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A constructive proof on the existence of globally exponentially attractive set and positive invariant set of general Lorenz family

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ABSTRACT

In this paper, we give a constructive proof on the existence of globally exponentially attractive set and positive invariant set of general Lorenz family, which contains four independent parameters and is more general than any Lorenz systems studied so far in the literature. The system considered in this paper not only contains the classical Lorenz system and the generalized Lorenz family as special cases, but also provides three new Lorenz systems, which do not belong to the generalized Lorenz system, but the general Lorenz system. The results presented in this paper contain all the existing relative results as special cases.

1. Introduction

Since the discovery of the Lorenz chaotic attractor [1] in 1963, many other chaotic systems have been found, including the well-known Rössler system [2], Chua's circuit [3], which have been served as models for study of chaos. In late 1990's, a new chaotic system was found, which is a dual system to the Lorenz system, and now known as the Chen system [4]. Due to its close relation to the Lorenz system and importance, the Chen system has been widely studied (e.g., see [5–7] and references therein).

The ultimate boundedness of a chaotic system is very important for the study of the qualitative behaviour of a chaotic system. If one can show that a chaotic system under consideration has a globally attractive set, then one knows that the system cannot have equilibrium points, periodic or quasi-periodic solutions, or other chaotic attractors existing outside the attractive set. This greatly simplifies the analysis of dynamics of the system. The ultimate boundedness also plays a very important role in the designs of chaos control and chaos synchronization. The ultimate boundedness property of the Lorenz system has been investigated by many researchers (e.g., see [8,9,11–15]). In particular, in [15] we generalized the boundedness or ultimate boundedness to the concept of globally exponentially attractive set and positive invariant set, and proved the existence of such set for a class of Lorenz family.

It, however, has been noticed that so far very little has been achieved on other chaotic systems, regarding the property of ultimate boundedness. A smooth Chua's circuit has been studied and estimation on its ultimate boundedness has been

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obtained [16]. For the Chen system, a recent article [17] investigated its property of ultimate boundedness, but the parameter values considered in this article does not cover the most interesting case of the Chen's chaotic attractor.

Consider the following general system:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - xz - \gamma y, \\ \dot{z} &= xy - \beta z, \end{aligned} \tag{1}$$

where the dot denotes differentiation with respect to time t; $\sigma > 0$, $\beta > 0$, $\gamma > 0$ and $\rho \in (-\infty, +\infty)$ are parameters. When

$$\sigma = 10, \quad \beta = \frac{8}{3}, \quad \gamma = 1, \quad \rho = 28, \tag{2}$$

system (1) is the classical Lorenz system; and when

$$\sigma = 25\alpha + 10, \quad \beta = \frac{8+\alpha}{3}, \quad \gamma = 1 - 29\alpha, \quad \rho = 28 - 35\alpha, \tag{3}$$

where $\alpha \in [0, \frac{1}{29})$, system (1) becomes the generalized Lorenz system.

As shown in Fig. 1, three new Lorenz systems are found, which do not belong to the classical Lorenz system, nor the generalized Lorenz system.

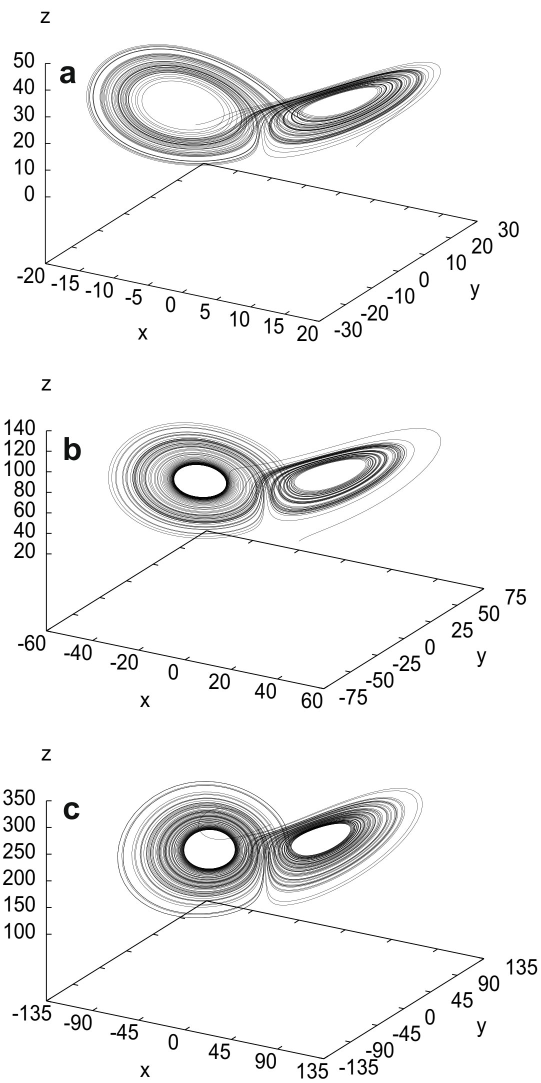

Fig. 1. The chaotic attractors of system (1) projected on the x–z plane: (a) the Lorenz attractor for $\sigma = 35, \beta = \frac{8}{3}, \gamma = 1, \rho = 28$; (b) a chaotic attractor for $\sigma = 30, \beta = 5, \gamma = 6, \rho = 80$; and (c) a chaotic attractor for $\sigma = 40, \beta = 5, \gamma = 20, \rho = 250$.

Let $P(\sigma,\beta,\gamma,\rho)$ be a point in the subspace $\widetilde{R}^4 \subseteq R^4$, defined as

$$\widetilde{R}^4 = R_+ \times R_+ \times R_+ \times R = (0,+\infty) \times (0,+\infty) \times (0,+\infty) \times (-\infty,+\infty),$$

then the classical Lorenz system corresponds to only one point $P(\sigma,\beta,\gamma,\rho) = P(10,\frac{8}{3},1,28)$ in $\widetilde{R}^4$, while the generalized Lorenz system or the Lorenz family corresponds to a small line segment in $\widetilde{R}^4$, implying that the four parameters given in (3) are not independent (linearly related).

In this paper, we remove the restriction on the parameters, given by (3) and assume generally that the four parameters σ,β,γ and ρ are free to change in the sub-space $\widetilde{R}^4$. We want to prove that there exist globally exponentially attractive set and positive invariant set for such system described by (1), and thus confirm the conjecture that a chaotic system has a global asymptotic stability in Lagrange sense, i.e., there exists a compact set $\Omega \subseteq R^3$ which is globally exponentially attractive and positive invariant.

The earliest work on ultimate boundedness of chaotic systems goes back to Leonov et al. [8] who proved that the classical Lorenz system (when $\sigma = 10, \beta = \frac{8}{3}, \gamma = 1, \rho = 28$) is ultimately bounded (i.e., the system is asymptotically stable in the sense of Lagrange). However, they did not obtain the globally exponentially Lagrange stability. Later, Liao [10] used the same generalized Lyapunov function, but different approaches to further investigate the ultimate boundedness of the classical Lorenz system, a and improved the results. Recently, Yu and Liao [12,14] simplified the proofs and studied the globally attractive set and positive invariant set of the classical Lorenz system and the generalized Lorenz system [14].

Very recently, Liao et al. [15] proposed globally exponentially attractive set and positive invariant set for the classical Lorenz system and the generalized system, and provided as constructive proof for the existence of such sets.

In this paper, we consider system (1) which is more general than the classical Lorenz system and the generalized Lorenz system. The main goals are as follows:

(1) The methodology developed in [15] will be extended to study the general Lorenz system (1) for which the parameter values have been generalized from finite region to infinite region. We shall prove that system (1) has a generic property: it is always globally exponentially Lagrange stable regardless whether it is chaotic or not. In other words, there always exist globally exponentially attractive set and positive invariant set $\Omega \subseteq R_3$ for such a system. The result contains all existing results as its special cases.
(2) When studying chaotic systems, people usually assume that the system under consideration is ultimately bounded. However, there does not exist a general proof for the ultimate boundedness of chaotic systems. Now, for the general Lorenz system (1), we have shown that such system has globally exponentially attractive set and positive invariant set.
(3) As is well-known, one of the most popular and useful approach used in chaos control and chaos synchronization is feedback control, such as linear feedback control, particularly diagonal linear feedback control. Such controls are easy to realize in practice or experiment. However, The control strategy is based on the assumption that the system is ultimately bounded. Such analysis based on an assumption may have some theoretical important, but is hard to realize in practice.

With the property of the ultimate boundedness established with explicit estimation in this paper, it becomes possible to use simple linear feedback controls to reach chaos control and chaos synchronization.

In next section, we present some definitions which are needed in the Section 3. Our main result and its proof for system (1) to have globally exponentially attractive set and positive invariant set are given in Section 3. We present two some applications in Section 4 and finally give a summery in Section 5.

2. Preliminaries

In this section, we present some basic definitions and two lemmas which are needed for proving all theorems in the next section. For convenience, let $X := (x,y,z)$ and $X(t) := X(t,t_0,X_0)$.

Definition 1. For the general chaotic system (1), if there exists compact (bounded and closed) set $\Omega \subset R^3$ such that $\forall X_0 \in R^3$, the following condition:

$$\rho(X(t),\Omega) := \inf_{Y\in\Omega} \|X(t) - Y\| \to 0 \quad \text{as} \quad t \to +\infty,$$

holds, then the set Ω is said to be globally attractive. That is, system (1) is ultimately bounded, namely, system (1) is globally stable in the sense of Lagrange or dissipative with ultimate bound.

Further, if $\forall X_0 \in \Omega_0 \subseteq \Omega \subset R^3$, $X(t,t_0,X_0) \subseteq \Omega_0$, then Ω_0 for $t \geqslant 0$ is called the positive invariant set of system (1).

Definition 2. For the general chaotic system (1), if there exists compact set $\Omega \subset R^3$ such that $\forall X_0 \in R^3$, and constants $M(X_0) > 0$, $\alpha > 0$ such that

$$\rho(X(t),\Omega) \leqslant M(X(t_0))e^{-\alpha(t-t_0)},$$

then system (1) is said to have globally exponentially attractive set, or system (1) is globally exponentially stable in the sense of Lagrange, and Ω is called the globally exponentially attractive set.

In general, from the definition we see that a globally exponential attractive set is not necessarily a positive invariant set. But our results obtained in the next section indeed show that a globally exponentially attractive set is a positive invariant set.

Note that it is difficult to verify the existence of Ω in Definition 2. Since the Lyapunov direct method is still a powerful tool in the study of asymptotic behaviour of non-linear dynamical systems, the following definition is more useful in applications.

Definition 3. For the general chaotic system (1), if there exists a positive definite and radially unbounded Lyapunov function $V(x)$, and positive numbers $L > 0, \alpha > 0$ such that the following inequality

$$(V(x(t) - L) \leqslant (V(X_0) - L)e^{-\alpha(t-t_0)}$$

is valid for $V(X(t) > L(t \geqslant t_0)$, then system (1) is said to be globally exponentially attractive or globally exponentially stable in the sense of Lagrange, and $\Omega := \{X|V(t) \leqslant L,\ t \geqslant t_0\}$ is called the globally exponentially attractive set.

3. Main result

In this section, we present our main result for the general Lorenz system (1). We use the same generalized Lyapunov function:

$$V_\lambda = \frac{1}{2}[\lambda x^2 + y^2 + (z - \lambda\sigma - \rho)^2], \tag{4}$$

which is obviously positive definite and radially unbounded. Here, $\lambda \geqslant 0$ is an arbitrary constant. Let $X = (x, y, z)$. We have the following result.

Theorem 1. *Suppose* $\beta > \min\{\sigma, \gamma\} = \xi$*, and let*

$$L_\lambda = \frac{(\lambda\sigma + \rho)^2\beta^2}{8(\beta - \xi)\xi}. \tag{5}$$

Then the estimation

$$(V_\lambda(X(t)) - L_\lambda) \leqslant (V_\lambda(X(t_0)) - L_\lambda)e^{-2\xi(t-t_0)} \tag{6}$$

holds, and thus $\Omega_\lambda = \{X|V_\lambda(X) \leqslant L_\lambda\}$ *is the globally exponentially attractive set and positive invariant set of system (1), i.e.,* $\overline{\lim}_{t\to+\infty} V_\lambda(X(t)) \leqslant L_\lambda$.

Proof. Differentiating the Lyapunov function V_λ in (4) with respect time t along the trajectory of system (1) yields

$$\begin{aligned}
\left.\frac{dV_\lambda}{dt}\right|_{(1)} &= \lambda x\dot{x} + y\dot{y} + (z - \lambda\sigma - \rho)\dot{z} \\
&= -\lambda\sigma x^2 + \lambda\sigma xy + \rho xy - xyz - \gamma y^2 + xyz - \lambda\sigma xy - \rho xy - \beta z^2 + \lambda\sigma\beta z + \rho\beta z \\
&= -\lambda\sigma x^2 - \gamma y^2 - \beta z^2 + \beta(\lambda\sigma + \rho)z \\
&\leqslant -\lambda\xi x^2 - \xi y^2 - \xi z^2 + 2\xi(\lambda\sigma + \rho)z - \xi(\lambda\sigma + \rho)^2 - (\beta - \xi)z^2 + (\beta - 2\xi)(\lambda\sigma + \rho)z + \xi(\lambda\sigma + \rho)^2 \\
&\leqslant -\lambda\xi x^2 - \xi y^2 - \xi(z - \lambda\sigma - \rho)^2 - (\beta - \xi)\left[z - \frac{(\beta - 2\xi)(\lambda\sigma + \rho)}{2(\beta - \xi)}\right]^2 + \frac{(\beta - 2\xi)^2(\lambda\sigma + \rho)^2}{4(\beta - \xi)} + \xi(\lambda\sigma + \rho)^2 \\
&\leqslant -2\xi V_\lambda + \frac{(\lambda\sigma + \rho)^2[(\beta - 2\xi)^2 + 4(\beta - \xi)\xi]}{4(\beta - \xi)} \\
&= -2\xi V_\lambda + \frac{(\lambda\sigma + \rho)^2\beta^2}{4(\beta - \xi)} \\
&\leqslant -2\xi(V_\lambda - L_\lambda).
\end{aligned} \tag{7}$$

Thus, we have

$$(V_\lambda(X(t)) - L_\lambda) \leqslant (V_\lambda(X(t_0)) - L_\lambda)e^{-2\xi(t-t_0)}$$

and

$$\overline{\lim}_{t\to+\infty} V_\lambda(X(t)) \leqslant L_\lambda,$$

which clearly shows that $\Omega_\lambda = \{X|V_\lambda(X) \leqslant L_\lambda\}$ is the globally exponentially attractive set and positive invariant set of system (1).

The proof is complete. □

Corollary 1. *When*

$$\sigma = a_\alpha = 25\alpha + 10, \quad \beta = b_\alpha = \frac{8+\alpha}{3}, \quad \rho = c_\alpha = 28 - 35\alpha, \quad \gamma = d_\alpha = 1 - 29\alpha,$$

$\xi = \min\{\sigma, \gamma\} = \min\{a_\alpha, d_\alpha\} = d_\alpha$. *Thus,*

$$L_\lambda = \frac{(\lambda\sigma + \rho)^2 \beta^2}{8(\beta - \xi)\xi} = \frac{(\lambda a_\alpha + c_\alpha) b_\alpha^2}{8(b_\alpha - d_\alpha) d_\alpha},$$

and the estimate (6) holds, implying that $\overline{\lim}_{t\to+\infty} V_\lambda(X(t)) \leqslant L_\lambda$, *i.e.,* $\Omega_\lambda = \{X | V_\lambda(X) \leqslant L_\lambda\}$ *is the globally exponentially attractive set and positive invariant set of system (1).*

Remark 1. Corollary 1 gives the result of Theorem 1 in [15], implying that Theorem 1 of [15] is a special case of Theorem 1 in this paper.

Corollary 2. *When* $\sigma \geqslant 1$, $\beta \geqslant 2$, $\gamma = 1$, $\bar{\xi} = \min\{\sigma, \gamma\} = 1$, *and thus*

$$\widehat{L}_\lambda = \frac{(\lambda\sigma + \rho)^2 \beta^2}{8(\beta - 1)}.$$

Therefore, the following:

$$(V_\lambda(X(t)) - \widehat{L}\lambda) \leqslant (V_\lambda(X(t_0)) - \widehat{L}_\lambda) e^{-2(t-t_0)}$$

holds and

$$\overline{\lim}_{t\to+\infty} V_\lambda(X(t)) \leqslant \widehat{L}_\lambda,$$

i.e., $\widehat{\Omega}_\lambda = \{X | V_\lambda(X) \leqslant \widehat{L}_\lambda\}$ *is the globally exponentially attractive set and positive invariant set of system (1).*

Remark 2. Corollary 2 is the result of Theorem 3 in [15], indicating that Theorem 3 of [15] is a special case of Theorem 1 in this paper.

Remark 3. It is seen from Theorem 1 that larger (smaller) value of ξ implies larger (smaller) value of L_λ, and thus fast (slow) speed for trajectories to converge to the globally exponentially attractive set of system (1).

Remark 4. To illustrate the globally exponentially attractive set and positive invariant set, we use the classical Lorenz system as an example. For this system, the parameter values are given in (2), and so

$$L_\lambda = \frac{(10\lambda + 28)^2 (8/3)^2}{8(5/3)} = \frac{32(5\lambda + 14)^2}{15},$$

which gives the following estimate of the ultimate bound:

$$\Omega_\lambda = \{X | V_\lambda(X) \leqslant L_\lambda\}$$

which is the globally exponentially attractive set and positive invariant set of the classical Lorenz system.

Remark 5. Note that the parameter values for the Lorenz attractor shown in Fig. 1(a) satisfy the condition in Theorem 1: $\frac{8}{3} = \beta > \min\{\sigma, \gamma\} = \min\{35, 1\} = 1$, that for the second and third chaotic attractors shown in Fig. 1b and c do not satisfy the condition in Theorem 1. Thus, we need new theorem when this condition does not hold.

Now we turn to consider the case that the condition in Theorem 1 is not satisfied.

Theorem 2. *Let* $\bar{\xi} = \min\{\sigma, \gamma, \frac{\beta}{2}\}$, *and*

$$\bar{L}_\lambda = \frac{(\lambda\sigma + \rho)^2 \beta}{4\bar{\xi}}. \tag{8}$$

Then the estimate $\overline{\lim}_{t\to+\infty} V_\lambda(X(t)) \leqslant \bar{L}_\lambda$ *holds, i.e.,* $\bar{\Omega}_\lambda = \{X | V_\lambda(X) \leqslant \bar{L}_\lambda\}$ *is the globally exponentially attractive set and positive invariant set of system (1).*

Proof. Again applying the positive definite and radially unbounded generalized Lyapunov function given in (4) and evaluating the derivative of $\frac{dV_\lambda}{dt}$ along the trajectory of system (1) leads to

$$\begin{aligned}\left.\frac{dV_\lambda}{dt}\right|_{(1)} &= \lambda x\dot{x} + y\dot{y} + (z-\lambda\sigma-\rho)\dot{z}\\ &= -\lambda\sigma x^2 - \gamma y^2 - \beta z^2 + \beta(\gamma\sigma+\rho)z\\ &= -\lambda\sigma x^2 - \gamma y^2 - \frac{\beta}{2}z^2 - \frac{\beta}{2}[z^2 - 2(\lambda\sigma+\rho)z + (\lambda\sigma+\rho)^2 - (\lambda\sigma+\rho)^2]\\ &\leqslant -\lambda\bar{\xi}x^2 - \bar{\xi}y^2 - \bar{\xi}z^2 - \bar{\xi}(z-\lambda\sigma-\rho)^2 + \frac{\beta}{2}(\lambda\sigma+\rho)^2\\ &\leqslant -\lambda\bar{\xi}x^2 - \bar{\xi}y^2 - \bar{\xi}(z-\lambda\sigma-\rho)^2 + 2\bar{\xi}\,\bar{L}_\lambda\\ &= -2\bar{\xi}(V_\lambda - \bar{L}_\lambda),\end{aligned} \tag{9}$$

which implies that

$$(V_\lambda(X(t)) - \bar{L}_\lambda) \leqslant (V_\lambda(X(t_0)) - \bar{L}_\lambda)e^{-2\bar{\xi}(t-t_0)}.$$

Thus,

$$\overline{\lim}_{t\to+\infty} V_\lambda(X(t)) \leqslant \bar{L}_\lambda,$$

suggesting that $\bar{\Omega}_\lambda = \{X|V_\lambda(X) \leqslant \bar{L}_\lambda\}$ is the globally exponentially attractive set and positive invariant set of system (1).
This completes the proof of Theorem 2. □

For the application of Theorem 2, let us consider the first two chaotic attractors depicted in Fig. 1a and b.

Example 1. For the first chaotic attractor shown in Fig. 1a, the parameters are $\sigma = 30, \beta = 5, \gamma = 5, \rho = 80$. Thus

$$\bar{\xi}_1 = \min\left\{\sigma, \gamma, \frac{\beta}{2}\right\} = \min\{30, 62.5\} = 2.5.$$

Taking $\lambda = 1$, we have

$$\bar{L}_1 = \frac{(\sigma+\rho)^2\beta}{4\bar{\xi}} = \frac{(30+80)^2 \times 5}{4 \times 2.5} = 6050,$$

and thus we have the following estimation:

$$(V_1(X(t)) - 6050) \leqslant (V_1(X(t_0)) - 6050)e^{-5(t-t_0)},$$

the globally exponentially attractive set and positive invariant set is given by

$$\Omega_1 = \{X|V_1(X) \leqslant 6050\}.$$

For the second chaotic attractor shown in Fig. 1b, the parameters are $\sigma = 40, \beta = 20, \gamma = 5, \rho = 250$. Thus

$$\bar{\xi}_1 = \min\left\{\sigma, \gamma, \frac{\beta}{2}\right\} = \min\{40, 202.5\} = 2.5.$$

Again taking $\lambda = 1$, we have

$$\bar{L}_1 = \frac{(\sigma+\rho)^2\beta}{4\bar{\xi}} = \frac{(40+250)^2 \times 5}{4 \times 2.5} = 42050.$$

So the following estimation:

$$(V_1(X(t)) - 42050) \leqslant (V_1(X(t_0)) - 42050)e^{-5(t-t_0)}$$

holds, and $\Omega_1 = \{X|V_1(X) \leqslant 6050\}$ is the globally exponentially attractive set and positive invariant set.

Remark 6. Although the proof of Theorem 2 is simple, it is conservative since the term $-\frac{\beta}{2}z^2$ is neglected. In the following, we give a theorem which is the result of generalization and improvement of Theorems 1 and 2. However, this new theorem has a parameter to be determined, while the parameters used in Theorems 1 and 2 are the system parameters.

Theorem 3. *Choose $\eta \in (0, \beta)$. Let $\xi^* = \min\{\sigma, \gamma, \eta\}$, and*

$$L^*_\lambda = \frac{(\lambda\sigma+\rho)^2\beta^2}{8(\beta-\xi^*)\xi^*}. \tag{10}$$

Then when $V_\lambda(X(t)) > L^*_\lambda$ and $V_\lambda(X(t_0)) > L^*_\lambda$, there exists the estimation

$$(V_\lambda(X(t)) - L^*_\lambda) \leqslant (V_\lambda(X(t_0)) - L^*_\lambda)e^{-2\xi^*(t-t_0)}, \tag{11}$$

and thus $\overline{\lim}_{t\to+\infty} V_\lambda(X(t)) \leqslant L_\lambda^*$, *i.e.*, $\Omega_\lambda^* = \{X | V_\lambda(X) \leqslant L_\lambda^*\}$ *is the globally exponentially attractive set and positive invariant set of system (1).*

Proof. Using the $V_\lambda(X)$ given in (2), we obtain

$$\begin{aligned}\left.\frac{dV_\lambda}{dt}\right|_{(1)} &= \lambda x\dot{x} + y\dot{y} + (z-\lambda\sigma-\rho)\dot{z}\\ &= -\lambda\sigma x^2 - \gamma y^2 - \beta z^2 + \beta(\lambda\sigma+\rho)z\\ &\leqslant -\lambda\sigma x^2 - \gamma y^2 - \xi^* z^2 + 2\xi^*(\lambda\sigma+\rho)z - \xi^*(\lambda\sigma+\rho)^2 - (\beta-\xi^*)z^2 + (\beta-2\xi^*)(\lambda\sigma+\rho)z + \xi^*(\lambda\sigma+\rho)^2\\ &\leqslant -\lambda\xi^* x^2 - \xi^* y^2 - \xi^*(z-\lambda\sigma-\rho)^2 - (\beta-\xi^*)\left[z - \frac{(\beta-2\xi^*)(\lambda\sigma+\rho)}{2(\beta-\xi^*)}\right]^2 + \frac{(\beta-2\xi^*)^2(\lambda\sigma+\rho)^2}{4(\beta-\xi^*)} + \xi^*(\lambda\sigma+\rho)^2\\ &\leqslant -2\xi^* V_\lambda + \frac{(\lambda\sigma+\rho)^2\beta^2}{4(\beta-\xi^*)}\\ &\leqslant -2\xi^*(V_\lambda - L_\lambda^*).\end{aligned} \tag{12}$$

Therefore, when $V_\lambda(X(t)) > L_\lambda^*$ and $V_\lambda(X(t_0)) > L_\lambda^*$, we have

$$(V_\lambda(X(t)) - L_\lambda^*) \leqslant (V_\lambda(X(t_0)) - L_\lambda^*)e^{-2\xi^*(t-t_0)},$$

and thus $\Omega_\lambda^* = \{X | V_\lambda(X) \leqslant L_\lambda^*\}$ is the globally exponentially attractive set and positive invariant set of system (1). □

Remark 7. When $\beta > \min\{\sigma,\gamma\}$, we can take $\xi^* = \min\{\sigma,\gamma\}$, leading to the conclusion of Theorem 1; when $\beta \leqslant \min\{\sigma,\gamma\}$, we can choose $\eta = \frac{\beta}{2}$, and so $\xi^* = \min\{\sigma,\gamma,\frac{\beta}{2}\} = \frac{\beta}{2}$, resulting in the result of Theorem 2. Note that Theorem 3 does not need to discuss β and $\min\{\sigma,\gamma\}$, and thus is more general and applicable; while the conditions given in Theorems 1 and 2 are explicitly expressed in terms of system parameters, which does not need to determine ξ, and thus easier to be used in application.

In this following, we may separate the variable x form the variables y and z, and obtain another result as follows. Let

$$V_0 = \frac{1}{2}[y^2 + (z-\rho)^2] \quad \text{and} \quad L_0 = \frac{\beta^2\rho^2}{8(\beta-\xi)\xi} \quad \text{where} \quad \xi = \begin{cases}\min\{\sigma,\gamma\}, & \text{if} \quad \sigma \neq 2\gamma,\\ \gamma+\varepsilon \\ (\varepsilon>0), & \text{if} \quad \sigma = 2\gamma.\end{cases} \tag{13}$$

Theorem 4. *If* $\beta > \min\{\sigma,\gamma\} = \xi$, *then*

$$\begin{cases}(V_0(X(t)) - L_0) \leqslant (V_0(X(t_0)) - L_0)e^{-\xi(t-t_0)},\\ (x^2(t) - 2L_0) \leqslant (x^2(t_0)) - 2L_0)e^{-2\xi(t-t_0)};\end{cases} \tag{14}$$

$$\begin{cases}\overline{\lim}_{t\to+\infty}(y^2(t) + (z(t)-\rho)^2) \leqslant \frac{\beta^2\rho^2}{4(\beta-\xi)\xi},\\ \overline{\lim}_{t\to+\infty} x^2(t) \leqslant \frac{\beta^2\rho^2}{4(\beta-\xi)\xi};\end{cases} \tag{15}$$

and

$$\Omega_0 := \left\{X \left| \begin{matrix} V_0 \leqslant L_0 \\ x^2 \end{matrix}\right.\right\} = \begin{cases} y^2 + (z-\rho)^2 \leqslant \frac{\beta^2\rho^2}{4(\beta-\xi)\xi},\\ x^2 \leqslant \frac{\beta^2\rho^2}{4(\beta-\xi)\xi}\end{cases} \tag{16}$$

is the globally exponentially attractive set and positive invariant set of system (1).

Proof. Take $\lambda = 0$ in Theorem 1. Similar to the proof for Theorem 1, we obtain

$$\begin{aligned}\frac{dV_0}{dt} &= y(\rho z - xz - \gamma y) + (z-\rho)(xy - \beta z)\\ &= -\gamma y^2 - \beta z^2 + \rho\beta z\\ &\leqslant -\xi y^2 - \xi z^2 + 2\xi\rho z - \xi\rho^2 - (\beta-\xi)z^2 + (\beta-2\xi)\rho z + \xi\rho^2\\ &\leqslant -2\xi V_0 + \frac{\beta^2\rho^2}{4(\beta-\xi)}\\ &\leqslant -2\xi(V_0 - L_0),\end{aligned}$$

which, in tern, results in

$$(V_0(X(t)) - L_0) \leqslant (V_0(X(t_0)) - L_0)e^{-2\xi(t-t_0)}$$

and so

$$\overline{\lim}_{t\to+\infty}(y^2(t)+(z(t)-\rho)^2)\leqslant\frac{\beta^2\rho^2}{4(\beta-\xi)\xi}.$$

Next, for the first equation of (1), construct the positive definite and radially unbounded Lyapunov function:

$$\widetilde{V}_0=\frac{1}{2}x^2,$$

and thus

$$\begin{aligned}\frac{d\widetilde{V}_0}{dt}&=-\sigma x^2+\sigma xy\\&\leqslant-\sigma x^2+\sigma|x||y|\\&\leqslant-\sigma x^2+\frac{1}{2}\sigma x^2+\frac{1}{2}\sigma y^2\\&\leqslant-2\sigma\widetilde{V}_0+\sigma\widetilde{V}_0+\sigma(V_0(x_0)-L_0)e^{-2\xi(t-t_0)}\end{aligned}$$

from which we have

$$(\widetilde{V}_0(x(t))-L_0)\leqslant(\widetilde{V}_0(x_0)-L_0)e^{-\sigma(t-t_0)}+\int_{t_0}^{t}e^{-\sigma(t-\tau)}\sigma(V_0(x_0)-L_0)e^{-2\xi(\tau-t_0)}d\tau.$$

Now we estimate the integral in the above inequality:

$$\begin{aligned}&\sigma(V_0(x_0)-L_0)e^{-\sigma t+2\xi t_0}\int_{t_0}^{t}e^{(\sigma-2\xi)\tau}d\tau\\&=(V_0(x_0)-L_0)e^{-\sigma t+2\xi t_0}\left(\frac{\sigma}{\sigma-2\xi}\right)[e^{(\sigma-2\xi)t}-e^{(\sigma-2\xi)t_0}]\\&=(V_0(x_0)-L_0)\left(\frac{\sigma}{\sigma-2\xi}\right)[e^{-2\xi(t-t_0)}-e^{-\sigma(t-t_0)}]\\&\leqslant\begin{cases}(V_0(x_0)-L_0)\left(\frac{\sigma}{\sigma-2\xi}\right)e^{-2\xi(t-t_0)} & \text{when } \sigma>2\xi,\\(V_0(x_0)-L_0)\left(\frac{\sigma}{2\xi-\sigma}\right)e^{-\sigma(t-t_0)} & \text{when } \sigma<2\xi.\end{cases}\end{aligned}$$

Hence, $\overline{\lim}_{t\to+\infty}\widetilde{V}_0(X(t))\leqslant\widetilde{L}_0$, i.e., $\overline{\lim}_{t\to+\infty}\frac{1}{2}x^2\leqslant\widetilde{L}_0$.

Therefore, summarizing the above two parts shows that any trajectory of system (1) globally exponentially converges to Ω_0, namely, Ω_0 is the globally exponentially attractive set and positive invariant set.

The proof is complete. □

Let $\bar{\xi}_1=\min\{\gamma,\frac{\beta}{2}\}$, $\widetilde{L}_0=\frac{\rho^2\beta}{4\bar{\xi}_1}$. Again choose $V_0(y,z)=\frac{1}{2}[y^2+(z-\rho)^2]$. Then we have

Theorem 5. *When $V_0(X(t))>\widetilde{L}_0$ and $V_0(X(t_0))>\widetilde{L}_0$, the following estimate:*

$$(V_0(X(t))-\widetilde{L}_0)\leqslant(V_0(X(t_0))-\widetilde{L}_0)e^{-2\bar{\xi}_1(t-t_0)}$$

holds with

$$\overline{\lim}_{t\to+\infty}\widetilde{V}_0(X(t))\leqslant\widetilde{L}_0\quad\text{and}\quad\overline{\lim}_{t\to+\infty}\frac{1}{2}x^2(t)\leqslant\widetilde{L}_0,$$

i.e.,

$$\widetilde{\Omega}_0=\left\{X\left|\begin{array}{l}V_0(X(t))\leqslant\widetilde{L}_0\\\frac{1}{2}x^2\leqslant\widetilde{L}_0\end{array}\right.\right\}$$

is the globally exponentially attractive set and positive invariant set of system (1).

Proof. In Theorem 2, taking $\lambda=0$ yields

$$\frac{dV_0}{dt}=-\bar{\xi}_1[y^2+(z-\rho)^2]+2\bar{\xi}_1\widetilde{L}_0\leqslant-2\bar{\xi}_1(V_0-\widetilde{L}_0),$$

and thus

$$(V_0(X(t))-\widetilde{L}_0)\leqslant(V_0(X(t_0))-\widetilde{L}_0)e^{-2\bar{\xi}_1(t-t_0)}$$

which implies that $\overline{\lim}_{t\to+\infty}\widetilde{V}_0(X(t)) \leqslant \widetilde{L}_0$. Further, by Theorem 4, we have $\overline{\lim}_{t\to+\infty}\frac{1}{2}x^2(t) \leqslant \widetilde{L}_0$. Combining these two results shows that

$$\widetilde{\Omega}_0 = \left\{ X \middle| \begin{array}{l} V_0(X(t)) \leqslant \widetilde{L}_0 \\ \frac{1}{2}x^2 \leqslant \widetilde{L}_0 \end{array} \right\}$$

is the globally exponentially attractive set and positive invariant set of system (1). □

Example 2. Let us apply the above theorem to re-consider the first two chaotic attractors shown in Figs. 1a and b. For the first chaotic attractor, $\sigma = 30, \beta = 5, \gamma = 5, \rho = 80$,

$$\bar{\xi}_1 = \min\left\{\sigma, \gamma, \frac{\beta}{2}\right\} = \min\{30, 5, 2.5\} = 2.5.$$

Thus

$$\widetilde{L}_0 = \frac{\rho^2\beta}{4\bar{\xi}_1} = \frac{80^2 \times 5}{10} = 3200,$$

giving a better estimation than that obtained in Example 1.

For the second chaotic attractor, $\sigma = 40, \beta = 20, \gamma = 5, \rho = 250$. $\bar{\xi}_1 = \min\{\gamma, \frac{\beta}{2}\} = 2.5$, and thus

$$\widetilde{L}_0 = \frac{(\sigma+\rho)^2\beta}{4\bar{\xi}_1} = \frac{(250)^2 \times 5}{10} = 31250,$$

which is sharp than that given in Example 1.

Theorem 6. *Choose $\eta \in [0, \beta]$ and let $\min\{\gamma, \eta\} = \xi^*$. Let*

$$L_0^* = \frac{\rho^2\beta^2}{8(\beta - \xi^*)\xi^*}.$$

Then, when $V_0(X(t)) > L_0^*$ and $V_0(X(t_0)) > L_0^*$, the following estimate

$$(V_0(X(t)) - L_0^*) \leqslant (V_0(X(t_0)) - L_0^*)e^{-2\xi^*(t-t_0)}$$

holds with

$$\overline{\lim}_{t\to+\infty}\widetilde{V}_0(X(t)) \leqslant L_0^* \quad \text{and} \quad \overline{\lim}_{t\to+\infty}\frac{1}{2}x^2(t) \leqslant L_0^*,$$

i.e.,

$$\Omega^* = \left\{ X \middle| \begin{array}{l} \frac{1}{2}y^2 + \frac{1}{2}(z-\rho)^2 \leqslant L_0^* \\ \frac{1}{2}x^2 \leqslant L_0^* \end{array} \right\}$$

is the globally exponentially attractive set and positive invariant set of system (1).

Proof. In Theorem 3 take $\lambda = 0$, then following the proofs given for Theorems 3 and 5, we can similarly prove this theorem, and the details are omitted for simplicity. □

At the end of this section, we give an application of Theorem 6 to globally exponentially stabilize the origin $(0, 0, 0)$ using a simple feedback control.

4. Application

In this section, we only present two simple examples to illustrate the application of the theoretical results established in the previous section for the globally exponential attracting set and positive invariant set. More applications in chaos control and chaos synchronization will be discussed in other papers.

4.1. Application to stability and stabilization

Theorem 7. *When $\rho \leqslant 0$, the equilibrium point $(0, 0, 0)$ of system (1) is globally exponentially stable, and is thus unique. When $\rho > 0$, there exist many feedback control laws to stabilize the equilibrium point $(0, 0, 0)$, and the simplest one is to add a linear feedback control $-cx$ to the second equation of (1), where $c \geqslant \rho$.*

Proof. When $\rho < 0$, in Theorem 2 we choose $\lambda\sigma + \rho = 0$, namely $\lambda = -\frac{\rho}{\sigma}$. Thus the L_λ in Theorem 2 becomes $L_\lambda = L_{-\frac{\rho}{\sigma}} = 0$. This shows that the globally exponential attracting set is reduced to the minimum, i.e., $\Omega_\lambda = \Omega_{\frac{\rho}{\sigma}} = \{0\}$. Therefore, the equilibrium point $(0,0,0)$ of system (1) is globally exponentially stable.

When $\rho = 0$, by using Theorem 4, we have $L_0 = \frac{\beta^2\rho^2}{8(\beta-\xi)\xi} = 0$, which implies that

$$\lim_{t\to+\infty} y^2(t) + z^2(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} x^2(t) = 0.$$

Hence, the minimum globally exponential attracting set $\Omega_0 = \{0\}$, i.e., the equilibrium point $(0,0,0)$ of system (1) is globally exponentially stable.

When $\rho > 0$, adding the feedback control $-cx$ to the second equation of system (1) yields the following controlled system:

$$\begin{aligned}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= \rho x - xz - \gamma y - cx = \tilde{\rho} x - xz - \gamma y, \\
\dot{z} &= xy - \beta z,
\end{aligned}$$

where $\tilde{\rho} = \rho - c \leqslant 0$. From the above discussions for the cases $\rho < 0$ and $\rho = 0$, we know that the conclusion is also true for $\rho > 0$. □

4.2. Qualitative study on the trajectory of system (1) in the complementary set of the globally exponential attracting set Ω

The second example is to consider the qualitative behaviour of the trajectories of system (1) in the complementary set of the globally exponential attracting set Ω, R^3/Ω.

Theorem 8. *In the complementary set R^3/Ω, (i.e., outside of the globally exponential attracting set Ω), there do not exist chaotic attractor, or equilibrium point, or periodic solution, or quasi-periodic motion, or any other type of positive invariant sets.*

Proof. First of all, all trajectories in R^3/Ω move according to the exponential decay, converging towards the globally exponential attracting set Ω. Obviously, there cannot exist any strange attractor like the Lorenz attractor. Thus, in R^3/Ω there are no fundamental differences between chaotic and non-chaotic systems.

Because the equilibrium points, periodic solutions and quasi-periodic solutions of system (1) are all positive invariant sets of system (1). Thus, we prove for general case of positive invariant set.

Suppose this is not true. Without loss of generality, assume Q is a positive invariant set of R^3/Ω. Then $Q \bigcup \Omega = \phi$, where ϕ denotes the empty set. This implies that Q and Ω do not intersect. So we have

$$\inf_{X\in\Omega,\overline{X}\in Q} \|X - \overline{X}\| > 0.$$

From the definition of positive invariant set, we know that $X(t,t_0,X_0) \in Q \ (t \geqslant t_0)$ as long as $X_0 \in Q$. Thus,

$$\inf_{X\in\Omega,X(t,t_0,X_0)\in Q,t\geqslant t_0} \|X - X(t,t_0,X_0)\| > 0.$$

On the other hand, since Ω is a globally exponential attracting set, we have that $\forall X_0 \in R^3$, $X(t,t_0,X_0) \to \Omega$, which implies

$$\inf_{X\in\Omega,X(t,t_0,X_0)\in Q,t\geqslant t_0} \|X - X(t,t_0,X_0)\| = 0,$$

leading to a contradiction to the above inequality. This shows that the conclusion of Theorem 8 is true. □

Remark 8. The Ω used in Theorem 8 is a general notation. The conclusion is always true when Ω is chosen as any of the $\Omega, \Omega_0, \widetilde{\Omega}_0, \Omega_1, \Omega_\lambda, \overline{\Omega}_\lambda$ and Ω_λ^*, which are used in the previous section.

5. Conclusion

In this paper, we have extended the method developed in [15] to study the globally exponentially attractive set and positive invariant set for a more general Lorenz family. It has been shown that such system indeed has globally exponentially attractive set and positive invariant set, and contains all the existing relative results as special cases. Exponential estimation is explicitly derived. The approach presented in this paper may be applied to study other chaotic systems.

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