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EIGHT LIMIT CYCLES AROUND A CENTER IN QUADRATIC HAMILTONIAN SYSTEM WITH THIRD-ORDER PERTURBATION

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In this paper, we show that generic planar quadratic Hamiltonian systems with third degree polynomial perturbation can have eight small-amplitude limit cycles around a center. We use higher-order focus value computation to prove this result, which is equivalent to the computation of higher-order Melnikov functions. Previous results have shown, based on first-order and higher-order Melnikov functions, that planar quadratic Hamiltonian systems with third degree polynomial perturbation can have five or seven small-amplitude limit cycles around a center. The result given in this paper is a further improvement.

Keywords: Hilbert's 16th problem; limit cycle; near-Hamiltonian system; center bifurcation; higher order Melnikov function; focus value.

1. Introduction

The second part of the well-known Hilbert's 16th problem [Hilbert, 1902] can be described as follows: What is the upper bound of the number of limit cycles of the following system:

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ denote nth degree polynomials of x and y, and the bound only depends on n? This number is denoted by $H(n)$, called Hilbert number. The finiteness of $H(n)$ has not been solved even for quadratic systems. Later, a so-called *weak* Hilbert's 16th problem was proposed by Arnold [1983], which asks for the maximal number of isolated zeros of the Abelian integral or Melnikov function:

$$M(h, \delta) = \int_{H(x,y)=h} Q_n dx - P_n dy, \tag{2}$$

where $H(x, y), P_n$ and Q_n are all real polynomials of x and y with $\deg H = n + 1$, and $\max\{\deg P_n, \deg Q_n\} \le n$. The weak Hilbert's 16th problem itself is closely related to the following near-Hamiltonian system [Han, 2006]:

$$\begin{aligned} \dot{x} &= H_y(x, y) + \varepsilon p_n(x, y, \varepsilon), \\ \dot{y} &= -H_x(x, y) + \varepsilon q_n(x, y, \varepsilon), \end{aligned} \tag{3}$$

where $H(x, y)$, $p_n(x, y)$ and $q_n(x, y)$ are all polynomial functions of x and y, and $0 < \varepsilon \ll 1$ is a small perturbation. Studying the bifurcation of

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limit cycles for such a system can be transformed to investigating the zeros of the Melnikov function as follows:

$$M(h) = \oint_{L_h} q_n dx - p_n dy$$
$$= \iint_{D(h)} \left(\frac{\partial p_n}{\partial x} + \frac{\partial q_n}{\partial y} \right) dxdy, \qquad (4)$$

where L_h is a contour around a singular point, and $D(h)$ is the region bounded by the contour. It should be noted that more precisely, the above Melnikov function is called the first order Melnikov function.

If the Hilbert's 16th problem is restricted to a neighborhood of isolated fixed points with Hopf singularity, then the problem becomes the study of degenerate Hopf bifurcations, associated with computation of focus values or normal forms. Many results have been obtained in this direction (e.g. see [Bautin, 1952; Kukles, 1944; Li & Liu, 1991; Malkin, 1964]). In 1952, Bautin [1952] proved that the general quadratic system (1) ($n = 2$) can have three small-amplitude limit cycles around a fine focus point or a center. For cubic systems, many results have been obtained, showing that in the vicinity of a singular point the number of small-amplitude limit cycles can be five [Christopher & Lloyd, 1990], six [Lloyd *et al.*, 1988], seven [Li & Bai, 1989; Lloyd *et al.*, 1988; Sadovskii, 2003], eight [James & Lloyd, 1991; Yu & Corless, 2009], and nine [Yu & Corless, 2009; Yu & Han, 2012]. When considering multiple singular points, it has been shown that cubic planar polynomial systems can have limit cycles (not necessarily small): ten [Han *et al.*, 2004], eleven [Li, 2003; Zhang *et al.*, 2004], twelve [Yu & Han, 2004; Yu & Han, 2005a, 2005b], and thirteen [Li *et al.*, 2009; Li & Liu, 2010; Yang *et al.*, 2010]. It should be pointed out that the nine small-amplitude limit cycles given in [Yu & Corless, 2009] were obtained by perturbing an elementary center (linear center) of general cubic systems, while that given in [Yu & Han, 2012] were obtained by perturbing a center of an integrable system with cubic polynomials.

In this paper, we consider the bifurcation of limit cycles in quadratic Hamiltonian system with cubic degree polynomial perturbation, and pay particular attention to the limit cycles bifurcating from a center. We may assume, without loss of generality, that system $(3)_{\varepsilon=0}$ has a center at the origin $(x, y) = (0, 0)$. To determine the number of limit cycles in the vicinity of the origin, we may use either the Melnikov function method (e.g. see [Han, 2006; Han *et al.*, 2009]) or the focus value method (or the normal form method, see [Yu, 1998]). In this paper, we will apply the method of focus value computation to study the bifurcation of limit cycles. In general, the focus values of system (3) evaluated at the origin can be written in the form of

$$v_i = \tilde{v}_{i0} + \varepsilon\tilde{v}_{i1} + \varepsilon^2\tilde{v}_{i2} + \varepsilon^3\tilde{v}_{i3}$$
$$+ \varepsilon^4\tilde{v}_{i4} + \cdots, \quad i = 0, 1, 2, \ldots, \qquad (5)$$

where $\tilde{v}_{i0} = 0$, $i = 0, 1, 2, \ldots$ since the origin is a center when $\varepsilon = 0$. Thus, for sufficiently small ε, we can use the leading focus values $\tilde{v}_{i1}$ to determine the bifurcation of small limit cycles. If all $\tilde{v}_{i1} = 0$, then use $\tilde{v}_{i2}$, and so on.

For the quadratic Hamiltonian system with second degree polynomial perturbation, it has been shown that such perturbed systems can have maximal two limit cycles [Horozov & Iliev, 1994; Gavrilov, 2001], i.e. $H_2(2) = 2$, representing the number of limit cycles bifurcating from closed orbits of quadratic Hamiltonian systems under second order perturbation. It is easy to show that $\tilde{H}_2(2) = 2$, where the tilde denotes small-amplitude limit cycles around singular points. Recently, it has been shown [Han *et al.*, 2009] that quadratic Hamiltonian systems with third order polynomial perturbation can have five small-amplitude limit cycles in the vicinity of a center, i.e. $\tilde{H}_2(3) \geq 5$. The result given in [Han *et al.*, 2009] was obtained by using the first order Melnikov function. More recently, bifurcation of limit cycles in quadratic Hamiltonian systems with up to 20th degree polynomial perturbations has been studied using the first order Melnikov functions [Yu & Han, 2011]. Iliev, on the other hand, studied a so-called Bogdanov–Takens Hamiltonian system under various degree of polynomial perturbations using various order of Melnikov functions [Iliev, 2000]. The Bogdanov–Takens Hamiltonian system is given by

$$\begin{aligned} \dot{x} &= y + \varepsilon p_n(x, y, \varepsilon), \\ \dot{y} &= -x + x^2 + \varepsilon q_n(x, y, \varepsilon), \end{aligned} \qquad (6)$$

and the corresponding unperturbed system has Hamiltonian

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3, \qquad (7)$$

which is called the Bogdanov–Takens unfolding, known from the unfolding of a cusp singularity.

It has been shown in [Iliev, 2000] that the Bogdanov–Takens Hamiltonian system with third degree polynomial perturbation can have seven limit cycles upon using fourth order Melnikov function, i.e. $H_2(3) \geq 7$. Certainly, these seven limit cycles are not necessarily small.

Very recently, we consider perturbing the following generic quadratic Hamiltonian system:

$$\begin{aligned} \dot{x} &= y + a_1xy + a_2y^2, \\ \dot{y} &= -x + x^2 - \frac{1}{2}a_1y^2, \end{aligned} \tag{8}$$

and obtain seven small-amplitude limit cycles around the center $(0, 0)$. The derivation of system (8) from a general quadratic system can be found in [Yu & Han, 2011]. The Hamiltonian of system (8) is given by

$$\begin{aligned} H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 \\ + \frac{1}{2}a_1xy^2 + \frac{1}{3}a_2y^3. \end{aligned} \tag{9}$$

It is noted that system (8) has a center at the origin $(0, 0)$ and another singularity at $(1, 0)$. Since we are interested in the limit cycles bifurcating from the center $(0, 0)$, we will ignore whether the singular point $(1, 0)$ is a center or a saddle point.

The perturbed Hamiltonian system or near-Hamiltonian system of (8) can be generally written as

$$\begin{aligned} \dot{x} &= y + a_1x\,y + a_2y^2 + \varepsilon p_n(x, y), \\ \dot{y} &= -x + x^2 - \frac{1}{2}a_1y^2 + \varepsilon q_n(x, y), \end{aligned} \tag{10}$$

where the general perturbing polynomial functions $p_n(x, y, \varepsilon)$ and $q_n(x, y, \varepsilon)$ can be written as

$$\begin{aligned} p_n(x, y, \varepsilon) &= \sum_{1 \leq i+j \leq n} [a_{ij1} + \varepsilon a_{ij2} + \varepsilon^2 a_{ij3} \\ &\quad + \varepsilon^3 a_{ij4} + \cdots]x^i y^j \\ q_n(x, y, \varepsilon) &= \sum_{1 \leq i+j \leq n} [b_{ij1} + \varepsilon b_{ij2} + \varepsilon^2 b_{ij3} \\ &\quad + \varepsilon^3 b_{ij4} + \cdots]x^i y^j. \end{aligned} \tag{11}$$

We have found that such a perturbed system can have seven small-amplitude limit cycles around the origin, under the assumption: $a_{ijk} = 0$ (i.e. $p_n(x, y) \equiv 0$), and $b_{i0k} = 0$. This in general may result in missing possibly more limit cycles. Therefore, in this paper, we assume that

$$a_{301}a_{121}b_{304} \neq 0, \tag{12}$$

and obtain eight limit cycles around the origin, i.e. $\tilde{H}_2(3) \geq 8$.

In order to distinguish the order of focus value with the order of ε, we call ε^n-order focus values with respect to the nth order Melnikov function. The rest of the paper is organized as follows. In the next section, for completeness, we apply the ε-order focus values to reinvestigate this case to confirm $\tilde{H}_2(3) \geq 5$. The results obtained from ε^2-, ε^3-, ε^4- and ε^5-order focus values will be presented in Secs. 3–6, respectively. Conclusion is drawn in Sec. 7.

2. Perturbed Quadratic Hamiltonian Systems and $\tilde{H}_2^1(3) = 5$ Based on ε-Order Focus Values

In this section, we use ε-order focus values to re derive the result $\tilde{H}_2^1(3) = 5$, where the superscript "1" indicates the result based on ε-order focus values. The case for quadratic Hamiltonian systems with second degree polynomial perturbation is straightforward. It can be shown that for any ε^n-order focus values, $\tilde{H}_2(2) = 2$, agreeing with $H_2(2) = 2$. We have the following theorem.

Theorem 1. *With the ε-order focus values, $\tilde{H}_2^1(3) = 5$.*

Proof. In order for the origin $(0, 0)$ to be a linear center under perturbation up to ε order, we set $b_{011} = 0$ under which $\tilde{v}_{01} = 0$. Then, applying the Maple program to system (10) yields

$$\begin{aligned} \tilde{v}_{11} = \frac{1}{16}[6b_{031} + 2b_{211} - (a_1 - 2)b_{111} \\ - 4a_2b_{021} + 6a_{301} + 2a_{121}]. \end{aligned} \tag{13}$$

Solving (13) for b_{031} results in

$$\begin{aligned} b_{031} = \frac{1}{6}[(a_1 - 2)b_{111} + 4a_2b_{021} - 2b_{211} \\ - 6a_{301} - 2a_{121}]. \end{aligned} \tag{14}$$

Then we obtain $\tilde{v}_{21}$ as follows:

$$\begin{aligned} \tilde{v}_{21} = -\frac{1}{48}a_2(5a_1 - 2)(a_1b_{021} - b_{121}) \\ - \frac{1}{192}[20a_2^2 + (3a_1 + 10)(a_1 - 2)] \\ \times (b_{111} + b_{211} + 3a_{031}). \end{aligned} \tag{15}$$

First, assume $a_1 \neq \frac{2}{5}$. Then we solve b_{121} from $\tilde{v}_{21} = 0$ to obtain

$$\begin{aligned} b_{121} = a_1 b_{021} + \frac{20a_2^2 + (3a_1+10)(a_1-2)}{4a_2(5a_1-2)} \\ \times (b_{111} + b_{211} + 3a_{031}), \end{aligned} \tag{16}$$

under which $\tilde{v}_{31}, \tilde{v}_{41}$, etc., are simplified as

$$\begin{aligned} \tilde{v}_{31} &= \frac{35}{3072} Q_1 F_1(a_1, a_2), \\ \tilde{v}_{41} &= -\frac{7}{73728} Q_1 M_1(a_1, a_2), \\ \tilde{v}_{51} &= -\frac{7}{84934656} Q_1 N_1(a_1, a_2), \\ &\vdots \end{aligned} \tag{17}$$

where

$$\begin{aligned} Q_1 &= \frac{1}{(5a_1-2)}(b_{111} + b_{211} + 3a_{301}) \\ &\quad \times [(a_1+1)(a_1-2)^2 - 4a_2^2], \\ F_1 &= 3a_1^2 + 12a_1 - 4 - 4a_2^2, \\ M_1 &= 27a_1^4 - 90a_1^3 - 1308a_1^2 + 1608a_1 - 256 \\ &\quad + 4a_2^2(105a_1^2 + 402a_1 - 344 - 64a_2^2), \\ N_1 &= 19683a_1^6 + 343116a_1^5 - 124524a_1^4 \\ &\quad - 6168672a_1^3 + 7612368a_1^2 + 1585344a_1 \\ &\quad - 1071424 + 4a_2^2[422145a_1^4 + 1867608a_1^3 \\ &\quad + 119640a_1^2 - 5068704a_1 + 1265424 \\ &\quad - 4a_2^2(101127a_1^2 - 99084a_1 \\ &\quad - 316356 + 66964a_2^2)], \\ &\vdots \end{aligned} \tag{18}$$

It is noted that all the expressions $\tilde{v}_{41}, \tilde{v}_{51}, \ldots$ contain a common factor Q_1. Setting $Q_1 = 0$ results in all the ε-order focus values to be zero. Hence, in order for this factor to be nonzero, we may use a_2 to solve $F_1 = 0$, yielding

$$a_2 = \pm\frac{1}{2}\sqrt{3(a_1+2)^2 - 16}. \tag{19}$$

To guarantee a_2 being real, the value of a_1 must be taken from the following intervals:

$$\begin{aligned} a_1 \in \left(-\infty, -\frac{4\sqrt{3}}{3} - 2\right) \cup \left(\frac{4\sqrt{3}}{3} - 2, \infty\right) \\ \approx (-\infty, -4.309401077) \cup (0.309401077, \infty). \end{aligned} \tag{20}$$

Under the condition (19), $\tilde{v}_{41}$ and $\tilde{v}_{51}$ are further simplified as

$$\begin{aligned} \tilde{v}_{41} &= -\frac{7}{4096(5a_1-2)}(a_1+2)^2(a_1^2 - 8a_1 + 4) \\ &\quad \times (b_{111} + b_{211} + 3a_{301}) \\ &\quad \times (11a_1^3 + 46a_1^2 - 84a_1 + 24), \\ \tilde{v}_{51} &= \frac{7a_1}{12288(5a_1-2)}(a_1+2)^2 \\ &\quad \times (a_1^2 - 8a_1 + 4)(b_{111} + b_{211} + 3a_{301}) \\ &\quad \times (11a_1^3 + 72a_1^2 - 175a_1 + 50). \end{aligned} \tag{21}$$

It is easy to see from (21) that the only possibility for $\tilde{v}_{41} = 0$ but $\tilde{v}_{51} \neq 0$ is to choose the roots of the polynomial

$$F_1(a_1) = 11a_1^3 + 46a_1^2 - 84a_1 + 24, \tag{22}$$

which has three real roots, since its discriminant $d = -\frac{86528}{11979} < 0$, given by

$$\begin{aligned} a_1 = -5.6118538340\ldots, 0.3650705869\ldots, \\ 1.0649650652\ldots, \end{aligned} \tag{23}$$

which are all located in the interval given in (20). The above results show that there exist in total six solutions.

It is easy to see that the two third degree polynomials in $\tilde{v}_{41}$ and $\tilde{v}_{51}$ do not have common roots. That is, the solutions given in (23) do not yield the third degree polynomial in $\tilde{v}_{51}$ to be zero. Further, a direct calculation yields

$$\begin{aligned} D_1 &= \det\begin{bmatrix} \frac{\partial \tilde{v}_{31}}{\partial a_1} & \frac{\partial \tilde{v}_{31}}{\partial a_2} \\ \frac{\partial \tilde{v}_{41}}{\partial a_1} & \frac{\partial \tilde{v}_{41}}{\partial a_2} \end{bmatrix} \\ &= -\frac{245a_2(a_1+2)^2(a_1^2 - 8a_1 + 4) \times (b_{111} + b_{211} + 3a_{301})^2}{1572864(5a_1-2)^3} \end{aligned}$$

$$\times(165a_1^6 - 382a_1^5 - 4568a_1^4 + 4896a_1^3 + 688a_1^2 - 1696a_1 + 384) \neq 0,$$

if $b_{111} + b_{211} + 3a_{301} \neq 0$, where $\tilde{v}_{31}$ and $\tilde{v}_{41}$ are given in (17), evaluated at the critical point determined by 19) and (23), since none of the solutions given in (23) is a root of the sixth degree polynomial in the above expression. Thus $D_1 \neq 0$, implying that proper perturbations on a_1 and a_2 can be found such that $|\tilde{v}_{31}| \ll |\tilde{v}_{41}| \ll |\tilde{v}_{51}| < \epsilon$ and $\tilde{v}_{31}\tilde{v}_{41} < 0, \tilde{v}_{41}\tilde{v}_{51} < 0$. Three other perturbations on $b_{121}, b_{031}, b_{011}$ can be easily obtained one by one, satisfying $|\tilde{v}_{j1}| \ll |\tilde{v}_{(j+1)1}|$ and $\tilde{v}_{j1}\tilde{v}_{(j+1)1} < 0$ for $j = 0, 1, 2, 3$. Hence, for this case, five small-amplitude limit cycles can bifurcate from the origin (the center).

The remaining case is $a_1 = \frac{5}{2}$, for which b_{031} and b_{121} become

$$\begin{aligned} b_{031} &= \frac{1}{12}(b_{111} + 8a_2b_{021} - 4b_{211} - 12a_{301} - 4a_{121}), \\ b_{121} &= \frac{5}{2}b_{021} + \frac{5}{168a_2}(16a_2^2 + 7) \times (b_{111} + b_{211} + 3a_{301}), \end{aligned} \tag{24}$$

under which $\tilde{v}_{11} = \tilde{v}_{21} = 0$, and

$$\begin{aligned} \tilde{v}_{31} &= \frac{5}{147456}(32a_2^2 - 7)(b_{111} + b_{211} + 3a_{301}) \times (16a_2^2 - 179), \\ \tilde{v}_{41} &= \frac{1}{84934656}(32a_2^2 - 7)(b_{111} + b_{211} + 3a_{301}) \times (4096a_2^4 - 84304a_2^2 + 76201), \\ \tilde{v}_{51} &= \frac{1}{782757787696}(32a_2^2 - 7)(b_{111} + b_{211} + 3a_{301}) \times (68571136a_2^6 + 69609216a_2^4 - 8963283984a_2^2 + 797929021), \\ &\vdots \end{aligned} \tag{25}$$

Hence, the only solution is $a_2 = \pm\frac{1}{4}\sqrt{179}$, for which $\tilde{v}_{41} = \frac{2302911}{262144}(b_{111} + b_{211} + 3a_{301})$, showing that when $a_1 = \frac{5}{2}$, the system can have at most four small limit cycles around the origin.

In conclusion, we have shown that $\tilde{H}_2^1(3) = 5$ based on the ε-order focus values. ■

Remark 1

(i) The method and formulas presented in this section for proving Theorem 1 are different from that in [Han *et al.*, 2009], and the computation is simpler.
(ii) The coefficients a_1 and a_2 are used here to obtain two additional limit cycles towards $\tilde{H}_2^1(3) = 5$, while they do not play any role for quadratic systems in determining $\tilde{H}_2^1(2) = 2$.

3. $\tilde{H}_2^2(3) = 6$ Based on ε^2-Order Focus Values

In this section, we use the ε^2-order focus value to consider the limit cycles around the origin of the quadratic near-Hamiltonian system (10) when all the ε-order focus values equal zero. We have the following result.

Theorem 2. *When all the ε-order focus values vanish, $\tilde{H}_2^2(3) = 6$ based on the analysis of the ε^2-order focus values.*

Proof. First, we need to find the conditions under which all the ε-order focus values, $\tilde{v}_{i1}$, vanish. It is seen from the factor Q_1 given in (18) that there are two cases: (1) $b_{111} + b_{211} + 3a_{301} = 0$; (2) $(a_1 + 1)(a_1 - 2)^2 - 4\,a_2^2 = 0$. Under these two conditions, b_{031} and b_{121} are given by (14) and (16) accordingly. The complete conditions are given below:

Case 1

$$b_{111} = -b_{211} - 3a_{301}, \quad b_{121} = a_1\,b_{021},$$

$$b_{031} = \frac{1}{6}(4a_2b_{021} - a_1b_{211} - 3a_1a_{301} - 2a_{121});$$

Case 2

$$\begin{aligned} a_2 &= \pm\frac{1}{2}(a_1^3 - 3a_1^2 + 4)^{1/2}, \\ b_{121} &= a_1b_{021} + \frac{a_1}{4a_2}(a_1 - 2)(b_{211} + b_{111} + 3a_{301}), \\ b_{031} &= \frac{1}{6}[(a_1 - 2)b_{111} - 2b_{211} + 4a_2b_{021} - 6a_{301} - 2a_{121}]. \end{aligned} \tag{26}$$

In the following, we consider the two cases in detail. It should be noted that the special case $a_1 = \frac{5}{2}$ discussed in the previous section for ε-order focus values is included in the two general cases.

Case 1. This case can have six small-amplitude limit cycles near the origin. Similarly, in order for the origin $(0, 0)$ to be a linear center under perturbation up to ε^2 order, let $b_{012} = 0$. Then under the first set of conditions given in (26), the first ε^2-order focus value is given by

$$\tilde{v}_{12} = \frac{1}{16}[6b_{032} - (a_1 - 2)b_{112} + 2b_{212} - 4a_2b_{022} - 2b_{201}(b_{211} + 3a_{301})]. \tag{27}$$

Solving b_{032} from the equation $\tilde{v}_{12} = 0$ yields

$$b_{032} = \frac{1}{6}[(a_1 - 2)b_{112} - 2b_{212} + 4a_2b_{022} + 2b_{201}(b_{211} + 3a_{301})]. \tag{28}$$

Next, solving $\tilde{v}_{22} = 0$ for b_{122} gives

$$\begin{aligned} b_{122} = a_1b_{022} + \frac{1}{12a_2(5a_1 - 2)} &\times \{3(3a_1^2 + 4a_1 - 20 + 20a_2^2)(b_{112} + b_{212}) \\ &- 2[10a_1a_2b_{211} + (9a_1^2 - 6a_1 + 20a_2^2)b_{021} \\ &+ 3(3a_1 - 10)b_{301} + 6a_2(5a_1 - 3)a_{301} \\ &- 10a_2a_{121}](b_{211} + 3a_{301}) - 12[(3a_1 - 2)a_{121} \\ &+ (3a_1 - 10)a_{301}]b_{021}\}. \end{aligned} \tag{29}$$

Then, similarly solving $\tilde{v}_{32} = 0$ for b_{212} yields

$$\begin{aligned} b_{212} = -b_{112} - \frac{1}{21(a_1^3 - 3a_1^2 + 4 - 4a_2^2)(3a_1^2 + 12a_1 - 4 - 4a_2^2)}&\{2(21a_1^2 - 36a_1 + 20 - 28a_2^2)(b_{211} + 3a_{301}) \\ \times [a_1a_2(2b_{211} - 3a_{301}) - (3a_1^3 - 4a_2^2)b_{021} - 2a_2a_{121}] &- 3[57a_1^4 - 300a_1^3 + 168a_1^2 + 336a_1 - 112 \\ + a_2^2(84a_1^2 - 480a_1 + 80)](b_{211}b_{301} + 3b_{301}a_{301} + 2a_{301}b_{021}) & \\ + 2a_1^2(21a_1^2 - 36a_1 + 20 - 28a_2^2)a_{121}b_{021}\}. & \end{aligned} \tag{30}$$

Then we obtain

$$\begin{aligned} \tilde{v}_{42} &= \frac{1}{36864}Q_{21}F_{21}(a_1, a_2), \\ \tilde{v}_{52} &= -\frac{1}{10616832}Q_{21}M_{21}(a_1, a_2), \\ \tilde{v}_{62} &= \frac{1}{7134511104}Q_{21}N_{21}(a_1, a_2), \\ &\vdots \end{aligned} \tag{31}$$

where

$$\begin{aligned} Q_{21} = \frac{1}{3a_1^2 + 12a_1 - 4 - 4a_2^2}&\{b_{211}[4a_1a_2b_{211} - 2(3a_1^3 - 4a_2^2)b_{021} - 2a_2(2a_{121} - 3a_1a_{301}) \\ &+ 3(a_1^3 - 4a_2^2)b_{301}] - 12a_{121}(a_1^2b_{021} + a_2a_{301}) - [6a_1(2a_1^2b_{021} + 3a_2a_{301}) \\ &- 9(a_1^3 - 4a_2^2)b_{301}]a_{301}\}, \end{aligned} \tag{32}$$

$$F_{21} = 81a_1^4 - 648a_1^3 - 648a_1^2 + 1632a_1 - 880 - 8a_2^2(63a_1^2 - 204a_1 - 212 + 110a_2^2), \tag{33}$$

$$\begin{aligned} M_{21} = 6075a_1^6 &- 92340a_1^5 + 196020a_1^4 + 250272a_1^3 - 1051056a_1^2 + 854208a_1 - 207680 \\ &+ 4a_2^2[3(3123a_1^4 - 7800a_1^3 - 78840a_1^2 + 75552a_1 + 8752) - 4a_2^2(2511a_1^2 \\ &- 53388a_1 - 6564 + 12980a_2^2)], \end{aligned} \tag{34}$$

$$N_{21} = 404595a_1^8 - 35108640a_1^7 + 338936400a_1^6 - 742732416a_1^5 - 1033811424a_1^4 + 3914076672a_1^3 - 2826056448a_1^2 + 116090880a_1 + 446800640 - 16a_2^2\{6003963a_1^6 - 49334184a_1^5 - 43124940a_1^4 + 66313152a_1^3 + 180447696a_1^2 - 288936576a_1 + 236774080 - 2a_2^2[18200889a_1^4 - 36734544a_1^3 - 195474600a_1^2 + 144468288a_1 + 191492880 + 40a_2^2(1092537a_1^2 + 90696a_1 - 2959676 + 349063a_2^2)]\}. \quad (35)$$

Eliminating a_2 from the equations $F_{21}(a_1, a_2) = M_{21}(a_1, a_2) = 0$ yields the solution for a_2:

$$a_2^2 = G_{21}(a_1) = \frac{G_{21}^N}{4G_{21}^D}, \quad (36)$$

where

$$G_{21}^N = 10179a_1^6 - 81864a_1^5 - 179172a_1^4 + 204992a_1^3 - 32496a_1^2 - 124032a_1 + 66880, \quad (37)$$
$$G_{21}^D = 5109a_1^4 + 12076a_1^3 - 75936a_1^2 - 167664a_1 + 48944,$$

and a resultant equation: $R_{21}(a_1) = (a_1 + 2)R_{21}^*(a_1) = 0$, where

$$R_{21}^* = 77571a_1^9 - 1561014a_1^8 + 9024720a_1^7 - 9985760a_1^6 - 33089760a_1^5 + 106013376a_1^4 - 124646144a_1^3 + 66931200a_1^2 - 14081280a_1 + 924160. \quad (38)$$

There are two cases: $G_{21}^D = 0$ and $G_{21}^D \neq 0$. When $G_{21}^D = 0$, a_2 becomes a free parameter, and F_{21} and M_{21} are reduced to (by a Groebner basis reduction)

$$F_{21}^* = -\frac{1}{1703}(1429596a_1^3 + 946728a_1^2 + 7306224a_1 - 2820128) - a_2^2(504a_1^2 - 1632a_1 - 1696 + 880a_2^2),$$

$$M_{21}^* = -\frac{1}{1899636895}(3882430557112320a_1^3 + 744175894966272a_1^2 - 20372431117787136a_1 + 5134141771333632). \quad (39)$$

Obviously, $M_{21}^* \neq 0$ when $G_{21}^D = 0$. Thus, the case $G_{21}^D = 0$ gives less number of limit cycles.

Now we assume $G_{21}^D \neq 0$. Then for a_2 given in (37), N_{21} becomes

$$N_{21}^* = \frac{912384}{(G_{21}^D)^4}(a_1 + 2)^4(222054091780227759a_1^{20} - 6373884288696994836a_1^{19} + 57872670509850006330a_1^{18} - 95008467211017717240a_1^{17} - 1070901986288903038080a_1^{16} + 3405647993468931379872a_1^{15} + 7502901541666624628032a_1^{14} - 30682840424726820986240a_1^{13} - 18272114454422946840320a_1^{12} + 141218579798872808061440a_1^{11} - 50908318975749623584768a_1^{10} - 332515317146691984467968a_1^9 + 422613421162028272250880a_1^8 + 123263528407830337413120a_1^7 - 673363362757398717480960a_1^6 + 660101849483275511496704a_1^5 - 341539074886857685467136a_1^4 + 109461214641140744192000a_1^3 - 21805360740053211545600a_1^2 + 2410620747661516472320a_1 - 108965872871895203840.$$

Since $G_{21}(-2) = -4 < 0$, $a_1 = -2$, which satisfies $R_{21}(a_1) = 0$ is not a solution. We only need to consider the real solutions of $R_{21}^* = 0$. First, it is easy to see that $N_{21}^* \neq 0$ when $R_{21}^* = 0$, under the condition $G_{21}^D \neq 0$. Therefore, there exist parameter values such that (actually $R_{21}^* = 0$ has six real solutions) $\tilde{v}_{i1}, i = 0, 1, \ldots, 5$, but $\tilde{v}_{i6} \neq 0$. Thus, we know that Case 1 can have at most six small-amplitude limit cycles around the origin. Furthermore, using (31)–(35) and (36) it can be shown that

$$D_{21} = \det \begin{bmatrix} \dfrac{\partial \tilde{v}_{42}}{\partial a_1} & \dfrac{\partial \tilde{v}_{42}}{\partial a_2} \\ \dfrac{\partial \tilde{v}_{52}}{\partial a_1} & \dfrac{\partial \tilde{v}_{52}}{\partial a_2} \end{bmatrix} = -\frac{1}{97844723712}\left[\frac{\partial(Q_{21}F_{21})}{\partial a_1}\frac{\partial(Q_{21}M_{21})}{\partial a_2} - \frac{\partial(Q_{21}F_{21})}{\partial a_2}\frac{\partial(Q_{21}M_{21})}{\partial a_1}\right]$$

$$= -\frac{1}{97844723712}\left[\left(\frac{\partial Q_{21}}{\partial a_1}F_{21} + Q_{21}\frac{\partial F_{21}}{\partial a_1}\right)\left(\frac{\partial Q_{21}}{\partial a_2}M_{21} + Q_{21}\frac{\partial M_{21}}{\partial a_2}\right)\right.$$
$$\left. - \left(\frac{\partial Q_{21}}{\partial a_2}F_{21} + Q_{21}\frac{\partial F_{21}}{\partial a_2}\right)\left(\frac{\partial Q_{21}}{\partial a_1}M_{21} + Q_{21}\frac{\partial M_{21}}{\partial a_1}\right)\right]$$

$$= -\frac{Q_{21}^2}{97844723712}\left(\frac{\partial F_{21}}{\partial a_1}\frac{\partial M_{21}}{\partial a_2} - \frac{\partial F_{21}}{\partial a_2}\frac{\partial M_{21}}{\partial a_1}\right) \quad \text{(due to } F_{21} = M_{21} = 0 \text{ at the critical point)}$$

$$= -\frac{11a_2(a_1+1)(a_1+2)^2 Q_{21}^2}{131072(G_{21}^D)^3}(1249408869579a_1^{16} - 25349207372424a_1^{15} + 133358276098944a_1^{14}$$
$$+ 189343131380256a_1^{13} - 1946402075623936a_1^{12} - 4007107219164032a_1^{11} + 6336827152556032a_1^{10}$$
$$+ 5811037111648768a_1^{9} - 26195046324699648a_1^{8} - 27920651374812979 2a_1^{7}$$
$$+ 310501457384177664a_1^{6} + 311258045221625856a_1^{5} - 802642255883698176a_1^{4}$$
$$+ 590462898498666496a_1^{3} - 201499555600531456a_1^{2} + 32823637655552000a_1 - 2062302224384000)$$
$$\neq 0,$$

since $Q_{21} \neq 0$ and $G_{21}^D \neq 0$, and the 16th degree polynomial factor in the above expression does not contain any real roots of $R_{21}^*(a_1) = 0$.

Summarizing the above results shows that when all the ε-order focus values, $\tilde{v}_{i1}$, equal zero, one can perturb $b_{012}, b_{032}, b_{122}, b_{212}, a_2$ and a_1 backwards to generate

$$|\tilde{v}_{j2}| \ll |\tilde{v}_{(j+1)2}| \quad \text{and} \quad \tilde{v}_{j2}\,\tilde{v}_{(j+1)2} < 0 \quad \text{for } j = 0, 1, \ldots, 5.$$

This shows that for Case 1 there can be indeed six small-amplitude limit cycles in the vicinity of the origin, based on the analysis of ε^2-order focus values.

Case 2. For this case, there exist five small-amplitude limit cycles in the neighborhood of the origin. Again let $b_{012} = 0$ under which the origin $(0, 0)$ becomes a linear center up to ε^2 order. Then under the second set of conditions given in (26), solving b_{032} from the equation $\tilde{v}_{12} = 0$ results in

$$b_{032} = \frac{1}{6}\{(a_1 - 2)b_{112} - 2b_{212} + 4a_2 b_{022} - [2b_{021} + 2b_{201} - (a_1 - 4)b_{101}]b_{111} - 2(b_{211} + 3a_{301})b_{101}\}. \tag{40}$$

Next, solving $\tilde{v}_{22} = 0$ for b_{122} yields

$$b_{122} = a_1 b_{022} + \frac{a_1 a_2}{(a_1+1)(a_1-2)}(b_{112} + b_{212}) - \frac{1}{6(a_1+1)(5a_1-2)(a_1-2)^2}$$
$$\times \{(a_1 - 2)[(10a_1^3 - 33a_1^2 + 6a_1 + 28)b_{111}^2 + 6a_1 b_{211}^2 - ((a_1 - 2)(23a_1 + 14)b_{211}$$
$$- 2(5a_1^2 - 11a_1 - 10)a_{121} + 3(17a_1^2 - 14a_1 - 16)a_{301})b_{111} - 6a_1(2a_{121}b_{211} + 6a_{121}a_{301} + 9a_{301}^2)]$$
$$- 2a_2[2((a_1 + 1)(5a_1^2 - 2a_1 - 28)b_{111} + 3(a_1 - 4)(3a_1 + 4)b_{211} - 6(3a_1 - 2)a_{121}$$

$$+3(9a_1^2-30a_1-28)a_{301})b_{021}+3((5a_1^3-7a_1^2+8a_1-60)b_{101}+(5a_1^3-10a_1^2+4a_1-40)b_{201}$$
$$+2(3a_1-16)b_{301})b_{111}+3((5a_1^2+4a_1-60)b_{101}+2(a_1^2b_{201}-20b_{201}-6b_{301}))(b_{211}+3a_{301})]\}. \quad (41)$$

Further, we solve $\tilde{v}_{32}=0$, but its expression shows that it does not contain any more b_{ij2} coefficients. Thus, we may use some b_{ij1} coefficients to solve the equation $\tilde{v}_{32}=0$. For example, solving for b_{021} leads to

$$\begin{aligned}
b_{301} = {} & \frac{2a_2}{3(a_1+1)(a_1-2)^2[21(a_1-2)(a_1^2+2a_1+2)(a_1^2-8a_1+4)b_{111} - 2(21a_1^4+19a_1^3-30a_1^2-324a_1+152)(b_{211}+3a_{301})]} \\
& \times\{(a_1-2)[21(a_1+1)(a_1-2)(a_1^2-2)(a_1^2-8a_1+4)b_{111}^2-(14a_1^5+7a_1^4-226a_1^3+100a_1^2 \\
& +56a_1-32)b_{211}^2+((7a_1^6-105a_1^5+33a_1^4+744a_1^3-528a_1^2-528a_1+368)b_{211} \\
& +6a_1(7a_1^4-84a_1^3+104a_1^2+160a_1-112)a_{121}+3(28a_1^6-294a_1^5+327a_1^4+656a_1^3-336a_1^2 \\
& -672a_1+336)a_{301})b_{111}+2((14a_1^5-203a_1^4+248a_1^3+424a_1^2-256a_1+16)a_{121} \\
& -3(5a_1-2)(a_1^2+2a_1+2)(7a_1^2-20a_1+4)a_{301})b_{211}+3(2(14a_1^5-203a_1^4+248a_1^3+424a_1^2 \\
& -256a_1+16)a_{121}-3a_1(56a_1^4-95a_1^3+30a_1^2-332a_1+152)a_{301})a_{301}] \\
& -2a_2[21(a_1-2)(a_1^2-8a_1+4)(3(a_1+2)^2b_{101}+(a_1+2)(a_1^2+2a_1+4)b_{201} \\
& +4(a_1+1)^2b_{021})b_{111}+21(a_1^2-4)(a_1^2-8a_1+4)(b_{211}+3a_{301})(3(a_1+2)b_{101} \\
& +(a_1^2+2a_1+4)b_{201})+2((14a_1^6-21a_1^5-390a_1^4+138a_1^3+1116a_1^2+24a_1-368)b_{211} \\
& +6a_1^2(7a_1^3-42a_1^2+36a_1+8)a_{121}+6(7a_1^6-21a_1^5-132a_1^4+29a_1^3+462a_1^2 \\
& +84a_1-168)a_{301})b_{021}]\}. \qquad (42)
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\tilde{v}_{42} &= -\frac{7}{6134}Q_{22}F_{22}(a_1,a_2), \\
\tilde{v}_{52} &= \frac{7}{1769472}Q_{22}M_{22}(a_1,a_2), \\
\tilde{v}_{62} &= \frac{1}{169869312}Q_{22}N_{22}(a_1,a_2), \\
&\vdots
\end{aligned} \qquad (43)$$

where

$$\begin{aligned}
Q_{22} = {} & \frac{a_2(a_1+2)(b_{111}+b_{211}+3a_{301})}{(a_1+1)(a_1-2)[21(a_1-2)(a_1^2+2a_1+2)(a_1^2-8a_1+4)b_{111} - 2(21a_1^4+19a_1^3-30a_1^2-324a_1+152)(b_{211}+3a_{301})]} \\
& \times\{(a_1-2)[(2a_1-1)(a_1+2)^3b_{211}^2-(a_1+2)((a_1^4-6a_1^3-8a_1^2+4a_1+8)b_{111} \\
& +2(2a_1-1)(a_1^2-2a_1-2)a_{121}-6(2a_1-1)(a_1^2+2a_1+2)a_{301})b_{211}
\end{aligned}$$

$$+3(2a_1^2(a_1^2+2a_1+2)a_{121}-(a_1^5-a_1^4-14a_1^3-10a_1^2+8a_1+8)a_{301})b_{111}$$
$$-3(a_1+2)(2a_1-1)(2(a_1^2-2a_1-2)a_{121}-3a_1^2a_{301})a_{301}]$$
$$+2a_2[(a_1+2)(3a_1^2-4)(3(a_1+2)b_{101}+(a_1^2+2a_1+4)b_{201})(b_{211}+3a_{301})$$
$$+2((a_1+2)(2a_1^4+9a_1^3+4a_1^2-8a_1-8)b_{211}+6(a_1^5+5a_1^4+8a_1^3-a_1^2-8a_1-4)a_{301}$$
$$+6a_1^2(a_1^2+2a_1+2)a_{121})b_{021}]\}, \tag{44}$$

$$F_{22}=55a_1^4-424a_1^3+1104a_1^2-832a_1+16, \tag{45}$$

$$M_{22}=3245a_1^6-46164a_1^5+247836a_1^4-625888a_1^3+754608a_1^2-365376a_1+25664, \tag{46}$$

$$N_{22}=1745315a_1^8-13055560a_1^7-1637032a_1^6+308566208a_1^5-1248116224a_1^4$$
$$+2205908864a_1^3-1880103808a_1^2+661809152a_1-34712320. \tag{47}$$

It is easy to see that $Q_{22}=0$ yields $\tilde{v}_{42}=\tilde{v}_{52}=\tilde{v}_{62}=\cdots=0$. Thus, to obtain maximal number of limit cycles, let $F_{22}=0$, but $M_{22}\neq 0$. In fact, the equation $F_{22}=0$ has two real solutions for which $M_{22}\neq 0$. Further, properly perturbing $b_{012}, b_{032}, b_{122}, b_{301}$ and a_1 backwards, we obtain five small-amplitude limit cycles for Case 2 based on the analysis of ε^2-order focus values.

The proof of Theorem 2 is complete. ■

Remark 2. It is easy to see from (31) and (33), as well as (43) and (45) that the special case $a_1=\frac{5}{2}$ gives one less limit cycle for each case, that is, when $a_1=\frac{5}{2}$, Case 1 has five small-amplitude limit cycles, while Case 2 yields four limit cycles.

4. Seven Limit Cycles Obtained from ε^3-Order Focus Values

In this section and the following two sections, we will not investigate all possible cases, but instead we find one case for each order of focus value, showing seven limit cycles from ε^3-order focus values, eight limit cycles from ε^4-order focus values, and eight limit cycles from ε^5-order focus values.

Theorem 3. *When all the ε- and ε^2-order focus values vanish, there exist seven limit cycles obtained from the analysis of the ε^3-order focus values.*

Proof. Suppose the conditions given in (26) for Case 1 are satisfied, under which all the ε-order focus values vanish. Further assume the following conditions hold, for which all the ε^2-order focus values become zero.

$$b_{021}=\frac{b_{211}+3a_{301}}{2[(3a_1^3-4a_2^2)b_{211}+6a_1^2a_{121}+6a_1^3a_{301}]}\{3a_1^3b_{301}-2a_2[2a_{121}-a_1(2b_{211}-3a_{301})+6a_2b_{301}]\},$$

$$b_{212}=\frac{1}{[(3a_1^3-4a_2^2)b_{211}+6a_1^2a_{121}+6a_1^3a_{301}]}\{[(3a_1^3-4a_2^2)b_{301}+4a_1a_2a_{301}]b_{211}^2$$
$$+6a_1a_2a_{301}^2(b_{211}-3a_{301})+[2(3a_1^2b_{301}-2a_2a_{301})a_{121}+3(3a_1^3-4a_2^2)a_{301}b_{301}](b_{211}+3a_{301})$$
$$-(3a_1^3-4a_2^2)b_{211}(b_{112}-3b_{301}a_{301})-6a_1^2(a_{121}+a_1a_{301})b_{112}\},$$

$$b_{122}=-\frac{1}{2[(3a_1^3-4a_2^2)b_{211}+6a_1^2a_{121}+6a_1^3a_{301}]}\{[2a_1^2(a_1+1)b_{211}-4a_1a_2(a_1+1)b_{301}+2a_1(a_1+1)a_{121}$$
$$+(16a_1^3+7a_1^2-12a_2^2)a_{301}]b_{211}^2+3[(14a_1^3-a_1^2-12a_2^2)b_{211}-12a_1a_2(a_1+1)b_{301}+2a_1(4a_1-5)a_{121}$$
$$+6a_1^2(2a_1-1)a_{301}]a_{301}^2-2[6a_2(a_1+1)(a_{121}+2a_1a_{301})b_{301}-(-2a_1a_{121}-2a_{121}-2a_1a_{301}$$
$$+7a_{301}a_1^2)a_{121}]b_{211}-2a_1[(3a_1^3-4a_2^2)b_{211}+6a_1^2a_{121}+6a_1^3a_{301}]b_{022}$$
$$-12(a_1+1)(a_{121}+3a_2b_{301})a_{121}a_{301}\},$$

$$b_{032} = -\frac{1}{6[(3a_1^3 - 4a_2^2)b_{211} + 6a_1^2 a_{121} + 6a_1^3 a_{301}]}\{[-4a_1a_2b_{211} + (3a_1^3 + 4a_2^2)b_{301} + 4a_2a_{121} - 10a_1a_2a_{301}]b_{211}^2 + 3[4a_1a_2b_{211} + 3(3a_1^3 + 4a_2^2)b_{301} + 4a_2a_{121} + 6a_1a_2a_{301}]a_{301}^2 + 2[6a_1^2(b_{211} + 3a_{301})b_{301} + 8a_2a_{301}b_{211} - 3a_1^3b_{112} - 12a_1^2a_2b_{022}]a_{121} + 6[(3a_1^3 + 4a_2^2)b_{211}b_{301} - a_1^4b_{112} - 4a_1^3a_2b_{022}]a_{301} - (3a_1^3 - 4a_2^2)(a_1b_{112} + 4a_2b_{022})b_{211}\}.$$

Here, it is assumed that $(3a_1^3 - 4a_2^2)b_{211} + 6a_1^2a_{121} + 6a_1^3a_{301} \neq 0$. Similarly, first let $b_{013} = 0$ under which the origin $(0, 0)$ becomes a linear center under perturbation up to ε^3 order. Then, solving $\tilde{v}_{13} = 0$ for b_{033} yields

$$b_{033} = -\frac{1}{6[(3a_1^3 - 4a_2^2)b_{211} + 6a_1^2a_{121} + 6a_1^3a_{301}]}\{[((3a_1^3 - 4a_2^2)((a_1 - 2)b_{102} - 2b_{202} - 2b_{022}) + 4a_1a_2b_{112})b_{211} + (3(a_1^3 - 4a_2^2)b_{301} - 2a_2(2a_{121} - 3a_1a_{301}))b_{112} + 3(2a_1^2a_{121} + (5a_1^3 - 4a_2^2)a_{301}) \times (a_1b_{102} - 2b_{102} - 2b_{202} - 2b_{022}) - (3a_1^3 - 4a_2^2)(a_1b_{113} - 2b_{113} - 2b_{213} + 4a_2b_{023})]b_{211} + 3[(3(a_1^3 - 4a_2^2)b_{301} - 4a_2a_{121} - 6a_1a_2a_{301})b_{112} + 6a_1^2(a_{121} + a_1a_{301})(a_1b_{102} - 2b_{102} - 2b_{202}) - 12a_1^2(a_{121} + a_1a_{301})b_{022} - 2a_1^3(a_1b_{113} - 2b_{113} + 4a_2b_{023} - 2b_{213})]a_{301} - 6a_1^2(a_1b_{113} - 2b_{113} + 4a_2b_{023} - 2b_{213})a_{121}\}. \tag{48}$$

Further, solving $\tilde{v}_{23} = 0$ for b_{123}, $\tilde{v}_{33} = 0$ for b_{213}, and $\tilde{v}_{43} = 0$ for b_{302} yields (the lengthy expressions of $b_{123}, b_{213}, b_{302}$ are omitted here for simplicity)

$$\tilde{v}_{53} = -\frac{77}{294912}Q_{31}F_{31}, \quad \tilde{v}_{63} = \frac{11}{14155776}Q_{31}M_{31}, \quad \tilde{v}_{73} = -\frac{55}{10871635968}Q_{31}N_{31}, \ldots \tag{49}$$

where

$$Q_{31} = \frac{b_{211} + 3a_{031}}{[(3a_1^3 - 4a_2^2)b_{211} + 6a_1^2a_{121} + 6a_1^3a_{301}] \times [81a_1^4 - 648a_1^3 - 648a_1^2 + 1632a_1 - 880 - 8a_2^2(63a_1^2 - 204a_1 - 212 + 110a_2^2)]}, \tag{50}$$

and F_{31}, M_{31} and N_{31} are functions of $a_1, a_2, b_{211}, b_{301}, a_{121}$ and a_{301}. $b_{211} + 3a_{031} = 0$ (i.e. $Q_{31} = 0$) yields $\tilde{v}_{5i} = 0, i = 3, 4, \ldots$. So, in order to obtain maximal number of limit cycles, assume $b_{211} + 3a_{031} \neq 0$. Then, eliminating b_{211} from the equations $F_{31} = 0$ and $M_{31} = 0$ results in an expression $b_{211}^{(1)}$ and a resultant equation $F_{32} = 0$; similarly, eliminating b_{211} from the equations $F_{31} = 0$ and $N_{31} = 0$ results in another expression $b_{211}^{(2)}$ and a resultant equation $M_{32} = 0$; it is found that $b_{211}^{(1)} \equiv b_{211}^{(2)}$, and so let $b_{211} = b_{211}^{(1)}$. The two resultant equations are given by

$$F_{32} = Q_{32}F_{33}(a_1, a_2), \quad M_{32} = Q_{32}M_{33}(a_1, a_2), \tag{51}$$

where

$$Q_{32} = -a_2(a_1^3 - 3a_1^2 - 4a_2^2 + 4)(2a_{121} + 3a_1a_{301} - 2a_1b_{211})\{[2(a_1^3 - 2a_2^2)(2a_1^2(a_1^2a_{121} + 2a_2^2a_{301})a_{121} - a_2^2(a_1^3 - 4a_2^2)a_{301}^2)b_{211} - a_2^2(2a_1(11a_1^4a_{301} - 76a_1a_2^2a_{301} - 58a_1^3a_{121} + 8a_2^2a_{121})a_{121} + (3a_1^6 + 16a_1^3a_2^2 - 80a_2^4)a_{301}^2)a_{301} + 8a_1^3(4a_1^3 + a_2^2)a_{121}^3]b_{211} - 3[a_1((3a_1^6a_{301} - 4a_1^5a_{121} + 4a_1^3a_2^2a_{301} - 32a_1^4a_{301} - 72a_1^2a_2^2a_{121})a_{121} - 2a_1a_2^2(a_1^3 - 36a_2^2)a_{301}^2)a_{121}$$

$$- a_2^2(3a_1^6 - 12a_1^3a_2^2 - 32a_2^4)a_{301}^3]a_{301} + 60a_1^5a_{121}^4\}\{[(a_1(a_1^3 - 4a_2^2)(a_1^3 - 2a_2^2)b_{211}$$
$$+ 6(a_1^3 - 4a_2^2)(a_1^3 + a_2^2)a_{121} + 3a_1(3a_1^6 - 7a_1^3a_2^2 - 4a_2^4)a_{301})b_{211} + 12(a_1^2(a_1^3 + 4a_2^2)a_{121}$$
$$+ (3a_1^6 + 8a_1^3a_2^2 - 4a_2^4)a_{301})a_{121} + 9a_1(3a_1^6 + 4a_1^3a_2^2 - 8a_2^4)a_{301}^2]b_{211} + (2a_{121} + 3a_1a_{301})$$
$$\times [4a_1((a_1^3 + 16a_2^2)a_{121} + 3a_1(a_1^3 + 10a_2^2)a_{301})a_{121} + 9(a_1^3 + a_2^2)(a_1^3 + 4a_2^2)a_{301}^2]\}, \quad (52)$$

$$F_{33} = 405a_1^4 + 6264a_1^3 + 6264a_1^2 - 5664a_1 + 1360 - 8a_2^2(99a_1^2 + 708a_1 + 524 - 170a_2^2), \quad (53)$$

$$M_{33} = 8991a_1^6 + 96228a_1^5 - 890028a_1^4 - 397728a_1^3 + 2094480a_1^2 - 1063872a_1 + 177344$$
$$+ 4a_2^2[14553a_1^4 + 269064a_1^3 + 425592a_1^2 - 328032a_1 - 89328$$
$$- 4a_2^2(8307a_1^2 + 66492a_1 + 22332 - 11084a_2^2)]. \quad (54)$$

Eliminating a_2 from the equations $F_{33}(a_1, a_2) = M_{33}(a_1, a_2) = 0$ yields the solution for a_2:

$$a_2^2 = G_3(a_1) = \frac{G_3^N}{4G_3^D}, \quad (55)$$

where

$$G_3^N = 105165a_1^6 + 1945512a_1^5 + 4062996a_1^4 + 620224a_1^3 - 605712a_1^2 + 294528a_1 - 70720, \quad (56)$$

$$G_3^D = 17499a_1^4 + 78836a_1^3 + 664464a_1^2 + 1162416a_1 - 72176,$$

and a resultant equation: $R_3(a_1) = (a_2+2)R_3^*(a_1) = 0$, where

$$R_3^* = 224181a_1^9 + 7250742a_1^8 + 51634440a_1^7 - 140047504a_1^6 - 106276800a_1^5 + 502223232a_1^4 - 398093440a_1^3 + 109666560a_1^2 - 6539520a_1 + 86528. \quad (57)$$

Similarly, we can discuss two cases: $G_3^D = 0$ and $G_3^D \neq 0$, and can show that the case $G_3^D = 0$ yields one less limit cycle than the case $G_3^D \neq 0$, so we assume $G_3^D \neq 0$. Since $G_3(-2) = -4 < 0$, we only need to consider the real roots of G_3^*, which consist of three real solutions:

$$a_1 = -2.0772478597\ldots, 0.0187162703\ldots, 0.0563411398\ldots, \quad (58)$$

all of them satisfy $G_3 > 0$. However, all of them give $F_{31} \neq 0, M_{31} \neq 0$ and $N_{31} \neq 0$, indicating that there are no parameter values which can be chosen to obtain $\tilde{v}_{53} = \tilde{v}_{63} = \tilde{v}_{73} = 0$. Therefore, eight limit cycles are not possible for this case.

In other words, under the conditions given in Case 1 of (26) and that given at the beginning of this section, the system can have at most seven small limit cycles. By proper perturbations on the parameters $b_{013}, b_{033}, b_{123}, b_{213}, b_{302}, a_2$ and a_1 we can obtain seven limit cycles. It should be noted that there exist an infinite number of choices to have seven limit cycles since there is a free parameter. We can ignore the equation $N_{31} = 0$ and only need to find solutions satisfying $F_{32} = 0$. For simplicity, let

$$a_{121} = a_{301} = b_{303} = b_{124} = 0.$$

Thus, the conditions under which all the ε- and ε^2-order focus values vanish become

$$b_{111} = -b_{211}$$

$$b_{121} = \frac{a_1[4a_1a_2b_{211} + 3(a_1^3 - 4a_2^2)b_{301}]}{2(3a_1^3 - 4a_2^2)},$$

$$b_{031} = -\frac{(a_1^3 - 4a_2^2)(a_1b_{211} - 2a_2b_{301})}{2(3a_1^3 - 4a_2^2)},$$

$$b_{021} = \frac{4a_1a_2b_{211} + 3(a_1^3 - 4a_2^2)b_{301}}{2(3a_1^3 - 4a_2^2)},$$

$$b_{212} = -b_{112} + b_{301}b_{211},$$

$$b_{122} = a_1b_{022} - \frac{a_1(a_1 + 1)(a_1b_{211} - 2a_2b_{301})b_{211}}{3a_1^3 - 4a_2^2},$$

$$b_{032} = \frac{1}{6}(a_1b_{112} + 4a_2b_{022}) + \frac{b_{211}[4a_1a_2b_{211} - (3a_1^3 + 4a_2^2)b_{301}]}{63a_1^3 - 4a_2^2}.$$

Then, having found b_{033} from $\tilde{v}_{13} = 0$, b_{123} from $\tilde{v}_{23} = 0$, b_{213} from $\tilde{v}_{33} = 0$, and b_{302} from $\tilde{v}_{43} = 0$, we obtain $\tilde{v}_{53}, \tilde{v}_{63}$ and $\tilde{v}_{73}$, given in (49), with the new expressions for Q_{31}, F_{31}, M_{31}:

$$Q_{31} = \frac{\{-2a_1[a_1(a_1^3-4a_2^2)b_{211}+2a_2(a_1^3+4a_2^2)b_{301}]b_{211}+(9a_1^6-8a_1^3a_2^2+16a_2^4)b_{301}^2\}b_{211}}{2(3a_1^3-4a_2^2)[81a_1^4-648a_1^3-648a_1^2+1632a_1-880-8a_2^2(63a_1^2-204a_1-212+110a_2^2)]}, \tag{59}$$

$$\begin{aligned} F_{31} = {} & 3a_1^3(a_1-2)^4(27a_1^3+378a_1^2+468a_1-104)-2a_2^2\{(a_1-2)^2(720a_1^6+6915a_1^5+558a_1^4 \\ & -8632a_1^3+4432a_1^2+11440a_1-2080)+4a_2^2[117a_1^7-51a_1^6+3769a_1^5-12074a_1^4 \\ & +15112a_1^3+6368a_1^2-21232a_1-2080-4a_2^2(13a_1^5+208a_1^4-473a_1^3+4034a_1^2 \\ & -3196a_1-3496-4a_2^2(13a_1^2+91a_1-422))]\}, \end{aligned} \tag{60}$$

$$\begin{aligned} M_{31} = {} & 3a_1^3(a_1-2)^4(4617a_1^5+54270a_1^4-164376a_1^3-201744a_1^2+286416a_1-40352) \\ & -2a_2^2\{(a_1-2)^2(30510a_1^8+24813a_1^7-2540934a_1^6+2709732a_1^5+3627528a_1^4-4240528a_1^3 \\ & -6627104a_1^2+6535360a_1-807040)-4a_2^2[1053a_1^9-4923a_1^8-596022a_1^7+6131316a_1^6 \\ & -14693784a_1^5+10560048a_1^4+5270496a_1^3-12060736a_1^2-1279360a_1+3660800 \\ & -4a_2^2(3(4706a_1^7+2051a_1^6+179168a_1^5-682532a_1^4+1704256a_1^3-1476144a_1^2-1175936a_1 \\ & +738880)-4a_2^2(1261a_1^5+22945a_1^4-47054a_1^3+498292a_1^2-838664a_1 \\ & -164480-4a_2^2(1261a_1^2+8827a_1-52154)))]\}, \end{aligned} \tag{61}$$

$$N_{31} = \cdots$$

Now eliminating a_2 from the equations $F_{31}=0$ and $M_{31}=0$ yields a solution $a_2^2 = G_{31}(a_1)$ and a resultant equation: $R_{31}(a_1) = a_1(a_1-1)(a_1-2)(a_1+2)R_{31}^*(a_1) = 0$, where

$$\begin{aligned} R_{31}^* = {} & (4455a_1^8+7236a_1^7+313672a_1^6-2213168a_1^5+6903296a_1^4-12532544a_1^3+13646720a_1^2 \\ & -7033088a_1+1112320)(13351908375a_1^{17}+265409355510a_1^{16}+1081982776752a_1^{15} \\ & +9569452329504a_1^{14}-39169526239872a_1^{13}-127208995583l808a_1^{12}+1357649044542540 8a_1^{11} \\ & -50519661087526400a_1^{10}+86886455726370304a_1^9+37595181861387264a_1^8 \\ & -339423155743125504a_1^7+464992256852189184a_1^6-282090445527318528a_1^5 \\ & +75653277426057216a_1^4-3285541114019840a_1^3-2141390834892800a_1^2 \\ & +183591357317120a_1+21921657651200). \end{aligned}$$

Solving $R_{31}(a_1) = 0$, we obtain seven real solutions of a_1, which satisfy $G_{31}(a_1) > 0$. Furthermore, for the solution $a_2^2 = G_{31}(a_1)$, we, with the aid of (49), (59)–(61), obtain the following determinant:

$$\begin{aligned} D_{31} = {} & \det\begin{bmatrix} \frac{\partial \tilde{v}_{53}}{\partial a_1} & \frac{\partial \tilde{v}_{53}}{\partial a_2} \\ \frac{\partial \tilde{v}_{63}}{\partial a_1} & \frac{\partial \tilde{v}_{63}}{\partial a_2} \end{bmatrix} = -\frac{77Q_{31}^2}{4174708211712}\left(\frac{\partial F_{31}}{\partial a_1}\frac{\partial M_{31}}{\partial a_2} - \frac{\partial F_{31}}{\partial a_2}\frac{\partial M_{31}}{\partial a_1}\right) \\ = {} & -\frac{77a_2Q_{31}^2}{521838526464}\{9a_1^3(a_1-2)^6(5275530a_1^{10}+120407715a_1^9+662209668a_1^8-253452168a_1^7 \\ & -2297545488a_1^6+476154816a_1^5+2172215616a_1^4-960958336a_1^3-775560448a_1^2+315244800a_1 \\ & -36341760)+2a_2^2[3(a_1-2)^4(13533156a_1^{14}+140321484a_1^{13}-363560805a_1^{12}+1941075063a_1^{11} \end{aligned}$$

$$+9300506472a_1^{10}+7846977804a_1^{9}-22064876352a_1^{8}-31397354272a_1^{7}+45004347904a_1^{6}$$
$$+14378089344a_1^{5}-43470117120a_1^{4}+18162170624a_1^{3}+14755420160a_1^{2}-5787059200a_1$$
$$+726835200)-2a_2^2((a_1-2)^2(133290846a_1^{14}+1031602554a_1^{13}-3336210504a_1^{12}$$
$$+10721006703a_1^{11}+63250405830a_1^{10}-63211186044a_1^{9}-332773753944a_1^{8}+303623111776a_1^{7}$$
$$+723292838848a_1^{6}-429781438848a_1^{5}-907565685504a_1^{4}+404989385472a_1^{3}+480346553856a_1^{2}$$
$$-163481144320a_1+48647505920)-4a_2^2(492804a_1^{15}+189417150a_1^{14}+1257308766a_1^{13}$$
$$+1091200455a_1^{12}-6911221320a_1^{11}-53511530364a_1^{10}+170729575280a_1^{9}-12997675776a_1^{8}$$
$$-556070255616a_1^{7}+618355580032a_1^{6}+499206518784a_1^{5}-854373933312a_1^{4}-105393625088a_1^{3}$$
$$+237202461696a_1^{2}+110981984256a_1+172705095680+4a_2^2(11329929a_1^{13}-24180273a_1^{12}$$
$$+250285566a_1^{11}-2939960835a_1^{10}+10735983315a_1^{9}-41206928154a_1^{8}+78611654640a_1^{7}$$
$$+35625521792a_1^{6}-251704829280a_1^{5}+36530564160a_1^{4}+255492287744a_1^{3}-25643866368a_1^{2}$$
$$-175674537216a_1-100844423680-4a_2^2(1554462a_1^{11}+25684035a_1^{10}-55825172a_1^{9}$$
$$+945083700a_1^{8}-4386102786a_1^{7}+15904459144a_1^{6}-26385529560a_1^{5}-23846931840a_1^{4}$$
$$+59245274400a_1^{3}+7351979904a_1^{2}-65788578432a_1-35236972544-4a_2^2(81965a_1^{9}$$
$$+3184467a_1^{8}+13001586a_1^{7}-82564118a_1^{6}+690326559a_1^{5}-2945002254a_1^{4}+3732548152a_1^{3}$$
$$+2137222128a_1^{2}-10504933200a_1-6469603936-4a_2^2(114751a_1^{6}-3072147a_1^{4}+1974258a_1^{5}$$
$$-57535244a_1^{3}+160049340a_1^{2}-468735936a_1-549537056$$
$$-52a_2^2(2a_1+7)(1261a_1^2+8827a_1-97034)))))))]\}\neq 0,$$

for the roots of R_{31}^*. Therefore, we have shown that when all the ε- and ε^2-order focus values equal zero, one can choose $b_{013}, b_{033}, b_{123}, b_{213}, b_{112}, a_2$ and a_1 such that $\tilde{v}_{i3}=0, i=0,1,\ldots,6$, but $\tilde{v}_{73}\neq 0$. Further, we can perturb these coefficients backwards to generate

$$|\tilde{v}_{j3}|\ll|\tilde{v}_{(j+1)3}| \quad \text{and} \quad \tilde{v}_{j3}\tilde{v}_{(j+1)3}<0 \quad \text{for } j=0,1,\ldots,6.$$

This shows that for the case considered in this section, seven small-amplitude limit cycles exist in the vicinity of the origin based on the analysis of ε^3-order focus values.

This completes the proof of Theorem 3. ■

5. Eight Limit Cycles Obtained from ε^4-Order Focus Values

In this section, we will present a case for the near-Hamiltonian system (10) when all the ε-, ε^2- and ε^3-order focus values vanish.

Theorem 4. *When all the ε-, ε^2-, and ε^3-order focus values vanish, there exist eight limit cycles obtained from the analysis of the ε^4-order focus values.*

Proof. The conditions under which all the ε^3-order focus values become zero are given below:

$$b_{301}=0,\quad b_{211}=\frac{1}{2a_1}(2a_{121}+3a_1a_{301}),$$

$$b_{033}=\frac{1}{36a_1(a_1^3-4a_2^2)}\{6a_1(a_1^3-4a_2^2)(a_1b_{113}+4a_2b_{023})-a_1(2a_{121}+9a_1a_{301})[3(a_1^3-4a_2^2)b_{102}$$
$$+8a_2b_{112}+9(2a_{121}-3a_1a_{301})a_{301}]-2[2(15a_1^3+4a_2^2)a_{121}+9a_1(3a_1^3+4a_2^2)a_{301}]b_{022}\},$$

$$b_{123} = a_1 b_{023} + \frac{1}{12a_1^2(a_1^3 - 4a_2^2)}\{8a_1^2 a_2(a_1+1)(8a_{121} + 9a_1 a_{301})b_{022} + 4(a_1+1)[a_1^3 b_{112} + 9a_2(a_{121} + 3a_1 a_{301})a_{301}]a_{121} + 9a_1^2[2(2a_1^3 + a_1^2 - 4a_2^2)b_{112} - 27a_2(a_1+1)a_{301}^2]a_{301}\},$$

$$b_{213} = -b_{113} + \frac{1}{12a_1(a_1^3 - 4a_2^2)}\{(2a_{121} + 9a_1 a_{301})[6(a_1^3 - 4a_2^2)(b_{102} + b_{202}) + a_1(8a_2 b_{112} + 9a_{301}(2a_{121} - 3a_1 a_{301}))] + 4[2(9a_1^3 - 4a_2^2)a_{121} + 9a_1(3a_1^3 - 4a_2^2)a_{301}]b_{022}\},$$

$$b_{302} = \frac{1}{6(a_1^3 - 4a_2^2)(2a_{121} + 9a_1 a_{301})}\{4[2(9a_1^3 - 4a_2^2)a_{121} + 3a_1(7a_1^3 - 4a_2^2)a_{301}]b_{022} + a_1(2a_{121} + 9a_1 a_{301})[8a_2 b_{112} + 9(2a_{121} - 3a_1 a_{301})a_{301}]\}.$$

Then, the conditions given at the beginning of the proof for Theorem 3 under which all the ε^2-order focus values vanish, and the conditions given in Case 1 of (26) under which all the ε-order focus values become zero can be simplified.

Similarly, we first let $b_{014} = 0$ under which the origin $(0, 0)$ becomes a linear center under perturbation up to ε^4 order. Then, solving $\tilde{v}_{14} = 0$ for b_{034} yields

$$b_{034} = \frac{1}{12a_1}[2(2a_{121} + 9a_1 a_{301})(b_{203} + b_{023}) - (a_1 - 2)(2a_{121} + 9a_1 a_{301})b_{103} + 2a_1(a_1 - 2)(b_{114} + b_{112}b_{102}) - 4a_1(b_{214} + b_{112}b_{202}) + 4a_1(2a_2 b_{024} - b_{112}b_{022})]. \tag{62}$$

Then, solving $\tilde{v}_{24} = 0$ for b_{124}, $\tilde{v}_{34} = 0$ for b_{214}, $\tilde{v}_{44} = 0$ for b_{303}, and $\tilde{v}_{54} = 0$ for b_{022} yields

$$\tilde{v}_{64} = -\frac{143}{294912}Q_4 F_4, \quad \tilde{v}_{74} = \frac{715}{113246208}Q_4 M_4, \quad \tilde{v}_{84} = -\frac{715}{97844723712}Q_4 N_4, \ldots \tag{63}$$

where Q_4 is a lengthy quotient expression of $a_1, a_2, b_{112}, a_{121}$ and a_{301}, and F_4, M_4 and N_4 are polynomials of a_1 and a_2, given by

$$F_4 = 405a_1^4 + 6264a_1^3 + 6264a_1^2 - 5664a_1 + 1360 - 8a_2^2(99a_1^2 + 708a_1 + 524 - 170a_2^2), \tag{64}$$

$$M_4 = 8991a_1^6 + 96228a_1^5 - 890028a_1^4 - 397728a_1^3 + 2094480a_1^2 - 1063872a_1 + 177344 + 4a_2^2[14553a_1^4 + 269064a_1^3 + 425592a_1^2 - 328032a_1 - 89328 - 4a_2^2(8307a_1^2 + 66492a_1 + 22332 - 11084a_2^2)], \tag{65}$$

$$N_4 = 9082611a_1^8 + 24786000a_1^7 - 1425803472a_1^6 + 5718703680a_1^5 + 3690544032a_1^4 - 16670271744a_1^3 + 6941193984a_1^2 - 229020672a_1 - 325899520 - 16a_2^2\{22470453a_1^6 + 657728100a_1^4 + 349839324a_1^5 - 712114848a_1^3 - 1609038864a_1^2 + 855025344a_1 - 313210688 - 2a_2^2[3(6779079a_1^4 + 45423144a_1^3 + 80547480a_1^2 - 142504224a_1 - 133284304) - 8a_2^2(4266441a_1^2 + 894612a_1 - 19575668 + 1273045a_2^2)]\}. \tag{66}$$

It is easy to check that the solutions of $F_4 = M_4 = 0$ do not satisfy $N_4 = 0$. Therefore, for the case considered in this section, the system can have at most eight small-amplitude limit cycles around the origin. As a matter of fact, eliminating a_2 from the equations $F_4 = M_4 = 0$ yields the solution for a_2, given by (55), and the resultant equation given by $R_3(a_1) = (a_1 + 2)R_3^*(a_1)$, where $R_3^*(a_1)$ is given in (57). Thus, following the analysis given in the previous section, we know that there are three real solutions of a_1, given in (58), with corresponding values of a_2, satisfying $F_4 = M_4 = 0$, but $N_4 \neq 0$.

To show the existence of eight limit cycles around the origin, we can use (63)–(65) to calculate the following determinant at the above critical point (determined from $R_3^*(a_1) = 0$), yielding

$$D_4 = \det \begin{bmatrix} \frac{\partial \tilde{v}_{64}}{\partial a_1} & \frac{\partial \tilde{v}_{64}}{\partial a_2} \\ \frac{\partial \tilde{v}_{74}}{\partial a_1} & \frac{\partial \tilde{v}_{74}}{\partial a_2} \end{bmatrix} = -\frac{102245 Q_4^2}{33397665693696} \left(\frac{\partial F_4}{\partial a_1}\frac{\partial M_4}{\partial a_2} - \frac{\partial F_4}{\partial a_2}\frac{\partial M_4}{\partial a_1} \right)$$

$$\begin{aligned} &= -\frac{102245 a_2 (a_1+1) Q_4^2}{274877906944 (G_3^D)^3} (229351763907682011 a_1^{18} + 2699927727044834952 a_1^{17} \\ &\quad + 8758376732173987530 0 a_1^{16} + 247748283932991865344 0 a_1^{15} + 2889428467733106539769 6 a_1^{14} \\ &\quad + 2878262159719922833136 64 a_1^{13} + 2083536090616271308366 080 a_1^{12} \\ &\quad + 1130674558783098990036 5824 a_1^{11} + 4990868250203088681496 4224 a_1^{10} \\ &\quad + 1399273128810785847066 09152 a_1^{9} + 1713848705357719919431 00416 a_1^{8} \\ &\quad - 3889339278872412137639 1168 a_1^{7} - 2158135588443008120129 61792 a_1^{6} \\ &\quad - 6765674196055619728441 344 a_1^{5} + 9814653415253853636270 4896 a_1^{4} \\ &\quad - 3428194917942009678476 0832 a_1^{3} + 4029745321332325625298 944 a_1^{2} \\ &\quad - 1966457664091358007132 16 a_1 + 3440013069016973443072) \neq 0. \end{aligned}$$

which implies that for the case considered in this section, eight small-amplitude limit cycles exist in the vicinity of the origin based on the analysis of ε^4-order focus values.

Theorem 4 is proved. ■

6. Eight Limit Cycles Obtained from ε^5-Order Focus Values

In this section, we assume that all the ε^k-order $(k = 0, 1, 2, 3, 4)$ focus values vanish and consider a case based on ε^5-order focus values. However, this attempt does not give more limit cycles, but we still obtain eight limit cycles.

Theorem 5. *When all the ε^k-order $(k = 0, 1, 2, 3, 4)$ focus values vanish, there exist eight limit cycles obtained from the analysis of the ε^5-order focus values.*

Proof. The conditions under which all the ε^k-order $(k = 0, 1, 2)$ focus values become zero are given in the previous sections. Further, we have the following conditions under which all the ε^4-order focus values vanish:

$$b_{112} = \frac{3a_2}{a_1^3(2a_{121} + 3a_1 a_{301})} a_{121} a_{301} (2a_{121} - 3a_1 a_{301}),$$

$$b_{022} = \frac{3}{8a_1^2(2a_{121} + 3a_1 a_{301})} a_{301} (2a_{121} - 3a_1 a_{301})(2a_{121} + 9a_1 a_{301}),$$

$$\begin{aligned} b_{303} = &-\frac{1}{3a_1^3(a_1^3 - 4a_2^2)(2a_{121} + 9a_1 a_{301})} \{2a_1^3 a_2 (2a_{121} + 9a_1 a_{301})^2 b_{102} + 18 a_1^4 a_2 (2a_{121} + 9a_1 a_{301}) b_{202} a_{301} \\ &- 4a_1^4 a_2 (2a_{121} + 9a_1 a_{301}) b_{113} - 2a_1^3 [2(9a_1^3 - 4a_2^2) a_{121} + 3a_1 (7a_1^3 - 4a_2^2) a_{301}] b_{023}\} \\ &+ \frac{a_2 a_{301} (2a_{121} - 3a_1 a_{301})}{a_1^3 (a_1^3 - 4a_2^2)(2a_{121} + 9a_1 a_{301})(2a_{121} + 3a_1 a_{301})^2} \{8a_{121}^4 + 729 a_1^5 a_{301}^4 \\ &+ 9a_{121} a_{301} [2(2a_1 a_{121} + (22a_1^3 - a_1^2 - 4a_2^2) a_{301}) a_{121} + 3a_1 (32a_1^3 - 3a_1^2 + 4a_2^2) a_{301}^2]\}, \end{aligned}$$

$$
\begin{aligned}
b_{214} &= -b_{114} + \frac{1}{6a_1}(2a_{121} + 9a_1a_{301})(3b_{103} + 3b_{203} + 2b_{023}) + \frac{2\,a_2}{3(a_1^3 - 4a_2^2)}(2a_{121} + 9a_1a_{301})b_{113} \\
&\quad + \frac{2a_1^2}{3(a_1^3 - 4a_2^2)}(8a_{121} + 9a_1a_{301})b_{023} - \frac{1}{12a_1^5(a_1^3 - 4a_2^2)(2a_{121} + 3a_1a_{301})^2} \\
&\quad \times \{4a_1^2a_2(2a_{121} + 3a_1a_{301})[8a_1^2a_{121}^3 + 243a_1^5a_{301}^3 + 3a_{121}a_{301}(2(17a_1^3 - 12a_2^2)a_{121} \\
&\quad + 9a_1(9a_1^3 + 4a_2^2)a_{301})]b_{102} + 36a_1^2a_2a_{301}b_{202}(2a_{121} + 3a_1a_{301})[2(3a_1^3 - 4a_2^2)a_{121}^2 \\
&\quad + 27a_1^5a_{301}^2 + 3a_1(7a_1^3 + 4a_2^2)a_{121}a_{301}] - 3a_2a_{301}(2a_{121} - 3a_1a_{301})[16a_1a_{121}^4 + 1458a_1^6a_{301}^4 \\
&\quad + 9a_{121}a_{301}(4((a_1^3 + 2a_1^2 - 4a_2^2)a_{121} + a_1^3(21a_1 - 1)a_{301})a_{121} + 9a_1^2(21a_1^3 - 2a_1^2 + 4a_2^2)a_{301}^2)]\}, \\
b_{214} &= a_1b_{024} + \frac{2a_2(a_1 + 1)}{3(a_1^3 - 4a_2^2)}(8a_{121} + 9a_1a_{301})b_{023} - \frac{1}{24a_1^4(a_1^3 - 4a_2^2)(2a_{121} + 3a_1a_{301})^2} \\
&\quad \times \{2a_1^3(2a_{121} + 9a_1a_{301})(2a_{121} + 3a_1a_{301})^2[a_1(a_1 + 1)(2a_{121} + 9a_1a_{301})b_{102} + 9(a_1^2 + 4a_2^2)a_{301}b_{202}] \\
&\quad - 4a_1^4(2a_{121} + 3a_1a_{301})^2[2a_1(a_1 + 1)a_{121} + 9(2a_1^3 + a_1^2 - 4a_2^2)a_{301}]b_{113} \\
&\quad - 3a_{301}(2a_{121} - 3a_1a_{301})[8a_1(a_1 + 1)a_{121}^4 + 729a_1^4(a_1^2 + 4a_2^2)a_{301}^4 \\
&\quad + 9a_{121}a_{301}(2(2(3a_1^3 + a_1^2 - 8a_2^2)a_{121} + a_1(29a_1^3 - a_1^2 + 84a_1a_2^2 - 36a_2^2)a_{301})a_{121} \\
&\quad + 9a_1^2(13a_1^3 - a_1^2 + 44a_1a_2^2 - 12a_2^2)a_{301}^2)]\}, \\
b_{304} &= \frac{1}{6}a_1b_{114} + \frac{2a_2}{3}b_{024} - \frac{1}{12}(2a_{121} + 9a_1a_{301})b_{103} - \frac{a_2}{9(a_1^3 - 4a_2^2)}(2a_{121} + 9a_1a_{301})(2b_{113} - 9a_{301}b_{202}) \\
&\quad + \frac{1}{72a_1^5(a_1^3 - 4a_2^2)(2a_{121} + 3a_1a_{301})^2}\{4a_1^3a_2(2a_{121} + 3a_1a_{301})[16a_1a_{121}^3 + 486a_1^4a_{301}^3 \\
&\quad + 3a_{121}a_{301}(2(3a_1^3 + 28a_1^2 - 12a_2^2)a_{121} - 9a_1(a_1^3 - 20a_1^2 - 4a_2^2)a_{301})]b_{102} \\
&\quad - 4a_1^4(2a_{121} + 3a_1a_{301})^2[2(15a_1^3 + 4a_2^2)a_{121} + 9a_1(3a_1^3 + 4a_2^2)a_{301}]b_{023} \\
&\quad - 3a_2a_{301}(2a_{121} - 3a_1a_{301})[32a_1a_{121}^4 + 2916a_1^6a_{301}^4 + 9a_{121}a_{301}(4((a_1^3 + 4a_1^2 - 4a_2^2)a_{121} \\
&\quad + a_1(39a_1^3 - 2a_1^2 + 12a_2^2)a_{301})a_{121} + 9a_1^2(45a_1^3 - 4a_1^2 - 4a_2^2)a_{301}^2)]\}.
\end{aligned}
$$

Then, similarly we first let $b_{015} = 0$ under which the origin $(0, 0)$ becomes a linear center under perturbation up to ε^5 order. Then, solving $\tilde{v}_{15} = 0$ for b_{035}, $\tilde{v}_{25} = 0$ for b_{125}, $\tilde{v}_{35} = 0$ for b_{215}, $\tilde{v}_{45} = 0$ for b_{304}, and $\tilde{v}_{55} = 0$ for b_{023} yields

$$
\begin{aligned}
\tilde{v}_{65} &= -\frac{143}{2959296}Q_5F_5, \\
\tilde{v}_{75} &= \frac{715}{905969664}Q_5M_5, \\
\tilde{v}_{85} &= -\frac{715}{782757789696}Q_5N_5, \ldots
\end{aligned}
\tag{67}
$$

where Q_5 is a lengthy quotient expression of $a_1, a_2, b_{102}, b_{202}, b_{113}, a_{121}$ and a_{301}, and $F_5 = F_4, M_5 = M_4$ and $N_5 = N_4$, given in (64)–(66). Therefore, from the analysis given in the previous section, we know that the solutions of $F_5 = M_5 = 0$ do not satisfy $N_5 = 0$, and so nine limit cycles is not possible for this case. The remaining part of proving the existence of eight limit cycles is almost exactly the same as that given in the proof of Theorem 4. ∎

7. Conclusion

In this paper, we have studied bifurcation of limit cycles from a center of quadratic Hamiltonian systems with third order polynomials perturbation. It is shown using higher-order focus values computation that such a system can have at least eight small-amplitude limit cycles in the vicinity of a center. This result, obtained by perturbing a general quadratic Hamiltonian system, is better than the existing seven limit cycles, obtained by perturbing the quadratic Bogdanov–Takens Hamiltonian system. What is maximal number of limit cycles for the system considered in this paper is still open.

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