Two- and three-dimensional tori bifurcating from interactions of dynamic modes

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The sequence of bifurcations into two- and three-dimensional tori associated with a non-linear autonomous system is investigated in the vicinity of a compound critical point—at which the jacobian of the system involves a double zero of index two and a pair of purely imaginary eigenvalues. This investigation is complementary to an earlier work in which incipient and secondary bifurcations associated with this system were explored.

1. Introduction

It is known that the interactions of static and dynamic bifurcations in the vicinity of a compound critical point of a non-linear autonomous system may lead to bifurcations into invariant tori associated with quasi-periodic solutions. Depending on the nature of the corresponding critical point and the structure of the associated jacobian, the problem may be of co-dimension 2 (see, e.g., Langford 1979, Guckenheimer and Holmes 1983), co-dimension 3 (see, e.g., Medved 1984, Arneodo et al. 1985, Yu and Huseyin 1988 a, b), or of a higher co-dimension.

Recently, the bifurcation and stability properties of an autonomous system in the vicinity of a specific compound critical point—at which the system exhibits a pair of purely imaginary eigenvalues and a double zero eigenvalue of index two—have been studied (Yu and Huseyin 1988 a). However, attention in the study of Yu and Huseyin (1988 a) was focused on the static and dynamic stability boundaries (incipient bifurcations) and secondary bifurcations. Although the possibility of bifurcations into invariant tori was observed, this more complex phenomenon was not explored. The aim of this paper is to study in detail the critical conditions leading to two-dimensional (2-D) and three-dimensional (3-D) tori. The analysis is based on a combination of the ‘intrinsic harmonic balancing’ (Huseyin 1986) and the ‘unification technique’ (Yu and Huseyn 1986, 1988 a, b) which enables one to obtain simplified differential equations governing the local dynamics. This approach also facilitates the stability analysis of the solutions.

Hence, this paper is complementary to the earlier work by Yu and Huseyin (1988 a), and is illustrated using the same example involving a control system.

2. 2-D and 3-D tori generated by interactions of incipient Hopf bifurcations

For convenience, the formulation of an earlier paper (Yu and Huseyin 1988 a) will first be summarized. The autonomous system under consideration is assumed to be

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described by
\[ \frac{dz^i}{dt} = Z_i(z^i; \eta^\beta) \quad (i, j = 1, 2, 3, 4 \quad \text{and} \quad \beta = 1, 2, 3) \] (1)

where the \( z^i \) are the components of the state vector \( z \), and the \( \eta^\beta \) are certain independent parameters. It is further assumed that the functions \( Z_i \) are analytic, at least in the region of interest.

Now suppose that the system has a single-valued equilibrium surface in the region of interest, which is expressed as \( z^i = f_\lambda(\eta^\beta) \). Next, introduce the non-singular transformation
\[ z^i = f_\lambda(\eta^\beta) + T_{ij} w^j \] (2)

into (1), such that the resulting system
\[ \frac{dw^j}{dt} = W_j(w^j; \eta^\beta) \] (3)

has a jacobian matrix evaluated at the critical point \( c \) (where \( \eta^\beta = \eta_c^\beta \)) of the form
\[
\begin{bmatrix}
0 & \omega_c \\
-\omega_c & 0
\end{bmatrix}
\] (4)

It follows from the transformation (2) that the new system (3) has the following properties
\[ W_i(0; \eta^4) = W_{ip}(0; \eta^5) = W_{ipj}(0; \eta^6) = \ldots = 0 \] (5)

where the subscripts on the \( W_i \)'s indicate differentiations with respect to the corresponding parameters.

It was shown (see Yu and Huseyn 1988 a, Appendix C) that, in the vicinity of \( c \), the differential equations (3) may be approximated by
\[
\begin{align*}
\frac{dy^1}{dt} &= W_{11\beta}\mu^\beta y^1 + W_{12\beta}\mu^\beta y^2 + \frac{1}{2} W_{111}(y^1)^2 + W_{112}y^1 y^2 + \frac{1}{2} W_{122}(y^2)^2 + \frac{1}{2} l_1 \rho^2 \\
\frac{dy^2}{dt} &= W_{21\beta}\mu^\beta y^1 + W_{22\beta}\mu^\beta y^2 + \frac{1}{2} W_{211}(y^1)^2 + W_{212}y^1 y^2 + \frac{1}{2} W_{222}(y^2)^2 + \frac{1}{2} l_2 \rho^2 \\
\frac{d\rho}{dt} &= \alpha^\beta \rho + \alpha_1 y^1 \rho + \alpha_2 y^2 \rho
\end{align*}
\] (6)

and
\[ \frac{d\theta}{dt} = \omega_c + \omega^\beta \mu^\beta + \omega_1 y^1 + \omega_2 y^2 \] (7)

up to second-order terms. Here, \( y^1, y^2, \rho \) and \( \theta \) are the new variables and the constants \( l_1, l_2, \alpha^\beta, \omega^\beta, \alpha_1, \alpha_2, \omega_1 \) and \( \omega_2 \) are given in the above-mentioned paper. \( \rho \) and \( \theta \) emerge as polar co-ordinates upon introduction of a Fourier series representation for periodic solutions (where \( \rho \) is the amplitude).

It is easy to verify, on the basis of (6), that incipient bifurcations from the initial equilibrium solution \( y^i = \rho = 0 \) occur along three critical surfaces which were designated as \( S_1, S_2 \) and \( S_3 \) by Yu and Huseyn (1988 a). \( S_1 \) is associated with static
bifurcations, while $S_2$ and $S_3$ lead to two distinct families of Hopf bifurcations. These can readily be obtained as

$$S_2: (W_{11\beta} + W_{22\beta})\mu^\beta = 0, \quad S_3: \alpha^\beta \mu^\beta = 0$$  \hfill (8)

The eigenvalues of the jacobian of (6), evaluated on the intersection of $S_2$ and $S_3$, consist of a zero and a purely imaginary pair (this corresponds to two purely imaginary pairs in the original system (3)), and the system exhibits interesting behaviour characteristics in the vicinity of this intersection. In order to study these features of the system and to transform the jacobian matrix of (6) evaluated on (8) into a canonical form, consider the transformation

$$\begin{bmatrix}
y_1 \\
y_2 \\
p
\end{bmatrix} = \begin{bmatrix}
\omega_t W_{11\beta} \mu^\beta_c & \omega_t^2 & 0 \\
\omega_t W_{21\beta} \mu^\beta_c & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}$$  \hfill (9)

where $\bar{c}$ is a point on the intersection and $\omega_t$, given by

$$\omega_t = [(W_{11\beta} W_{22\gamma} - W_{12\beta} W_{21\gamma}) \mu^\beta \mu^\gamma]^{1/2}$$  \hfill (10)

is the frequency of the Hopf family associated with $S_2$ (note that $\omega_t$ is the frequency associated with $S_3$).

If (9) is substituted into (6) and (7), we obtain

$$\frac{dv_1}{dt} = V_1(v^1, v^2, v^3; \mu^1, \mu^2, \mu^3)$$

$$= \frac{1}{W_{21\beta} \mu^\beta_c} (W_{11\beta} \mu^\beta_c W_{22\beta} + W_{21\beta} \mu^\beta_c W_{22\beta}) \mu^\beta v^1 + \frac{\omega_t}{W_{21\beta} \mu^\beta_c} W_{21\beta} \mu^\beta v^2$$

$$+ \frac{\omega_t^2}{2W_{21\beta} \mu^\beta_c} (W_{211} W_{11\beta} W_{21\gamma} + 2W_{212} W_{11\beta} W_{21\gamma} + W_{222} W_{21\beta} W_{21\gamma}) \mu^\beta_c (v^1)^2$$

$$+ \frac{1}{W_{21\beta} \mu^\beta_c} (W_{211} W_{11\beta} + W_{212} W_{21\beta}) \mu^\beta_c v^3 v^2 + \frac{\omega_t^2}{2W_{21\beta} \mu^\beta_c} (W_{211} (v^3)^2) + \frac{l^2}{2\omega_t W_{21\beta} \mu^\beta_c} (v^3)^2$$  \hfill (11a)

$$\frac{dv_2}{dt} = V_2(v^1, v^2, v^3; \mu^1, \mu^2, \mu^3)$$

$$= -\frac{1}{\omega_t} \left[ (W_{11\beta} \mu^\beta_c W_{22\beta} - W_{21\beta} \mu^\beta_c W_{11\gamma}) + \frac{W_{21\beta} \mu^\beta_c (W_{11\beta} \mu^\beta_c W_{21\beta} W_{21\gamma} - W_{21\beta} \mu^\beta_c W_{11\gamma})}{W_{21\beta} \mu^\beta_c} \right] \mu^\beta v^1$$

$$+ \frac{1}{W_{21\beta} \mu^\beta_c} (W_{211} \mu^\beta_c W_{11\beta} - W_{11\beta} \mu^\beta_c W_{11\gamma}) \mu^\beta v^2$$

$$+ \frac{1}{2} \left[ (W_{111} W_{11\beta} W_{11\gamma} + 2W_{112} W_{11\beta} W_{21\gamma} + W_{122} W_{21\beta} W_{21\gamma})
- \frac{W_{211} \mu^\beta_c (W_{211} W_{11\beta} W_{11\gamma} + 2W_{212} W_{11\beta} W_{21\gamma} + W_{222} W_{21\beta} W_{21\gamma})}{W_{21\beta} \mu^\beta_c} \right] \mu^\beta_c (v^1)^2$$

$$+ \omega_t \left[ (W_{111} W_{11\beta} + W_{112} W_{21\beta}) - \frac{W_{11\beta} \mu^\beta_c (W_{211} W_{11\beta} + W_{212} W_{21\beta})}{W_{21\beta} \mu^\beta_c} \right] \mu^\beta_c v^1 v^2$$

$$= V_2(v^1, v^2, v^3; \mu^1, \mu^2, \mu^3)$$

$$= -\frac{1}{\omega_t} \left[ (W_{11\beta} \mu^\beta_c W_{22\beta} - W_{21\beta} \mu^\beta_c W_{11\gamma}) + \frac{W_{21\beta} \mu^\beta_c (W_{11\beta} \mu^\beta_c W_{21\beta} W_{21\gamma} - W_{21\beta} \mu^\beta_c W_{11\gamma})}{W_{21\beta} \mu^\beta_c} \right] \mu^\beta v^1$$

$$+ \frac{1}{W_{21\beta} \mu^\beta_c} (W_{211} \mu^\beta_c W_{11\beta} - W_{11\beta} \mu^\beta_c W_{11\gamma}) \mu^\beta v^2$$

$$+ \frac{1}{2} \left[ (W_{111} W_{11\beta} W_{11\gamma} + 2W_{112} W_{11\beta} W_{21\gamma} + W_{122} W_{21\beta} W_{21\gamma})
- \frac{W_{211} \mu^\beta_c (W_{211} W_{11\beta} W_{11\gamma} + 2W_{212} W_{11\beta} W_{21\gamma} + W_{222} W_{21\beta} W_{21\gamma})}{W_{21\beta} \mu^\beta_c} \right] \mu^\beta_c (v^1)^2$$

$$+ \omega_t \left[ (W_{111} W_{11\beta} + W_{112} W_{21\beta}) - \frac{W_{11\beta} \mu^\beta_c (W_{211} W_{11\beta} + W_{212} W_{21\beta})}{W_{21\beta} \mu^\beta_c} \right] \mu^\beta_c v^1 v^2$$

$$= V_2(v^1, v^2, v^3; \mu^1, \mu^2, \mu^3)$$
\[
\begin{align*}
&+ \frac{\omega^2}{2W_{21\beta}} (W_{11\beta} - W_{21\beta}) \mu^\beta (v^3)^2 + \frac{1}{2\omega_c^2 W_{21\beta}} \mu^\beta \\
&\times (l_1 W_{21\beta} - l_2 W_{11\beta}) \mu^\beta (v^3)^2 \\
&= V_\alpha (v^1, v^2, v^3; \mu^1, \mu^2, \mu^3) \\
&= v^3 \left[ \mu^\beta \mu^\beta + \omega_c (\alpha_1 W_{11\beta} + \alpha_2 W_{21\beta}) \mu^\beta v^1 + \alpha_1 \omega_c^2 v^2 \right] \\
\end{align*}
\]

(11 b)

\[
\frac{dv^3}{dt} = V_\alpha (v^1, v^2, v^3; \mu^1, \mu^2, \mu^3) \\
= v^3 \left[ \mu^\beta \mu^\beta + \omega_c (\alpha_1 W_{11\beta} + \alpha_2 W_{21\beta}) \mu^\beta v^1 + \alpha_1 \omega_c^2 v^2 \right] \\
\]

(11 c)

and

\[
\frac{d\theta}{dt} = \omega_c + \omega^\beta \mu^\beta + \omega_c (\alpha_1 W_{11\beta} + \alpha_2 W_{21\beta}) \mu^\beta v^1 + \alpha_1 \omega_c^2 v^2 \\
\]

(12)

where the constant coefficients of the original system (6) are retained for convenience in applications. It is easy to verify that the Jacobian of (11) evaluated on the initial equilibrium solution \( v^1 = 0 \) (i.e. \( w^1 = 0 \)) along the intersection (8) is in the canonical form

\[
f = \begin{bmatrix} 0 & \omega_c & 0 \\ -\omega_c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

(13)

It is noted that due to (5), system (11) has the following properties:

\[
V_i (0; \mu^i) = V_{ip} (0; \mu^i) = V_{ip} (0; \mu^i) = \ldots = 0
\]

(14)

The differential equations given by (11) are still not in the most appropriate form for the analysis of the tori. Indeed, a representation totally in polar co-ordinates will facilitate bifurcation analyses considerably. Such a representation can be achieved by introducing an additional Fourier series with the frequency \( \omega_0 \) (which is equal to \( \omega_c \) at \( \epsilon \)),

\[
v^i (\bar{\tau}, \sigma^a) = \sum_{m=0}^M \left[ q_{im} (\sigma^a) \cos m \bar{\tau} + s_{im} (\sigma^a) \sin m \bar{\tau} \right], \quad \bar{\tau} = \omega_0 t
\]

(15)

where the \( \sigma^a \) (\( a = 1, 2, 3 \)) are certain unidentified parameters and \( q_{im} (0) = s_{im} (0) = 0 \) (for all \( i, m \)) since \( v^i (\bar{\tau}, 0) = 0 \).

Applying the 'intrinsic harmonic balancing' and the 'unification procedure' to (11) (see Appendix A) eventually yields the rate equations

\[
\begin{align*}
\frac{d\rho}{dt} &= \dot{\rho} \left[ \frac{1}{2} (W_{11\beta} + W_{22\beta}) \phi^\beta + A \dot{\rho}^2 + C \rho^2 \right] \\
\frac{d\rho}{dt} &= \rho (\mu^\beta \phi^\beta + B \dot{\rho}^2 + D \rho^2) \quad (\rho \equiv v^3)
\end{align*}
\]

(16)

where \( \phi^\beta = \mu^\beta - \mu_0^\beta \), and \( \dot{\rho} \) represents the amplitude of the periodic solution (15).

The bifurcations associated with system (16) are now readily explored. First, one has:

(i) The initial equilibrium solution, given by

\[
\rho = \rho_0 = 0 \quad \text{(i.e. } v^1 = 0 \text{ or } w^1 = 0) \]

(17)
(ii) A family of limit cycles, expressed by
\[
\rho^2 = \frac{1}{2A} (W_{1,1\beta} + W_{2,2\beta}) \phi^\beta \\
\rho = 0
\]
\[
\left\{ \begin{array}{l}
\rho^2 = -\frac{1}{D} \frac{\alpha^\theta \phi^\theta}{\phi^\beta} \\
\rho = 0
\end{array} \right\}
\tag{18}
\]
bifurcates from the initial equilibrium solution (17) along \(S_2\), in the vicinity of \(\bar{c}\).

(iii) Another family of limit cycles, given by
\[
\rho^2 = -\frac{1}{D} \frac{\alpha^\theta \phi^\theta}{\phi^\beta}
\]
\[
\left\{ \begin{array}{l}
\rho^2 = \frac{1}{2(AD - BC)} \left[ 2C\alpha^\theta - D(W_{1,1\beta} + W_{2,2\beta}) \right] \phi^\beta \\
\rho = 0
\end{array} \right\}
\tag{19}
\]
bifurcates from (17) along \(S_3\), in the vicinity of \(\bar{c}\).

(iv) Finally, one can readily obtain the quasi-periodic solution, described by
\[
\rho^2 = \frac{1}{2(AD - BC)} \left[ 2A\alpha^\theta - B(W_{1,1\beta} + W_{2,2\beta}) \right] \phi^\beta \\
\rho = 0
\]
\[
\left\{ \begin{array}{l}
\rho^2 = -\frac{1}{2(AD - BC)} \left[ 2A\alpha^\theta - B(W_{1,1\beta} + W_{2,2\beta}) \right] \phi^\beta \\
\rho = 0
\end{array} \right\}
\tag{20}
\]
where \(AD - BC \neq 0\), which represents an invariant 2-D torus if the ratio \(\omega_c/\omega_z\) is irrational.

The stability of the steady-state solutions (i)–(iv) can be determined from the jacobian of (16), which is in the form of
\[
J = \begin{bmatrix}
\frac{1}{2} (W_{1,1p} + W_{2,2p}) \phi^\beta + 3A\rho^2 + C\rho^2 & 2C\rho^2 \\
2B\rho^2 & \alpha^\theta \phi^\theta + B\rho^2 + 3D\rho^2
\end{bmatrix}
\tag{21}
\]

Evaluating the jacobian on the initial equilibrium solution \(\rho = 0\) results in
\[
J(0) = \begin{bmatrix}
\frac{1}{2} (W_{1,1p} + W_{2,2p}) \phi^\beta & 0 \\
0 & \alpha^\theta \phi^\theta
\end{bmatrix}
\tag{22}
\]

which in turn gives the stability conditions
\[
(W_{1,1p} + W_{2,2p}) \phi^\beta < 0 \quad \text{and} \quad \alpha^\theta \phi^\theta < 0
\tag{23}
\]
for the initial equilibrium solution. The two inequalities in (23) give two critical surfaces which are recognized as \(S_2\) and \(S_3\), respectively, obtained by Yu and Huseyin (1988 a). Incipient Hopf bifurcations (18) and (19) take place along \(S_2\) and \(S_3\), respectively.

Evaluating the jacobian (21) on the family of limit cycles (18) yields
\[
J_{(ii)} = \begin{bmatrix}
-(W_{1,1p} + W_{2,2p}) \phi^\beta & 0 \\
0 & \left( \alpha^\theta - \frac{B}{2A} (W_{1,1p} + W_{2,2p}) \right) \phi^\beta
\end{bmatrix}
\tag{24}
\]
which results in the stability conditions

\[(W_{11\beta} + W_{22\beta})\phi^\theta > 0 \quad \text{and} \quad \left(\alpha^\theta - \frac{B}{2A} (W_{11\beta} + W_{22\beta})\right)\phi^\theta < 0 \quad (25)\]

for the bifurcating limit cycles (18).

The second inequality of (25) implies that the orbitally stable limit cycles (18) become unstable at the critical surface

\[S_9: [2\alpha^\theta - B(W_{11\beta} + W_{22\beta})]\phi^\theta = 0 \quad (W_{11\beta} + W_{22\beta})\phi^\theta > 0 \quad (26)\]

and a new family of limit cycles bifurcates from the family of limit cycles (18) along \(S_9\), leading to an invariant 2-D torus if the ratio \(\omega_{\beta}/\omega_c\) is irrational. The quasi-periodic solutions of this family of 2-D tori are described by (20).

Similarly, the stability conditions of the family of the limit cycles (19) can be obtained by evaluating the Jacobian (21) on (19), which is given by

\[J_{\text{multi}} = \begin{bmatrix}
\frac{1}{2} (W_{11\beta} + W_{22\beta}) - \frac{C}{D} \alpha^\theta \\
0
\end{bmatrix} \phi^\theta \quad (27)\]

Thus, the inequalities

\[\left(\frac{1}{2} (W_{11\beta} + W_{22\beta}) - \frac{C}{D} \alpha^\theta\right)\phi^\theta < 0 \quad \text{and} \quad \alpha^\theta \phi^\theta > 0 \quad (28)\]

must be satisfied for the stability of the limit cycles (19).

It is observed from (28) that along the critical surface

\[S_{10}: [2C\alpha^\theta - D(W_{11\beta} + W_{22\beta})]\phi^\theta = 0 \quad (\alpha^\theta \phi^\theta > 0) \quad (29)\]

a secondary Hopf bifurcation occurs from the family of limit cycles (19), which leads to the same family of 2-D tori (20).

Finally, evaluating the Jacobian (21) on the quasi-periodic solution (20) (the 2-D tori) yields

\[J_{\text{mono}} = \begin{bmatrix}
2A\rho^2 & 2C\rho \\
2B\rho & 2D\rho^2
\end{bmatrix} \quad (30)\]

which in turn results in the stability conditions for the family of the 2-D tori (20) as trace \(J < 0\) and det \(J > 0\), i.e.

\[\left[2A(C - D)\alpha^\theta - D(A - B)(W_{11\beta} - W_{22\beta})\right]\phi^\theta < 0 \quad \text{and} \quad AD - BC > 0 \quad (31)\]

which results in another critical surface,

\[S_{11}: [2A(C - D)\alpha^\theta - D(A - B)(W_{11\beta} + W_{22\beta})]\phi^\theta = 0 \quad AD - BC > 0 \quad (32)\]

along which the family of 2-D tori (20) loses its stability and may bifurcate into a family of 3-D tori, with the third frequency

\[\omega_\gamma = (AD - BC)^{1/2} |\rho\| \quad (33)\]

where \(\gamma^*\) represents a point on the critical surface \(S_{11}\).

Note that bifurcations from \(S_2\), away from the intersection (8), are governed by the
simpler equations
\[
\bar{\rho} \left[ \frac{1}{2} (W_{1,\beta} + W_{2,\beta}) \phi^\beta + A \bar{\rho}^2 \right] = 0
\]
\[
\alpha^\beta \mu^\beta \rho = 0
\]
(34)

since \( \alpha^\beta \mu^\beta \neq 0 \) (see Appendix A). The solution of (34) is identical with (18), but the stability conditions are given by
\[
(W_{1,\beta} + W_{2,\beta}) \phi^\beta > 0 \quad \text{and} \quad \alpha^\beta \mu^\beta < 0
\]
(35)

3. 2-D and 3-D tori generated by interactions of secondary Hopf bifurcations

In analogy with § 2, we can consider the intersection of secondary Hopf bifurcations. In Yu and Huseyin (1988 a), two critical surfaces given by
\[
S_5: (W_{1,1,\beta} + W_{2,2,\beta}) \mu^\beta + (W_{1,1,1} + W_{2,1,1}) y_1^\beta + (W_{1,1,2} + W_{2,2,2}) y_2^\beta = 0
\]
\[
S_6: \alpha^\beta \mu^\beta + \alpha_1 y_1^\beta + \alpha_2 y_2^\beta = 0
\]
(36)

were obtained. Here, \( y_i^\beta \) represent the static bifurcation solutions (see Appendix B). These surfaces lead to two distinct secondary Hopf bifurcations from the incipient static bifurcation solution (see Also Fig. 1). In the vicinity of the intersection of \( S_5 \) and \( S_6 \) we can expect to find bifurcations into tori, as before.

Following a similar procedure to that in § 2, we obtain the rate equations (see Appendix B)
\[
\frac{d\rho}{dt} = \bar{\rho} \left[ \frac{1}{2} (\bar{W}_{1,1,\beta} + \bar{W}_{2,2,\beta}) \phi^\beta + \bar{A} \bar{\rho}^2 + \bar{C} \rho^2 \right]
\]
\[
\frac{d\rho}{dt} = \rho \left[ \bar{z}^\beta \phi^\beta + \bar{B} \bar{\rho}^2 + \bar{B} \rho^2 \right], \quad (\rho \equiv v^3)
\]
(37)

The bifurcation solutions can be obtained from (37) as follows:

(a) \( \bar{\rho} = \rho = 0 \), \( (\rho \equiv v^3) \)
\[
(\rho = 0)
\]
(b) \( \bar{\rho}^2 = \frac{1}{2A} (\bar{W}_{1,1,\beta} + \bar{W}_{2,2,\beta}) \phi^\beta \)
\[
(\rho = 0)
\]
(c) \( \bar{\rho} = 0 \)
\[
(\rho^2 = \frac{1}{B} \bar{z}^\beta \phi^\beta)
\]
(d) \( \bar{\rho}^2 = \frac{1}{2 \bar{A} \bar{D} - \bar{B} \bar{C}} \left[ 2 \bar{C} \bar{z}^\beta - \bar{D} (\bar{W}_{1,1,\beta} + \bar{W}_{2,2,\beta}) \right] \phi^\beta \)
\[
(\rho^2 = \frac{1}{2 \bar{A} \bar{D} - \bar{B} \bar{C}} \left[ 2 \bar{A} \bar{z}^\beta - \bar{B} (\bar{W}_{1,1,\beta} + \bar{W}_{2,2,\beta}) \right] \phi^\beta)
\]
where \( \bar{A} \bar{D} - \bar{B} \bar{C} \neq 0 \).

Here (a) describes the incipient static bifurcation, while (b) and (c) represent two secondary Hopf bifurcations along \( S_5 \) and \( S_6 \), with the frequencies \( \omega_s \) and \( \omega_r \),
respectively. On the other hand, (d) expresses a quasi-periodic solution on an invariant 2-D torus.

By using (25) and (28), we can directly obtain the stability conditions for the families of bifurcating limit cycles (39) and (40) as

\[
(W_{11,\theta} + W_{22,\theta}) \phi^\theta > 0 \quad \text{and} \quad \left[ \ddot{z}^\theta - \frac{\ddot{B}}{2A} (W_{11,\theta} + W_{22,\theta}) \right] \phi^\theta < 0 \quad (42)
\]

for (39), and

\[
\left[ \frac{1}{2} (W_{11,\theta} + W_{22,\theta}) - \frac{C}{D} \ddot{z}^\theta \right] \phi^\theta < 0 \quad \text{and} \quad \ddot{z}^\theta \phi^\theta > 0 \quad (43)
\]

for (40), respectively.

Two critical surfaces may now be identified from (42) and (43) as

\[
S_{12}: \{ 2 \ddot{A} \ddot{z}^\theta - \ddot{B} (W_{11,\theta} + W_{22,\theta}) \} \phi^\theta = 0 \quad (\ddot{W}_{11,\theta} + \ddot{W}_{22,\theta}) \phi^\theta > 0 \quad (44)
\]

and

\[
S_{13}: \{ 2 \dddot{C} \ddot{z}^\theta - \ddot{C} (W_{11,\theta} + W_{22,\theta}) \} \phi^\theta = 0 \quad \dddot{z}^\theta \phi^\theta > 0 \quad (45)
\]

respectively. Along these two critical surfaces, two distinct secondary Hopf bifurcations take place from the families of limit cycles (39) and (40), respectively, leading to the same quasi-periodic motion on the 2-D tori (41).

Finally, from (31) we obtain the stability conditions for the family of the 2-D tori (41) as

\[
\{ 2 \dddot{A} (\dddot{C} - \dddot{B}) \ddot{z}^\theta - \dddot{B} (\dddot{A} - \dddot{B}) (W_{11,\theta} + W_{22,\theta}) \} \phi^\theta < 0 \quad \text{and} \quad \dddot{A} \dddot{B} - \dddot{B} \dddot{C} > 0 \quad (46)
\]

which in turn gives the critical surface

\[
S_{14}: \{ 2 \ddot{A} (\dddot{C} - \dddot{B}) \ddot{z}^\theta - \dddot{B} (\dddot{A} - \dddot{B}) (W_{11,\theta} + W_{22,\theta}) \} \phi^\theta = 0 \quad \dddot{A} \dddot{B} - \dddot{B} \dddot{C} > 0 \quad (47)
\]

along which the family of the 2-D tori (41) loses stability and bifurcates into a family of 3-D tori with the third frequency

\[
\omega^* = (\ddot{A} \dddot{B} - \dddot{B} \dddot{C})^{1/2} |\dot{\rho}| \quad (48)
\]

where \( \omega^* \) denotes a point on the critical surface \( S_{14} \).

If we consider the family of limit cycles bifurcating along \( S_3 \) (not near \( S_6 \)), we still obtain the same solution (39) for this family of limit cycles, but the stability conditions are slightly different from (42) and are given by

\[
(W_{11,\theta} + W_{22,\theta}) \phi^\theta > 0 \quad \text{and} \quad \dddot{z}^\theta \mu^\theta < 0 \quad (49)
\]

The bifurcation flow chart for system (3) is illustrated in Fig. 1, where the abbreviations are as follows:

- I.S. Initial solution
- S.B.S. Static bifurcation solution
- H.B.S. Hopf bifurcation solution
- S.S.B.S. Secondary static bifurcation solution
- S.H.B.S. Secondary Hopf bifurcation solution
- Q.P.S. Quasi-periodic solution
- 2P.P.I.E. Two pairs of pure imaginary eigenvalues
Example. In this section the electrical network used in [1] is considered, attention here being focused on the sequence of bifurcations into 2D and 3D tori. The system under consideration consists of an inductor L, three capacitors $C_1$, $C_2$, and $C_3$, three resistors $R_1$, $R_2$, and $R_3$, and two tunnel diodes. Suppose $L$, $C_1$, $C_2$, $C_3$, and $R_1$, $R_2$, and $R_3$ are linear components and $\alpha$ may be varied. On the other hand, the two tunnel diodes are non-linear elements and they both are controlled. The characteristics of the two tunnel diodes are given by

$$i = \frac{v_i}{R} - \frac{\alpha}{2} \left( v_i^2 + \frac{v_i^4}{\alpha^2} \right).$$

The parallel dashed lines indicate the connected Hopf bifurcation solutions which have the same frequency. In view of the rich behaviour exhibited by the system under consideration—and in particular, the succession of bifurcations into periodic and quasi-periodic solutions—a chaotic behaviour may be triggered at some stage. However, no attempt has been made here to identify any such possible behaviour.

![Bifurcation flow chart](image)

**Figure 1. Bifurcation flow chart.**
and
\[ i_2 = \eta^2 v_2 - \frac{1}{3}(v_2)^3 + b(v_2)^3 \]  \hspace{1cm} (51)
respectively, where \( \eta^1 \) and \( \eta^2 \) are certain independent control parameters, and \( a \) and \( b \) are positive constants.

Figure 2. Non-linear electrical network.

The current \( i_L \), through the inductor and the voltages \( v_{C_1}, v_{C_2} \), and \( v_{C_3} \), across the capacitors \( C_1, C_2 \) and \( C_3 \), respectively, are chosen as the state variables and denoted by \( z^1, z^2, z^3 \) and \( z^4 \), respectively. Furthermore, assuming that \( C_1, C_2, C_3, L, R_1 \) and \( R_2 \) have the values 1, 1, 1, 100 and 50 in corresponding units, respectively, we obtain the following state variable equations:

\[
\begin{align*}
\frac{dz^1}{dt} &= (\eta^1 - 0.01) z^1 + z^4 - \frac{2}{3}(z^1)^3 + a(z^1)^3 \\
\frac{dz^2}{dt} &= (\eta^2 - 0.02) z^2 + z^4 - \frac{1}{3}(z^2)^2 + b(z^2)^3 \\
\frac{dz^3}{dt} &= z^4 \\
\frac{dz^4}{dt} &= -z^1 - z^2 - z^3 - \eta^3 z^4
\end{align*}
\]  \hspace{1cm} (52)

where \( R_3 = \eta^3 \) is treated as a third control parameter (\( \eta^3 \geq 0 \)).

The initial equilibrium solution is described by \( z^i = 0 \) (since \( z^i = 0 \) yield \( dz^i/dt = 0 \) for all values of \( \eta^1, \eta^2 \) and \( \eta^3 \)). The jacobian matrix of (52) evaluated on the initial equilibrium solution is in the form

\[
J = [W_{ij}]_{i=1}^{n} = \begin{bmatrix}
\eta^1 - 0.01 & 0 & 0 & 1 \\
0 & \eta^2 - 0.02 & 0 & 1 \\
0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -\eta^3
\end{bmatrix}
\]  \hspace{1cm} (53)

It can be demonstrated that at the critical point \( c \) (defined by \( \eta^1_c = 0.01, \eta^2_c = 0.02 \) and \( \eta^3_c = 0 \)) the jacobian has a double zero of index two and a pair of pure imaginary eigenvalues. In order to use the results and formulae obtained in the previous sections, we can introduce two transformations (for details see Yu and Huseyin 1988 a) to obtain
the simplified rate equations

\[
\begin{align*}
\frac{dy_1}{dt} &= \frac{1}{3}(\mu^1 + 2\mu^2)y_1 + \frac{1}{3}\mu^1 y_2 - \frac{4}{9}y_1 y_2 - \frac{2}{9}(y_1^2) \\
\frac{dy_2}{dt} &= \frac{1}{3}(\mu^1 - \mu^2)y_1 + \frac{1}{3}\mu^1 y_2 + \frac{1}{3}(y_1^2) - \frac{4}{9}y_1 y_2 - \frac{2}{9}(y_2^2) - \frac{1}{6}\rho^2 \\
\frac{d\rho}{dt} &= \frac{1}{6}(\mu^1 + \mu^2 - 3\mu^3)\rho - \frac{4}{9}y_2\rho
\end{align*}
\]

and

\[
\frac{d\theta}{dt} = \sqrt{3} + 0[(y_1, y_2, \rho^2)]^2
\]  

For this example, the incipient bifurcation solutions can be expressed explicitly in terms of the control parameters. Except for the initial equilibrium solution \(y_1 = y_2 = \rho = 0\), there exist two static bifurcation solutions and one Hopf bifurcation solution. One static bifurcation solution is described by

(i) \(y_1 = -3\mu^2, \ y_2 = 3\mu^2, \ \rho = 0\)

and the other one is given by

(ii) \(y_1 = -3\mu^2, \ y_2 = \frac{2}{3}\mu^1 + 3\mu^2, \ \rho = 0\)

On the other hand, the family of the limit cycles is represented by

\[
\begin{align*}
y_2 &= \frac{2}{3}(\mu^1 + \mu^2 - 3\mu^3) \\
y_1 &= \frac{(2y_2^3 - 3\mu^1)y_2}{(3\mu^1 + 6\mu^2 - 4y_2^2)} \\
\rho^2 &= -2y_1^2(y_1^3 + 3\mu^2)
\end{align*}
\]

The critical surfaces \(S_1, S_2, \ldots, S_8\), along which the incipient as well as the secondary static and Hopf bifurcations occur, can be obtained as in Yu and Huseyin (1988 a). In addition, it can be shown that the bifurcations which take place along \(S_2\) and \(S_3\) are unstable. For convenience, we list the critical surfaces \(S_2, S_3, S_4\) and \(S_6\) below:

\[
\begin{align*}
S_2 &: \mu^1 + \mu^2 = 0 \\
S_3 &: \mu^1 + \mu^2 - 3\mu^3 = 0 \\
S_4^{(1)} &: \mu^1 + \mu^2 = 0 \\
S_4^{(2)} &: \mu^1 - \mu^2 - 3\mu^3 = 0
\end{align*}
\]

and

\[
\begin{align*}
S_6^{(1)} &: \mu^1 - \mu^2 = 0 \\
S_6^{(2)} &: 3\mu^1 + 7\mu^2 + 3\mu^3 = 0
\end{align*}
\]

where the superscripts (1) and (2) denote the critical surfaces corresponding to the static bifurcation solutions (1) and (2), respectively.
Now, based on (54)–(61), we can apply the formulation obtained in the previous sections directly, to obtain the following results.

First, consider the bifurcations taking place in the vicinity of the intersection of $S_2$ and $S_3$. We can use (A 23)–(A 26) to obtain the rate equations

\[
\begin{align*}
\frac{d\bar{p}}{dt} &= \bar{\rho} \left[ \frac{1}{3} (\phi^1 + \phi^2) + \frac{128}{729} (\mu^2)^3 \bar{\rho}^2 + \frac{2}{27\mu^2} \rho^2 \right] \\
\frac{d\rho}{dt} &= \rho \left[ \frac{1}{6} (\phi^1 + \phi^2 - 3\phi^3) + \frac{8}{243} (\mu^2)^2 \bar{\rho}^2 + \frac{2}{27\mu^2} \rho^2 \right]
\end{align*}
\]  

(62)

for this case. Here, $\bar{c}$ represents a point on the intersection line, given by

\[
\mu^1 + \mu^2 = 0, \quad \mu^3 = 0
\]  

(63)

which follows from (59).

Using (18) and (19) with Appendix A, we can obtain two families of limit cycles, described by

\[
\bar{\rho}^2 = -\frac{243}{128(\mu^2)^3}(\phi^1 + \phi^2) \quad (\rho = 0)
\]  

(64)

and

\[
\rho^2 = -\frac{2}{5\mu^2}(\phi^1 + \phi^2 - 3\phi^3) \quad (\bar{\rho} = 0)
\]  

(65)

respectively, bifurcating from the initial equilibrium solution $\bar{p} = \rho = 0$, along $S_2$ and $S_3$, respectively. A family of 2-D tori, represented by (20), which is now in the form

\[
\begin{align*}
\bar{\rho}^2 &= -\frac{243}{206(\mu^2)^3}(\phi^1 + \phi^2 + 3\phi^3) \\
\rho^2 &= -\frac{36}{103} \mu^2 (5\phi^1 + 5\phi^2 - 24\phi^3)
\end{align*}
\]  

(66)

occurs either from the family of limit cycles (64) along the critical surface

\[
S_9: 5\phi^1 + 5\phi^2 - 24\phi^3 = 0
\]  

(67)

or from the family of limit cycles (65) along another critical surface

\[
S_{10}: \phi^1 + \phi^2 + 3\phi^3 = 0
\]  

(68)

Since $AD - BC = \frac{206}{5\mu^2}(\mu^2)^2 > 0$ (see (31)), we can use (32) to obtain the critical surface

\[
S_{11}: \phi^1 + \phi^2 = 0
\]  

(69)

along which a family of 3-D tori may occur. However, it can be shown that the originally stable family of the 2-D tori (66) becomes unstable before crossing the critical surface $S_{11}$.

Next, we can use the results and formulation obtained in § 3 to analyse the bifurcations in the vicinity of the intersection of $S_{11}$ and $S_{11}^{(1)}$, and $S_{11}^{(2)}$ and $S_{11}^{(2)}$. Since the application is straightforward, we will not repeat the analysis and only list the results as follows.

In the vicinity of intersection of $S_{11}$ and $S_{11}^{(1)}$, the rate equations can be expressed
2-D and 3-D tori bifurcations

by

\[
\begin{align*}
\frac{d\bar{\rho}}{dt} &= \bar{\rho} \left[ 1 \left( \phi^1 + \phi^2 \right) - \frac{2}{3\mu_4^2} \rho^2 \right] \\
\frac{d\rho}{dt} &= \rho \left[ 1 \left( \phi^1 - 3\phi^2 - 3\phi^3 \right) + \frac{2}{9\mu_5^2} \rho^2 \right]
\end{align*}
\]

(70)

A family of limit cycles, given by

\[
\rho^2 = -2\mu_4^2 \left( \phi^1 - 3\phi^2 - 3\phi^3 \right) \quad (\bar{\rho} = 0)
\]

(71)

bifurcates from the incipient static bifurcation solution (56) along \( S_0^{(1)} \). This family of limit cycles becomes unstable when the critical surface

\[
S_1^{(1)}: 5\phi^1 - 7\phi^2 - 9\phi^3 = 0
\]

(72)

is reached and may bifurcate into a family of 2-D tori. Since \( \tilde{A}\tilde{D} - \tilde{B}\tilde{C} = 0 \), searching for the solution of the family of the 2-D tori as well as the bifurcations into 3-D tori needs higher-order terms to supplement (70).

Similarly, we can obtain the rate equations governing the local dynamics in the vicinity of the intersection of \( S_0^{(2)} \) and \( S_0^{(2)} \) as

\[
\begin{align*}
\frac{d\bar{\rho}}{dt} &= \bar{\rho} \left[ -\frac{1}{3} \left( \phi^1 - \phi^2 \right) - \frac{10}{27\mu_5^2} \rho^2 \right] \\
\frac{d\rho}{dt} &= \rho \left[ -\frac{1}{3} \left( \phi^1 + 3\phi^2 + 3\phi^3 \right) + \frac{2}{27\mu_5^2} \rho^2 \right]
\end{align*}
\]

(73)

A family of limit cycles, described by

\[
\rho^2 = \frac{9}{2} \mu_5^2 \left( \phi^1 + 3\phi^2 + 3\phi^3 \right) \quad (\bar{\rho} = 0)
\]

(74)

bifurcates from the incipient static bifurcation solution (56) along \( S_0^{(2)} \). Furthermore, along the critical surface

\[
S_1^{(2)}: 6\phi^1 + 14\phi^2 + 15\phi^3 = 0
\]

(75)

this family of limit cycles loses stability and may bifurcate into a family of 2-D tori. It is necessary to supplement (73) with higher-order terms in order to investigate the solution of this family of 2-D tori and the bifurcations into 3-D tori.

**Appendix A**

Here, the simplified rate equations (16) approximating (11) (i.e. the initial system (3)) up to second-order terms are derived via the 'unification technique' coupled with the 'intrinsic harmonic balancing' as follows.

First, suppose the steady-state (stationary and periodic) solutions of (11) in the vicinity of \( \bar{c} \) are in the parametric form

\[
\nu' = \nu'(\bar{\tau}, \sigma^*), \quad \mu^k = \mu^k(\sigma^*), \quad \bar{\omega} = \bar{\omega}(\sigma^*)
\]

(A 1)

where \( \bar{\tau} = \bar{\omega}t \), \( \bar{\omega} \) is the frequency. Furthermore, assume that the solutions can be expressed in a more explicit Fourier series

\[
\nu'(\bar{\tau}, \sigma^*) = \sum_{m=0}^{M} \left[ q_{lm}(\sigma^*) \cos m\bar{\tau} + s_{lm}(\sigma^*) \sin m\bar{\tau} \right], \quad \bar{\tau} = \bar{\omega}t
\]

(A 2)
Substituting the assumed solution (A 1) into (11) yields the identity
\[ \bar{\sigma}(\sigma^a) \frac{d \bar{\sigma}^a}{d \bar{t}} = V_i[v_i'(\bar{t}, \sigma^a), \mu^a(\sigma^a)] \]  
(A 3)

which is then differentiated successively with respect to the perturbation parameters \( \sigma^a \) to generate a sequence of perturbation equations
\[ \bar{\omega}^a v^a_i + \bar{\omega} v^a_i = V_i \nu^a_i + V_i \mu^a \]  
(A 4)

and
\[ \bar{\omega}^a v^a_i + \bar{\omega}^a v^a_i + \bar{\omega}^a v^a_i + \bar{\omega}^a v^a_i = \]
\[ = V_i \nu_i v^a_i + V_i v^a_i \mu^a + V_i \nu_i \mu^a + V_i v^a_i \mu^a + V_i \nu_i \mu^{a,a} + V_i \nu_i \mu^{a,b} \]  
(A 5)

where \( i, j, k, \gamma = 1, 2, 3; \beta, \gamma = 1, 2, 3; a, b = 1, 2, \) subscripts on the functions \( V_i \) denote differentiations with respect to the corresponding variables, and summation convention applies. For clarity, differentiations of variables with respect to the \( \sigma^a \) are indicated by superscripts after a comma.

Evaluating the first-order perturbation equation (A 4) at the critical point \( \bar{t} \) with the aid of (14) yields
\[ \omega_i v^a_i = \]  
(A 6)

where \( V_i \) is given by (13). Substituting the Fourier series (A 2) into (A 6) and comparing the coefficients of \( \cos m \bar{t} \) and \( \sin m \bar{t} \) for each \( m \) in the resulting equation, and recognizing that we can assume
\[ s_{11}^{(a)} = 0 \]  
(A 7)

without loss of generality (since the system is autonomous), results in
\[
\begin{align*}
q_{11}^a = -s_{11}^a \\
q_{21} = s_{11}^a = 0 \\
q_{im} = s_{im}^a = 0 \quad (for \ m \neq 1; \ i = 1, 2; \ a = 1, 2)
\end{align*}
\]  
(A 8)

Next, evaluating the second-order perturbation equation (A 5) at the critical point \( \bar{t} \) with the aid of (14), introducing (A 2), and comparing the coefficients of \( \cos m \bar{t} \) and \( \sin m \bar{t} \) yields
\[ V_{jk}(q_{j0}^b q_{k0}^b + \frac{1}{2} q_{j1}^b q_{k1}^b + \frac{1}{2} s_{j1}^a s_{k1}^a) + V_{jk}(q_{j0}^b \mu^{b,b} + q_{j0}^b \mu^{b,a}) + V_{jk} q_{j0}^{ab} = 0 \]  
(A 9)

for \( m = 0 \)
\[ \bar{\omega}^a q_{j1}^b + \bar{\omega}^a q_{k1}^b + \omega_i s_{j1}^a = V_{jk}(q_{j0}^b q_{k1}^b + q_{j1}^b q_{k0}^b) + V_{ij}(q_{j1}^b \mu^{b,b} + q_{j1}^b \mu^{b,a}) + V_{ij} q_{j1}^{ab} \]  
\[ -(\bar{\omega}^a q_{j1}^b + \bar{\omega}^a q_{k1}^b + \omega_i s_{j1}^a) = V_{jk}(q_{j0}^b s_{k1}^a + s_{j1}^a q_{k0}^b) + V_{ij}(s_{j1}^a \mu^{b,b} + s_{j1}^a \mu^{b,a}) + V_{ij} s_{j1}^{ab} \]  
(A 10)

for \( m = 1, \) and
\[ 2\omega_i s_{j2}^{ab} = \frac{1}{2} V_{jk}(q_{j1}^b s_{k1}^a + s_{j1}^a q_{k1}^b) + V_{ij} s_{j2}^{ab} \]
\[ -2\omega_i q_{j2}^{ab} = \frac{1}{2} V_{jk}(q_{j1}^a s_{k1}^b + s_{j1}^a q_{k1}^b) + V_{ij} s_{j2}^{ab} \]  
(A 11)

for \( m = 2. \)
It is noted that (A 9)–(A 11) represent a total of 45 equations. Since the polar formulation here involves 3 key variables \( p_{11}, p_{30} \) and \( \tilde{\omega} \), and 3 parameters \( \mu^1, \mu^2 \) and \( \mu^3 \) (in fact, two independent parameters—co-dimension 2), it can be shown that we only need 9 perturbation equations out of the 45 equations for constructing the governing bifurcation equations. These equations consist of (A 10) (for \( i = 1, 2 \)) and (A 9) (for \( i = 3 \)). For clarity, they are written in a more explicit form as follows:

\[
\begin{align*}
2(\tilde{\omega}_3 q_{30}^1 + \tilde{\omega}^\beta q_{11}^1) q_{11}^1 &= 0 \\
(\tilde{\omega}_3 q_{30}^2 + \tilde{\omega}^\beta q_{11}^2) q_{11}^1 + (2\tilde{\omega}_3 q_{30}^1 + \tilde{\omega}^\beta q_{11}^1) q_{11}^2 &= 0 \\
2(\tilde{\omega}_3 q_{30}^3 + \tilde{\omega}^\beta q_{11}^3) q_{11}^2 &= 0 \\
2\tilde{\omega}^{-1} q_{11}^1 &= 2(\tilde{\omega}_3 q_{30}^1 + \tilde{\omega}^\beta q_{11}^1) q_{11}^1 \\
\tilde{\omega}^{-1} q_{11}^1 + \tilde{\omega}^{-2} q_{11}^2 &= (\tilde{\omega}_3 q_{30}^1 + \tilde{\omega}^\beta q_{11}^1) q_{11}^1 + (2\tilde{\omega}_3 q_{30}^1 + \tilde{\omega}^\beta q_{11}^1) q_{11}^2 \\
2\tilde{\omega}^{-2} q_{11}^2 &= 2(\tilde{\omega}_3 q_{30}^3 + \tilde{\omega}^\beta q_{11}^3) q_{11}^2
\end{align*}
\]  

and

\[
\begin{align*}
V_{333}(q_{30}^1)^2 + T_5(q_{11}^1)^2 + 2V_{333}(q_{30}^1) q_{30}^3 = 0 \\
V_{333}(q_{30}^1)(q_{30}^2) + T_5(q_{11}^1)(q_{11}^2) + V_{333}(q_{30}^1)(q_{30}^3) + V_{333}(q_{30}^2)(q_{30}^3) = 0 \\
V_{333}(q_{30}^2)^2 + T_5(q_{11}^1)^2 + 2V_{333}(q_{30}^2) q_{30}^3 = 0
\end{align*}
\]  

where the superscripts '1' and '2' denote differentiations with respect to \( \sigma^1 \) and \( \sigma^2 \), respectively, and

\[
\begin{align*}
\tilde{T}_i &= \frac{1}{2}(V_{11i} + V_{22i}) \\
\tilde{\omega}_j &= \frac{1}{2}(V_{11j} + V_{22j}), \quad \tilde{\omega}^\beta = \frac{1}{2}(V_{11\beta} + V_{22\beta})
\end{align*}
\]  

and

\[
\begin{align*}
\tilde{\omega}_j &= \frac{1}{2}(V_{12j} - V_{21j}), \quad \tilde{\omega}^\beta = \frac{1}{2}(V_{12\beta} - V_{21\beta})
\end{align*}
\]  

Instead of trying to solve the above equations for the unknown derivatives, the unification technique will now be used to derive a set of governing relationships. Thus, multiplying the first equation of (A 12) by \((\sigma^1)^2/2\), the second equation by \(\sigma^1\sigma^2\) and the third one by \((\sigma^2)^2/2\), and adding them together yields

\[
(\tilde{\omega}_3 q_{30} + \tilde{\omega}^\beta) p_{11} = 0
\]  

upon considering appropriate Taylor expansions of the functions in (A 1). The same procedure can be applied to (A 14) and (A 13) to obtain

\[
V_{333}(q_{30})^2 + \frac{1}{2} V_{333}(q_{30})^2 + \frac{1}{2} T_5(q_{11})^2 = 0
\]  

and

\[
(\tilde{\omega}_3 q_{30} + \tilde{\omega}^\beta - \bar{\Omega}) q_{11} = 0
\]

respectively, where \( \bar{\Omega} = \tilde{\omega} - \omega_c \).

Since for system (11), \( \tilde{\omega}_3 = V_{333} = \tilde{T}_3 = \tilde{\omega}_3 = 0 \), we have to use the third-order perturbation equations to supplement (A 18) and (A 19). To this end, differentiating
with respect to $\sigma^c$ for a third time and evaluating at the critical point $\tilde{c}$ yields

$$\begin{align*}
\ddot{\omega}^{ab}v_{i,ab}^{c} + \ddot{\omega}^{ab}v_{i,ab}^{c,ab} + \ddot{\omega}^{ab}v_i^c &+ \ddot{\omega}^{ab}v_{i,ab}^{a,be} + \ddot{\omega}^{ae}v_i^b + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe} \\
= V_{\mu
u}(v_{\mu,ab}v_{i,ab}^{c} + \ddot{\omega}^{ab}v_{i,ab}^{c,ab} + \ddot{\omega}^{ab}v_i^c &+ \ddot{\omega}^{ab}v_{i,ab}^{a,be} + \ddot{\omega}^{ae}v_i^b + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe}) \\
+ V_{\mu
u}(v_{\mu,ab}v_{i,ab}^{c,ab} + \ddot{\omega}^{ab}v_i^c &+ \ddot{\omega}^{ab}v_{i,ab}^{a,be} + \ddot{\omega}^{ae}v_i^b + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe}) \\
+ V_{\mu
u}(v_{\mu,ab}v_{i,ab}^{c} + \ddot{\omega}^{ab}v_i^c &+ \ddot{\omega}^{ab}v_{i,ab}^{a,be} + \ddot{\omega}^{ae}v_i^b + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe}) + V_{\mu
u}(v_{\mu,ab}v_{i,ab}^{c,ab} + \ddot{\omega}^{ab}v_i^c &+ \ddot{\omega}^{ab}v_{i,ab}^{a,be} + \ddot{\omega}^{ae}v_i^b + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe} + \ddot{\omega}^{ab}v_i^{abe}) \quad (A\ 21)
\end{align*}$$

where $a, b, c = 1, 2$.

Substituting for $v_i^c$ from (A 2), comparing the coefficients of the same order harmonics—$\cos m\tilde{c}$ and $\sin m\tilde{c}$—and following the procedure that led to (A 18) and (A 19), we eventually obtain the governing equations

$$\begin{align*}
\frac{1}{2}(W_{1,11} + W_{2,22})\phi_{1,11} + A(q_{11})^{3} + Cq_{11}(q_{30})^{2} &+ 0 \\
A_{11}^{\phi} \phi_{1,11} + B(q_{11})^{2}q_{30} + D(q_{30})^{3} &+ 0 = 0 \\
\alpha_{11}^{\phi} \phi_{1,11} + B(q_{11})^{2}q_{30} + D(q_{30})^{3} &+ 0 = 0 \\
\end{align*} \quad (A 22)$$

where

$$A = W_{2,11} \mu_{1}^{\phi}(W_{2,11} - W_{2,22}) + W_{2,21}(W_{2,21} + W_{1,11}) + (W_{2,21})^{2} - (W_{2,11})^{2}$$

$$\times W_{1,12} \mu_{1}^{\phi} W_{2,21}^{2} + [W_{1,22}(W_{2,21} + W_{1,11}) + W_{1,12}(W_{2,12} + W_{2,21})$$

$$+ (W_{1,12})^{2} - (W_{2,21})^{2}]W_{1,11} \mu_{1}^{\phi} W_{2,21} \mu_{1}^{\phi}$$

$$- W_{2,11}(W_{1,11} + W_{2,21})(W_{1,22} \mu_{1}^{\phi})^{2} + W_{2,22}(W_{2,22} + W_{1,12})(W_{2,21} \mu_{1}^{\phi})^{2}$$

$$+ (W_{1,11} W_{1,12} - W_{2,22} W_{2,21}) \{[\alpha_{11}^{\phi} + 3(W_{1,11} \mu_{1}^{\phi})^{2}]\} \quad (A 23)$$

$$B = -\frac{1}{\mu_{1}^{\phi}}(\alpha_{11} W_{1,11} - \alpha_{21} W_{2,11})(W_{1,11} W_{1,12} - 2W_{1,12} W_{1,11} - W_{1,12} W_{2,11})$$

$$- (\alpha_{11} W_{1,12} - \alpha_{21} W_{2,12})(W_{2,11} W_{1,12} - 2W_{2,12} W_{1,11} - W_{2,22} W_{2,21}) \mu_{1}^{\phi}$$

$$\quad (A 24)$$

$$C = \frac{1}{4\alpha_{11}^{\phi}} [(W_{1,11} W_{1,12} - 2W_{1,12})W_{2,11} + W_{1,12} W_{2,12})$$

$$- (W_{1,12} W_{2,22} - 2W_{2,12} W_{1,12})W_{2,11} \mu_{1}^{\phi}$$

$$\quad (A 25)$$

and

$$D = \frac{1}{2\alpha_{11}^{\phi}} (\alpha_{11} W_{1,11} - \alpha_{21} W_{1,12}) - \alpha_{21} (W_{1,12} W_{2,12}) \mu_{1}^{\phi}$$

$$\quad (A 26)$$

It can be shown (see the proof in Yu and Huseyn 1988 a, Appendix C) that the rate equations can be obtained directly from (A 22) as

$$\begin{align*}
\frac{d\bar{\rho}}{dt} &= \bar{\rho} \left[ \frac{1}{2}(W_{1,11} + W_{2,22})\phi_{1,11} + A\bar{\rho}^{2} + C\bar{\rho}^{2} \right] \\
\frac{d\rho}{dt} &= \rho [\alpha_{11}^{\phi} \phi_{1,11} + B\bar{\rho}^{2} + D\bar{\rho}^{2}] \quad (\rho = \omega^{3})
\end{align*} \quad (A 27)$$

where $\phi_{1,11}$, $\mu_{1}^{\phi}$, and $\bar{\rho}$ is linked to the amplitude $q_{11}$.

Note that if the above procedure is applied to consider the local dynamic behaviour in the vicinity of $S_2$ away from the intersection of $S_2$ and $S_3$, we can obtain
the rate equations for this case as

\[
\frac{d\bar{\rho}}{dt} = \bar{\rho} \left[ \frac{1}{2} (W_{11} \rho + W_{22} \rho^2) + A\bar{\rho}^2 \right]
\]

\[
\frac{d\rho}{dt} = \alpha \mu \rho
\]

(\ref{rate_equations})

**Appendix B**

Here, we briefly describe how to obtain the rate equations (\ref{rate_equations}) as follows.

First, the incipient static bifurcation solution of system (\ref{static_equations}) can be found in the implicit form

\[
\begin{align*}
W_{11} \mu \rho y_1 + W_{22} \rho^2 + \frac{1}{2} W_{111} (y_1)^2 + W_{112} y_1 y_2 + \frac{1}{2} (W_{122} (y_2)^2 &= 0 \\
W_{22} \mu \rho y_2 + W_{22} \rho^2 + \frac{1}{2} W_{221} (y_1)^2 + W_{212} y_1 y_2 + \frac{1}{2} W_{222} (y_2)^2 &= 0
\end{align*}
\]

(\ref{static_solution})

where \((y_1, y_2, 0)\) represents the static bifurcation solution.

Next, use a translation

\[
\begin{align*}
y_1 &= y_1' + \bar{y}_1 \\
y_2 &= y_2' + \bar{y}_2 \\
\rho &= \rho
\end{align*}
\]

(\ref{translation})

to transform (\ref{static_equations}) and (\ref{rate_equations}) into

\[
\begin{align*}
\frac{d\bar{y}_1}{dt} &= \bar{W}_{11} \mu \rho \bar{y}_1 + \bar{W}_{12} \mu \rho \bar{y}_2 + \frac{1}{2} \bar{W}_{111} (\bar{y}_1)^2 + \bar{W}_{112} \bar{y}_1 \bar{y}_2 + \frac{1}{2} \bar{W}_{122} (\bar{y}_2)^2 + \frac{1}{2} l_1 \rho^2 \\
\frac{d\bar{y}_2}{dt} &= \bar{W}_{21} \mu \rho \bar{y}_1 + \bar{W}_{22} \mu \rho \bar{y}_2 + \frac{1}{2} \bar{W}_{211} (\bar{y}_1)^2 + \bar{W}_{212} \bar{y}_1 \bar{y}_2 + \frac{1}{2} \bar{W}_{222} (\bar{y}_2)^2 + \frac{1}{2} l_2 \rho^2 \\
\frac{d\rho}{dt} &= \bar{\alpha} \mu \rho + \alpha_1 \bar{y}_1 \rho + \alpha_2 \bar{y}_2 \rho
\end{align*}
\]

(\ref{transformed_equations})

and

\[
\frac{d\theta}{dt} = \omega_1 + \bar{\omega} \mu \rho + \omega_1 \bar{y}_1 + \omega_2 \bar{y}_2
\]

(\ref{transformed_angle})

respectively, where

\[
\begin{align*}
\bar{W}_{11} \mu &= W_{11} \mu + W_{111} \bar{y}_1 + W_{112} \bar{y}_2 \\
\bar{W}_{12} \mu &= W_{12} \mu + W_{112} \bar{y}_1 + W_{112} \bar{y}_2 \\
\bar{W}_{21} \mu &= W_{21} \mu + W_{211} \bar{y}_1 + W_{212} \bar{y}_2 \\
\bar{W}_{22} \mu &= W_{22} \mu + W_{212} \bar{y}_1 + W_{222} \bar{y}_2 \\
\bar{\alpha} \mu &= \alpha \mu + \alpha_1 \bar{y}_1 + \alpha_2 \bar{y}_2 \\
\bar{\omega} \mu &= \omega \mu + \omega_1 \bar{y}_1 + \omega_2 \bar{y}_2
\end{align*}
\]

(\ref{transformed_variables})
Now, (B 3) and (B 4) are in the same form as (6) and (7), respectively, with the initial equilibrium solution $\eta = \varphi = 0$. Thus, the formulation obtained in Appendix A can be directly applied here. The rate equations (A 27) for this case are now in the form of

\[
\begin{align*}
\frac{d\bar{\rho}}{dt} &= \bar{\rho} \left[ \frac{1}{2} (\tilde{W}_{11}\phi + \tilde{W}_{22}\phi) + \tilde{A}\varphi^2 + \tilde{C}\varphi^2 \right] \\
\frac{d\rho}{dt} &= \rho[\tilde{Z}^{2}\phi + \tilde{B}\varphi^2 + \tilde{D}\varphi^2], \quad (\rho = \varphi^3)
\end{align*}
\]

(B 6)

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ can still be expressed by (A 23), (A 24), (A 25) and (A 26), respectively, provided $\tilde{W}_{11}\mu$, $\tilde{W}_{22}\mu$, etc. are used to replace $W_{11}\mu$, $W_{22}\mu$, etc., respectively, in those equations.

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