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On the study of globally exponentially attractive set of a general chaotic system

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Abstract

In this paper, we prove that there exists globally exponential attractive and positive invariant set for a general chaotic system, which does not belong to the known Lorenz system, or the Chen system, or the Lorenz family. We show that all the solution orbits of the chaotic system are ultimately bounded with exponential convergent rates and the convergent rates are explicitly estimated. The method given in this paper can be applied to study other chaotic systems.

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Keywords: Chaotic attractor; Globally exponentially attractive set; Ultimate boundedness of chaos

1. Introduction

Since the discovery of the Lorenz chaotic attractor [1] in 1963, many other chaotic systems have been found, including the well-known Rössler system [2], Chua's circuit [3], which have been served as models for study of chaos. In late 1990s, a new chaotic system was found, which is a dual system to the Lorenz system, and now known as the Chen system [4]. Due to its close relation to the Lorenz system and importance, the Chen system has been widely studied (e.g., see [5–7] and references therein).

The ultimate boundedness of a chaotic system is very important for the study of the qualitative behaviour of a chaotic system. If one can show that a chaotic system under consideration has a globally attractive set, then one knows that the system cannot have equilibrium points, periodic or quasi-periodic solutions, or other chaotic attractors existing outside the attractive set. This greatly simplifies the analysis of dynamics of the system. The ultimate boundedness also plays a very important role in the designs of chaos control and chaos

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synchronization. The ultimate boundedness property of the Lorenz system has been investigated by many researchers (e.g., see [8–14]). However, so far very little has been achieved on other chaotic systems, regarding the property of ultimate boundedness. A smooth Chua's circuit has been studied and estimation on its ultimate boundedness has been obtained [15]. For the Chen system, a recent article [16] investigated its property of ultimate boundedness, but the parameter values considered in this article does not cover the most interesting case of the Chen's chaotic attractor.

Consider the following general system:

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= dx - xz + cy, \\ \dot{z} &= xy - bz, \end{aligned} \tag{1}$$

where the dot denotes differentiation with respect to time t; a, b, c and d are parameters. This general system contains many famous chaotic systems, including the Lorenz system ($c = -1$), the Chen system ($d = c - a$) and the Lü system ($d = 0$). In this paper, we generally assume that $a > 0$, $c > 0$ and $d \in (-\infty, \infty)$. Since for the known chaotic attractors, the values of b are $b = \frac{8}{3}$ for the Lorenz attractor, $b = 3$ for the Chen attractor, $b = \frac{44}{15}$ for the Lü attractor, and $b \in [\frac{8}{3}, 3]$ for the Lorenz family, we pay particular attention in this paper to

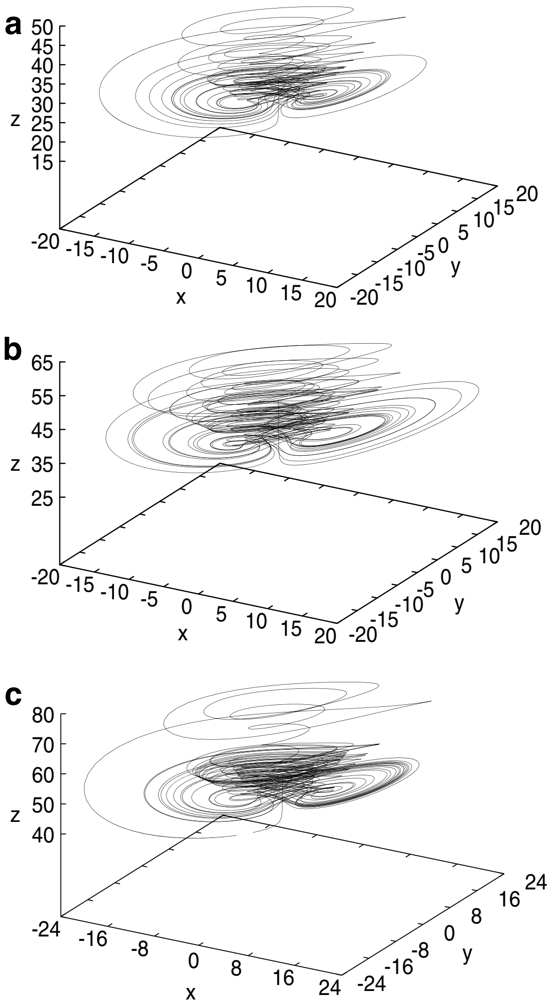

Fig. 1. The chaotic attractors of system (1) for $a = 40$, $b = 0.5$, $c = 35$, projected on the x–z plane, when: (a) $d = -10$; (b) $d = 0$; and (c) $d = 10$.

$b \in (0,1]$. To illustrate that the system can exhibit chaos for $b \in (0,1]$ and for different values of d, three chaotic attractors of system (1) are shown in Fig. 1 when $a = 40$, $b = 0.5$, $c = 35$ while $d = -10$, 0, 10. This clearly shows that these three chaotic attractors do not belong to any of the Lorenz, Chen and Lü systems, and thus it is necessary to investigate the dynamical property of this system. Some definitions and lemmas are given in Section 2 which will be used for proving the main theorem in Section 3. The conclusion is drawn in Section 4.

2. Preliminaries

In this section, we present some basic definitions and two lemmas which are needed for proving the main theorem in the next section. For convenience, let $X := (x,y,z)$ and $X(t) := X(t,t_0,X_0)$.

Definition 1. For the general chaotic system (1), if there exists compact (bounded and closed) set $\Omega \subset R^3$ such that $\forall X_0 \in R^3$, the following condition:

$$\rho(X(t),\Omega) := \inf_{y\in\Omega} \|X(t) - y\| \to 0 \quad \text{as } t \to +\infty$$

holds, then the set Ω is said to be globally attractive. That is, system (1) is ultimately bounded, namely, system (1) is globally stable in the sense of Lagrange or dissipative with ultimate bound.

Further, if $\forall X_0 \in \Omega_0 \subseteq \Omega \subset R^3$, $X(t,t_0,X_0) \subseteq \Omega_0$, then Ω_0 is called the positive invariant set of system (1).

Definition 2. For the general chaotic system (1), if there exists compact set $\Omega \subset R^3$ such that $\forall X_0 \in R^3$, and constants $M(X_0) > 0$, $\alpha > 0$ such that

$$\rho(X(t),\Omega) \leqslant M(X(t_0))\mathrm{e}^{-\alpha(t-t_0)},$$

then system (1) is said to have globally exponentially attractive set, or system (1) is globally exponentially stable in the sense of Lagrange, and Ω is called the globally exponentially attractive set.

Both globally attractive set and globally exponentially attractive set are positive invariant sets.

Note that it is difficult to verify the existence of Ω in Definition 2. Since the Lyapunov direct method is still a powerful tool in the study of asymptotic behaviour of nonlinear dynamical systems, the following definition is more useful in applications.

Definition 3. For the general chaotic system (1), if there exists a positive definite and radially unbounded Lyapunov function $V(x)$, and positive numbers $L > 0$, $\alpha > 0$ such that the following inequality

$$(V(x(t) - L) \leqslant (V(X_0) - L)\mathrm{e}^{-\alpha(t-t_0)}$$

is valid for $V(X(t) > L$ $(t \geqslant t_0)$, then system (1) is said to be globally exponentially attractive or globally exponentially stable in the sense of Lagrange, and $\Omega := \{X | V(t) \leqslant L,\ t \geqslant t_0\}$ is called the globally exponentially attractive set.

In order to give a complete proof (in the next section) for the globally exponential boundedness of the general chaotic system, we need the following lemmas.

Lemma 1. *Let $X(t) = (x(t), y(t), z(t))$ be any solution of system (1), and ϵ be an arbitrary small positive number $(0 < \epsilon \ll 1)$. Then, we have the following results.*

(i) *The boundedness of $y(t)$ implies the boundedness of $x(t)$, i.e., if $\lim_{t\to\infty}|y(t)| \leqslant M$ $(M > 0)$ but $|x(t)| > M$, then $|x(t)|$ monotonically decreases as t increases, and*

$$|x(t)| \leqslant M + \epsilon \tag{2}$$

holds for sufficiently large t.

(ii) *If $|y(t)| \geqslant M > 0$ but $|x(t)| < M$, then $|x(t)|$ monotonically increases as t increases, and*

$$|x(t)| \geqslant M - \epsilon \tag{3}$$

holds for sufficiently large t.

Proof.

(i) For the first equation of system (1), construct the positive definite and radially unbounded Lyapunov function

$$V = |x|. \tag{4}$$

Then,

$$D^+V = D^+|x| \leqslant -a|x| + a|y| \leqslant -a|x| + aM = -a(|x| - M) < 0 \quad \text{when } |x| > M,$$

from which we obtain the differential equation: $\frac{\mathrm{d}(|x|-M)}{|x|-M} \leqslant -a\mathrm{d}t$ $(x \neq 0)$ and its solution is given by

$$|x(t)| \leqslant M + (|x(t_0)| - M)\mathrm{e}^{-at}, \tag{5}$$

where $|x(t_0)| > M$. Hence, $|x(t)| \leqslant M + \epsilon$ for sufficiently large t.
When $x = 0$, it is easy to see from the first equation of (1) that any trajectory of the system cannot stay on the y–z plane $(x = 0)$ and must cross the plane, as long as $y \neq 0$.

(ii) Suppose $y(t) \geqslant M > 0$, but $|x(t)| < M$. Then from the first equation of system (1) we obtain

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -ax(t) + ay(t) \geqslant -ax(t) + aM = a(M - x(t)) > 0 \quad \text{when } |x(t)| < M,$$

which, in turn, results in the differential equation $\frac{\mathrm{d}(M-x(t))}{\mathrm{d}t} \leqslant -a(M - x(t))$ or $\frac{\mathrm{d}(M-x(t))}{M-x(t)} \leqslant -a\mathrm{d}t$. Thus, the solution of $x(t)$ is

$$M - x(t) \leqslant (M - x(t_0))\mathrm{e}^{-at},$$

i.e.,

$$x(t) \geqslant M - (M - x(t_0))\mathrm{e}^{-at}. \tag{6}$$

This implies that $x(t) \geqslant M - \epsilon$ for sufficiently large t, since $|x(t_0)| < M$.
If, on the other hand, $y(t) \leqslant -M < 0$, but $|x(t)| < M$. Then, similarly from the first equation of system (1) we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax(t) + ay(t) \leqslant -ax(t) - aM = -a(x(t) + M) < 0 \quad \text{when } |x(t)| < M,$$

which yields $\frac{\mathrm{d}(x(t)+M)}{\mathrm{d}t} \leqslant -a(x(t) + M)$, or $\frac{\mathrm{d}(x(t)+M)}{x(t)+M} \leqslant -a\mathrm{d}t$. The solution is $x(t) + M \leqslant (x(t_0) + M)\mathrm{e}^{-at}$, i.e.,

$$-x(t) \geqslant M - (x(t_0) + M)\mathrm{e}^{-at}. \tag{7}$$

This clearly indicates that $-x(t) \geqslant M - \epsilon$ for sufficiently large t. Combining the above two results shows that the conclusion of (ii) is true. □

Lemma 2. *$|y|$ reaches its maximum value on the ellipse*:

$$\mathrm{E}: \quad y^2 + (z - d)^2 - y(z - d) = R_1^2, \tag{8}$$

as $|y|_{\max} = \frac{2}{\sqrt{3}} R_1$.

Proof. A straightforward calculation shows that

$$y^2 + (z - d)^2 - y(z - d) = \left[(z - d) - \frac{1}{2}y\right]^2 + \frac{3}{4}y^2 = R_1^2.$$

Thus, at the two points $(y, z) = \left(\pm\frac{2}{\sqrt{3}}R_1, d \pm \frac{2}{\sqrt{3}}R_1\right)$, the maximum value of $|y|$ is reached as $|y|_{\max} = \frac{2}{\sqrt{3}}R_1$. □

Remark 1. Based on Lemma 1(i), we only need to show that $y(t)$ and $z(t)$ are ultimately bounded, and then the boundedness of the variable $x(t)$ follows. By Lemma 1(ii), we know that $|x(t)|$ can be sufficiently large as long as $|y(t)|$ is sufficiently large.

3. The main theorem and its proof

In this section, based on a combination of geometric and algebraic methods, we prove the main theorem for the ultimate boundedness of the general chaotic system (1).

Theorem 1. *When $a > 0$, $0 < b \leqslant 1$, $c > 0$ and $d \in (-\infty, \infty)$, the general chaotic system (1) has globally exponentially attractive set, i.e., all solution orbits of system (1) are ultimately bounded.*

Proof. Based on Lemma 1(i), we only need to prove that y and z are ultimately bounded. Furthermore, it is noted that system (1) is symmetric with respect to (x, y), i.e., substituting (x, y) with $(-x, -y)$ does not change the system. Therefore, we only need to consider $x > 0$ and the case $x < 0$ can be directly concluded from the symmetry.

Let $H = \sqrt{\frac{2c}{b} + 1}$. For convenience, we define five lines:

$$L_{1,2}: \ z = \pm Hy + |d|, \quad L_{3,4}: \ z = \pm Hy - |d|, \quad L_5: \ z = \frac{1}{2}y. \tag{9}$$

Note that $H > 1$ and that the five lines $L_1, L_2, \ldots, L_5$, together with the y and z axes, divide the y–z plane into eight regions: (I), (I′), (II), (II′), (III), (III′), (IV) and (IV′). A total of six positive definite and radially unbounded Lyapunov functions are needed for these regions (see Fig. 2):

$$\begin{aligned}
&V_1 = \frac{1}{2}[y^2 + (z-d)^2] && \text{for (I) and (I}'),\\
&V_2 = y + \frac{1}{3}z && \text{for (II)},\\
&V_2' = -V_2 = -y - \frac{1}{3}z && \text{for (II}'),\\
&V_3 = \frac{1}{2}[y^2 + (z-d)^2 - y(z-d)] && \text{for (III) and (III}'),\\
&V_4 = y - 2Hz && \text{for (IV)},\\
&V_4' = -V_4 = -y + 2Hz && \text{for (IV}').
\end{aligned} \tag{10}$$

In the following, we follow the order of regions given in Eq. (10) to complete the proof. For convenience, denote the points $P_i(y_i, z_i)$ and $P_i'(y_i', z_i')$ by P_i and P_i', respectively.

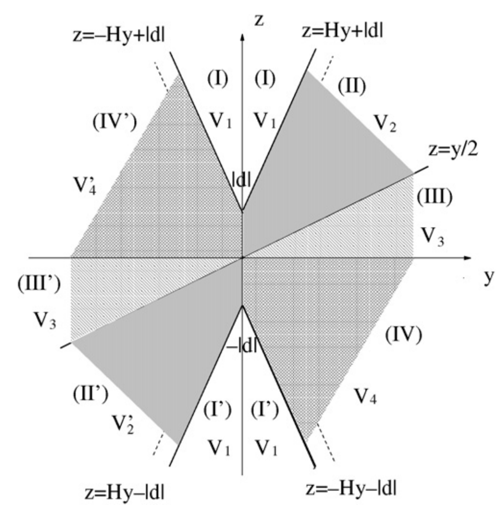

Fig. 2. Geometric distribution with Lyapunov functions.

Region (I) $\cup$ (I′): $\{z \pm Hy \geqslant |d|\} \cup \{z \pm Hy \leqslant -|d|\}$. The region is shown in Fig. 2. Consider the positive definite and radially unbounded Lyapunov function

$$V_1 = \frac{1}{2}[y^2 + (z-d)^2],$$

which represents a family of circles on the y–z plane centered at the point $(0, d)$. Then, differentiating V_1 with respect to time t and evaluating the derivative on the solution of system (1) yields

$$\begin{aligned}
\left.\frac{\mathrm{d}V_1}{\mathrm{d}t}\right|_{(1)} &= y\dot{y} + (z-d)\dot{z} = dxy - xyz + cy^2 + (xy - bz)(z-d) = cy^2 - bz^2 + bdz \\
&= -\frac{b}{2}[y^2 + (z-d)^2] + \frac{b}{2}[y^2 + (z-d)^2] + cy^2 - bz^2 + bdz \\
&= -bV_1 - \frac{b}{2}\left[z^2 - \left(\frac{2c}{b}+1\right)y^2 - d^2\right] \\
&\leqslant -bV_1 \quad \text{when} \quad z^2 - \left(\frac{2c}{b}+1\right)y^2 \geqslant d^2.
\end{aligned} \tag{11}$$

Note that the equation

$$z^2 - \left(\frac{2c}{b}+1\right)y^2 = d^2$$

represents a hyperbola with two asymptotes: $z = \pm Hy$ (see Figs. 2 and 3).

Next, consider the following circle:

$$\mathrm{C}: \; y^2 + (z-d)^2 = R^2 \geqslant 4d^2, \tag{12}$$

and let P_1 and P_{10} be the intersection points of the circle with the z-axis, and P_2, P'_2, P_9 and P'_9 be the intersection points of the circle with the four lines L_1, L_2, L_3 and L_4. Then, the coordinates of the six intersection points are given by

$$\begin{aligned}
&P_1(y_1, z_1) = (0, d+R), \quad P_{10}(y_{10}, z_{10}) = (0, d-R), \\
&P_2(y_2, z_2) = P_2\left(\frac{-(|d|-d)H + \sqrt{(1+H^2)R^2 - (|d|-d)^2}}{1+H^2}, \frac{|d| + dH^2 + H\sqrt{(1+H^2)R^2 - (|d|-d)^2}}{1+H^2}\right), \\
&P_9(y_9, z_9) = P_9\left(\frac{-(|d|+d)H + \sqrt{(1+H^2)R^2 - (|d|+d)^2}}{1+H^2}, \frac{-|d| + dH^2 - H\sqrt{(1+H^2)R^2 - (|d|+d)^2}}{1+H^2}\right), \\
&P'_2(y'_2, z'_2) = P'_2(-y_2, z_2), \quad P'_9(y'_9, z'_9) = P'_9(-y_9, z_9).
\end{aligned} \tag{13}$$

It is obvious that the points P_2 and P'_2, the points P_9 and P'_9 are symmetric, respectively, about the z-axis (see Fig. 3).

Thus, in Region (I) $\cup$ (I′), when $V_1(X(t)) > \frac{R^2}{2}$ $(t \geqslant t_0)$, there exists exponential estimation given as follows:

$$V_1(X(t)) \leqslant V_1(X(t_0))\mathrm{e}^{-b(t-t_0)}. \tag{14}$$

Now define the set

$$\Omega_1 := \{(y,z) | (z \pm Hy \geqslant |d|) \cup (z \pm Hy \leqslant -|d|), \; y^2 + (z-d)^2 \leqslant R^2\}. \tag{15}$$

Then any trajectory located in Region (I) $\cup$ (I′), but outside Ω_1 must enter either into Ω_1 or into the neighboring Region (II) $\cup$ (II′) $\cup$ (IV) $\cup$ (IV′), as shown in Fig. 3.

Remark 2.

The case $d = 0$ becomes simpler since the four half lines $z = \pm H \pm |d|$ emerge into two lines passing through the origin. Then the circle C is reduced to a point (the origin) when $R = d$, see Fig. 3(c).

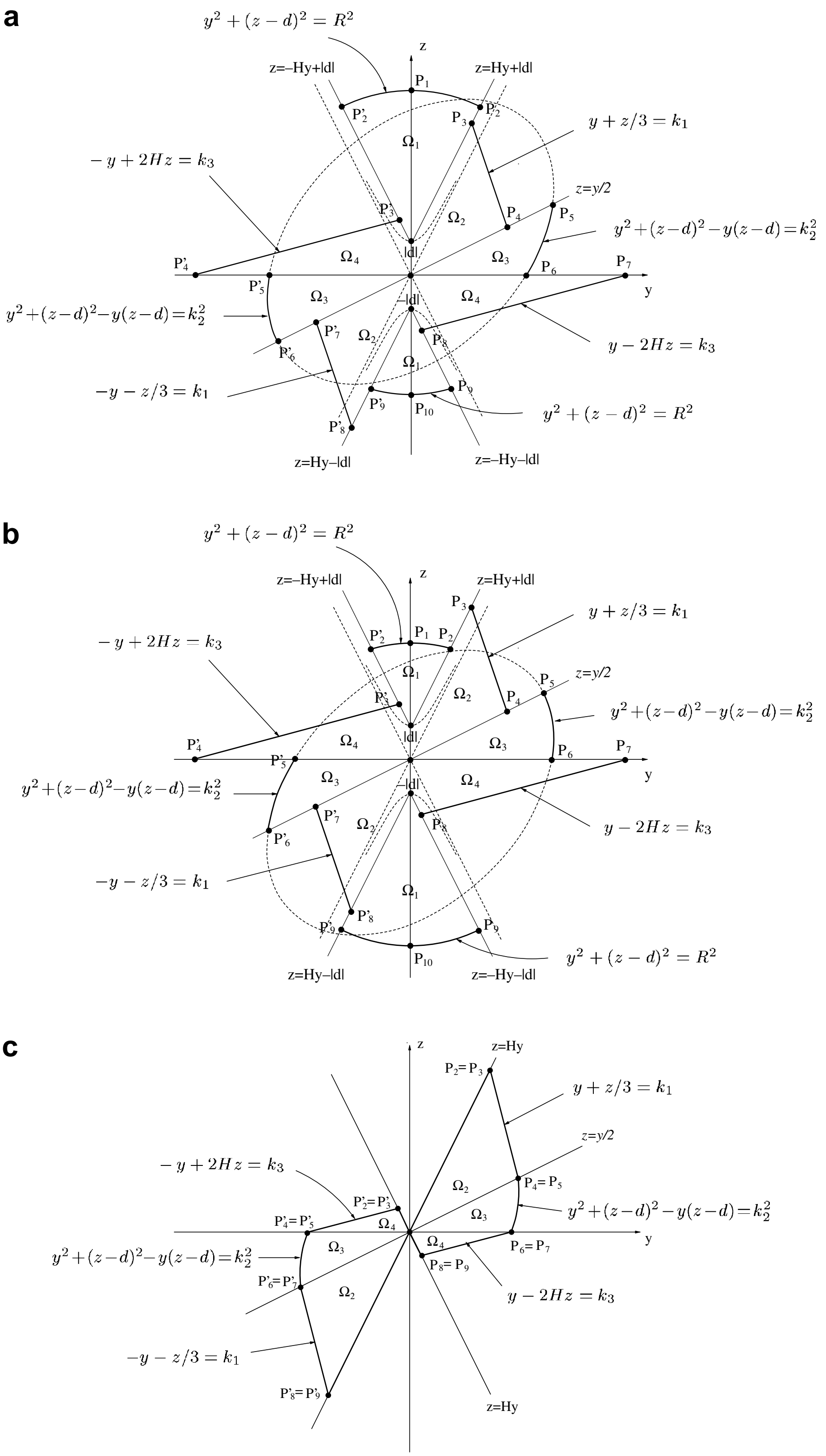

Fig. 3. Trapping regions for $x > 0$: (a) $d > 0$; (b) $d < 0$; and (c) $d = 0$.

Next, we consider region (II) $\cup$ (II). To simplify the analysis for this region and the region (IV) $\cup$ (IV′), define

$$\delta = \begin{cases} +1 & \text{for the regions (II) and (IV),} \\ -1 & \text{for the regions (II}') \text{ and (IV}'). \end{cases} \tag{16}$$

Region (II) $\cup$ (II′): $\left\{yz \geqslant 0,\ \frac{1}{2}|y| \leqslant |z| \leqslant H|y| + |d|\right\}$. Let

$$\xi_1 = 2[3(c+1+|d|) + (H+1)(1-b)] \quad \text{and} \quad \eta_1 = \max\{\xi_1 + 1, |d|\}. \tag{17}$$

So, by Lemma 1(ii) we know that when $|y| \geqslant \eta_1$, $x \geqslant \xi_1$ as $t \gg 1$.

Consider the positive definite and radially unbounded Lyapunov function

$$V_2 = \delta\left(y + \frac{1}{3}z\right),$$

from which we obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}V_2}{\mathrm{d}t}\right|_{(1)} &= \delta\left(\dot{y} + \frac{1}{3}\dot{z}\right) = \delta\left[-\left(y + \frac{1}{3}z\right) + \left(y + \frac{1}{3}z\right) + dx - xz + cy + \frac{1}{3}(xy - bz)\right] \\
&= -\delta\left(y + \frac{1}{3}z\right) + \delta dx - x\delta z + (c+1)\delta y + \frac{1}{3}x\delta y + \frac{1}{3}(1-b)\delta z \\
&= -V_2 + \delta dx - x|z| + (c+1)|y| + \frac{1}{3}x|y| + \frac{1}{3}(1-b)|z| \\
&\leqslant -V_2 + |d|x - \frac{1}{2}x|y| + (c+1)|y| + \frac{1}{3}x|y| + \frac{1}{3}(1-b)(H|y| + |d|) \\
&\leqslant -V_2 + \left[-\frac{1}{6}x + |d|\left(\frac{x}{|y|}\right) + (c+1) + \frac{1}{3}H(1-b) + \frac{1}{3}(1-b)\left(\frac{|d|}{|y|}\right)\right]|y| \\
&\leqslant -V_2 - \left[\frac{1}{6}x - (c+1+|d|) - \frac{1}{3}(H+1)(1-b)\right]|y| \\
&= -V_2 - \frac{1}{6}(x - \xi_1)|y| \leqslant -V_2 \quad \text{when } x \geqslant \xi_1.
\end{aligned} \tag{18}$$

This implies that when $|y| \geqslant \eta_1$ (and so $x \geqslant \xi_1$), $V_2(X(t)) > 0$ $(t \geqslant t_0)$, there exists exponential estimation for Region (II) $\cup$ (II′):

$$V_2(X(t)) \leqslant V_2(X(t_0))\mathrm{e}^{-(t-t_0)}. \tag{19}$$

Denote the intersection points of the two lines $\delta(y + \frac{1}{3}z) = k_1$ $(k_1 > 0)$ with the lines $z = H\ y \pm |d|$ and $z = \frac{1}{2}y$ as P_3, P_4, P_7' and P_8', which are given by

$$y_3 = \frac{3k_1 - |d|}{3 + H} > 0, \quad y_4 = \frac{6}{7}k_1, \quad y_7' = -y_4 = -\frac{6}{7}k_1, \quad y_8' = -y_3 = -\frac{3k_1 - |d|}{3 + H}.$$

It is clearly shown in Fig. 3 that the minimal value of $|y_3|$, $|y_4|$, $|y_7'|$ and $|y_8'|$, is $|y_3| = |y_8'| = \frac{3k_1 - |d|}{3+H}$. Letting $\frac{3k_1 - |d|}{3+H} = \eta_1$ leads to

$$k_1 = \left(1 + \frac{1}{3}H\right)\eta_1 + \frac{1}{3}|d|, \tag{20}$$

and thus the four intersection points are

$$\begin{aligned}
&P_3(y_3, z_3) = P_3(\eta_1, H\eta_1 + |d|), \\
&P_4(y_4, z_4) = P_4\left(\frac{2}{7}((3+H)\eta_3 + |d|),\ \frac{1}{7}((3+H)\eta_3 + |d|)\right), \\
&P_7'(y_7', z_7') = P_7'\left(-\frac{2}{7}((3+H)\eta_3 + |d|),\ -\frac{1}{7}((3+H)\eta_3 + |d|)\right), \\
&P_8'(y_8', z_8') = P_8'(-\eta_1, -H\eta_1 - |d|).
\end{aligned} \tag{21}$$

Then, define the set

$$\Omega_2 := \left\{ (y,z) | yz \geqslant 0,\ \frac{1}{2}|y| \leqslant |z| \leqslant H|y| + |d|,\ \delta\left(y + \frac{1}{3}z\right) \leqslant k_1 \right\} \tag{22}$$

for Region (II) $\cup$ (II$'$). Therefore, any trajectory located in Region (II) $\cup$ (II$'$), but outside Ω_2 must enter either into Ω_2 or into the neighboring Region (I) $\cup$ (I$'$) $\cup$ (III) $\cup$ (III$'$) (see Fig. 3).

Region (III) $\cup$ **(III)**: $\{yz \geqslant 0,\ |z| \leqslant \frac{1}{2}|y|\}$.

Let

$$\xi_2 = \frac{2}{3}(6c - 2b + 13 + 4|d|) \quad \text{and} \quad \eta_2 = \max\left\{\xi_2 + 1, |d|, \frac{2R}{\sqrt{3}}\right\}. \tag{23}$$

Then, it follows from Lemma 1(ii) that $x \geqslant \xi_2$ when $|y| \geqslant \eta_2$ as $t \gg 1$.

For Region (III) $\cup$ (III$'$), construct the positive definite and radially unbounded Lyapunov function

$$V_3 = \frac{1}{2}[y^2 + (z-d)^2 - y(z-d)],$$

which denotes a family of ellipses on the y–z plane centered at the point $(0,d)$. Then, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}V_3}{\mathrm{d}t}\right|_{(1)} &= y\dot{y} + (z-d)\dot{z} - \frac{1}{2}(z-d)\dot{y} - \frac{1}{2}y\dot{z} = cy^2 - bz^2 + bdz - \frac{1}{2}(z-d)(dx - xz + cy) - \frac{1}{2}y(xy - bz) \\
&= -[y^2 + (z-d)^2 - y(z-d)] + d^2 + [y^2 + (z-d)^2 - y(z-d)] - d^2 + cy^2 + bdz - \frac{1}{2}xy^2 + \frac{1}{2}xz^2 \\
&\quad + \frac{d^2}{2}x - dxz + \frac{cd}{2}y + \frac{b}{2}yz - bz^2 - \frac{c}{2}yz \\
&= -2V_3 + d^2 - \frac{1}{2}xy^2 + \frac{1}{2}xz^2 + \frac{d^2}{2}x - dxz + \frac{(c+2)d}{2}y \\
&\quad + \frac{b}{2}yz - (2-b)dz + (c+1)y^2 + (1-b)z^2 - \frac{c+2}{2}yz \\
&\leqslant -2V_3 + d^2 - \frac{1}{2}xy^2 + \frac{1}{2}xz^2 + \frac{d^2}{2}x + |d|x|z| + \frac{(c+2)}{2}|d||y| \\
&\quad + \frac{b}{2}yz + (2-b)|d||z| + (c+1)y^2 + (1-b)z^2 \\
&\leqslant -2V_3 + d^2 + \left[-\frac{1}{2}x + \frac{1}{8}x + \frac{d^2}{2|y|}\left(\frac{x}{|y|}\right) + \frac{|d|}{2}\left(\frac{x}{|y|}\right) + \frac{(c+2)|d|}{2|y|} + \frac{b}{4}\right. \\
&\qquad \left. + (2-b)\left(\frac{|d|}{2|y|}\right) + (c+1) + \frac{1-b}{4}\right]y^2 \\
&\leqslant -2V_3 + d^2 + \left[-\frac{3}{8}x + \frac{|d|}{2} + \frac{|d|}{2} + \frac{(c+2)}{2} + \frac{b}{4} + \frac{2-b}{2} + c + 1 + \frac{1-b}{4}\right]y^2 \\
&\leqslant -2V_3 + d^2 - \frac{3}{8}\left[x - \frac{2}{3}(6c - 2b + 13 + 4|d|)\right]y^2 \\
&= -2V_3 + d^2 - \frac{3}{8}(x - \xi_2)y^2 \\
&\leqslant -2\left(V_3 - \frac{1}{2}d^2\right) \quad \text{when} \quad x \geqslant \xi_2.
\end{aligned} \tag{24}$$

Therefore, in Region (III) $\cup$ (III$'$), when $|y| \geqslant \eta_2$ (and so $x \geqslant \xi_2$), $V_3(X(t)) > \frac{d^2}{2}(t \geqslant t_0)$, we obtain the exponential estimation:

$$V_3(X(t)) - \frac{d^2}{2} \leqslant \left(V_3(X(t_0)) - \frac{d^2}{2}\right)\mathrm{e}^{-2(t-t_0)}. \tag{25}$$

Now we need to determine the four intersection points of the ellipse $y^2 + (z-d)^2 - y(z-d) = k_2^2 \geqslant d^2$ with the line $z = \frac{1}{2}y$ and the y-axis: $P_5(y_5, z_5)$, $P_6(y_6, z_6)$, $P_5'(y_5', z_5')$ and $P_6'(y_6', z_6')$ such that the minimum of the absolute values of the four y-coordinates is not less than η_2. It is easy to see that this ellipse has two intersection points with the y-axis (one positive and one negative) since $k_2^2 \geqslant d^2$, and thus it also has two intersection points with the line $z = \frac{1}{2}y$, with one in the first quadrant and the other in the third quadrant. The y coordinates of these four intersection points are

$$y_5 = \frac{2}{\sqrt{3}}\sqrt{k_2^2 - d^2}, \quad y_6 = \frac{1}{2}\left[-d + \sqrt{4k_2^2 - 3d^2}\right],$$
$$y_5' = \frac{1}{2}\left[-d - \sqrt{4k_2^2 - 3d^2}\right], \quad y_6' = -\frac{2}{\sqrt{3}}\sqrt{k_2^2 - d^2}.$$

Then choosing the smallest one as η_2 yields

$$\min\left\{\frac{2}{\sqrt{3}}\sqrt{k_2^2 - d^2},\ \frac{1}{2}\left[\sqrt{4k_2^2 - 3d^2} - |d|\right]\right\} = \eta_2,$$

from which we obtain

$$k_2^2 = \max\left\{\eta_2^2 + d^2 + \eta_2|d|,\ \frac{3}{4}\eta_2^2 + d^2\right\} = \eta_2^2 + d^2 + \eta_2|d|. \tag{26}$$

Hence, the four intersection points are given by

$$\begin{aligned}
P_5(y_5, z_5) &= \left(\frac{2}{\sqrt{3}}\sqrt{\eta_2^2 + \eta_2|d|},\ \frac{1}{\sqrt{3}}\sqrt{\eta_2^2 + \eta_2|d|}\right),\\
P_6(y_6, z_6) &= \left(\eta_2 + \frac{1}{2}(|d| - d), 0\right),\\
P_5'(y_5', z_5') &= \left(-\eta_2 - \frac{1}{2}(|d| + d), 0\right),\\
P_6'(y_6', z_6') &= \left(-\frac{2}{\sqrt{3}}\sqrt{\eta_1^2 + \eta_1|d|},\ -\frac{2}{\sqrt{3}}\sqrt{\eta_2^2 + \eta_2|d|}\right).
\end{aligned} \tag{27}$$

Define a set inside Region (III) $\cup$ (III$'$) as

$$\Omega_3 := \left\{(y,z)|yz \geqslant 0,\ |z| \leqslant \frac{1}{2}|y|,\ y^2 + (z-d)^2 - y(z-d) \leqslant k_2^2\right\}. \tag{28}$$

Then, any trajectory located in Region (III) $\cup$ (III$'$), but outside Ω_3 must ultimately enter either into Ω_3 or into the neighboring Region (II) $\cup$ (II$'$) $\cup$ (IV) $\cup$ (IV$'$). (See Fig. 3.)

Region (IV) $\cup$ (IV$'$): $\{y\, z \leqslant 0, |z| \leqslant H|y| + |d|\}$.

Let

$$\xi_3 = \frac{c + 1 + 2|d| + 2H(H+1)(1-b)}{H} \quad \text{and} \quad \eta_3 = \max\{\xi_3 + 1, |d|\}. \tag{29}$$

So, when $|y| \geqslant \eta_3$, $x \geqslant \xi_3$ holds as $t \gg 1$ due to Lemma 1(ii).

Consider the positive definite and radially unbounded Lyapunov function

$$V_4 = \delta(y - 2Hz).$$

Then, we have

$$\begin{aligned}
\left.\frac{dV_4}{dt}\right|_{(1)} &= \delta(\dot{y} - 2H\dot{z}) = \delta[(-y + 2Hz) + (y - 2Hz) + dx - xz + cy - 2Hxy + 2Hbz]\\
&= -\delta(y - 2Hz) + \delta dx - x\delta z + (c+1)\delta y - 2Hx\delta y - 2H(1-b)\delta z\\
&\leqslant -V_4 + |d|x + x|z| - 2Hx|y| + (c+1)|y| + 2H(1-b)|z|\\
&\leqslant -V_4 + |d|x + x(H|y| + |d|) - 2Hx|y| + (c+1)|y| + 2H(1-b)(H|y| + |d|)\\
&\leqslant -V_4 + \left[-Hx + 2|d|\left(\frac{x}{|y|}\right) + (c+1) + 2H^2(1-b) + 2H(1-b)\left(\frac{|d|}{|y|}\right)\right]|y|\\
&\leqslant -V_4 - [Hx - 2|d| - (c+1) - 2H(H+1)(1-b)]|y|\\
&= -V_4 - H(x - \xi_3)|y| \leqslant -V_4 \quad \text{when } x \geqslant \xi_3.
\end{aligned} \tag{30}$$

This clearly indicates that when $|y| \geqslant \eta_3$ (and so $x \geqslant \xi_3$), $V_4(X(t)) > 0$ $(t \geqslant t_0)$, there exists estimation for Region $(\mathrm{IV}) \cup (\mathrm{IV}')$:

$$V_4(X(t)) \leqslant V_4(X(t_0))\mathrm{e}^{-(t-t_0)}. \tag{31}$$

Let the intersection points of the two lines $\delta(y - 2H\,z) = k_3$ $(k_3 > 0)$ with the lines $z = -Hy \pm |d|$ and the y-axis be P_7, P_8, P_3' and P_4' (see Fig. 3), then a similar discussion as that for Region $(\mathrm{II}) \cup (\mathrm{II}')$ leads to

$$k_3 = (1 + 2H^2)\eta_3 + 2H|d|, \tag{32}$$

and then the four points are described as

$$\begin{aligned}
&P_7(y_7, z_7) = P_7((1 + 2H^2)\eta_3 + 2H|d|,\ 0),\\
&P_8(y_8, z_8) = P_8(\eta_3,\ -H\eta_3 - |d|),\\
&P_3'(y_3', z_3') = P_3'(-\eta_3,\ H\eta_3 + |d|),\\
&P_4'(y_4', z_4') = P_4'(-(1 + 2H^2)\eta_3 - 2H|d|,\ 0).
\end{aligned} \tag{33}$$

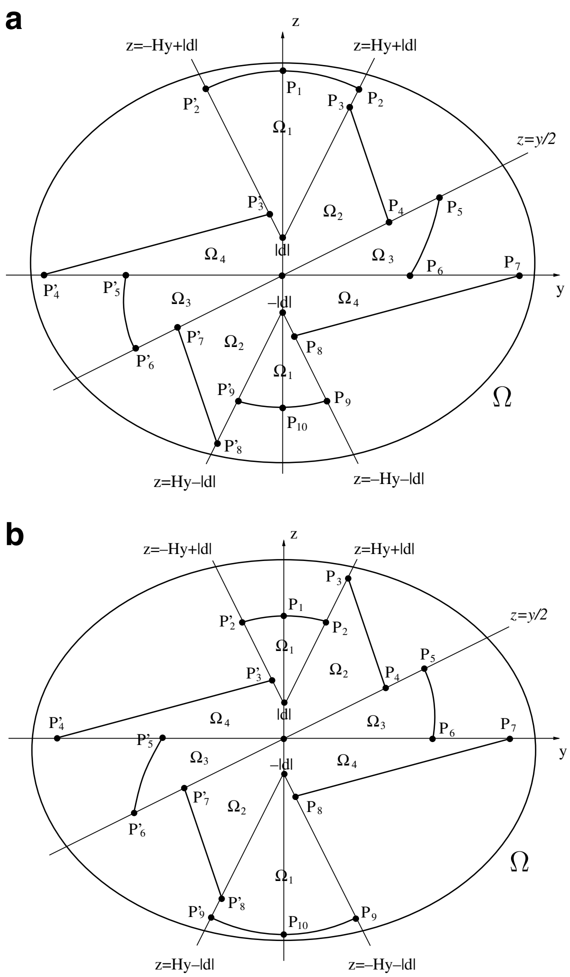

Fig. 4. Closed trapping region Ω for $x > 0$: (a) $d > 0$ and (b) $d < 0$.

Similarly, we define the set

$$\Omega_4 := \{(y,z) | yz \leqslant 0, \ |z| \leqslant H|y| + |d|, \ \delta(y - 2Hz) \leqslant k_3\} \tag{34}$$

for Region (IV) ∪ (IV′). Then, any trajectory located in Region (IV) ∪ (IV′), but outside Ω_3 must either enter into Ω_3 or into the neighboring Region (I) ∪ (I′) ∪ (III) ∪ (III′), as shown in Fig. 3.

The remaining task is to obtain a closed trapping region from the above defined regions Ω_k, $k = 1, 2, 3, 4$. It should be pointed out that Fig. 3 is only a general illustration. When the parameters a, b, c and d are varying, the intersection points P_n, $n = 1, 2, \ldots, 10, 2', 3', \ldots, 9'$ may change their relative positions. The boundaries include arcs of circles, arcs of ellipses, and line segments. Since the slopes of the lines $\overline{P_3P_4}$, and $\overline{P'_7P'_8}$ are fixed as -3 and that for the lines $\overline{P'_3P'_4}$ and $\overline{P_7P_8}$ are fixed as $\frac{1}{2H}$, in general one cannot adjust them to get all the boundary cures (lines) to be connected. However, it is easy to show that if the two slopes of the lines are allowed to be varied, then one can easily make these non-closed boundaries to be closed by first choosing the farthest point among the 18 points and then determine all the other points accordingly. However, the easiest way to obtain a closed trapping region is to draw a closed curve which encloses all the ten boundary curves (lines), as shown in Fig. 4 for $d < 0$ and $d > 0$. When $d = 0$, the distribution of the regions are symmetric about the origin, and thus the closed boundary can be easily constructed, as shown in Fig. 3(c).

Define the closed trapping region as Ω (see Fig. 4). Then, all solution orbits of system (1) outside Ω must move towards Ω and finally enter Ω for sufficiently large t.

Thus proof of Theorem 1 is complete. □

4. Conclusion

In this paper, by combining geometric and algebraic methods, we have shown that the general chaotic system (1) is ultimately bounded, for an interval of $b \in (0, 1]$ which has not been investigated. Six Lyapunov functions are constructed according to different geometric configurations. Thus, for any combination of the system parameter values: $a > 0$, $0 < b \leqslant 1$, $c > 0$ and $d \in (-\infty, \infty)$, this chaotic system has globally exponentially attractive set. The exponential estimation is explicitly derived. This combination method can be applied to study other chaotic systems. The case $b > 1$ is under investigation and the results will be presented in a forthcoming paper.

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