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Normal forms of vector fields with perturbation parameters and their application

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Abstract

This paper is concerned with the normal forms of vector fields with perturbation parameters. Usually, such a normal form is obtained via two steps: first find the normal form of a "reduced" system of the original vector field without perturbation parameters, and then add an unfolding to the normal form. This way, however, it does not yield the relation (transformation) between the original system and the normal form. A study is given in this paper to consider the role of near-identity transformations in the computation of normal forms. It is shown that using only near-identity transformations cannot generate the normal form with unfolding as expected. Such normal forms, which contain many nonlinear terms involving perturbation parameters, are not very useful in bifurcation analysis. Therefore, additional transformations are needed, resulting in a further reduction of normal forms – the simplest normal form. Examples are presented to illustrate the theoretical results.

1. Introduction

Normal form theory has been playing an important role in the study of nonlinear dynamical systems. The theory can be traced back to the original work of one hundred years ago on differential equations, and most credit should be given to Poincaré [1]. The method of normal forms has been proved useful to transform a system to its simpler form, which is qualitatively equivalent to the original system in the vicinity of a fixed point. This greatly simplifies the study for the complex dynamical behaviour of the original system (e.g. see [2–7]). The basic idea of normal form (NF) theory is using the linear singularity of a system at a fixed point to form a Lie bracket operator and then repeatedly employing the operator to remove higher order nonlinear terms as many as possible. Recently, explicit computation of the NF and NT with the aid of computer algebra systems such as Maple, Mathematica, Reduce, etc., have been developed (e.g., see [8–14]). However, so far most of NF computations have been restricted to systems which do not contain perturbation parameters. Such a NF may be called "reduced" normal form. In order to use the "reduced" NF for bifurcation

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and stability analysis, universal unfolding is usually added to the "reduced" form, or approximation of an unfolding is simply derived from the original system using linear analysis. Although the "two-step" approach greatly simplifies NF computation, it has raised some questions particularly related to applications. How to derive a normal form directly from the original system which contains perturbation parameters? Can one just use near-identity transformations to obtain the normal form with unfolding? If the answer to the second question is no, then what kind of transformations should be used to compute the normal form?

A method for directly computing normal forms from the original system with perturbation parameters has been mentioned by few researchers (e.g., see [3,6]). The basic idea of this direct computation is to extend the dimension of the original system to include trivial differential equations for the parameters. Then apply the conventional normal form theory to the extended system with the near-identity transformation given in terms of the state and parameter variables. It is clear that such a near-identity transformation contains the state variables as well as parameter variables, implying that such a transformation is an implicit function of time.

It is well known that the (conventional) normal form (CNF) is not unique, nor is the simplest form, which can be further simplified using a similar nonlinear transformation (e.g., see [12,15–27]). The further simplified form is called "unique normal form", "minimal normal form" or "the simplest normal form" (SNF). The basic idea of the SNF can be roughly explained as follows: unlike the CNF computation that only uses the kth order nonlinear transformation to simplify the kth order nonlinear terms of the system, the SNF computation applies not only the kth order, but also lower order ($<k$) NTs to eliminate the kth order nonlinear terms. Computation of the SNF is much more complicated and tedious than that of the CNF. Recently, particular attention has been given to develop efficient methods and algorithms for computing the SNF (e.g., see [28–30]). However, the research work on the SNF computation has so far been mainly restricted to the systems which do not contain perturbation parameters (unfolding), except for two simple cases – single zero singularity [28] and Hopf bifurcation [31]. Normal forms with unfolding for these two cases have been explicitly calculated from the original system.

It should be mentioned that Kokubu [32] followed the idea of the SNF [15] to consider the normal forms with perturbation parameters. Some simpler forms were obtained and applied to study the bifurcations of a system of reaction diffusion equations with the Brusselator type singularity. However, these *simpler* forms are not the *simplest* forms since the SNF obtained in the paper [32] still involve many nonlinear terms which contain perturbation parameters.

In this paper, we shall first show that with the "extended system" approach the NF with unfolding cannot be obtained using only near-identity transformations. In particular, three cases: a single zero, Hopf and a double zero singularities are considered in detail. To obtain the SNF with unfolding, additional transformations such as time rescaling and parameter rescaling need be introduced, illustrated by the SNF of Hopf bifurcation. The rest of the paper is organized as follows. In the next section, the general formula for computing NF is derived. Section 3 is devoted to show that the NF with unfolding cannot be obtained using only near-identity transformations. In Section 4, the SNF of Hopf bifurcation is presented. Examples are then given in Section 5 to demonstrate the theoretical results. Finally, conclusion is drawn in Section 6.

2. Normal form computation

Before presenting the general formula for computing normal forms, we use Hopf bifurcation as an example to discuss the computation of normal forms with perturbation parameters. Consider the system, described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \ \mu \in \boldsymbol{R}, \ \boldsymbol{f} : \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n, \tag{1}$$

where the dot denotes differentiation with respect to time t; $\boldsymbol{f}$ satisfies $\boldsymbol{f}(\boldsymbol{0}, \mu) = \boldsymbol{0}$ for any feasible real values of μ, implying that $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium point of system (1). The Jacobian of the system evaluated at the critical point $(\boldsymbol{x}, \mu) = (\boldsymbol{0}, 0)$ is assumed to include, without loss of generality, a purely imaginary pair, $\pm$i, and is given in the Jordan canonical form

$$J_c = \operatorname{diag}\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_p \begin{bmatrix} \alpha_{p+1} & \omega_1 \\ -\omega_1 & \alpha_{p+1} \end{bmatrix} \begin{bmatrix} \alpha_{p+2} & \omega_2 \\ -\omega_2 & \alpha_{p+2} \end{bmatrix} \cdots \begin{bmatrix} \alpha_{p+q} & \omega_q \\ -\omega_q & \alpha_{p+q} \end{bmatrix}\right], \tag{2}$$

where $\alpha_j < 0$, $j = 1, 2, \ldots, p + q$; $\omega_k > 0$, $k = 1, 2, \ldots, q$ and $2 + p + 2q = n$.

It is well known that the NF of the "reduced" system of (1) (when $\mu = 0$) can be expressed in polar coordinates as follows:

$$\begin{aligned} \dot{r} &= b_{13} r^3 + b_{15} r^5 + b_{17} r^7 + \cdots, \\ \dot{\theta} &= 1 + b_{23} r^2 + b_{25} r^4 + b_{27} r^6 + \cdots, \end{aligned} \tag{3}$$

where r and θ represent the amplitude and phase of motion, respectively. The coefficients b_{ij}'s are explicitly given in terms of the coefficients of the original function $\boldsymbol{f}$. When $b_{13} \neq 0$, it is called Hopf bifurcation. For Hopf bifurcation, one may find the universal unfolding such that the NF of system (1) becomes

$$\dot{r} = \beta_1(\mu)r + b_{13}r^3 + b_{15}r^5 + b_{17}r^7 + \cdots,$$
$$\dot{\theta} = 1 + \beta_2(\mu) + b_{23}r^2 + b_{25}r^4 + b_{27}r^6 + \cdots, \tag{4}$$

where the two added terms $\beta_1(\mu)r$ and $\beta_2(\mu)r$ are called universal unfolding. Note that all the coefficients b_{ij}'s are same as that of Eq. (3), i.e., Eq. (4) is obtained by adding the unfolding to Eq. (3). $\beta_1(\mu)$ and $\beta_2(\mu)$ are, in general, cannot be found explicitly. If the original function $\boldsymbol{f}$ is assumed analytic, then $\beta_1(\mu)$ and $\beta_2(\mu)$ are also analytic and can be expanded in Taylor series

$$\beta_i(\mu) = \beta_{i1}\mu + \beta_{i2}\mu^2 + \beta_{i3}\mu^3 + \cdots \quad (i = 1, 2), \tag{5}$$

where the two linear coefficients β_{11} and β_{21} can be easily determined from the derivatives of $\boldsymbol{f} = (f_1, f_2)^{\mathrm{T}}$. Suppose that the unfolding only takes the linear terms, then it has been proved (e.g., see [6]) that when $b_{13} \neq 0$, the two "linear" unfolding terms $\beta_{11}\mu r$ and $\beta_{21}\mu r$ are indeed the universal unfolding. That is, the following normal form:

$$\dot{r} = \beta_{11}\mu r + b_{13}r^3 + \cdots, \qquad \dot{\theta} = 1 + \beta_{21}\mu + \cdots, \tag{6}$$

where β_{11} and β_{21} are constants, captures all the dynamical behaviour of system (1) in the vicinity of the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ for small μ.

Naturally we want to ask the following questions:

(i) How to derive the NF given by Eq. (4) directly from system (1)? In general, we may extend the dimension of system (1) to $n + 1$ by adding the equation $\dot{\mu} = 0$ to the system. In other words, we consider the "extended" system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \quad \dot{\mu} = 0, \ \ \boldsymbol{x} \in \boldsymbol{R}^n, \ \ \mu \in \boldsymbol{R}, \ \ \boldsymbol{f} : \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n, \tag{7}$$

and apply the near-identify transformations

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, v), \qquad \mu = v + p(\boldsymbol{y}, v), \tag{8}$$

to obtain the new system

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}, v), \qquad \dot{v} = 0. \tag{9}$$

Then the question is: will the first equation of (9) be the normal form, as the sample as Eq. (6)? That is, can one just use the near-identity transformation (8) to obtain the normal form of system (1)?

(ii) If the answer to question (i) is no, then what kind of transformations should be used to obtain the NF of system (1), given in the form of (6)?

Note that the transformation (8) involves both state variable $\boldsymbol{y}$ and parameter v, indicating that μ is implicitly a function of time. In practice, instead of the second equation of (8), the parameter rescaling

$$\mu = v + p(v) \tag{10}$$

is also often used. It is clear that the second equation of (8) has more freedom in choosing the nonlinear coefficients than that of Eq. (10). For some singularities, for example, Hopf bifurcation, we can show that the second equation of (8) does not provide any more choices and can be reduced to Eq. (10). That is, the near-identity transformation (8) is indeed, for some singularities, given in the form of

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, v), \qquad \mu = v + p(v). \tag{11}$$

It is clear that the idea of universal unfolding is to make a "reduced" NF be able to capture the local dynamical behaviours of the original system near a critical point. Further, it is assumed that the normal form obtained through the "two-step" approach is equivalent to the original system. However, it does not provide the transformation between the original system and the NF. The explicit computation of the unfolding as well as the transformations are not only important from the computational point of view: in practice one usually needs to know how to transform back to the original system once a solution is obtained from the NF; but is also important from the theoretical point of view: why is the NF qualitatively equivalent to the original system? This question can be answered if one can prove the existence of a transformation between the original system and the NF.

In the following, we consider more general systems with any kind of singularities, and present a formula for efficiently computing the coefficients of the NF and NT. We consider

$$\dot{\boldsymbol{x}} = L\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\mu}), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \ \ \boldsymbol{\mu} \in \boldsymbol{R}^m, \tag{12}$$

where $\boldsymbol{x}$ and $\boldsymbol{\mu}$ represent the n-dimensional state variable and m-dimensional parameter variable, respectively. It is assumed that $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium point of the system for any real values of $\boldsymbol{\mu}$, i.e., $\boldsymbol{f}(\boldsymbol{0},\boldsymbol{\mu}) \equiv \boldsymbol{0}$. Note that $L\boldsymbol{x} \triangleq \boldsymbol{v}_1(\boldsymbol{x})$ represents the linear part and L is the Jacobian matrix evaluated at the critical point $(\boldsymbol{x},\boldsymbol{\mu}) = (\boldsymbol{0},\boldsymbol{0})$. It is further assumed that all eigenvalues of L have zero real parts and, without loss of generality, L is given in Jordan canonical form. $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\mu})$ denotes the nonlinear terms of the system.

To find the NF of system (12), one may expand the dimension of system (12) from n to $n+m$, by adding the equation $\dot{\boldsymbol{\mu}} = \boldsymbol{0}$ to system (12) to obtain a new system

$$\dot{\boldsymbol{x}} = L\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\mu}), \quad \dot{\boldsymbol{\mu}} = \boldsymbol{0} \ \ \boldsymbol{x} \in \boldsymbol{R}^n, \ \ \boldsymbol{\mu} \in \boldsymbol{R}^m. \tag{13}$$

Then one may assume that the near-identity ST for system (13) is given by

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y},\boldsymbol{v}), \qquad \boldsymbol{\mu} = \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{y},\boldsymbol{v}), \tag{14}$$

and then the normal form is

$$\dot{\boldsymbol{y}} = L\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y},\boldsymbol{v}), \qquad \dot{\boldsymbol{v}} = \boldsymbol{0}. \tag{15}$$

Now we shall use Eqs. (13)–(15) to consider the computation of NF. Assume that the nonlinear function $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\mu})$ is analytic with respect to $\boldsymbol{x}$ and $\boldsymbol{\mu}$, and thus we may expand $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\mu})$, $\boldsymbol{h}(\boldsymbol{y},\boldsymbol{v})$, $\boldsymbol{p}(\boldsymbol{y},\boldsymbol{v})$ and $\boldsymbol{g}(\boldsymbol{y},\boldsymbol{v})$ as

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\mu}) &= \boldsymbol{f}_2(\boldsymbol{x},\boldsymbol{\mu}) + \boldsymbol{f}_3(\boldsymbol{x},\boldsymbol{\mu}) + \cdots + \boldsymbol{f}_k(\boldsymbol{x},\boldsymbol{\mu}) + \cdots, \\
\boldsymbol{h}(\boldsymbol{y},\boldsymbol{v}) &= \boldsymbol{h}_2(\boldsymbol{y},\boldsymbol{v}) + \boldsymbol{h}_3(\boldsymbol{y},\boldsymbol{v}) + \cdots + \boldsymbol{h}_k(\boldsymbol{y},\boldsymbol{v}) + \cdots, \\
\boldsymbol{p}(\boldsymbol{y},\boldsymbol{v}) &= \boldsymbol{p}_2(\boldsymbol{y},\boldsymbol{v}) + \boldsymbol{p}_3(\boldsymbol{y},\boldsymbol{v}) + \cdots + \boldsymbol{p}_k(\boldsymbol{y},\boldsymbol{v}) + \cdots, \\
\boldsymbol{g}(\boldsymbol{y},\boldsymbol{v}) &= \boldsymbol{g}_2(\boldsymbol{y},\boldsymbol{v}) + \boldsymbol{g}_3(\boldsymbol{y},\boldsymbol{v}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y},\boldsymbol{v}) + \cdots,
\end{aligned} \tag{16}$$

where $\boldsymbol{f}_k(\boldsymbol{x},\boldsymbol{\mu})$, $\boldsymbol{h}_k(\boldsymbol{y},\boldsymbol{v})$, $\boldsymbol{p}_k(\boldsymbol{y},\boldsymbol{v})$ and $\boldsymbol{g}_k(\boldsymbol{y},\boldsymbol{v})$ are the kth-degree vector homogeneous polynomials in their arguments.

To show the basic idea of the NF theory, we rewrite Eqs. (13)–(15) as follows:

$$\dot{\tilde{\boldsymbol{x}}} = \widetilde{L}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}), \quad \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{y}} + \tilde{\boldsymbol{h}}(\tilde{\boldsymbol{y}}), \qquad \dot{\tilde{\boldsymbol{y}}} = \widetilde{L}\tilde{\boldsymbol{y}} + \tilde{\boldsymbol{g}}(\tilde{\boldsymbol{y}}), \tag{17}$$

where

$$\tilde{\boldsymbol{x}} = (\boldsymbol{x},\boldsymbol{\mu})^{\mathrm{T}}, \quad \tilde{\boldsymbol{y}} = (\boldsymbol{y},\boldsymbol{v})^{\mathrm{T}}, \quad \tilde{\boldsymbol{h}} = (\boldsymbol{h},\boldsymbol{p})^{\mathrm{T}}, \tag{18}$$

and $\tilde{\boldsymbol{f}} = (\boldsymbol{f},\boldsymbol{0})^{\mathrm{T}}$, $\tilde{\boldsymbol{g}} = (\boldsymbol{g},\boldsymbol{0})^{\mathrm{T}}$ and $\widetilde{L} = \mathrm{diag}[L\ 0]$. Then the basic idea of NF theory is to find a near-identity NT: $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{y}} + \tilde{\boldsymbol{h}}(\tilde{\boldsymbol{y}})$ such that the resulting system $\dot{\tilde{\boldsymbol{y}}} = \widetilde{L}\tilde{\boldsymbol{y}} + \tilde{\boldsymbol{g}}(\tilde{\boldsymbol{y}})$ becomes as simple as possible.

According to Takens normal form theory [33], we may first define an operator as follows:

$$L_k : \mathscr{H}_k \mapsto \mathscr{H}_k, \quad U_k \in \mathscr{H}_k \mapsto L_k(U_k) = [U_k, \tilde{\boldsymbol{v}}_1] \in \mathscr{H}_k,$$

where $\tilde{\boldsymbol{v}}_1 = (\boldsymbol{v}_1,\boldsymbol{0})^{\mathrm{T}}$, $\mathscr{H}_k$ denotes a linear vector space consisting of the kth-degree homogeneous vector polynomials. The operator $[U_k, \tilde{\boldsymbol{v}}_1]$ is called Lie bracket, defined by

$$[U_k, \tilde{\boldsymbol{v}}_1] = \widetilde{L}U_k - DU_k\tilde{\boldsymbol{v}}_1. \tag{19}$$

Next, define the space $\mathscr{R}_k$ as the range of L_k, and $\mathscr{K}_k$ as the complementary space of $\mathscr{R}_k$. Thus, $\mathscr{H}_k = \mathscr{R}_k \oplus \mathscr{K}_k$, and one can then choose bases for $\mathscr{R}_k$ and $\mathscr{K}_k$. Consequently, a vector homogeneous polynomial $\tilde{\boldsymbol{f}}_k \in \mathscr{H}_k$ can be split into two parts: one of them can be spanned by the basis of $\mathscr{R}_k$ and the other by that of $\mathscr{K}_k$. Normal form theory shows that the part belonging to $\mathscr{R}_k$ can be eliminated while the part belonging to $\mathscr{K}_k$ must be retained in the normal form. In fact, the "form" of the NF $\boldsymbol{g}_k$ depends upon the basis of the complementary space $\mathscr{K}_k$, which is induced by the linear vector $\tilde{\boldsymbol{v}}_1$.

With the equations given in (17), one can find linear algebraic equations order by order, which are used to solve for the coefficients of the NF and NT. It has been noted that obtaining the exact kth order algebraic equation from Eq. (14) is not an easy task and very time consuming. Most of existing symbolic computation methods cannot handle the computation efficiently. Recently, an efficient approach has been developed for computing the kth order algebraic equation which only involves the kth order terms [29,30].

The explicit recursive formula for computing $\boldsymbol{g}_k$, $\boldsymbol{h}_k$ and $\boldsymbol{p}_k$ can be derived from Eqs. (13)–(15) as follows. Differentiating the first equation of (14) yields $\dot{\boldsymbol{x}} = \dot{\boldsymbol{y}} + D\boldsymbol{h}_y(\boldsymbol{y}, \boldsymbol{v})\dot{\boldsymbol{y}} = (I + D\boldsymbol{h})\dot{\boldsymbol{y}}$ which in turn results in

$$\begin{aligned}\sum_{i=2}^{\infty} \boldsymbol{g}_i(\boldsymbol{y},\boldsymbol{v}) &= \sum_{i=2}^{\infty} \boldsymbol{f}_i(\boldsymbol{y},\boldsymbol{v}) + L\boldsymbol{h}(\boldsymbol{y},\boldsymbol{v}) - D\boldsymbol{h}(\boldsymbol{y},\boldsymbol{v})L\boldsymbol{y} - D\boldsymbol{h}(\boldsymbol{y},\boldsymbol{v})\sum_{i=2}^{\infty} \boldsymbol{g}_i(\boldsymbol{y},\boldsymbol{v}) \\ &\quad + \sum_{i=2}^{\infty} D\boldsymbol{f}_i(\boldsymbol{y},\boldsymbol{v})\tilde{\boldsymbol{h}}(\tilde{\boldsymbol{y}}) + \frac{1}{2!}\sum_{i=2}^{\infty} D^2\boldsymbol{f}_i(\boldsymbol{y},\boldsymbol{v})\tilde{\boldsymbol{h}}^2(\tilde{\boldsymbol{y}}) + \frac{1}{3!}\sum_{i=3}^{\infty} D^3\boldsymbol{f}_i(\boldsymbol{y},\boldsymbol{v})\tilde{\boldsymbol{h}}^3(\tilde{\boldsymbol{y}}) + \cdots \\ &= \sum_{i=2}^{\infty} \boldsymbol{f}_i(\boldsymbol{y},\boldsymbol{v}) + \sum_{i=2}^{\infty} [\boldsymbol{h}_i(\boldsymbol{y},\boldsymbol{v}), \boldsymbol{v}_1(\boldsymbol{y})] + \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} \{D\boldsymbol{f}_i(\boldsymbol{y},\boldsymbol{v})\tilde{\boldsymbol{h}}_j(\tilde{\boldsymbol{y}}) - D\boldsymbol{h}_j(\boldsymbol{y},\boldsymbol{v})\boldsymbol{g}_i(\boldsymbol{y})\} \\ &\quad + \sum_{i=2}^{\infty} \frac{1}{i!}\sum_{j=i}^{\infty} D^i\boldsymbol{f}_j(\boldsymbol{y},\boldsymbol{v})\left\{\sum_{l=2}^{\infty} \tilde{\boldsymbol{h}}_l(\tilde{\boldsymbol{y}})\right\}^i. \end{aligned} \tag{20}$$

Then taking the kth order terms from the above equation, together with the equation obtained from the second equation of (14) yields the following result.

Theorem 1. *The recursive formula for computing the coefficients of the normal form and associated nonlinear transformation is given by*

$$\boldsymbol{g}_k = \boldsymbol{f}_k + [\boldsymbol{h}_k, \boldsymbol{v}_1] + \sum_{i=2}^{k-1}(D\boldsymbol{f}_i\tilde{\boldsymbol{h}}_{k-i+1} - D\boldsymbol{h}_{k-i+1}\boldsymbol{g}_i) + \sum_{i=2}^{[k/2]}\frac{1}{i!}\sum_{j=i}^{k-i} D^i\boldsymbol{f}_j \sum_{\substack{l_1+l_2+\cdots+l_i=k-(j-i)\\ 2\leqslant l_1,l_2,\cdots,l_i\leqslant k+2-(i+j)}} \tilde{\boldsymbol{h}}_{l_1}\tilde{\boldsymbol{h}}_{l_2}\cdots\tilde{\boldsymbol{h}}_{l_i} \tag{21}$$

for $k = 2, 3, \ldots,$ *and*

$$\boldsymbol{0} = \sum_{j=1}^{k} D\boldsymbol{p}_{k-j+1}\boldsymbol{g}_j \tag{22}$$

for $k = 1, 2, \ldots,$ *where* $[\cdot,\cdot]$ *denotes the Lie bracket.*

The proof can follow a similar approach of [31] and is omitted here for briefly. Note in Eqs. (21) and (22) that the variables $\boldsymbol{y}$ and $\boldsymbol{v}$ in $\boldsymbol{g}_j$, $\boldsymbol{f}_j$, $\boldsymbol{h}_j$ and $\boldsymbol{p}_j$ have been dropped for simplicity, and $\tilde{\boldsymbol{h}}$ is defined in Eq. (18). $\boldsymbol{f}_j(\boldsymbol{y}, \boldsymbol{v})$, $\boldsymbol{g}_j(\boldsymbol{y}, \boldsymbol{v})$, $\boldsymbol{g}_j(\boldsymbol{h}, \boldsymbol{v})$ and $\boldsymbol{p}_j(\boldsymbol{y}, \boldsymbol{\mu})$ are all jth-degree vector homogeneous polynomials in their arguments. The notation $D^i\boldsymbol{f}_j\tilde{\boldsymbol{h}}_{l_1}\tilde{\boldsymbol{h}}_{l_2}\cdots\tilde{\boldsymbol{h}}_{l_i}$ denotes the ith order terms of the Taylor expansion of the jth-degree vector homogeneous polynomial $\boldsymbol{f}_j(\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{v}), \boldsymbol{v}+ \boldsymbol{p}(\boldsymbol{y}, \boldsymbol{v})) \equiv \boldsymbol{f}_j(\tilde{\boldsymbol{y}} + \tilde{\boldsymbol{h}}(\tilde{\boldsymbol{y}}))$ about $\tilde{\boldsymbol{y}}$. More precisely,

$$D^i\boldsymbol{f}_j(\tilde{\boldsymbol{y}} + \tilde{\boldsymbol{h}}(\tilde{\boldsymbol{y}})) = D(D(\cdots D((D\boldsymbol{f}_j)\tilde{\boldsymbol{h}}_{l_1})\tilde{\boldsymbol{h}}_{l_2})\cdots\tilde{\boldsymbol{h}}_{l_i-1})\tilde{\boldsymbol{h}}_{l_i}, \tag{23}$$

where each differential operator D affects only function $\boldsymbol{f}_j$, not $\tilde{\boldsymbol{h}}_{l_m}$ (i.e., $\tilde{\boldsymbol{h}}_{l_m}$ is treated as a constant vector in the process of the differentiation), and thus $i \leqslant j$.

3. The NF computation

In this section, we shall use the state transformation (ST) defined in the previous section to compute the NF with unfolding for three singularities: Single zero, Hopf singularity and a double zero. We only consider "function" unfolding since it includes the "linear" unfolding. For simplicity, we assume that the system is described on center manifold. It should be emphasized that in the following NF computation, we use the kth order NT to only solve for the kth order algebraic equation. If the coefficients of lower order nonlinear transformations are allowed to solve higher order algebraic equations, then we call it the SNF computation, which will be discussed in the next section. Eq. (21) will be applied here to compute the normal form.

3.1. Single zero

For this case, $L = 0$ and system (12) becomes

$$\dot{x} = \sum_{j=1}^{\infty} a_{1j} x \mu^j + \sum_{j=0}^{\infty} a_{2j} x^2 \mu^j + \sum_{j=0}^{\infty} a_{3j} x^3 \mu^j + \cdots, \tag{24}$$

and the near-identity NTs are given by

$$\begin{aligned} x &= y + \sum_{j=1}^{\infty} h_{1j} y v^j + \sum_{j=0}^{\infty} h_{2j} y^2 v^j + \sum_{j=0}^{\infty} h_{3j} y^3 v^j + \cdots, \\ \mu &= v + \sum_{j=1}^{\infty} p_{1j} y v^j + \sum_{j=0}^{\infty} p_{2j} y^2 v^j + \sum_{j=0}^{\infty} p_{3j} y^3 v^j + \cdots, \end{aligned} \tag{25}$$

which are slightly different from those given in Eq. (16).

Consider the generic case: $a_{11} \neq 0$ and $a_{20} \neq 0$. We want to determine the NT such that the NF is given in the form of

$$\dot{y} = L_1 v y + g_2 y^2 + g_3 y^3 + g_4 y^4 + \cdots, \tag{26}$$

where $L_1 v y$ denotes the universal unfolding for the generic case. The NF given by Eq. (26) is based on the fact that the NF of system (24) without the parameter v is given by $\dot{x} = a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + \cdots$ In other words, in the final NF all the terms involving μ, except for the universal unfolding $L_1 v y$, should be removed.

It is straightforward to use Eqs. (21) and (22) to determine the coefficients of the near-identity transformations. When $k = 1$, Eq. (22) yields $Dvg_1 = 0$ since $Dv = 0$ and $g_1 = Ly = 0$, and so (22) is automatically satisfied.

When $k = 2$, Eq. (25) results in $(2p_{20}y + p_{11}v)g_1 + (Dv)g_2y^2 = 0$ which is again automatically satisfied, while Eq. (24) gives the equation: $L_1yv + g_2y^2 - a_{11}yv - a_{20}y^2 = 0$, which in turn results in $L_1 = a_{11}$ and $g_2 = a_{20}$, as expected. This indicates that no second order terms can be eliminated.

Similarly, we can use Eqs. (21) and (22) to find the third order algebraic equations

$$\begin{aligned} &(g_3 - a_{30} - a_{11}p_{20})y^3 + (a_{20}h_{11} + a_{11}h_{20} - a_{21} - a_{11}p_{11})y^2 v - (a_{11}p_{02} + a_{21})yv^2 = 0, \\ &2p_{20}y^3 + (2a_{11}p_{20} + a_{20}p_{11})y^2 v + a_{11}p_{11}yv^2 = 0. \end{aligned} \tag{27}$$

The second equation of (27) gives $p_{20} = p_{11} = 0$, and then the first equation results in

$$g_3 = a_{30}, \quad p_{02} = \frac{a_{21}}{a_{11}} \quad \text{and} \quad a_{20}h_{11} + a_{11}h_{20} = a_{21}. \tag{28}$$

Eq. (28) clearly indicates that one has to use one of the second-order transformation coefficient h_{20} or h_{11} to solve the equation. Therefore, for the single zero singularity, the CNF – which uses only the kth order NT coefficients to eliminate the kth order nonlinear terms of the system – contains the the third-order term y^2v (if $a_{21} \neq 0$). It is easy to show that more such terms involving the parameter v appear in higher order terms. In fact, if the linear part is null, i.e., $\mathbf{v}_1 = \mathbf{0}$ (or $L = 0$), then the CNF approach cannot be used to simplify the system. However, if lower order nonlinear transformation coefficients are allowed to solve higher order equations, then it leads to the SNF computation. For example, in this case, one can use the second-order coefficient h_{20} or h_{11} to remove the third-order term y^2v. This will be discussed in more detail in the next section.

The above results are obtained based on the generic conditions: $a_{11} \neq 0$ and $a_{20} \neq 0$. For other non-generic cases, such as $a_{11} \neq 0$, but $a_{20} = 0$, $a_{30} \neq 0$, etc., we may follow the similar procedure to show that the above conclusion is still true.

3.2. Hopf singularity

For Hopf singularity, the L may be assumed in the real form $L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and the two-dimensional system can be assumed as

$$\begin{aligned} \dot{x}_1 &= x_2 + \sum_{k=1}^{\infty} \mu^k (a_{110k}x_1 + a_{101k}x_2) + \sum_{k=0}^{\infty} \mu^k \sum_{\substack{i+j=2 \\ i,j \geqslant 0}}^{\infty} a_{1ijk} x_1^i x_2^j, \\ \dot{x}_2 &= -x_1 + \sum_{k=1}^{\infty} \mu^k (a_{210k}x_1 + a_{201k}x_2) + \sum_{k=0}^{\infty} \mu^k \sum_{\substack{i+j=2 \\ i,j \geqslant 0}}^{\infty} a_{2ijk} x_1^i x_2^j. \end{aligned} \tag{29}$$

We first prove that for Hopf bifurcation the general near-identity transformation (8) is reduced to the transformation (11), by using Eq. (22) in which $p_1(\boldsymbol{y}, v) = v$ and $g_1(\boldsymbol{y}) = (y_2, -y_1)^{\mathrm{T}}$. Therefore, for $k = 1$, $Dp_1(\boldsymbol{y}, v)g_1 = 0$, is automatically satisfied, since the operator D is with respect to $\boldsymbol{y}$. When $k = 2$,

$$Dp_2(\boldsymbol{y}\ , v)g_1 + Dp_1(\boldsymbol{y}, v)g_2 = Dp_2(\boldsymbol{y}, v)g_1 = 0 \ \Rightarrow\ Dp_2(\boldsymbol{y}, v) = 0.$$

Then, for $k = 3$, one has

$$Dp_3(\boldsymbol{y}\ , v)g_1 + Dp_2(\boldsymbol{y}, v)g_2 + Dp_1(\boldsymbol{y}, v)g_3 = Dp_3(\boldsymbol{y}, v)g_1 = 0 \ \Rightarrow\ Dp_3(\boldsymbol{y}, v) = 0.$$

It is easy to apply the method of mathematical induction to show that $Dp_j(\boldsymbol{y}, v) = 0$ for $j = 1, 2, 3, \ldots$ which implies that $p(\boldsymbol{y}, v) = \sum_{j=1}^{\infty} p_j(\boldsymbol{y}, v) = p(v)$, independent of $\boldsymbol{y}$. Hence, for Hopf bifurcation, we only need Eq. (21), and the near-identity NTs can be expressed as

$$\begin{aligned}
x_1 &= y_1 + \sum_{k=1}^{\infty} v^k (h_{110k}x_1 + h_{101k}x_2) + \sum_{k=0}^{\infty} v^k \sum_{\substack{i+j=2 \\ i,j\geqslant 0}}^{\infty} h_{1ijk} y_1^i y_2^j, \\
x_2 &= y_2 + \sum_{k=1}^{\infty} v^k (h_{210k}x_1 + h_{201k}x_2) + \sum_{k=0}^{\infty} v^k \sum_{\substack{i+j=2 \\ i,j\geqslant 0}}^{\infty} h_{2ijk} y_1^i y_2^j, \\
v &= v + \sum_{j=2}^{\infty} p_j v^j.
\end{aligned} \tag{30}$$

Further, with normal form theory, the expected NF may be assumed in the form of

$$\begin{aligned}
\dot{y}_1 &= y_2 + \beta_1 v y_1 + \beta_2 v y_2 + b_{13}(y_1^2 + y_2^2)y_1 + b_{15}(y_1^2 + y_2^2)^2 y_1 + b_{17}(y_1^2 + y_2^2)^3 y_1 + \cdots + b_{23}(y_1^2 + y_2^2)y_2 \\
&\quad + b_{25}(y_1^2 + y_2^2)^2 y_2 + b_{27}(y_1^2 + y_2^2)^3 y_2 + \cdots, \\
\dot{y}_2 &= -y_1 + \beta_1 v y_2 - \beta_2 v y_1 + b_{13}(y_1^2 + y_2^2)y_2 + b_{15}(y_1^2 + y_2^2)^2 y_2 + b_{17}(y_1^2 + y_2^2)^3 y_2 + \cdots - b_{23}(y_1^2 + y_2^2)y_1 \\
&\quad - b_{25}(y_1^2 + y_2^2)^2 y_1 - b_{27}(y_1^2 + y_2^2)^3 y_1 - \cdots,
\end{aligned} \tag{31}$$

where $\beta_1 v$ and $\beta_2 v$ denote the universal unfolding.

Now we apply formula (21) to find the algebraic equations for solving the NT coefficients. When $k = 2$, we obtain the following 10 algebraic equations:

$$\begin{aligned}
&-h_{2200} - h_{1110} - a_{1200} = 0, \\
&-h_{2110} + 2h_{1200} - 2h_{1020} - a_{1110} = 0, \\
&-h_{2020} + h_{1110} - a_{1020} = 0, \\
&h_{1200} - h_{2110} - a_{2200} = 0, \\
&h_{1110} + 2h_{2200} - 2h_{2020} - a_{2110} = 0, \\
&h_{1020} + h_{2110} - a_{2020} = 0, \\
&\beta_1 - h_{2101} - h_{1011} - a_{1101} = 0, \\
&\beta_2 - h_{2011} + h_{1101} - a_{1011} = 0, \\
&-\beta_2 + h_{1101} - h_{2011} - a_{2101} = 0, \\
&\beta_1 + h_{1011} + h_{2101} - a_{2011} = 0.
\end{aligned} \tag{32}$$

It is easy to show that the six $\boldsymbol{h}$ coefficients h_{1200}, h_{1110}, h_{1020}, h_{2200}, h_{2110} and h_{2020} can be used to uniquely solve the first six equations of (32) which are derived from the terms without the parameter v. From the last four equations of (32) which are obtained from the terms involving v, one finds $\beta_1 = \frac{1}{2}(a_{1101} + a_{2011})$, $\beta_2 = \frac{1}{2}(a_{1011} - a_{2101})$, and only two of the four $\boldsymbol{h}$ coefficients in the last four equations are independent, and thus two of the four $\boldsymbol{h}$ coefficients can be set zero. This shows that all the second order terms of the original system (29) can be eliminated, as expected. This is the same as the computation for the NF without unfolding.

From the third order equations, similarly we can apply Eq. (21) to find 18 algebraic equations, among them eight equations are obtained from the non-v coefficients and 10 equations are obtained from the v coefficients. They can be put in the following matrix forms:

$$[M_1|N_1]\begin{pmatrix} h_{2300} \\ h_{2210} \\ h_{2120} \\ h_{2030} \\ h_{1300} \\ h_{1210} \\ h_{1120} \\ h_{1030} \\ --- \\ b_{13} \\ b_{23} \end{pmatrix} = \begin{pmatrix} a_{1300}+A_{1300} \\ a_{1220}+A_{1220} \\ a_{1120}+A_{1120} \\ a_{1030}+A_{1030} \\ a_{2300}+A_{2300} \\ a_{2210}+A_{2210} \\ a_{2120}+A_{2120} \\ a_{2030}+A_{2030} \end{pmatrix}, \quad [M_2]\begin{pmatrix} h_{2201} \\ h_{2021} \\ h_{2111} \\ h_{2102} \\ h_{2012} \\ h_{1201} \\ h_{1021} \\ h_{1111} \\ h_{1102} \\ h_{1012} \end{pmatrix} = \begin{pmatrix} a_{1201}+A_{1201} \\ a_{1021}+A_{1021} \\ a_{1111}+A_{1111} \\ a_{1102}+A_{1102} \\ a_{1012}+A_{1012} \\ a_{2201}+A_{2201} \\ a_{2021}+A_{2021} \\ a_{2111}+A_{2111} \\ a_{2102}+A_{2102} \\ a_{2012}+A_{2012} \end{pmatrix}, \tag{33}$$

where A_{ijk0}'s are given explicitly in the second-order known coefficients a_{ijkl}'s, and

$$M_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the rank of M_1 is 6 and the rank of N_1 is 2, b_{13} and b_{23} must be used with the $\boldsymbol{h}$ coefficients to solve the first equation of (33). This indicates that the terms associated with the coefficients b_{13} and b_{23} must be retained in the normal form, as expected from the CNF theory. It is easy to see that the fourth, fifth, and the last two rows of M_2 are not independent, indicating that the rank of M_2 is 8. Thus, the second equation of (33) cannot be solved completely, and the two terms y_1v^2 and y_2v^2 must be retained in the normal form.

For the fourth order equations, 10 algebraic equations can be obtained from the non-v coefficients and 18 equations from the v coefficients. All the 10 equations can be solved using the h coefficients, again like the CNF. The 18 equations can be written as $M_3\boldsymbol{h}_4 = \boldsymbol{A}_4$ which has the similar form of (33), where $\boldsymbol{h}_4$ is a 18-dimensional vector containing all the fourth order h coefficients whose last index of the subscripts are non-zero, $\boldsymbol{A}_4$ is a 18-dimensional vector expressed in the known a coefficients, and M_3 is 18×18 matrix, given by

$$M_3 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{34}$$

from which we observe that similar to M_2, its eighth, ninth and the last two rows are not independent, and thus the two terms $y_1 v^3$ and $y_2 v^3$ must be retained in the normal form. Moreover, it can be shown that the rank of M_3 is 14, which implies that additional two terms must be retained in the normal form. A further consideration reveals that four possibilities can be chosen: $(y_1^3 v, y_1^2 y_2 v)$, $(y_1^3 v, y_2^3 v)$, $(y_1 y_2^2 v, y_1^2 y_2 v)$ and $(y_1 y_2^2 v, y_2^3 v)$. Therefore, the NF for Hopf bifurcation of system (28) cannot be obtained in the form (31) using only near-identity transformations.

It should be noted that for Hopf bifurcation it is assumed that $\beta_1 \neq 0$ and $b_{13} \neq 0$, where b_{13} is given explicitly by

$$b_{13} = \frac{1}{8}[3(a_{1300} + a_{2030}) + a_{2210} + a_{1120} + 2(a_{1200}a_{2200} - a_{1020}a_{2020}) + a_{2110}(a_{2020} + a_{2200}) - a_{1110}(a_{1020} + a_{1200})]. \tag{35}$$

If either one or both of the two conditions are not satisfied, then we have a generalized Hopf bifurcation. For such a case, we may follow the similar procedure to prove that the conclusion is still true.

3.3. A double zero

We now turn to a double zero singularity, usually called Takens–Bogdanov singularity. For this case, we can still use Eq. (28) to describe the system but now the linear part should be $\boldsymbol{L x} = (x_2, 0)^{\mathrm{T}}$, and the near-identity NT for the generic case can be assumed as

$$\begin{aligned} x_1 &= y_1 + \sum_{\substack{i+j+k+l=2 \\ i,j,k,l \geqslant 0}} h_{1ijkl} y_1^i y_2^j v_1^k v_2^l \\ x_1 &= y_1 + \sum_{\substack{i+j+k+l=2 \\ i,j,k,l \geqslant 0}} h_{2ijkl} y_1^i y_2^j v_1^k v_2^l \\ \mu_1 &= v_1 + \sum_{\substack{i+j=2 \\ i,j \geqslant 0}}^{\infty} p_{1ij} v_1^i v_2^j, \qquad \mu_2 = v_2 + \sum_{\substack{i+j=2 \\ i,j \geqslant 0}}^{\infty} p_{2ij} v_1^i v_2^j, \end{aligned} \tag{36}$$

The normal form, for the generic case, may be assumed in the form of

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = \alpha_1 v_1 y_1 + \alpha_2 v_2 y_2 + \sum_{k=2}^{\infty} (b_{1k} y_1^k + b_{2k} y_1^{k-1} y_2), \tag{37}$$

where $\alpha_1 v_1 y_1 + \alpha_2 v_2 y_2$ denotes the universal unfolding. It is assumed that $b_{12} b_{22} \neq 0$ for the generic case.

From the second order equations, we can use Eq. (21) to find the following solutions:

$$\begin{aligned}
&h_{2200} = -a_{1200}, \quad h_{2020} = -a_{1020}, \quad h_{2110} = a_{2020}, \\
&h_{1200} = \frac{1}{2}(a_{2020} + a_{1110}), \quad b_{12} = a_{2200}, \quad b_{22} = 2a_{1200} + a_{2110}, \\
&h_{2101} = -a_{1101}, \quad h_{2011} = -a_{1011}, \quad \alpha_{11} = a_{2101}, \quad \alpha_{21} = a_{1101} + a_{2011},
\end{aligned} \tag{38}$$

and h_{1020}, h_{1110} h_{1101} and h_{1011} can be set zero since they are not used at this order.

From the third order equations we can find a total of 18 algebraic equations, 10 of them are obtained from the v coefficients while the other eight from the non-v coefficients. The non-v equations are given by

$$\begin{aligned}
&-h_{2300} = a_{1300} + a_{1200}a_{2020}, \\
&-h_{2210} + 3h_{1300} = a_{1210} + \frac{1}{2}a_{1110}^2 + \frac{3}{2}a_{1110}a_{2020} - 2a_{1020}a_{1200}, \\
&-h_{2120} + 2h_{1210} = a_{1120} + 2a_{1020}a_{2020} - a_{1110}a_{1020}, \\
&-h_{2030} + h_{1120} = a_{1030} - 2a_{1020}^2, \\
&b_{13} = a_{2300} + a_{1110}a_{2200} - a_{2110}a_{1200}, \\
&b_{23} + 3h_{2300} = a_{2210} + \frac{1}{2}a_{2110}(a_{1110} + a_{2020}) + 2a_{1020}a_{2200} - 4a_{1200}a_{2020}, \\
&2h_{2210} = a_{2120} + 2a_{2020}^2 + a_{2110}a_{1020} + 4a_{1020}a_{1200}, \\
&h_{2120} = a_{2030} - a_{1020}a_{2020}.
\end{aligned} \tag{39}$$

It is easy to see from Eq. (39) that the eight equations can be solved using the two b coefficients and the seven h coefficients, with one of the two coefficients h_{2030} and h_{1120} being set zero.

The 10 v equations are

$$\begin{aligned}
&-h_{2201} = a_{1201} - a_{1011}a_{1200} + \frac{1}{2}a_{1101}(a_{2020} - a_{1110}), \\
&-h_{2021} + h_{1111} = a_{1021} - 3a_{1020}a_{1011}, \\
&-h_{2111} + 2h_{1201} = a_{1111} + a_{1011}(a_{2020} - a_{1110}) - 2a_{1020}a_{1101}, \\
&-h_{2102} = a_{1102} - a_{1101}a_{1011}, \\
&-h_{2012} + h_{1102} = a_{1012} - a_{1011}^2, \\
&0 = a_{2201} + \frac{1}{2}a_{2101}(a_{1110} - a_{2020}) + a_{1011}a_{2200} - a_{2011}a_{1200} - a_{1101}a_{2110}, \\
&h_{2111} = a_{2021} + a_{1020}(a_{2011} + 2a_{1101}) - 2a_{1011}a_{2020}, \\
&2h_{2201} = a_{2111} + 2(a_{1020}a_{2101} + a_{1200}a_{1011}) - 3a_{1101}a_{2020}, \\
&\alpha_{12} = a_{2102} + a_{1011}a_{2101} - a_{1101}a_{2011}, \\
&\alpha_{22} + h_{2102} = a_{2012} + a_{1011}a_{1101},
\end{aligned} \tag{40}$$

which clearly show that even with the two extra coefficients α_{12} and α_{22}, there are still two equations (four equations if using the "linear" unfolding) which are not solvable. Hence, the NF for Takens–Bogdanov singularity cannot be obtained using near-identity transformation. Similarly, we can follow the same procedure to prove that this is also true for non-generic cases.

Finally summarizing the results obtained in this section gives the following theorem.

Theorem 2. *The normal form with unfolding for system (9) cannot be obtained by only using near-identity transformations, at least for the three cases: Single zero, Hopf bifurcation and Takens–Bogdanov singularity.*

4. The SNF computation

In the previous sections, we have shown that the normal forms with unfolding (perturbation parameters) cannot be obtained by only using near-identity NT. Then questions arise: How to compute the normal form with unfolding? Or more specifically, what kind of transformations need to be used for computing the normal form with unfolding? These questions are related to further reduction of normal forms.

Actually, particular attention has been paid to further reduction of normal forms in the past two decades (e.g., see [12,15–24,26,27,30]). Such a reduced form is usually called ''unique normal form'', ''minimal normal form'' or ''the simplest normal form'' (SNF). The fundamental difference between the SNF and CNF is that the CNF method uses the kth order nonlinear transformation to possibly remove only the kth order nonlinear terms of the system, while the SNF approach applies not only the kth order, but also lower order ($<k$) NTs to eliminate the kth order nonlinear terms. However, the research work on the SNF computation has so far been mainly restricted to the systems which do not contain perturbation parameters (unfolding), except for two simple cases – single zero singularity [28] and Hopf bifurcation [31]. Normal forms with unfolding for the two cases have been explicitly calculated from the original system. For illustration, first consider the normal form of Hopf bifurcation without perturbation parameters for a numerical example with randomly choosing coefficients up to seventh order

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1^2 + \frac{1}{2}x_1x_2 + 2x_2^2 + 2x_1^3 + \frac{1}{7}x_1^2x_2 + \frac{5}{3}x_1x_2^2 + \frac{1}{2}x_2^3 + 5x_1^4 + \frac{1}{3}x_1^3x_2 - 15x_1^2x_2^2 + \frac{7}{3}x_1x_2^3 \\
&\quad + 2x_2^4 - 2x_1^5 + 5x_1^4x_2 + \frac{1}{4}x_1^3x_2^2 + x_1^2x_2^3 + \frac{7}{4}x_1x_2^4 + 20x_2^5 + \frac{1}{2}x_1^6 + \frac{2}{5}x_1^5x_2 - x_1^4x_2^2 + \frac{1}{3}x_1^3x_2^3 \\
&\quad + 2x_1^2x_2^4 + \frac{7}{5}x_1x_2^5 - 2x_2^6 + 2x_1^7 + x_1^6x_2 - 5x_1^5x_2^2 + \frac{1}{10}x_1^4x_2^3 + 3x_1^3x_2^4 + \frac{7}{2}x_1^2x_2^5 + 5x_1x_2^6 + x_2^7, \\
\dot{x}_2 &= -x_1 + 3x_1^2 + \frac{1}{4}x_1x_2 + 5x_2^2 + \frac{2}{5}x_1^3 + 3x_1^2x_2 + 10x_1x_2^2 + \frac{4}{7}x_2^3 + \frac{4}{3}x_1^4 - \frac{2}{3}x_1^3x_2 + 10x_1^2x_2^2 + 3x_1x_2^3 \\
&\quad + x_2^4 + 7x_1^5 - \frac{3}{5}x_1^4x_2 + 7x_1^2x_2^3 + \frac{3}{4}x_1x_2^4 + \frac{1}{8}x_2^5 - 2x_1^6 + 5x_1^5x_2 + \frac{1}{3}x_1^4x_2^2 + 7x_1^3x_2^3 + 4x_1^2x_2^4 \\
&\quad - \frac{1}{5}x_1x_2^5 + 3x_2^6 + x_1^7 + 5x_1^6x_2 + \frac{5}{3}x_1^5x_2^2 + \frac{1}{2}x_1^4x_2^3 - 3x_1^3x_2^4 + 7x_1^2x_2^5 + \frac{5}{8}x_1x_2^6 + 3x_2^7.
\end{aligned} \tag{41}$$

The normal form is obtained up to seventh order as follows:

$$\begin{aligned}
\dot{r} &= \boxed{-\frac{47}{336}r^3 - \frac{25,933,399}{1,354,752}r^5} - \frac{310,460,087,074,273}{249,707,888,640}r^7, \\
\dot{\theta} &= \boxed{1 - \frac{233,651}{13,440}r^2} - \frac{623,574,268,217}{2,167,603,200}r^4 - \frac{136,867,676,028,063,703}{16,647,192,576,000}r^6,
\end{aligned} \tag{42}$$

where the terms marked by the boxes denote the SNF. We can also employ the Maple program given in [34] to find the NT for the CNF, and the Maple program developed in [29] to obtain the NT for the SNF. It has been noted that the two nonlinear transformations have same terms, but most of them have different coefficients. Moreover, the NT for the SNF is more complicated than that of the CNF. Eq. (42) indicates that the SNF has only three terms for Hopf singularity. In fact, it has been shown that the SNF for Hopf singularity has only three terms up to any order [22], while the CNF has infinite terms.

Linear transformation, on the other hand, is usually used to transform a linear system (or the linear part of a nonlinear system) to Jordan canonical form. In the SNF computation, the linear transformation may be also used to eliminate higher order terms. For example, the CNF associated with a triple zero of index two cannot be further reduced if only nonlinear transformation (starting from second order) is used. However, if a linear parametric transformation [35] is first introduced, then two of the second order CNF terms can be removed using the two undetermined linear coefficients. This linear transformation can be in fact included in the general nonlinear transformation (not necessarily in a separate step like discussed above) and the general rule in the SNF computation still applies. In other words, ''using lower order nonlinear transformation to eliminate higher order normal form terms'' may start from linear transformation (in general the computation of the NT for the SNF begins with quadratic terms).

For Hopf and generalized Hopf bifurcations, suppose the CNF is given by Eq. (3). Then the corresponding SNFs have finite terms.

Theorem 3 [22]. *Suppose the general CNF of Hopf and generalized Hopf bifurcations is given by Eq. (3), then the corresponding SNF is given by*

$$\begin{aligned}
\dot{\rho} &= b_{1(2k+1)}\rho^{2k+1} + g_{1(4k+1)}\rho^{4k+1}, \\
\dot{\phi} &= 1 + b_{2(2k+1)}\rho^{2k},
\end{aligned} \tag{43}$$

when $b_{1(2k-1)} = 0$ $(k \geqslant 1)$ *and* $b_{1(2j-1)} = b_{2(2j-1)} = 0$ *for* $1 \leqslant j \leqslant k$; *or*

$$\begin{aligned} \dot{\rho} &= b_{1(2k+1)}\rho^{2k+1} + g_{1(4k+1)}\rho^{4k+1}, \\ \dot{\phi} &= 1 + g_{2(2j-1)}\rho^{2(j-1)} + g_{2(2(j+1)-1)}\rho^{2j} + \cdots + g_{2(2k+1)}\rho^{2k}, \end{aligned} \tag{44}$$

when $b_{13} = \cdots = a_{1(2k-1)} = 0$, $b_{1(2k+1)} \neq 0$ *and* $b_{23} = \cdots = b_{2(2j-3)} = 0$, $b_{2(2j-1)} \neq 0$ *for some* $2 \leqslant j \leqslant k$. *k is an arbitrary positive integer. Note that here the g coefficients are explicitly expressed in terms of b coefficients.*

The Takens–Bogdanov singularity (a double zero) has been considered by many researchers. Baider and Sanders [19] gave a detailed study and classified the normal form into three cases: (I) $\mu < 2v$, (II) $\mu > 2v$ and (III) $\mu = 2v$, where the μ and v are defined by the b coefficients of system (37) (where $\alpha_1(\mu)$ and $\alpha_2(\mu)$ have been assumed zero): $b_{12} = b_{13} = \cdots = b_{1\mu} = 0$ and $b_{22} = b_{23} = \cdots = b_{2v} = 0$. For example, the SNF for the generic case when $\mu = v = 1$ is given by

$$\dot{u}_1 = u_2, \qquad \dot{u}_2 = b_{12}u_1^2 + b_{22}u_1u_2 + g_{13}u_1^3 + \sum_{j=1}^{\infty}(g_{1(3j+1)} + g_{2(3j+1)}u_1)u_1^{3j+1}, \tag{45}$$

if $b_{12}b_{22} \neq 0$.

Baider and Sanders [19] gave a fair detailed study for the first two cases and obtained the "form" of the SNF for most of the sub-cases. Later, case (III) was considered by a number of authors (e.g., see [20,23]) and the "form" of the SNF was also obtained. However, some special sub-cases were still remained unsolved. Moreover, even for a classified case, certain non-algebraic number conditions must be satisfied in order for the algebraic equations to be solvable. Very recently, a technique called "matching pursuit" has been developed to overcome this difficulty [30]. With this approach and the "automatic" Maple program, one can find the SNF of Takens–Bogdanov singularity for any cases without making any algebraic assumptions. For example, consider the following system:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_1y_2 + \frac{229+\sqrt{44,881}}{630}y_1^3 + y_1^2y_2 + y_1^4 + y_1^3y_2 + \frac{2}{3}y_1^5 + y_1^4y_2 + \frac{1}{2}y_1^6 + \frac{1}{2}y_1^5y_2 + 5y_1^7 + 2y_1^6y_2 \\ &\quad + 7y_1^8 + 3y_1^7y_2 + \frac{3}{7}y_1^9 + 11y_1^8y_2 + \frac{2}{9}y_1^{10} + \frac{5}{9}y_1^9y_2 + \frac{1}{7}y_1^{11} + \frac{5}{11}y_1^{10}y_2 + 3y_1^{12} + \frac{2}{3}y_1^{11}y_2. \end{aligned} \tag{46}$$

Executing the computer programs such as those given in [23,26] fails to obtain the SNF since the following condition (using the notation of Eq. (37)):

$$315b_{13}^2 - 229b_{13}b_{22}^2 - 6b_{22}^4 \neq 0 \tag{47}$$

is violated at the eighth order at which a "zero division" problem occurs. However, one can apply the matching pursuit Maple program [30] to find the SNF

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= u_1u_2 + \frac{229+\sqrt{60,001}}{630}u_1^3 + u_1^2u_2 + u_1^4 + \frac{2}{3}u_1^5 + \frac{43\sqrt{60,001}-1790}{9450}u_1^6 + \frac{38,921,872-138,287\sqrt{60,001}}{782,775}u_1^7 \\ &\quad + \frac{290,685,973\sqrt{60,001}-68,546,927,567}{328,765,500}u_1^8 + \frac{3,355,418,332,083,737-13,698,517,799,633\sqrt{60,001}}{6,904,075,500}\boxed{u_1^7u_2} \\ &\quad + \frac{2,663,452,386,309,233,068-10,873,082,633,724,827\sqrt{60,001}}{6,524,351,347,500}u_1^{10} \\ &\quad + \frac{436,651,948,790,906,635,720,110,052,517-1,782,608,491,453,734,639,408,295,583\sqrt{60,001}}{4,435,977,661,838,451,225,000}u_1^{11} \\ &\quad + \frac{7,258,395,195,718,581,514,659,263,443,917\sqrt{60,001}-1,777,951,073,104,100,318,846,081,480,159,243}{3,220,519,782,494,715,589,350,000}u_1^{12}, \end{aligned} \tag{48}$$

where the term $u_1^7u_2$, marked by a box, will not present if the condition (47) is satisfied. In other words, when the condition (47) is not held, the SNF has an extra term.

Other singularities such as Hopf-zero (e.g., see [21,24]) and 1:2 double Hopf (e.g., see [27,36]) have also been studied and partial results are obtained.

Finally we turn to consider the SNF of systems with perturbation parameters. Certainly, one cannot use near-identity NT like the one given in Eq. (15) to find the SNF of such systems. Then what else possible transformations can be used? Recently, two singularities have been considered by using time rescaling and parameter rescaling. For the case of single zero, with the system described by system (23), the results are given as follows:

Theorem 4 [13]. *Suppose system* (24) *has a zero singularity at the equilibrium* $x = 0$ *for any real values of* μ *and* $a_{11} \neq 0$, *and the first non-zero coefficients of* a_{j0}*'s is* a_{k0}, *then the SNF of system* (24) *is given by*

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = a_{11}\mu u + a_{k0}u^k \quad (k \geqslant 2), \tag{49}$$

up to any order. Here both near-identity NT (25) *and the time rescaling*

$$t = [T_0 + T(u,\mu)]\tau = \left[1 + \sum_{i=1}^{\infty} t_{0i}\mu^i + \sum_{i=0}^{\infty} t_{1i}\mu^i u + \sum_{i=0}^{\infty} t_{2i}\mu^i u^2 + \cdots\right]\tau \tag{50}$$

have been used.

Other cases when $a_{11} = 0$ result in degenerate SNF. For example, suppose $a_{11} = a_{12} = 0$, but $a_{13} \neq 0$ and $a_{21} \neq 0$, then the SNF can be found as

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = a_{13}\mu^3 u + a_{21}\mu u^2 + a_{k0}u^k \quad (k \geqslant 2), \tag{51}$$

up to any order.

The computation of the SNF for Hopf bifurcation is more complicated. The main results obtained recently are summarized in the following theorem.

Theorem 5 [31]. *Under the following transformation*

$$\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{h}(\boldsymbol{u}, v) \triangleq \boldsymbol{u} + \boldsymbol{h}_2(\boldsymbol{u}, v) + \boldsymbol{h}_3(\boldsymbol{u}, v) + \cdots + \boldsymbol{h}_k(\boldsymbol{u}, v) + \cdots, \tag{52}$$

where v *is a new scaled parameter via the relation*

$$\mu = v + p(v) \triangleq v + p_2(v) + p_3(v) + \cdots + p_k(v) + \cdots \tag{53}$$

and the time rescaling

$$t = (T_0 + T(\boldsymbol{u}, v))\tau \triangleq (T_0 + T_1(\boldsymbol{u}, v) + T_2(\boldsymbol{u}, v) + \cdots + T_k(\boldsymbol{u}, v) + \cdots)\tau, \tag{54}$$

the SNF of system (29) *for Hopf bifurcation, given in polar coordinates, can be written as*

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}\tau} &= R(\alpha v + b_{13}R^2), \\ \frac{\mathrm{d}\Theta}{\mathrm{d}\tau} &= 1 + \beta v + b_{23}R^2 + b_{25}R^4 + \cdots + b_{2(2k+1)}R^{2k} + \cdots \end{aligned} \tag{55}$$

up to any order, where $b_{13} \neq 0$, *given by Eq.* (35).

It can be observed by comparing Eq. (55) with Eq. (43) (in which taking $k = 1$) that the amplitude equation of the SNF with unfolding is simpler than that of the SNF without parameter. The phase equation of the SNF with unfolding has *infinite* terms while that of the SNF without parameter has only *two* terms. However, the amplitude equation plays the most important role in determining bifurcation solutions and their stability conditions.

Later the SNF for codimension-2 generalized Hopf bifurcation was also obtained.

Theorem 6 [37]. *The SNF for codimension-two generalized Hopf bifurcation is given in polar coordinates*

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}\tau} &= R(\boldsymbol{\alpha}_1\boldsymbol{v} + \boldsymbol{\alpha}_2\boldsymbol{v}R^2 + b_{15}R^4), \\ \frac{\mathrm{d}\Theta}{\mathrm{d}\tau} &= 1 + b_{23}R^2 + b_{25}R^4 + \cdots \end{aligned} \tag{56}$$

up to any order, where $\boldsymbol{v} = (v_1, v_2)^{\mathrm{T}}$, *and* $\boldsymbol{\alpha}_1 = (\alpha_{11}, \alpha_{12})$ *and* $\boldsymbol{\alpha}_2 = (\alpha_{21}, \alpha_{22})$, *representing the coefficients of unfolding terms.*

5. Examples

In this section, three numerical examples and one physical example are given to demonstrate the applications of the theorems obtained in the previous sections. We only consider "function" unfolding since it includes the "linear" case. To demonstrate that the normal form of a vector field with perturbation parameters cannot be obtained by using only near-identity transformations, we only employ the near-identity transformations in the three examples.

5.1. Example 1: single zero

Consider the following one-dimensional equation up to third order terms:

$$\dot{x} = 2\mu x + x^2 + (4x^3 + 5\mu^2 x + 7\mu x^2). \tag{57}$$

We can use Eq. (21) to obtain the near-identity NT, given by

$$x = y - 7\mu y \tag{58}$$

up to fourth order, and the NF in the form of

$$\dot{y} = 2\mu y + y^2 + (4y^3 + 5\mu^2 y). \tag{59}$$

Comparing Eq. (57) with Eq. (59) shows that the NF is reduced by only one term, and that the third order term $\mu^2 y$ cannot be removed using the near-identity NT. Moreover, it is noted from Eq. (58) that many NT coefficients are not used since it has to obey the CNF "rule": The kth order NT coefficients are allowed to only eliminate the kth order terms. This is why conventional NF can be further simplified using lower order NT coefficients to remove higher order nonlinear terms (e.g., see [22]).

5.2. Example 2: Hopf singularity

Suppose the system is described by the following fourth order equations with randomly chosen coefficients:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \left(x_1 + \frac{1}{2}x_2\right)\mu + \left(\frac{2}{3}x_1 + 2x_2\right)\mu^2 - (3x_1 - x_2)\mu^3 + x_1^2 + 3x_1x_2 - 5x_2^2 - 6x_1^3 + \frac{1}{2}x_1^2x_2 + 3x_1x_2^2 \\
&\quad - \frac{1}{5}x_2^3 + 10x_1^4 + 7x_1^3x_2 - x_1^2x_2^2 - \frac{10}{3}x_1x_2^3 - 5x_2^4 + \left(\frac{5}{3}x_1^2 - x_1x_2 + 15x_2^2\right)\mu + (x_1^3 + 7x_1^2x_2 - \frac{7}{4}x_1x_2^2 + 3x_2^3)\mu \\
&\quad - \left(3x_1^2 - \frac{7}{9}x_1x_2 + 12x_2^2\right)\mu^2, \\
\dot{x}_2 &= -x_1 - (x_1 - 2x_2)\mu + \left(\frac{5}{2}x_1 + \frac{1}{2}x_2\right)\mu^2 + \left(\frac{1}{3}x_1 - 5x_2\right)\mu^3 + \frac{2}{3}x_1^2 - x_1x_2 + \frac{2}{5}x_2^2 - \frac{4}{3}x_1^3 + x_1^2x_2 + \frac{8}{5}x_1x_2^2 \\
&\quad - x_2^3 + \frac{1}{7}x_1^4 + 3x_1^3x_2 - \frac{11}{2}x_1^2x_2^2 - x_1x_2^3 - 9x_2^4 - \left(2x_1^2 + \frac{2}{7}x_1x_2 + x_2^2\right)\mu - \left(3x_1^3 + \frac{7}{3}x_1^2x_2 + 4x_1x_2^2 - \frac{1}{10}x_2^3\right)\mu \\
&\quad + \left(\frac{3}{8}x_1^2 - 9x_1x_2 - \frac{12}{7}x_2^2\right)\mu^2.
\end{aligned} \tag{60}$$

Similarly, Eq. (21) can be applied to find the NT (omitted here) and the NF (up to fourth order) in polar coordinates

$$\begin{aligned}
\dot{r} &= r\left\{\left(\frac{3}{2}\mu + \frac{7}{12}\mu^2 - 4\mu^3\right) - \frac{11}{120}r^2 + \left[\frac{1,348,483}{20,160}\sin 4\theta + \frac{567,331}{302,400}(1 - \cos 4\theta)\right]r^2\mu\right\}, \\
\dot{\theta} &= 1 + \left(\frac{3}{4}\mu - \frac{13}{32}\mu^2 + \frac{135}{128}\mu^3\right) - \frac{93,083}{10,800}r^2 + \left[\frac{1,348,483}{20,160}(1 - \cos 4\theta) - \frac{567,331}{302,400}\sin 4\theta\right]r^2\mu,
\end{aligned} \tag{61}$$

where r and θ represent the amplitude and phase of motion, respectively. Eq. (61) clearly shows that using near-identity NT cannot decouple the phase of motion from the amplitude due to the extra terms involved in the NF. However, these extra terms are usually ignored in the application of conventional NF theory. This example again confirms that using near-identity NT is not sufficient to eliminate higher order ($\geqslant 4$ for this case) terms containing the parameter μ.

5.3. Example 3: A double zero

For this case, we again use the system given by Eq. (60) but up to third order terms. Then one can use Eq. (21) to obtain the NF (up to third order)

$$\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -\mu y_1 + \left(3\mu + \frac{7}{6}\mu^2\right)y_2 + \frac{2}{3}y_1^2 + y_1y_2 + \frac{5}{3}y_1^3 - \frac{773}{30}y_1^2y_2 - \left(\frac{119}{30}y_1^2 - \frac{971}{105}y_1y_2\right)\mu.
\end{aligned} \tag{62}$$

Note that the last two terms in the second equation of (62) cannot be eliminated using near-identity NT. It again shows that using near-identity NT is not sufficient for removing higher order ($\geqslant 3$ for this case) terms which involve the perturbation parameters.

5.4. Example 4: Brusselator model

The above three examples seem to give us the impression that if the original system does not contain higher order μ terms, it might be possible to use near-identity NT to find the NF. To show that this is not true, consider the well-known Brusselator model, which is described by the following differential equations:

$$\dot{w}_1 = A - (1+B)w_1 + w_1^2 w_2, \qquad \dot{w}_2 = Bw_1 - w_1^2 w_2, \tag{63}$$

where $A, B > 0$ are parameters. It is easy to find that the system has a unique equilibrium: $w_{1e} = A,\ w_{2e} = \frac{B}{A}$. Evaluating the Jacobian of the system at the equilibrium shows that Hopf bifurcation occurs at the critical point $B = 1 + A^2$. Letting $B = 1 + A^2 + \mu$, where μ is a perturbation parameter, then the characteristic polynomial of the system has a pair of purely imaginary eigenvalues $\lambda = \pm A\mathrm{i}$. Suppose $A = 1$, and then introduce the transformation: $w_1 = w_{1e} + x_1$, $w_2 = w_{2e} - x_1 + x_2$ into Eq. (63) to obtain the new system

$$\dot{x}_1 = x_2 + \mu x_1 + 2x_1x_2 - x_1^3 + x_1^2 x_2, \qquad \dot{x}_2 = -x_1, \tag{64}$$

where only one μ-term, μx_1, is presented. We investigate what NF can be obtained for this special case. According to conventional NF theory, we may first find the NF of system (64) without parameter μ (i.e., setting $\mu = 0$ in Eq. (64)), and then add the "linear" unfolding to obtain the NF given in polar coordinates

$$\dot{r} = r\left(\frac{1}{2}\mu - \frac{3}{8}r^2\right), \qquad \dot{\theta} = 1 - \frac{1}{24}r^2. \tag{65}$$

Eq. (65) clearly indicates that the amplitude of motion is decoupled from the phase of motion, as expected if using the conventional NF theory.

However, if we directly apply Eq. (21) to the original system (64) we can obtain the following NF given in the form of

$$\begin{aligned} \dot{r} &= r\left\{\frac{1}{2}\mu - \frac{3}{8}r^2 + \left[\frac{5}{288}(1-\cos 4\theta) - \frac{9}{32}\sin 4\theta\right] r^2\mu\right\}, \\ \dot{\theta} &= 1 - \frac{1}{8}\mu^2 - \frac{1}{24}r^2 + \left[\frac{9}{32}(1-\cos 4\theta) - \frac{5}{288}\sin 4\theta\right] r^2\mu, \end{aligned} \tag{66}$$

which indicates that the amplitude of motion cannot be separated from the phase of motion even when just one μ-term, μx_1, is presented in the original system.

Comparing Eq. (66) with Eq. (65) shows that Eq. (66) contains one more term in the unfolding, which influences the solution even if the fourth order terms are neglected. Again this example shows that using only near-identity NT cannot remove higher order ($\geqslant 4$ for this example) terms involving the parameter μ.

6. Conclusion

The normal forms of vector fields associated with single zero, Hopf bifurcation and a double zero have been particularly investigated. It has been shown that "two step" approach used in common normal form applications does not establish correct relation between the original system and the normal form. Even when certain higher order terms are neglected from a normal form, the truncated form does not always agree with that obtained directly from the original system. The results reveal that *The normal form of a vector field with perturbation parameters cannot be obtained by using only near-identity transformations*. Further reduction of normal forms leading to the SNF is also discussed, which is the final goal in normal form computation.

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