles bifurcating from the two different closed orbits $L_{h_{5}^{*}}$ and $L_{h_{6}^{*}}$ of (70). One large limit cycle surrounding the center $(0,0)$ is shown in Fig. 10(a), while another large limit cycle enclosing the center $(1,0)$ with 2 small limit cycles is depicted in Fig. 10(b).

It is noted that all the four sets of values of $a_{1}$ and $a_{4}$ chosen above in (A), (B), (C) and (D) satisfy
$\frac{a_{1}+2 a_{4}}{a_{1}}=\frac{2 n}{m}, \quad \begin{array}{r}\text { where } n \text { is an integer and } \\ m \text { is an odd integer, }\end{array}$
so that a consistent integrating factor (and so a consistent Hamiltonian function for the whole transformed system) is obtained. However, this condition is not necessary since the singular line $1+a_{1} x=0$ divides the phase plane into two parts, and the analysis does not need the continuity on the singular line. To demonstrate this, in the following we present a case for which the condition (91) is not satisfied. Consider the (2,0)-distribution, and choose $a_{1}=-5$ and $a_{4}=-4$. The point $\left(a_{1}, a_{4}\right)=$ $(-5,-4)$ is marked by a dark circle in the $a_{1}-a_{4}$ parameter plane (see Fig. 1). Then,

$$
\frac{a_{1}+2 a_{4}}{a_{1}}=\frac{13}{5}, \quad b_{01}=a_{10}, \quad b_{11}=\frac{26}{3} a_{10}
$$



Fig. 10. Illustration of the existence of 4 limit cycles when $a_{1}=-\frac{4}{3}, a_{4}=-\frac{6}{5}$, and $a_{10}=1$, $b_{11}=\frac{1176}{65} a_{10}-\varepsilon_{1}$, $b_{01}=-\frac{513}{65} a_{10}+\varepsilon_{2}$, where $0<\varepsilon_{2} \ll \varepsilon_{1} \ll \varepsilon$ : (a) An unstable large limit cycle enclosing the center ( 0,0 ); and (b) Zoomed area around the center $(1,0)$ showing the existence of 1 large limit cycle and 2 small limit cycles.
and

$$
H(x, y)= \begin{cases}\frac{x^{2}+y^{2}}{2(1-5 x)^{8 / 5}}, & \forall h \in(0, \infty), \text { when } x<\frac{1}{5}, \\ -\frac{x^{2}+y^{2}}{2(1-5 x)^{8 / 5}}, & \forall h \in\left(-\infty,-\frac{1}{32} 2^{4 / 5}\right), \text { when } x>\frac{1}{5} .\end{cases}
$$

For this case, $\mu_{02}$ and $\mu_{10}$ become
$\mu_{02}=-\frac{130}{3} \pi a_{10}<0 \quad$ and $\quad \mu_{10}=-\frac{65}{12} \pi a_{10}<0$.
The computation result of $M_{00}(h)$ shows that $M_{00}(h)<0$ for $0<h \ll 1$, agrees with the sign of $\mu_{02}$. Moreover, $M_{00}(0.1)=0.0510077880>0$, implying that there exists $h=h_{7}^{*} \in(0,0.1)$ such that $M_{00}\left(h_{7}^{*}\right)=0$, and so a large limit cycle bifurcates from the closed orbit $L_{h_{7}^{*}}$ of (70). The result of $M_{10}(h)$ also shows that $M_{10}(h)<0$ for $0<$ $-\frac{1}{32} 2^{4 / 5}-h \ll 1$, agreeing with the sign of $\mu_{10}$, and that $M_{10}\left(-\frac{1}{32} 2^{4 / 5}-0.8\right)=7.4630743072>$ 0 , implying the existence $h=h_{8}^{*} \in\left(-\frac{1}{32} 2^{4 / 5}-\right.$ $\left.0.8,-\frac{1}{32} 2^{4 / 5}\right) \approx(-0.8544094102,-0.0544094102)$ such that $M_{10}\left(h_{8}^{*}\right)=0$. Thus, there exists another large limit cycle bifurcating from the closed orbit $L_{h_{8}^{*}}$ of (70). Therefore, this case exhibits 2 small limit cycles and 2 large limit cycles, leading to the existence of at least 4 limit cycles.

Another example is to choose the line $a_{4}=\frac{3}{4} a_{1}$. There exist many points on this line which exhibit 4 limit cycles. For example, choose $a_{1}=-4$, then
$a_{4}=-3$ for which 3 small limit cycles are around $(0,0)$, and 1 large limit cycle encloses $(1,0)$. On the other hand, taking $a_{1}=-5$, we have $a_{4}=-\frac{15}{4}$ for which 2 small limit cycles are in the neighborhood of $(0,0)$, and 2 large limit cycles exist with each enclosing $(0,0)$ and $(1,0)$. It should be noted that not all the points on this line $a_{4}=\frac{3}{4} a_{1}$ can generate 4 limit cycles. Identifying which parts of the line to have 4 limit cycles is not an easy task.

Summarizing the above results with the continuity of parameters $a_{1}$ and $a_{4}$ shows that at least for some regions in the $a_{1}-a_{4}$ parameter plane the reversible near-integrable system (67) can exhibit at least 4 limit cycles around the two singular points $(0,0)$ and $(1,0)$ with distribution either $(3,1)$ or $(1,3)$.

The proof of Theorem 3 is completed.

## 5. Conclusion

In this paper, we have proved that a quadratic non-Hamiltonian integrable system with two centers can have at least 4 limit cycles under quadratic
perturbations, with distributions either $(3,1)$ or $(1,3)$. This result gives a new record, answering the open problem of the existence of limit cycles in near-integrable quadratic systems. It is shown that such systems can have at least 4 limit cycles for some regions in the two-dimensional parameter plane, associated with the parameters of the integrable systems. Further research is needed on global analysis for all possible parameter values in the parameter plane.

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    ${ }^{\S}$ Author for correspondence

