Bifurcations associated with a double zero of index two and a pair of purely imaginary eigenvalues

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The bifurcation and instability behaviour of a non-linear autonomous system in the vicinity of a compound critical point, involving a pair of purely imaginary eigenvalues and a two-fold zero eigenvalue of index two, are investigated. The analysis is carried out via a systematic perturbation method which embraces a unification technique, leading to a simplified set of differential equations for the analysis of local behaviour. Incipient and secondary bifurcations are explored. The static and dynamic instability boundaries are established. A non-linear control system drawn from electrical network theory is analysed in detail.

1. Introduction

A variety of compound critical points may be exhibited by autonomous systems. These points are characterized by eigenvalues which cross the imaginary axis simultaneously and, in a non-linear system, generate complex interactions between static and dynamic bifurcation modes. The investigations carried out by Langford (1979) and others (Guckenheimer and Holmes 1983, Sprig 1983), for example, have demonstrated that a compound critical point characterized by a zero and a pair of purely imaginary eigenvalues of the jacobian lead to secondary bifurcations as well as bifurcations into invariant tori, in the vicinity of the critical point. On the other hand, a double zero of index one may lie on the intersection of dynamic and static bifurcation boundaries (Guckenheimer and Holmes 1983, Yu and Huseyn 1986, Takens 1973). If the jacobian has a two-fold zero eigenvalue in addition to a pair of purely imaginary eigenvalues, the system is liable to exhibit even more complicated phenomena in the neighbourhood of such a compound critical point. A two-fold zero eigenvalue, on the other hand, may arise in two distinct forms. Indeed, the repeated zero may either be associated with a single eigenvector or with two linearly independent eigenvectors. A canonical form of the jacobian corresponding to the former case is represented by a Jordan block of order two, and the repeated zero eigenvalues are said to be of index one. The latter case may be associated with a diagonal matrix with zero elements, and the eigenvalue is of index two. The aim of this paper is to explore the behaviour characteristics of the system in the vicinity of a compound critical point involving a pair of purely imaginary eigenvalues and a two-fold zero eigenvalue of index two.

The analysis is carried out through a systematic perturbation method which embraces a unification procedure and an intrinsic harmonic balancing technique (Huseyin 1986, Huseyn and Atadan 1983, Atadan and Huseyin 1984). Emphasis is on the primary and secondary bifurcation boundaries which may be linked to static and/or dynamic instabilities.

The results are illustrated by an example drawn from network theory.

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2. Formulation

Consider an autonomous system described by

$$\frac{dz^i}{dt} = Z_i(z^i; \eta^\beta)$$

(1)

where the $z^i$ are the components of the state vector $z$, and the $\eta^\beta$ are certain independent parameters. It is assumed that the functions $Z_i$ are analytic, at least in the region of interest. Attention in this paper will be focused on a critical equilibrium state where the Jacobian has a double zero and a pair of purely imaginary eigenvalues, and all the remaining eigenvalues have negative real parts. For simplicity, therefore, it is assumed that system (1) is a $4 \times 4$ system, with 'i' ranging from 1 to 4. It is also assumed that the system involves three independent parameters ($\beta = 1, 2, 3$).

Now suppose that the system has a single valued equilibrium surface in the region of interest, which is expressed as

$$z^i = f_i(\eta^\beta)$$

(2)

and $c$ is a critical point on this surface where the real parts of a pair of complex conjugate eigenvalues and two additional eigenvalues vanish simultaneously. It is assumed that the corresponding eigenvectors remain linearly independent.

Next, introduce the non-singular transformation

$$z^i = f_i(\eta^\beta) + T_i w^i$$

(3)

such that the resulting system

$$\frac{dw^i}{dt} = W_i(w^i; \eta^\beta)$$

(4)

has a Jacobian matrix at $c$ (where $\eta^\beta = \eta^\beta_c$) in the form

$$\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_c \\
0 & 0 & -\omega_c & 0
\end{bmatrix}$$

(5)

It follows from the transformation (3) that the new system has the following properties

$$W_i(0; \eta^\beta) = W_{i\beta}(0; \eta^\beta) = W_{i\beta\beta}(0; \eta^\beta) = \ldots = 0$$

(6)

where the subscripts to the $W_i$ indicate differentiations with respect to the corresponding parameters.

If the eigenvalues of the Jacobian of (4) in the vicinity of $c$ are denoted by $\lambda_1(\mu^\beta)$, $\lambda_2(\mu^\beta)$, and $\omega(\mu^\beta) = \omega_c$, then

$$\lambda_1(0) = \lambda_2(0) = 0, \quad \omega(0) = 0 \quad \text{and} \quad \omega(0) = \omega_c$$

where $\mu^\beta - \eta^\beta_c$, $\mu^\beta = 0$ giving the critical point $c$.

It is further assumed here that

$$\det \begin{bmatrix}
\frac{\partial W_{11}}{\partial \mu^1} & \frac{\partial W_{11}}{\partial \mu^2} & \frac{\partial W_{11}}{\partial \mu^3} \\
\frac{\partial W_{21}}{\partial \mu^1} & \frac{\partial W_{21}}{\partial \mu^2} & \frac{\partial W_{21}}{\partial \mu^3} \\
\frac{\partial \alpha}{\partial \mu^1} & \frac{\partial \alpha}{\partial \mu^2} & \frac{\partial \alpha}{\partial \mu^3}
\end{bmatrix} \neq 0$$

(7)
which represents a generalization of Hopf's transversality condition. It can be shown (see Appendix A) that

\[
\frac{\partial \alpha}{\partial \mu^a} \triangleq \alpha^a = \frac{1}{2} (W_{33a} + W_{44a}), \quad \frac{\partial \omega}{\partial \mu^a} \triangleq \omega^a = \frac{1}{2} (W_{34a} - W_{43a})
\]

(8)

It is known that a double zero eigenvalue of index one may lead to both static and dynamic bifurcations (Yu and Huseyn 1986, Huseyn 1986), and a pair of complex eigenvalues crossing the imaginary axis—while the rest of the eigenvalues remain on the left-hand side of the complex plane—results in a Hopf bifurcation. It is also known that the interactions between static and dynamic bifurcations may lead to an invariant torus in the vicinity of a coincident critical point at which a real eigenvalue and a pair of complex conjugate eigenvalues cross the imaginary axis simultaneously (Langford 1979). So the system with the Jacobian matrix of form (5) at the critical point may be expected to exhibit static bifurcation, dynamic bifurcation and a sequence of bifurcations leading to an invariant two-dimensional torus. Moreover, it is reasonable to expect that some more complicated situation, such as interactions between two distinct dynamic bifurcations leading to a three-dimensional invariant torus, may arise. It is therefore clear that for a comprehensive study of these phenomena the formulation must be capable of dealing with both static and dynamic bifurcations—preferably in a unified approach. Remarkably, intrinsic harmonic balancing and the unification technique can satisfy these requirements rather neatly, embracing both equilibrium and periodic solutions which may emerge under various combinations of the parameters.

Thus, suppose the steady-state solutions of (4) in the vicinity of \( c \) are in the parametric form

\[
w^i = w^i(\tau, \sigma^a), \quad \eta^a = \eta^a(\sigma^a) \quad \text{and} \quad \omega = \omega(\sigma^a)
\]

(9)

where \( \tau = \omega t \), \( \omega \) is the frequency, and the \( \sigma^a \) \((a = 1, 2, 3)\) are certain unidentified parameters.

Substituting the assumed solution (9) into (4) yields the identity

\[
\omega(\sigma^a) \frac{d}{dt} w^i(\tau, \sigma^a) \equiv W_i[w^i(\tau, \sigma^a); \eta^a(\sigma^a)]
\]

(10)

One may further assume that the Fourier series

\[
w^i(\tau, \sigma^a) = \sum_{m=0}^{M} [p_{im}(\sigma^a) \cos m\tau + r_{im}(\sigma^a) \sin m\tau]
\]

(11)

provides an appropriate approximation for the steady-state solutions in more explicit terms. Here, \( p_{im}(\sigma^a) \) and \( r_{im}(\sigma^a) \) are vectors corresponding to the \( m \)th harmonic, and they are functions of three variables. It is observed that

\[
p_{im}(0) = r_{im}(0) = 0 \quad (\forall i, m)
\]

(12)

since \( w^i(\tau, 0) = 0 \), and a solution given by \( p_{10}(\sigma^a) \neq 0 \) and/or \( p_{20}(\sigma^a) \neq 0 \), (with \( p_{im}(\sigma^a) = r_{im}(\sigma^a) = 0 \) for \( m > 0 \)) represents a post-critical equilibrium (stationary) solution—that is, a static bifurcation.

A sequence of perturbation equations is then generated by differentiating (10) with respect to the perturbation parameters \( \sigma^a \) \((a = 1, 2, 3)\) successively. Thus,

\[
\omega^a w^i_t + \omega \omega^a w^i_{\tau^a} = W_{ij}w^j_{\tau^a} + W_{ir}w^r_{\tau^a}
\]

(13)
and

\[ \omega^{ab} w_i^a + \omega^{ab} w_i^b + \omega^{ab} w_i^a + \omega^{ab} w_i^b = W_{ij} w_j^a + W_{ij} w_j^b + W_{ij} w_j^a + W_{ij} w_j^b + W_{ij} \eta^{a\beta} \eta^{b\alpha} + W_{ij} \eta^{a\beta} \eta^{b\alpha} \]  

(14)

e.g., where \( i, j, k = 1, 2, 3, 4 \); \( \beta, \gamma = 1, 2, 3 \) and \( \alpha, b = 1, 2, 3 \), the subscripts on the functions \( W_i \) denote differentiations with respect to the corresponding variables, and the usual summation convention applies here and will be used throughout this paper. For clarity, differentiations of variables with respect to the \( \sigma^a \) are indicated by superscripts after a comma.

Now we shall use the unification technique (Yu and Huseyn 1986) to derive the equivalent differential equations. First, evaluating the first-order perturbation equation (9) at the critical point \( c \) with the aid of (6) results in

\[ \omega_c w_i^a = W_{ij} w_j^a \]  

(15)

where \( W_{ij} \) is in the canonical form (5). Substituting the Fourier series (11) into (15) results in

\[ \omega_c \sum_{m=0}^{M} m(r_m^a \cos m \tau - p_m^a \sin m \tau) = W_{ij} \sum_{m=0}^{M} (p_m^a \cos m \tau + r_m^a \sin m \tau) \]  

(16)

where the superscripts \( a \) on the \( p \) and \( r \), together with commas, indicate differentiations with respect to \( \sigma^a \). Comparing the coefficients of \( \cos m \tau \) and \( \sin m \tau \) for each \( m \), and recognizing that one may assume

\[ r_{31}(\sigma^a) = 0 \]  

(17)

without loss of generality—since the system is autonomous—yields

\[ \left\{ \begin{array}{l}
p_{31}^1 = -r_{31}^1 \\
p_{31}^2 = r_{31}^2 = 0 \\
\end{array} \right. \]  

(18)

and

\[ p_m^a = r_m^a = 0 \quad \text{(for } m \neq 1; i = 1, 2, 3, 4; a = 1, 2, 3) \]  

Next, evaluating the second perturbation equation (14) at the critical point \( c \) with the aid of (6), introducing (11), and comparing the coefficients of \( \cos m \tau \) and \( \sin m \tau \) results in

\[ W_{ij} (p_{j0}^a p_{i0}^b + \frac{1}{2} p_{j1}^a p_{j0}^b + \frac{1}{2} r_{j1}^a r_{j0}^b) + W_{ij} (p_{j0}^a \eta^{b\alpha} + p_{j0}^a \eta^{b\alpha} + W_{ij} p_{j0}^a = 0 \]  

(for \( m = 0 \))  

(19)

\[ \omega_c r_{i1}^a + \omega_c r_{i1}^a + \omega_c r_{i1}^a = W_{ij} (p_{j0}^a p_{i0}^b + p_{j1}^a p_{i0}^b) + W_{ij} (p_{j0}^a \eta^{b\alpha} + p_{j1}^a \eta^{b\alpha}) \]

\[ + W_{ij} \eta^{a\beta} \eta^{b\alpha} + \omega_c p_{i1}^a \]  

(20)

and

\[ 2\omega_c r_{i2}^a = \frac{1}{2} W_{ij} (p_{j1}^a p_{i1}^b - r_{i1}^a r_{i1}^a) + W_{ij} p_{i2}^a \]  

(for \( m = 1 \))  

(21)
It is noted that (19), (20) and (21) represent a total of 120 equations. Since the system involves four key variables, it can be shown that one needs 24 perturbation equations to establish the relations between the key variables $p_{10}$, $p_{20}$, $p_{31}$ and $\omega$ and the parameters $\eta^1$, $\eta^2$ and $\eta^3$. These 24 equations consist of (19) (for $i = 1, 2$) and (20) (for $i = 3, 4$). For clarity, they are written in a more explicit form below:

\[2(W_{12}p_{10} + W_{13}p_{10})\eta^{i,1} + W_{11}(p_{10}^{2}) + 2W_{12}p_{10}p_{10} + W_{22}(p_{10})^{2} + l(p_{10})^{2} = 0\]
\[(W_{12}p_{10}^{2} + W_{13}p_{10})\eta^{i,1} + (W_{12}p_{10} + W_{13}p_{10})\eta^{i,3} + W_{11}(p_{10}^{2} + p_{10}^{2}) + W_{22}(p_{10})^{2} + l(p_{10})^{2} = 0\]
\[+ W_{22}p_{10}p_{10} + l(p_{10})^{2} = 0\]
\[\text{(22)}\]

\[2(W_{12}p_{10} + W_{13}p_{10})\eta^{i,2} + W_{11}(p_{10})^{2} + 2W_{12}p_{10}p_{10} + W_{22}(p_{10})^{2} + l(p_{10})^{2} = 0\]
\[(W_{12}p_{10}^{2} + W_{13}p_{10})\eta^{i,2} + (W_{12}p_{10} + W_{13}p_{10})\eta^{i,3} + W_{11}(p_{10}^{2} + p_{10}^{2}) + W_{22}(p_{10})^{2} + l(p_{10})^{2} = 0\]
\[+ W_{22}p_{10}p_{10} + l(p_{10})^{2} = 0\]
\[\text{(23)}\]

\[2(W_{12}p_{10} + W_{13}p_{10})\eta^{i,3} + W_{11}(p_{10})^{2} + 2W_{12}p_{10}p_{10} + W_{22}(p_{10})^{2} + l(p_{10})^{2} = 0\]
\[(W_{12}p_{10}^{2} + W_{13}p_{10})\eta^{i,3} + (W_{12}p_{10} + W_{13}p_{10})\eta^{i,1} + W_{11}(p_{10}^{2} + p_{10}^{2}) + W_{22}(p_{10})^{2} + l(p_{10})^{2} = 0\]
\[+ W_{22}p_{10}p_{10} + l(p_{10})^{2} = 0\]
\[\text{for } i = 1, 2\]

\[2(\alpha^2\eta^{i,1} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} = 0\]
\[(\alpha^2\eta^{i,2} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} + (\alpha^2\eta^{i,1} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} = 0\]
\[(\alpha^2\eta^{i,3} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} + (\alpha^2\eta^{i,1} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} = 0\]
\[2(\alpha^2\eta^{i,2} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} = 0\]
\[(\alpha^2\eta^{i,3} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} + (\alpha^2\eta^{i,2} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} = 0\]
\[2(\alpha^2\eta^{i,3} + \alpha_1 p_{10}^{2} + \alpha_2 p_{20}^{2})p_{31} = 0\]

and

\[2\omega^3 p_{31} = 2(\omega^2\eta^{i,1} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31}\]
\[\omega^1 p_{31}^{2} + \omega^2 p_{31}^{2} = (\omega^2\eta^{i,2} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31} + (\omega^2\eta^{i,1} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31}\]
\[\omega^1 p_{31}^{2} + \omega^3 p_{31}^{2} = (\omega^2\eta^{i,1} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31} + (\omega^2\eta^{i,1} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31}\]
\[2\omega^2 p_{31}^{2} = 2(\omega^2\eta^{i,2} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31}\]
\[\omega^2 p_{31}^{2} + \omega^3 p_{31}^{2} = (\omega^2\eta^{i,1} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31} + (\omega^2\eta^{i,2} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31}\]
\[2\omega^3 p_{31}^{2} = 2(\omega^2\eta^{i,3} + \omega_1 p_{10}^{2} + \omega_2 p_{20}^{2})p_{31}\]

Here the superscripts 1, 2 and 3 denote differentiations with respect to $\sigma^1$, $\sigma^2$ and $\sigma^3$, respectively, and

\[l_j = \frac{1}{4}(W_{33j} + W_{44j})\]
\[\alpha_j = \frac{1}{2}(W_{33j} + W_{44j}), \quad \sigma^\delta = \frac{1}{2}(W_{33\delta} + W_{44\delta})\]
and

\[ \omega_j = \frac{1}{2} (W_{34j} - W_{43j}), \quad \omega^\beta = \frac{1}{2} (W_{34\beta} - W_{43\beta}) \]  

(27)

where \( j = 1, 2, 3, 4 \) and \( \beta = 1, 2, 3 \).

Instead of trying to solve the above equations for the unknown derivatives, a unification technique will now be used to derive a set of governing relationships. Thus, multiplying the first equation of (22) by \((\sigma^1)^2/2\), the second equation by \(\sigma^1\sigma^2\), the third equation by \(\sigma^1\sigma^3\), the fourth equation by \((\sigma^2)^2/2\), the fifth equation by \(\sigma^2\sigma^3\) and the last equation by \((\sigma^3)^2/2\), and then adding them together yields

\[
W_{11\beta} \mu^\beta p_{10} + W_{12\beta} \mu^\beta p_{20} + \frac{1}{2} W_{111}(p_{10})^2 + W_{112} p_{10} p_{20} + \frac{1}{2} W_{122}(p_{20})^2
+ \frac{1}{2} l_i(p_{31})^2 = 0 \quad (i = 1, 2)
\]

(28)

upon considering appropriate Taylor expansions of the functions in (9), and introducing

\[ \eta^\beta = \eta^\beta_e + \mu^\beta \]

Similarly, the same procedure can be applied to (23) and (24) to obtain

\[ \alpha^\beta \mu^\beta p_{31} + \alpha_1 p_{10} p_{31} + \alpha_2 p_{20} p_{31} = 0 \]

(29)

and

\[ \Omega p_{31} = \omega^\beta \mu^\beta p_{31} + \omega_1 p_{10} p_{31} + \omega_2 p_{20} p_{31} \]

(30)

where \( \Omega = \omega - \omega_e \).

Equations (28), (29) and (30) govern bifurcation phenomena in the vicinity of the critical point c. Thus, one first observes that

\[ p_{10} = p_{20} = p_{31} = 0 \]

(31)

is a solution of these equations; indeed, (31) represents the fundamental equilibrium solution. Static bifurcations from the fundamental equilibrium surface occur for \( p_{31} = 0 \) and are characterized by the implicit equations

\[
W_{11\beta} \mu^\beta p_{10} + W_{12\beta} \mu^\beta p_{20} + \frac{1}{2} W_{111}(p_{10})^2 + W_{112} p_{10} p_{20} + \frac{1}{2} W_{122}(p_{20})^2 = 0
\]

\[
W_{21\beta} \mu^\beta p_{10} + W_{22\beta} \mu^\beta p_{20} + \frac{1}{2} W_{211}(p_{10})^2 + W_{212} p_{10} p_{20} + \frac{1}{2} W_{222}(p_{20})^2 = 0
\]

\[ p_{31} = 0 \]

(32)

which follow from (28).

On the other hand, \( p_{31} \neq 0 \) signifies the emergence of limit cycles, and the behaviour surface associated with this family (Hopf bifurcations) is expressed in the following implicit form

\[
W_{11\beta} \mu^\beta p_{10} + W_{12\beta} \mu^\beta p_{20} + \frac{1}{2} W_{111}(p_{10})^2 + W_{112} p_{10} p_{20} + \frac{1}{2} W_{122}(p_{20})^2 + \frac{1}{2} l_i(p_{31})^2 = 0
\]

\[
W_{21\beta} \mu^\beta p_{10} + W_{22\beta} \mu^\beta p_{20} + \frac{1}{2} W_{211}(p_{10})^2 + W_{212} p_{10} p_{20} + \frac{1}{2} W_{222}(p_{20})^2 + \frac{1}{2} l_i(p_{31})^2 = 0
\]

\[ \alpha^\beta \mu^\beta + \alpha_1 p_{10} + \alpha_2 p_{20} = 0 \]

(33)
Each point on this surface corresponds to a limit cycle. The frequencies of these limit cycles follows from (30) and are expressed as

$$\Omega = \left(\omega g\mu^g + \omega_1 p_{10} + \omega_2 p_{20}\right)$$  \hspace{1cm} (34)

The bifurcating steady-state solutions (both stationary and periodic) expressed by the Fourier series (11) may be constructed up to first- or second-order terms. Indeed, setting $$\sigma^1 = p_{10}$$, $$\sigma^2 = p_{20}$$ and $$\sigma^3 = p_{31}$$, and using the derivatives (18) in (11), one obtains the first-order approximation

\[
\begin{align*}
  w^1 &= p_{10} \\
  w^2 &= p_{20} \\
  w^3 &= p_{31} \cos \tau \\
  w^4 &= -p_{31} \sin \tau 
\end{align*}
\]  \hspace{1cm} (35)

A second-order approximation takes the form

\[
\begin{align*}
  w^1 &= p_{10} + p_{11} \cos \tau + r_{11} \sin \tau + p_{12} \cos 2\tau + r_{12} \sin 2\tau \\
  w^2 &= p_{20} + p_{21} \cos \tau + r_{21} \sin \tau + p_{22} \cos 2\tau + r_{22} \sin 2\tau \\
  w^3 &= p_{30} + p_{31} \cos \tau + p_{32} \cos 2\tau + r_{32} \sin 2\tau \\
  w^4 &= p_{40} + p_{41} \cos \tau + r_{41} \sin \tau + p_{42} \cos 2\tau + r_{42} \sin 2\tau 
\end{align*}
\]  \hspace{1cm} (36)

where the amplitudes $$p_{11}, r_{11}, p_{12}, \ldots$$ can be determined in terms of $$p_{10}, p_{20}, \ldots$$, by applying the unification technique to the remaining equations in (19), (20) and (21), which involve all derivatives, and solving the resulting equations for the unknown variables in terms of $$p_{10}, p_{20}, \ldots$$ The results—expressed up to second-order terms—are given in Appendix B. In particular, note that

$$r_{41} = -p_{31} + \frac{1}{2\Omega_c} \left[ (W_{341} + W_{431}) \mu^g + (W_{341} + W_{431}) p_{10} + (W_{342} + W_{432}) p_{20} \right] p_{31}$$

and $$r_{41} = -p_{31}$$ to first order, as indicated by (35).

It is also noted here that the perturbation procedure leading to (28), (29) and (30) could have been considerably simplified had one set $$\sigma^1 = p_{10}, \sigma^2 = p_{20}$$ and $$\sigma^3 = p_{31}$$ at the beginning of the analysis. The reader can readily recover these equations by adopting such an approach. Nevertheless, the suitability of $$p_{10}, p_{20}$$ and $$p_{31}$$ for the role of the $$\sigma^a$$ was not a priori evident, and a false choice may have led to wrong conclusions.

3. Bifurcation analysis

For simplicity of notation, let $$p_{10}, p_{20}$$ and $$p_{31}$$ be denoted by $$y^1, y^2$$ and $$\rho$$ respectively. It can then be shown that (the outline of the proof is given in Appendix C) the rate equations
\begin{equation}
\frac{dy^1}{dt} = W_{11\beta}\mu^\beta y^1 + W_{12\beta}\mu^\beta y^2 + \frac{1}{2} W_{111}(y')^2 + W_{112}y^1y^2 + \frac{1}{2} W_{122}(y')^2 + \frac{1}{2} l_1\rho^2 \right) \nonumber
\end{equation}

\begin{equation}
\frac{dy^2}{dt} = W_{21\beta}\mu^\beta y^1 + W_{22\beta}\mu^\beta y^2 + \frac{1}{2} W_{211}(y')^2 + W_{212}y^1y^2 + \frac{1}{2} W_{222}(y')^2 + \frac{1}{2} l_2\rho^2 \right) \nonumber
\end{equation}

\begin{equation}
\frac{d\rho}{dt} = \alpha_1 y^1 + \alpha_2 y^2 \nonumber
\end{equation}

and

\begin{equation}
\frac{d\theta}{dt} = \alpha + \omega + \omega_1 y^1 + \omega_2 y^2 \nonumber
\end{equation}

 characterize the dynamics in the vicinity of the critical point c.

Equation (38) can be inferred from (31) for \( p_3 \neq 0 \) and \( \frac{d\theta}{dt} = \omega \). On the other hand, (28) and (29) describe the steady states of the equation (37). The Jacobian of (37), given by

\begin{equation}
J = \begin{bmatrix}
W_{11\beta}\mu^\beta + W_{111} y^1 + W_{112} y^2 & W_{12\beta}\mu^\beta + W_{112} y^1 + W_{122} y^2 & l_1\rho \\
W_{21\beta}\mu^\beta + W_{211} y^1 + W_{212} y^2 & W_{22\beta}\mu^\beta + W_{212} y^1 + W_{222} y^2 & l_2\rho \\
\alpha_1 y^1 & \alpha_2 y^2 & \alpha + \omega_1 y^1 + \omega_2 y^2
\end{bmatrix}
\end{equation}

may be used to investigate the stability of the solutions and to explore the local bifurcation properties.

First of all, considering the initial equilibrium solution (31) and evaluating the Jacobian on this surface, one obtains

\begin{equation}
J = \begin{bmatrix}
W_{11\beta}\mu^\beta & W_{12\beta}\mu^\beta & 0 \\
W_{21\beta}\mu^\beta & W_{22\beta}\mu^\beta & 0 \\
0 & 0 & \alpha\mu^\beta
\end{bmatrix}
\end{equation}

which, in turn, yields the stable region for the initial equilibrium solution defined by

\begin{equation}
\begin{cases}
W_{11\beta}\mu^\beta W_{22\beta}y^1 - W_{12\beta}\mu^\beta W_{21\beta}y^1 > 0 \\
(W_{11\beta} + W_{22\beta}y^1)\mu^\beta < 0
\end{cases}
\end{equation}

On the basis of these conditions, one may envisage the following critical surfaces:

\begin{equation}
S_1: W_{11\beta}\mu^\beta W_{22\beta}y^1 - W_{12\beta}\mu^\beta W_{21\beta}y^1 = 0 \quad ((W_{11\beta} + W_{22\beta}y^1)\mu^\beta < 0, \quad \alpha\mu^\beta < 0)
\end{equation}

along which static bifurcations occur from the fundamental equilibrium surface \( y^1 = y^2 = \rho = 0 \). It is noted that (42) represents a second-order surface in the parameter space.

The second critical surface is described by

\begin{equation}
S_2: (W_{11\beta} + W_{22\beta}y^1)\mu^\beta = 0 \quad (W_{12\beta}\mu^\beta W_{22\beta}y^1 - W_{12\beta}\mu^\beta W_{21\beta}y^1 > 0, \quad \alpha\mu^\beta < 0)
\end{equation}
which describes the onset of dynamic instabilities, leading to Hopf bifurcations associated with the fundamental equilibrium surface. The frequency of the periodic solutions is given by

$$\omega_c = \sqrt{W_{11\theta} \mu^\phi \nu_{22\theta} \mu^\psi - W_{12\theta} \mu^\phi \nu_{21\theta} \mu^\psi}$$

(44)

The third critical surface is described by

$$S_3: \alpha^\phi \mu^\phi = 0 \quad (W_{11\theta} \mu^\phi W_{22\theta} \mu^\psi - W_{12\theta} \mu^\phi W_{21\theta} \mu^\psi < 0, \ (W_{11\theta} + W_{22\theta}) \mu^\phi < 0)$$

(45)

which leads to another family of Hopf bifurcations from the initial solution. The first-order approximation of the frequency of the periodic solutions is equal to \(\omega_c\).

Now, evaluating the jacobian (39) on the static bifurcation solution (32) results in the stability conditions

$$\begin{cases}
(W_{11\theta} + W_{22\theta}) \mu^\phi + (W_{111} + W_{221}) y^1 + (W_{112} + W_{222}) y^2 < 0 \\
(W_{11\theta} \mu^\phi + W_{111} y^1 + W_{222} y^2)(W_{22\theta} \mu^\phi + W_{211} y^1 + W_{222} y^2) \\
-(W_{12\theta} \mu^\phi + W_{112} y^1 + W_{222} y^2)(W_{21\theta} \mu^\phi + W_{211} y^1 + W_{222} y^2) > 0 \\
\alpha^\phi \mu^\phi + \alpha_1 y^1 + \alpha_2 y^2 < 0
\end{cases}$$

(46)

for the post-critical equilibrium solution.

By analogy with \(S_1, S_2\) and \(S_3\), three critical surfaces of this solution may now be identified. Thus, secondary static bifurcations may occur along the critical surface

$$S_4: (W_{11\theta} \mu^\phi + W_{111} y^1 + W_{112} y^2)(W_{22\theta} \mu^\phi + W_{211} y^1 + W_{222} y^2) \\
-(W_{12\theta} \mu^\phi + W_{112} y^1 + W_{222} y^2)(W_{21\theta} \mu^\phi + W_{211} y^1 + W_{222} y^2) = 0$$

(47)

which is subject to the constraints consistent with (46), as before. It is also noted that (47), together with (32), defines the critical surface in the parameter space. Secondary dynamic instabilities take place along

$$S_5: (W_{11\theta} + W_{22\theta}) \mu^\phi + (W_{111} + W_{221}) y^1 + (W_{112} + W_{222}) y^2 = 0$$

(48)

which may lead to the secondary Hopf bifurcations from (32). The frequency of the periodic solutions is equal to the square root of the left-hand side of (47).

Another family of limit cycles may emerge when

$$S_6: \alpha^\phi \mu^\phi + \alpha_1 y^1 + \alpha_2 y^2 = 0$$

(49)

In this case, the first-order approximation of the frequency is \(\omega_c\). It is noted here that \(S_6\) is the intersection of the post-critical surface (32) and the behaviour surface (33).

Finally, the characteristic polynomial of the jacobian evaluated on the Hopf bifurcation solution (33) may be obtained as

$$P(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$$

(50)
where

\[
\begin{align*}
a &= -[(W_{118} + W_{228})\mu^6 + (W_{111} + W_{221})y^1 + (W_{112} + W_{222})y^2] \\
b &= [(W_{118}\mu + W_{111}y^1 + W_{112}y^2)(W_{228}\mu + W_{221}y^1 + W_{222}y^2) \\
&
- (W_{128}\mu + W_{112}y^1 + W_{122}y^2)(W_{218}\mu + W_{211}y^1 + W_{212}y^2)] \\
&
- (\alpha_1 l_1 + \alpha_2 l_2)\rho^2 \\
c &= l_1 \rho^2[\alpha_1(W_{118}\mu + W_{221}y^1 + W_{222}y^2) - \alpha_2(W_{128}\mu + W_{111}y^1 + W_{122}y^2)] \\
&
+ l_2 \rho^2[\alpha_2(W_{118}\mu + W_{111}y^1 + W_{112}y^2) - \alpha_1(W_{128}\mu^2 + W_{112}y^1 + W_{122}y^2)] 
\end{align*}
\]

which leads to the stable region for the Hopf bifurcation solution (33), characterized by the conditions

\[a > 0, \ b > 0, \ c > 0 \text{ and } ab - c > 0\]  \hspace{1cm} (52)

One may now identify the critical surfaces:

\[S_7 : c = 0 \quad (a > 0, b > 0, ab - c > 0)\]  \hspace{1cm} (53)

along which a secondary bifurcation occurs from the first Hopf bifurcation solution (33), leading to a family of limit cycles with the same frequency \(\omega_c\).

On the other hand, the critical surface

\[S_8 : ab - c = 0 \quad (a > 0, b > 0, c > 0)\]  \hspace{1cm} (54)

Figure 1. Bifurcation flow chart.
describes the onset of another secondary Hopf bifurcation from the limit cycles (33), leading to a new family of limit cycles with frequency \( \omega_b = \sqrt{b_2} \).

Bifurcations into two- or three-dimensional tori may also occur; however, this requires further detailed analysis which is not presented here.

The bifurcation flow chart is sketched in Fig. 1.

4. Example
In this section, the non-linear electrical network shown in Fig. 2 is considered to demonstrate the applicability of the results and the formulae derived in the previous sections.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{network.png}
\caption{Non-linear electrical network.}
\end{figure}

The system under consideration consists of an inductor \( L \), three capacitors \( C_1, C_2 \) and \( C_3 \), three resistors \( R_1, R_2 \) and \( R_3 \), and two tunnel diodes. Suppose \( L, C_1, C_2, C_3, R_1, R_2 \) and \( R_3 \) are linear components; in addition, \( R_3 \) may be varied. On the other hand, the two tunnel diodes are non-linear elements, and they are both voltage controlled. The characteristics of the two tunnel diodes are given by

\[ i_1 = -\eta^1 v_1 - \frac{1}{3}(v_1)^2 + a(v_1)^3 \]  
(55)

and

\[ i_2 = -\eta^2 v_2 - \frac{1}{3}(v_2)^2 + b(v_2)^3 \]  
(56)

respectively, where \( \eta^1 \) and \( \eta^2 \) are certain independent control parameters, and \( a \) and \( b \) are some positive constants.

The current \( i_L \) in the inductor and the voltages \( v_{C_1}, v_{C_2} \) and \( v_{C_3} \) across the capacitors \( C_1, C_2 \) and \( C_3 \), respectively, are chosen as the state variables, and the state equations of the network are described by

\[
\begin{align*}
C_1 \frac{dv_{C_1}}{dt} &= \left( \eta^1 - \frac{1}{R_1} \right) v_{C_1} + i_L - \frac{2}{3}(v_{C_1})^2 - a(v_{C_1})^3 \\
C_2 \frac{dv_{C_2}}{dt} &= \left( \eta^2 - \frac{1}{R_2} \right) v_{C_2} + i_L - \frac{1}{3}(v_{C_1})^2 - b(v_{C_2})^3 \\
C_3 \frac{dv_{C_3}}{dt} &= i_L \\
L \frac{di_L}{dt} &= -v_{C_1} - v_{C_2} - v_{C_3} - R_3 i_L
\end{align*}
\]  
(57)
Denoting the state variables $v_{C_1}$, $v_{C_2}$, $v_{C_3}$ and $i_L$ by $z^1$, $z^2$, $z^3$ and $z^4$, respectively, and assuming that $C_1$, $C_2$, $C_3$, $L$, $R_1$ and $R_2$ have the values 1, 1, 1, 100 and 50 in corresponding units, respectively, one obtains the following equations:

\[
\begin{align*}
\frac{dz^1}{dt} &= (\eta^1 - 0.01)z^1 + z^4 - \frac{2}{3}(z^1)^2 - a(z^1)^3 \\
\frac{dz^2}{dt} &= (\eta^2 - 0.02)z^2 + z^4 - \frac{1}{3}(z^2)^2 - b(z^2)^3 \\
\frac{dz^3}{dt} &= z^4 \\
\frac{dz^4}{dt} &= -z^1 - z^2 - z^3 - \eta^3 z^4
\end{align*}
\]

(58)

where $R_3 \equiv \eta^3$ is treated as a third control parameter ($\eta^3 \geq 0$).

The initial equilibrium solution is described by $z' = 0$ (since $z' = 0$ yield $dz'/dt = 0$ for all values of $\eta^1$, $\eta^2$ and $\eta^3$). The jacobian matrix of (58) evaluated on the fundamental surface is in the form

\[
J = [W_{ij}]_{z'=0} = \begin{bmatrix}
\eta^1 - 0.01 & 0 & 0 & 0 \\
0 & \eta^2 - 0.02 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -\eta^3
\end{bmatrix}
\]

(59)

It can be demonstrated that at the critical point $c$, defined by $\eta^1_c = 0.01$, $\eta^2_c = 0.02$ and $\eta^3_c = 0$, the jacobian has a double zero of index two and a pair of purely imaginary eigenvalues. In order to use the formulae obtained in the theory, it is required to transform system (58) to a new system such that its jacobian will be in the canonical form (5). Thus, with the aid of the transformation of the variables

\[
\begin{bmatrix}
z^1 \\
z^2 \\
z^3 \\
z^4
\end{bmatrix} = \begin{bmatrix}
0 \\
-1 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 1 \\
0 & 0 & \sqrt{3} & 0
\end{bmatrix} \begin{bmatrix}
w^1 \\
w^2 \\
w^3 \\
w^4
\end{bmatrix}
\]

(60)

and the parameters

\[
\begin{bmatrix}
\eta^1 \\
\eta^2 \\
\eta^3
\end{bmatrix} = \begin{bmatrix}
0.01 \\
0.02 \\
0
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\mu^1 \\
\mu^2 \\
\mu^3
\end{bmatrix}
\]

(61)

one may transform (58) into
\[
\begin{align*}
\frac{dw^1}{dt} &= \frac{1}{3} (\mu^1 + 2\mu^2) w^1 + \frac{1}{3} \mu^1 w^2 + \frac{1}{3} (\mu^1 - 2\mu^2) w^4 - \frac{4}{9} w^1 w^2 - \frac{2}{9} (w^1)^2 \\
&\quad - \frac{2}{9} w^1 w^4 - \frac{4}{9} w^2 w^4 \\
\frac{dw^2}{dt} &= \frac{1}{3} (\mu^1 - \mu^2) w^1 + \frac{1}{3} \mu^1 w^2 + \frac{1}{3} (\mu^1 + \mu^2) w^4 - \frac{1}{3} (w^1)^2 - \frac{4}{9} w^1 w^3 - \frac{2}{9} (w^2)^2 \\
&\quad - \frac{1}{3} (w^4)^2 - \frac{4}{9} w^2 w^4 \\
\frac{dw^3}{dt} &= \sqrt{3} w^4 - \mu^3 w^3 \\
\frac{dw^4}{dt} &= -\sqrt{3} w^3 + \frac{1}{3} (\mu^1 - \mu^2) w^1 + \frac{1}{3} \mu^1 w^2 + \frac{1}{3} (\mu^1 + \mu^2) w^4 - \frac{1}{3} (w^1)^2 - \frac{4}{9} w^1 w^3 \\
&\quad - \frac{2}{9} (w^2)^2 - \frac{1}{3} (w^4)^2 - \frac{4}{9} w^2 w^4
\end{align*}
\]

where the third-order terms have been dropped, since they are not needed for the following bifurcation analysis.

Now, using (C 5) yields the transformation (see Appendix C)

\[
\begin{align*}
w^1 &= y^1 + \frac{1}{9\sqrt{3}} \left[ 3(\mu^1 - 2\mu^2) - 2y^1 - 4y^2 \right] \rho \cos \theta - \frac{1}{18} \rho^2 \cos 2\theta \\
w^2 &= y^2 + \frac{1}{9\sqrt{3}} \left[ 3(\mu^1 + \mu^2) - 4y^2 \right] \rho \cos \theta + \frac{1}{12\sqrt{3}} \rho^2 \sin 2\theta \\
w^3 &= \rho \cos \theta + \frac{1}{9\sqrt{3}} \left[ 3(\mu^1 - \mu^2)y^1 + 3\mu^1 y^2 - 3(y^1)^2 - 4y^1 y^2 - 2(y^2)^2 - \frac{3}{2} \rho^2 \right] \\
&\quad - \frac{1}{18\sqrt{3}} \rho^2 \cos 2\theta \\
w^4 &= -\rho \sin \theta + \frac{1}{18\sqrt{3}} \left[ 3(\mu^1 + \mu^2) - 4y^1 - 4y^2 \right] \rho \cos \theta + \frac{1}{6\sqrt{3}} (\mu^1 - \mu^2) \rho \sin \theta \\
&\quad - \frac{1}{9\sqrt{3}} \rho^2 \sin 2\theta
\end{align*}
\]

which, in conjunction with (37)–(38) and (25)–(27), results in the set of differential equations

\[
\begin{align*}
\frac{dy^1}{dt} &= \frac{1}{3} (\mu^1 + 2\mu^2) y^1 + \frac{1}{3} \mu^1 y^2 - \frac{4}{9} y^1 y^2 - \frac{2}{9} (y^1)^2 \\
\frac{dy^2}{dt} &= \frac{1}{3} (\mu^1 - \mu^2) y^1 + \frac{1}{3} \mu^1 y^2 - \frac{1}{3} (y^1)^2 - \frac{4}{9} y^1 y^2 - \frac{2}{9} (y^2)^2 - \frac{1}{6} \rho^2 \\
\frac{d\rho}{dt} &= \frac{1}{6} (\mu^1 + \mu^2 - 3\mu^3) \rho - \frac{4}{9} y^2 \rho
\end{align*}
\]
and
\[
\frac{d\rho}{dt} = \sqrt{3} + O((y^1, y^2, \rho)^2)
\] (65)

where \(\rho\) represents a measure of the amplitude of the periodic solutions, and \(y^1\) and \(y^2\) approximate \(w^1\) and \(w^2\) up to second-order terms.

Now, based on equations (64), the first-order approximations for the solutions are as follows.

The initial equilibrium solution is
\[
y^1 = y^2 = \rho = 0
\] (66)

The post-critical solutions are described by
\[
y^1 = -3\mu^2, \quad y^2 = 3\mu^2, \quad \rho = 0 \quad \text{(static bifurcation solution)}
\] (67)
\[
y^1 = -3\mu^2, \quad y^2 = \frac{3}{2}\mu^1 + 3\mu^2, \quad \rho = 0 \quad \text{(static bifurcation solution)}
\] (68)

and
\[
\begin{aligned}
y^2 &= \frac{3}{8}(\mu^1 + \mu^2 - 3\mu^3) \\
y^1 &= \frac{(2y^2 - 3\mu^1)y^2}{(3\mu^1 + 6\mu^2 - 4y^2)} \\
\rho^2 &= -2y^1(y^1 + 3\mu^2)
\end{aligned}
\] (Hopf bifurcation solution) (69)

Next, we apply the formulae obtained in §3 to analyse the bifurcations in the vicinity of the critical point for this example. It is noted that the results can be readily recovered by using the jacobian matrix of (64) directly and following the procedure described in §3.

First of all, (41) gives the stable region for the initial equilibrium solution (66) as
\[
\mu^1\mu^2 > 0, \quad \mu^1 + \mu^2 < 0, \quad \mu^1 + \mu^2 - 3\mu^3 < 0 \quad (\mu^3 \geq 0)
\] (70)

that is,
\[
\mu^1 < 0, \quad \mu^2 < 0, \quad \mu^3 \geq 0
\] (71)

The first inequality in (70) implies the critical surface \(S_1\) in the form \(\mu^1\mu^2 = 0\), i.e. \(\mu^1 = 0\) or \(\mu^2 = 0\) (or \(\mu^1 = \mu^2 = 0\) (degenerate)). However, \(S_1\) represents the intersection of the fundamental equilibrium surface with (67) or (68), and one therefore obtains the critical surface
\[
S_1: \mu^2 = 0 \quad (\mu^1 + \mu^2 < 0, \mu^3 \geq 0)
\] (72)
along which the static bifurcation (67) takes place from the fundamental equilibrium surface (66).

The critical surface \(S_2\)—expressed by (43)—along which Hopf bifurcations may occur, does not take place in the stable region because (43) implies \(\mu^1 + \mu^2 = 0\), which violates the condition \(\mu^1\mu^2 > 0\).

A family of limit cycles with the frequency \(\omega = \sqrt{3}\) may bifurcate from the fundamental equilibrium surface along the critical surface (45)
\[
S_3: \mu^1 + \mu^2 - 3\mu^3 = 0 \quad (\mu^1 < 0, \mu^2 < 0)
\] (73)
and the behaviour surface is expressed by (69).
Next, considering the stability of the static bifurcation solution (67) results in the stable region for this solution as
\[ \mu^1 \mu^2 < 0, \quad \mu^1 - \mu^2 < 0, \quad \mu^1 - 7\mu^2 - 3\mu^3 < 0 \] (74)
which, in turn, yields two critical surfaces. One of these surfaces is
\[ S^{(1)}_6 : \mu^1 = 0 \quad (\mu^1 - \mu^2 < 0, \mu^1 - 7\mu^2 - 3\mu^3 < 0) \] (75)
along which a secondary static bifurcation occurs from the first bifurcating solution (67) and this secondary bifurcating solution is described by (68), where the superscript (1) denotes the critical surface associated with the static bifurcation solution (67). A second critical surface is defined by
\[ S^{(1)}_6 : \mu^1 - 7\mu^2 - 3\mu^3 = 0 \quad (\mu^1 \mu^2 < 0, \mu^1 - \mu^2 < 0) \] (76)
along which a secondary Hopf bifurcation, approximated by (69) with the frequency \( \omega_s = \sqrt{3}, \) may take place from (67).
Similarly, investigating the stability conditions of the static bifurcation solution (68), one obtains the stable region for this solution:
\[ \mu^1 \mu^2 > 0, \quad \mu^1 + \mu^2 > 0, \quad 3\mu^1 + 7\mu^2 + 3\mu^3 > 0 \] (77)
which gives two critical surfaces.
The first critical surface is given by
\[ S^{(2)}_6 : \mu^1 = 0 \quad (\mu^1 + \mu^2 > 0, 3\mu^1 + 7\mu^2 + 3\mu^3 > 0) \] (78)
which may immediately be recognized as \( S^{(1)}_6, \) along which the solutions (67) and (68) locally intersect and exchange stabilities.
The second critical surface takes the form
\[ S^{(2)}_6 : 3\mu^1 + 7\mu^2 + 3\mu^3 = 0 \quad (\mu^1 \mu^2 > 0, \mu^1 + \mu^2 > 0) \] (79)
along which a secondary Hopf bifurcation emerges from (68). The behaviour surface associated with this solution is given by (69).
Finally, evaluating the jacobian of (64) on the Hopf bifurcation solution (69) results in the characteristic polynomial
\[ P(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c \] (80)
with the coefficients
\[
\begin{align*}
a &= -\frac{1}{2} [3(\mu^1 + \mu^2) - 2y^1 - 4y^2] \\
b &= \frac{1}{2} [(3\mu^1 - 4y^1 - 4y^2)(\mu^1 + 2y^1) - 4\rho^2] \\
c &= \frac{2}{3}(3\mu^1 + 6\mu^2 - 4y^2)\rho^2
\end{align*}
\] (81)
where \( y^1, y^2 \) and \( \rho \) are expressed by (69). It follows from the Hurwitz criterion that the stable region for the Hopf bifurcation solution (69) is described by
\[ a > 0, \quad b > 0, \quad c > 0, \quad ab - c > 0 \] (82)
which gives two critical surfaces. One of these is defined by \( c = 0, \) that is
\[ S'_6 : \mu^1 + 3\mu^2 + 3\mu^3 = 0 \quad (a > 0, b > 0, ab - c > 0) \] (83)
along which a secondary bifurcation takes place from the first bifurcating family of
limit cycles. The second critical surface is given by
\[ S_8: ab - c = 0 \quad (a > 0, b > 0, c > 0) \]  
which represents the onset of another secondary Hopf bifurcation, leading to a new family of limit cycles with the frequency
\[ \omega_c = \frac{\sqrt{3}}{2} \left[ (3\mu^1 - 4y^1 - 4y^2)(3\mu^2 + 2y^1) - 4\rho^2 \right]^{1/2} \]  

\[ \text{ACKNOWLEDGMENT} \]

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

\begin{appendix}

\section*{Appendix A}

In this appendix, it is shown that the derivatives of the real and imaginary parts of the complex eigenvalue, evaluated at the critical point \( c \), can be expressed in terms of the system coefficients.

In order to obtain the desired relationships, the Jacobian matrix \( W(\eta^\beta) = [W_{ij}(\eta^\beta)] \) is expanded into Taylor series in the vicinity of the compound critical point \( c \),
\[ W_{ij}(\eta^\beta) = W_{ij} + W_{ij,\mu}^\beta + \frac{1}{2} W_{ij,\mu^\beta}^\beta \mu^\beta + \ldots \]  
where \( W_{ij} \) is in the form (5) and \( \mu^\beta = \eta^\beta - \eta^\beta_c \). Suppose that the eigenvalues of the Jacobian matrix are given by \( \lambda_1(\mu^\beta), \lambda_2(\mu^\beta) \) and \( \lambda_3,4 = \alpha(\mu^\beta) \pm i\omega(\mu^\beta) \), where \( \lambda_1(\mu^\beta) \) and \( \lambda_3(\mu^\beta) \) may be real or complex conjugate; and \( \lambda_1(0) = \lambda_2(0) = \alpha(0) = 0 \).

Substituting (A 1) into the characteristic equation
\[ |W(\eta^\beta) - I\lambda(\eta^\beta)| = 0 \]  
one obtains
\[ W_{12,12}^\beta + O(\mu^\beta) - \lambda_1 + W_{12,12} + O(\mu^\beta) = 0 \]  

\begin{equation}
\begin{vmatrix}
W_{12} + O(\mu^\beta) & W_{13} + O(\mu^\beta) & W_{14} + O(\mu^\beta) \\
W_{23} + O(\mu^\beta) & W_{33} + O(\mu^\beta) & W_{34} + O(\mu^\beta) \\
W_{43} + O(\mu^\beta) & W_{44} + O(\mu^\beta) & W_{44} + O(\mu^\beta)
\end{vmatrix} = 0
\end{equation}

Differentiating (A 3) with respect to \( \mu^\beta (\beta = 1, 2, 3) \) and then evaluating at the critical point \( c \) yields
\[ \lambda(0) [\lambda^2(0) + \omega_c^2] (W_{11} + W_{22} - 2\lambda)^\beta + \lambda^2(0) (W_{33} + W_{44} - 2\lambda)^\beta - \omega_c (W_{34} - W_{43}) = 0 \]  

where \( \lambda^\beta = d\lambda/d\eta^\beta \). It is noted that \( \lambda_1^\beta \) and \( \lambda_2^\beta \) cannot be determined from (A 4).

Substituting \( \lambda = \lambda_3 = \alpha(\mu^\beta) + i\omega(\mu^\beta) \) into (A 4) yields
\[ \alpha^\beta = \frac{1}{\lambda} (W_{33} + W_{44}), \quad \omega^\beta = \frac{1}{\lambda} (W_{34} - W_{43}) \]  

Differentiating (A 3) with respect to \( \mu^\beta (\beta = 1, 2, 3) \) twice and evaluating at the critical point, after a lengthy calculation one may finally obtain.
\end{appendix}
\[ [\lambda^2(0) + \omega_c^2] [(W_{11\beta} - \lambda^2)(W_{22\beta} - \lambda^2) - W_{12\beta}W_{12\beta} - \frac{1}{2}\lambda(0)(W_{11\beta} + W_{22\beta} - 2\lambda^2)]
+ \lambda(0)[(W_{11\beta} + W_{22\beta} - 2\lambda^2)(W_{33\beta} + W_{44\beta} - 2\lambda^2) + (W_{33\beta} - \lambda^2)(W_{44\beta} - \lambda^2) - \frac{1}{2}\lambda(0)(W_{33\beta} + W_{44\beta} - 2\lambda^2) + \frac{1}{2}\omega_c(W_{33\beta} + W_{44\beta} - 2\lambda^2) + 2\omega_c(W_{33\beta} + W_{44\beta} - 2\lambda^2)]
- \lambda(0)[\omega_c(W_{11\beta} + W_{22\beta} - 2\lambda^2)(W_{34\beta} - W_{43\beta}) + (W_{23\beta}W_{31\beta} + W_{24\beta}W_{41\beta})]
+ \omega_c(W_{13\beta}W_{41\beta} - W_{14\beta}W_{31\beta} + W_{23\beta}W_{42\beta} - W_{24\beta}W_{32\beta})
+ \omega_c(W_{24\beta}W_{31\beta} - W_{23\beta}W_{41\beta}) = 0 \quad (A\ 6)\]

Substituting \(\lambda_1(\mu^\beta)\) or \(\lambda_2(\mu^\beta)\) into (A 6) and evaluating at the critical point, that is, setting \(\lambda(0) = 0\) results in

\[ (\lambda^2)^2 + A\lambda^2 + B = 0 \quad (A\ 7)\]

where

\[\begin{align*}
A &= -(W_{11\beta} + W_{22\beta}) \\
B &= (W_{11\beta}W_{22\beta} - W_{12\beta}W_{21\beta}) + \frac{1}{\omega_c}(W_{23\beta}W_{31\beta} - W_{24\beta}W_{41\beta})\end{align*} \quad (A\ 8)\]

For different values of \(A\) and \(B\), there are six typical cases of the first two eigenvalues of the Jacobian crossing the origin of the complex plane. These cases are sketched in Fig. 3.

![Diagram](image)

Figure 3. Locus of the eigenvalues of the Jacobian.

Similarly, replacing \(\lambda = \theta(\mu^\beta) + i\omega(\mu^\beta)\) into (A 6) gives

\[ \theta^\beta = \frac{1}{2}(W_{33\beta} + W_{44\beta}) - \frac{1}{\omega_c^2}(W_{23\beta}W_{31\beta} + W_{24\beta}W_{41\beta}) + \frac{1}{\omega_c}(W_{14\beta}W_{31\beta} - W_{13\beta}W_{41\beta} + W_{24\beta}W_{32\beta} - W_{23\beta}W_{42\beta}) \quad (A\ 9)\]
and

$$\omega^p = \frac{1}{2} (W_{34p_1} + W_{43p_1}) + \frac{1}{\omega_c^3} (W_{23p} W_{41p} - W_{24p} W_{31p})$$

$$- \frac{1}{\omega_c} \left[ \frac{1}{4} (W_{33p} - W_{44p})^2 + \frac{1}{4} (W_{34p} + W_{43p})^2 \right.$$  

$$+ (W_{13p} W_{31p} + W_{14p} W_{41p} + W_{23p} W_{32p} + W_{24p} W_{42p}) \left. \right]$$  \(\text{A 10}\)

**Appendix B**

In this appendix, the amplitudes of the harmonics are given in terms of \(p_{10}, p_{20}\) and \(p_{31}\) up to second-order terms:

$$p_{30} = \frac{1}{\omega_c} \left[ W_{41p} \mu^p p_{10} + W_{42p} \mu^p p_{20} + \frac{1}{2} W_{311}(p_{10})^2 + W_{412} p_{10} p_{20} + \frac{1}{2} W_{422}(p_{20})^2 \right.$$  

$$+ \frac{1}{4} (W_{333} + W_{444})(p_{31})^2 \left. \right]$$  \(\text{B 1}\)

$$p_{40} = -\frac{1}{\omega_c} \left[ W_{31p} \mu^p p_{10} + W_{32p} \mu^p p_{20} + \frac{1}{2} W_{311}(p_{10})^2 + W_{312} p_{10} p_{20} + \frac{1}{2} W_{322}(p_{20})^2 \right.$$  

$$+ \frac{1}{4} (W_{333} + W_{344})(p_{31})^2 \left. \right]$$  \(\text{B 2}\)

$$p_{11} = \frac{1}{\omega_c} (W_{14p} \mu^p + W_{114} p_{10} + W_{124} p_{20}) p_{31}$$  \(\text{B 3}\)

$$r_{11} = \frac{1}{\omega_c} (W_{13p} \mu^p + W_{113} p_{10} + W_{123} p_{20}) p_{31}$$  \(\text{B 4}\)

$$p_{21} = \frac{1}{\omega_c} (W_{24p} \mu^p + W_{214} p_{10} + W_{224} p_{20}) p_{31}$$  \(\text{B 5}\)

$$r_{21} = \frac{1}{\omega_c} (W_{23p} \mu^p + W_{213} p_{10} + W_{223} p_{20}) p_{31}$$  \(\text{B 6}\)

$$p_{41} = -\frac{1}{2\omega_c} [(W_{33p} - W_{44p}) \mu^p + (W_{331} - W_{441}) p_{10} + (W_{332} - W_{442}) p_{20}] p_{31}$$  \(\text{B 7}\)

$$r_{41} = -p_{31} + \frac{1}{2\omega_c} [(W_{34p} + W_{43p}) \mu^p + (W_{341} + W_{431}) p_{10} + (W_{342} + W_{432}) p_{20}] p_{31}$$  \(\text{B 8}\)

$$p_{12} = \frac{1}{4\omega_c} \left[ W_{134} - \frac{1}{4\omega_c} (W_{333} - W_{244}) \right] (p_{31})^2$$  \(\text{B 9}\)

$$r_{12} = \frac{1}{8\omega_c} \left[ (W_{133} - W_{144}) + \frac{1}{\omega_c} W_{234} \right] (p_{31})^2$$  \(\text{B 10}\)

$$p_{22} = \frac{1}{4\omega_c} W_{234} (p_{31})^2$$  \(\text{B 11}\)
Bifurcations, zeros and eigenvalues

\[ r_{22} = \frac{1}{8\omega_c}(W_{233} - W_{244})(p_{31})^2 \]  \hspace{1cm} (B 12)

\[ p_{32} = \frac{1}{12\omega_c}(W_{444} - W_{433} + 4W_{334})(p_{31})^2 \]  \hspace{1cm} (B 13)

\[ r_{32} = \frac{1}{6\omega_c}(W_{333} - W_{344} + W_{433})(p_{31})^2 \]  \hspace{1cm} (B 14)

\[ p_{42} = \frac{1}{12\omega_c}(W_{333} - W_{344} + 4W_{433})(p_{31})^2 \]  \hspace{1cm} (B 15)

\[ r_{42} = \frac{1}{6\omega_c}(W_{433} - W_{444} - W_{344})(p_{31})^2 \]  \hspace{1cm} (B 16)

Appendix C

In this appendix, a brief outline of an approach leading to the demonstration of the local equivalence of (37) and (38) to the original system is described. Thus, in order to demonstrate that the local dynamics and bifurcation behaviour of the original system (4) is embraced by the differential equations (37) and (38), consider the original system

\[ \frac{dw^l}{dt} = W_i(w^l; \eta^l) \]  \hspace{1cm} (C 1)

whose jacobian (evaluated at the critical point \( c \)) is in the canonical form

\[ J = [W_{ij}]_c = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_c \\
0 & 0 & -\omega_c & 0
\end{bmatrix} \]  \hspace{1cm} (C 2)

and the initial equilibrium solution is given by

\[ w^l = 0 \]  \hspace{1cm} (C 3)

First, expanding (C 1) into Taylor series around the point \((w^l; \eta^l) = (0; \eta^l)\), one obtains

\[
\frac{dw^1}{dt} = W_{11}\mu^\delta w^1 + W_{12}\mu^\delta w^2 + W_{13}\mu^\delta w^3 + W_{14}\mu^\delta w^4 + \frac{1}{2}W_{111}(w^1)^2
\]

\[
+ \frac{1}{2}W_{122}(w^2)^2 + \frac{1}{2}W_{133}(w^3)^2 + \frac{1}{2}W_{144}(w^4)^2 + W_{112}w^1w^2 + W_{113}w^1w^3 + W_{114}w^1w^4 + O((w^1; \mu^\delta)^2)
\]

\[
\frac{dw^2}{dt} = W_{21}\mu^\delta w^1 + W_{22}\mu^\delta w^2 + W_{23}\mu^\delta w^3 + W_{24}\mu^\delta w^4 + \frac{1}{2}W_{211}(w^1)^2
\]

\[
+ \frac{1}{2}W_{222}(w^2)^2 + \frac{1}{2}W_{233}(w^3)^2 + \frac{1}{2}W_{244}(w^4)^2 + W_{212}w^1w^2 + W_{213}w^1w^3 + W_{214}w^1w^4 + O((w^2; \mu^\delta)^2)
\]

\[
+ W_{214}w^1w^4 + W_{223}w^2w^3 + W_{224}w^2w^4 + W_{234}w^3w^4 + O((w^i; \mu^\delta)^2)
\]
\[
\frac{dw^3}{dt} = \omega_c w^2 + W_{313} \mu \theta w^1 + W_{323} \mu \theta w^2 + W_{333} \mu \theta w^3 + W_{343} \mu \theta w^4 + \frac{1}{2} W_{311} (w^1)^2 \\
+ \frac{1}{2} W_{322} (w^2)^2 + \frac{1}{2} W_{333} (w^3)^2 + \frac{1}{2} W_{344} (w^4)^2 + W_{312} w^1 w^2 + W_{313} w^1 w^3 \\
+ W_{314} w^1 w^4 + W_{323} w^2 w^3 + W_{324} w^2 w^4 + W_{334} w^3 w^4 + O((w^i, \mu \theta)^3)
\]
\[
\frac{dw^4}{dt} = -\omega_c w^3 + W_{413} \mu \theta w^1 + W_{423} \mu \theta w^2 + W_{433} \mu \theta w^3 + W_{443} \mu \theta w^4 + \frac{1}{2} W_{411} (w^1)^2 \\
+ \frac{1}{2} W_{422} (w^2)^2 + \frac{1}{2} W_{433} (w^3)^2 + \frac{1}{2} W_{444} (w^4)^2 + W_{412} w^1 w^2 + W_{413} w^1 w^3 \\
+ W_{414} w^1 w^4 + W_{423} w^2 w^3 + W_{424} w^2 w^4 + W_{434} w^3 w^4 + O((w^i, \mu \theta)^3)
\]

(C 4)

upon using (C 2).

Next, consider perturbing the steady-state solutions (36). To this end, suppose the time-invariant constants \( p_{10}, p_{20} \) and \( p_{31} \) are replaced by time-variant variables \( y^1, y^2 \) and \( \rho \), respectively. Using Appendix B, the solutions (36) take the form

\[
w^i = y^i + \frac{1}{\omega_c} (W_{143} \mu \theta + W_{144} y^1 + W_{144} y^2) \rho \cos \theta \\
+ \frac{1}{\omega_c} (W_{133} \mu \theta + W_{113} y^1 + W_{123} y^2) \rho \sin \theta \\
+ \frac{1}{4 \omega_c} (W_{134} - \frac{1}{\omega_c} (W_{233} - W_{244})) (\rho)^2 \cos 2\theta \\
+ \frac{1}{8 \omega_c} (W_{133} - W_{144}) + \frac{1}{\omega_c} W_{234} \rho^2 \sin 2\theta
\]  
(C 5 a)

\[
w^2 = y^2 + \frac{1}{\omega_c} (W_{243} \mu \theta + W_{244} y^1 + W_{244} y^2) \rho \cos \theta \\
+ \frac{1}{\omega_c} (W_{233} \mu \theta + W_{213} y^1 + W_{223} y^2) \rho \sin \theta \\
+ \frac{1}{4 \omega_c} W_{234} \rho^2 \cos 2\theta + \frac{1}{8 \omega_c} (W_{233} - W_{244}) \rho^2 \sin 2\theta
\]  
(C 5 b)

\[
w^3 = \rho \cos \theta + \frac{1}{\omega_c} \left[ W_{413} \mu \theta y^1 + W_{423} \mu \theta y^2 + \frac{1}{2} W_{411} (y^1)^2 + W_{412} y^1 y^2 + \frac{1}{2} W_{422} (y^2)^2 \\
+ \frac{1}{4} (W_{433} + W_{444}) \rho^2 \right] \\
+ \frac{1}{12 \omega_c} (W_{444} - W_{433} + 4 W_{334}) \rho^2 \cos 2\theta + \frac{1}{6 \omega_c} (W_{333} - W_{444} + W_{443}) \rho^2 \sin 2\theta
\]  
(C 5 c)
\[ \omega^2 = -\rho \sin \theta - \frac{1}{4\omega} \left[ W_{315} \rho^2 y^3 + W_{325} \rho^2 y^2 + \frac{1}{2} W_{311} (y^1)^2 + W_{312} y^1 y^2 + \frac{1}{2} W_{322} (y^2)^2 \right. \\
+ \frac{1}{4} (W_{333} + W_{344}) \rho^2 \left] \\
- \frac{1}{2\omega} \left[ (W_{333} - W_{444}) \rho^2 + (W_{221} - W_{331}) y^1 + (W_{332} - W_{442}) y^2 \right] \rho \cos \theta \\
+ \frac{1}{2\omega} \left[ (W_{343} + W_{433}) \rho^2 + (W_{341} + W_{431}) y^1 + (W_{432} + W_{432}) y^2 \right] \rho \sin \theta \\
+ \frac{1}{12\omega} (W_{333} - W_{444} + 4W_{444}) \rho^2 \cos 2\theta + \frac{1}{6\omega} (W_{433} - W_{444} - W_{334}) \rho^2 \sin 2\theta \]

(C 5 d)

where \( \tau \) is also replaced by \( \theta \).

Now, substituting (C 5) into (C 1) and solving for \( dy^1/dt, dy^2/dt, dp/dt \) and \( \rho d\theta/dt \), one obtains

\[ \frac{dy^1}{dt} = y^2 + W_{115} \rho^2 y^1 + W_{125} \rho^2 y^2 + \frac{1}{2} W_{111} (y^1)^2 + W_{112} y^1 y^2 + \frac{1}{2} W_{122} (y^2)^2 \\
+ \frac{1}{4} (W_{333} + W_{444}) \rho^2 + O((y^1, y^2, \rho, \mu \rho)^3) \]

\[ \frac{dy^2}{dt} = W_{215} \rho^2 y^1 + W_{225} \rho^2 y^2 + \frac{1}{2} W_{211} (y^1)^2 + W_{212} y^1 y^2 + \frac{1}{2} W_{222} (y^2)^2 \\
+ \frac{1}{4} (W_{333} + W_{444}) \rho^2 + O((y^1, y^2, \rho, \mu \rho)^3) \]

\[ \frac{dp}{dt} = \frac{1}{2} (W_{333} + W_{444}) \rho^2 + \frac{1}{2} (W_{331} + W_{441}) y^1 \rho + \frac{1}{2} (W_{332} + W_{442}) y^2 \rho \\
+ O((y^1, y^2, \rho, \mu \rho)^3) \]

\[ \frac{d\theta}{dt} = \rho \left[ \frac{1}{2} (W_{343} - W_{433}) \rho^2 + \frac{1}{2} (W_{341} - W_{431}) y^1 + \frac{1}{2} (W_{342} - W_{432}) y^2 \right] \\
+ O((y^1, y^2, \rho, \mu \rho)^3) \]

(C 6)

after lengthy but straightforward algebra. Truncating the high-order terms and using (27)-(29) results in (37) and (38).

REFERENCES


