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BIFURCATION CONTROL FOR A CLASS OF LORENZ-LIKE SYSTEMS

PEI YU*
Department of Applied Mathematics,
The University of Western Ontario London,
Ontario, Canada N6A 5B7
pyu@puwo.ca

JINHU LÜ
LSC, ISS, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing,
100080, P. R. China
School of Electrical and Computer Engineering,
RMIT University, Melbourne VIC 3001, Australia

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In this paper, a control method developed earlier is employed to consider controlling bifurcations in a class of Lorenz-like systems. Particular attention is focused on Hopf bifurcation control via linear and nonlinear stability analyses. The Lorenz system, Chen system and Lü system are studied in detail. Simple feedback controls are designed for controlling the stability of equilibrium solutions, limit cycles and chaotic motions. All formulas are derived in general forms including the system parameters. Computer simulation results are presented to confirm the analytical predictions.

Keywords: Lorenz-like systems; bifurcation control; feedback controller; limit cycles; chaos.

1. Introduction

Bifurcation and chaos control has been extensively studied in the past three decades, and many methodologies have been developed to solve physical and engineering problems (e.g. see [Abed & Fu, 1987; Nayfeh *et al.*, 1996; Yu & Huseyin, 1988; Laufenberg *et al.*, 1997; Wang & Abed, 1995; Ono *et al.*, 1998; Berns *et al.*, 2000; Chen *et al.*, 2000; Kang & Krener, 2000; Chen *et al.*, 2001; Lü *et al.*, 2002a; Lü *et al.*, 2002b; Chen & Lü, 2003; Lü & Lu, 2003; Yu & Chen, 2004]). In general, the aim of bifurcation control is to design a controller such that the bifurcation property of a nonlinear system undergoing bifurcation can be changed to achieve certain desirable dynamical behavior, such as changing stability of equilibrium solutions, making a Hopf bifurcation from subcritical to supercritical, eliminating chaotic motions, etc. Anti-bifurcation control and chaotification, on the other hand, are to purposefully create bifurcation or chaos to satisfy a particular purpose of design.

In this paper, our attention is focused on bifurcation control using nonlinear state feedback. A previously developed explicit formula [Yu & Chen, 2004] will be applied to consider a class

*Author for correspondence

of Lorenz-like systems, including the Lorenz system, Chen system and Lü system. This formula is given in the form of polynomials, and keeps the equilibria of the original system unchanged. This method has been used to study a simple Lorenz system (with only two independent parameters) and Rössler system [Yu & Chen, 2004] where explicit control designs are provided to show the applicability of the theory. To be more specific, consider the following general nonlinear system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}), \quad \mathbf{x} \in R^n, \ \ \boldsymbol{\mu} \in R^m, \ \ \mathbf{f}: R^{n+m} \to R^n, \tag{1}$$

where the dot denotes differentiation with respect to time t, $\mathbf{x}$ is an n-dimensional state vector and $\boldsymbol{\mu}$ is an m-dimensional parameter vector, which contains bifurcation parameters and control parameters. The function $\mathbf{f}$ is assumed analytic with respect to both $\mathbf{x}$ and $\boldsymbol{\mu}$.

Usually, the first step in the study of system (1) is to find its equilibrium solutions, which can be solved from the nonlinear algebraic equation $\mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{0}$, usually yielding multiple solutions. Let $\mathbf{x}^*(\boldsymbol{\mu})$ denote an equilibrium solution of the system, i.e. $\mathbf{f}(\mathbf{x}^*(\boldsymbol{\mu}), \boldsymbol{\mu}) \equiv \mathbf{0}$ for any value of $\boldsymbol{\mu}$. Further, suppose that the Jacobian of the system evaluated at the equilibrium solution $\mathbf{x}^*(\boldsymbol{\mu})$ has eigenvalues, $\lambda_1(\boldsymbol{\mu}), \lambda_2(\boldsymbol{\mu}), \ldots, \lambda_n(\boldsymbol{\mu})$, which may be real or complex. Assume that when $\boldsymbol{\mu}$ is varied, the real part of some eigenvalues becomes zero at the critical point $\boldsymbol{\mu} = \boldsymbol{\mu}^*$, leading to certain type of bifurcation, such as Hopf bifurcation, Hopf-zero bifurcation, etc.

The goal of bifurcation control here is to design a controller, given by

$$\mathbf{u} = \mathbf{u}(\mathbf{x}; \boldsymbol{\mu}), \tag{2}$$

such that the original equilibrium solution $\mathbf{x}^*$ is unchanged, but the bifurcation point $(\mathbf{x}^*, \boldsymbol{\mu}^*)$ is moved to a new position, $(\mathbf{x}^*, \tilde{\boldsymbol{\mu}})$, with $\tilde{\boldsymbol{\mu}} \neq \boldsymbol{\mu}$. Therefore, a necessary condition for the controller is

$$\mathbf{u}(\mathbf{x}^*; \boldsymbol{\mu}) = \mathbf{0}, \tag{3}$$

for all values of $\boldsymbol{\mu}$, in order not to change the original equilibrium solution $\mathbf{x}^*$. More precisely, suppose system (1) has k equilibria, given by

$$\mathbf{x}_i^*(\boldsymbol{\mu}) = (x_{1i}^*(\boldsymbol{\mu}), x_{2i}^*(\boldsymbol{\mu}), \ldots, x_{ni}^*(\boldsymbol{\mu})), \qquad i = 1, 2, \ldots, k, \tag{4}$$

satisfying $\mathbf{f}(\mathbf{x}_i^*(\boldsymbol{\mu}), \boldsymbol{\mu}) \equiv \mathbf{0}$ for $i = 1, 2, \ldots, k$. The general nonlinear state feedback control (2) is applied to system (1) to obtain a closed-loop control system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) + \mathbf{u}(\mathbf{x}, \boldsymbol{\mu}) \equiv \mathbf{F}(\mathbf{x}, \boldsymbol{\mu}). \tag{5}$$

In order for the controlled system (5) to keep all the original k equilibria unchanged under the control $\mathbf{u}$, it requires that the following conditions be satisfied:

$$\mathbf{u}(\mathbf{x}_i^*, \boldsymbol{\mu}) \equiv (u_1, u_2, \ldots, u_n)^T = \mathbf{0} \tag{6}$$

for $i = 1, 2, \ldots, k$.

A control law given in general polynomial function, satisfying condition (6), has been proposed [Yu & Chen, 2004]:

$$\begin{aligned} u_q(\mathbf{x}, \mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_k^*, \boldsymbol{\mu}) &= \sum_{i=1}^{n} A_{qi} \prod_{j=1}^{k} (x_i - x_{ij}^*) \\ &\quad + \sum_{i=1}^{n} \sum_{j=1}^{k} B_{qij} (x_i - x_{ij}^*) \prod_{p=1}^{k} (x_i - x_{ip}^*) \\ &\quad + \sum_{i=1}^{n} \sum_{j=1}^{k} C_{qij} (x_i - x_{ij}^*)^2 \prod_{p=1}^{k} (x_i - x_{ip}^*) \\ &\quad + \sum_{i=1}^{n} \sum_{j=1}^{k} D_{qij} (x_i - x_{ij}^*)^2 \prod_{p=1}^{k} (x_i - x_{ip}^*)^2 + \cdots \\ &\qquad (q = 1, 2, \ldots, n). \end{aligned} \tag{7}$$

It is easy to verify that $u_q(\mathbf{x}_i^*, \mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_k^*, \boldsymbol{\mu}) = 0$ for $i = 1, 2, \ldots, k$. It should be noted that although the formula (7) contains linear terms, purely linear feedback controls are not used since they only keep one of the equilibria unchanged.

Usually, terms given in Eq. (7) up to D_{qij} are enough for controlling a bifurcation if the singularity of the system is not highly degenerate. The coefficients A_{qi}, B_{qij}, C_{qij} and D_{qij}, which may be functions of $\boldsymbol{\mu}$, are determined from the stability of an equilibrium under consideration and that of the associated bifurcation solutions. More precisely, linear terms are determined by requiring the shift of an existing bifurcation (e.g. delaying an existing Hopf bifurcation). The nonlinear terms, on the other hand, can be used to change the stability of an existing bifurcation or create a new bifurcation (e.g. changing an existing subcritical Hopf bifurcation to supercritical). Note that not just A_{qi} terms may involve linear terms; B_{qij} terms, etc. may also contain linear terms.

Note that it is not necessary to take all the components u_q, $i = 1, 2, \ldots, n$, in the control. In most cases, using fewer components or just one component may be enough to satisfy the predesigned control objectives. It is preferable to have a simplest possible design for engineering applications. For example, if $x_{i1}^* = x_{i2}^* = \cdots = x_{ik}^*$ for some i, then one only needs to use these terms and omit the remaining terms in the control law. Moreover, lower-order terms related to these equilibrium components can be added. In fact, although the formula (7) looks complicated, in application many coefficients therein will be zero, as demonstrated in Secs. 3–5.

Stabilization of chaotic systems usually employ Lyapunov function to reach global stability, and linear feedback controls may be used. However, such a linear control does not keep all the equilibria of the system unchanged, but only stabilizes one of them. The Hopf bifurcation control studied in this paper guarantees all the equilibria of the system is kept unchanged, though the stability is usually not global.

In the next section, the general strategy of Hopf bifurcation control is discussed. Section 3 is devoted to study bifurcation control of the Lorenz system, and Secs. 4 and 5, to consider the Chen and Lü systems, respectively. Finally, concluding remarks are given in Sec. 6.

2. A Class of Lorenz-like Systems and Hopf Bifurcation Control

In this paper, we consider a class of Lorenz-like systems, which has rich complex dynamical behavior, including bifurcations to equilibrium solutions, periodic and quasi-periodic solutions and chaotic motions. This class of Lorenz-like systems can be generally described by

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= dx + cy - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{8}$$

where a, b, c and d are real parameters. System (8) is the Lorenz system when $c = -1$; the Chen system when $d = c - a$; and the Lü system when $d = 0$. The typical chaotic attractors for the three systems are shown in Fig. 1. It should be noted that the class of Lorenz-like systems, described by (8) are more general than the family of Lorenz systems [Lü & Chen, 2002; Chen & Lü, 2003] since in system (8) all the four parameters can be taken arbitrarily. However, in this paper we will focus on the study of

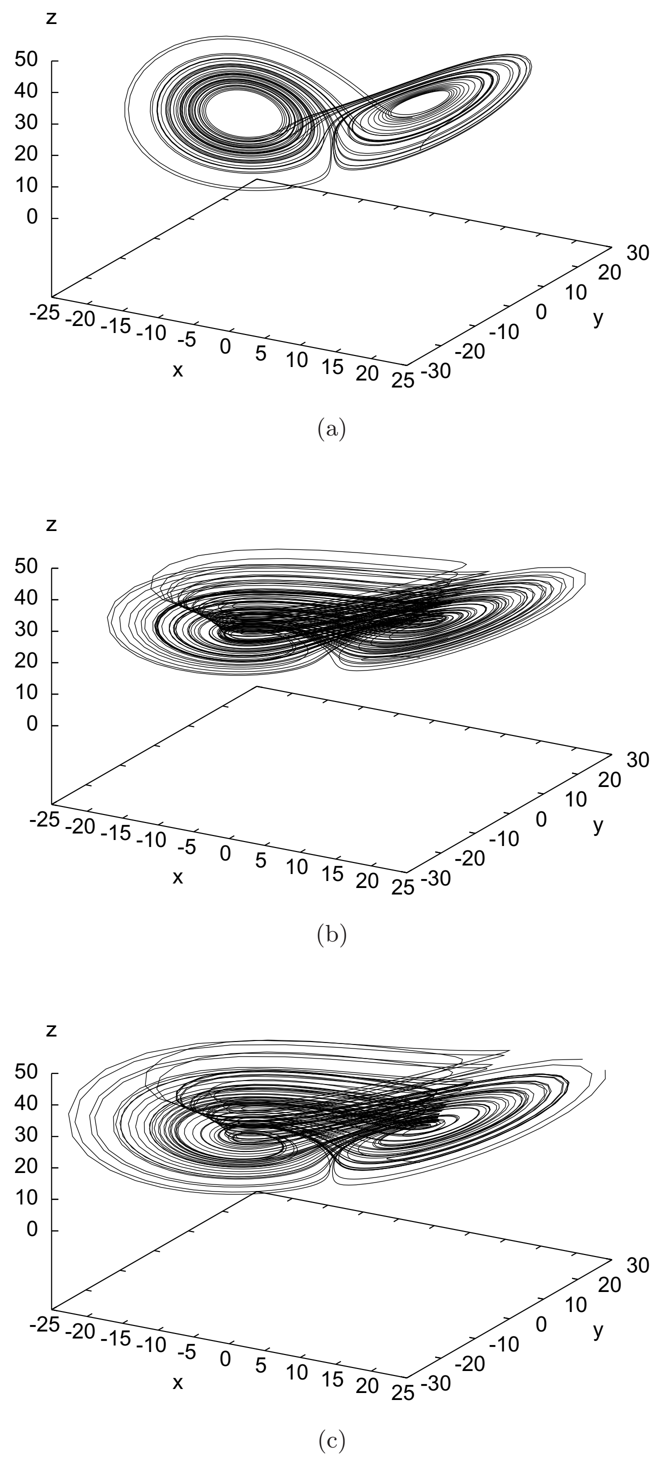

Fig. 1. Simulated trajectories for system (8); (a) the Lorenz attractor when $a = 10, b = 8/3, c = -1, d = 28$; (b) the Chen attractor when $a = 35, b = 3, c = 28, d = c - a = -7$ and (c) the Lü attractor when $a = 30, b = 44/15, c = 111/5$, $d = 0$.

bifurcation control for the three typical systems, i.e. the Lorenz, Chen and Lü systems.

For Hopf bifurcation control, we may use a controller $\mathbf{u}(\mathbf{x};\mu)$ in system (5), where μ is a scalar. Assume that the original system without control has an equilibrium $\mathbf{x}^*$ and Hopf bifurcation occurs at the critical point $(\mathbf{x}^*, \mu^*)$. Assume that when μ is varied, one pair of the complex conjugates, denoted by $\lambda_{1,2}(\mu)$ with $\lambda_1 = \overline{\lambda}_2 = \alpha(\mu)+i\omega(\mu)$, where $\alpha(\mu)$ and $\omega(\mu)$ represent the real and imaginary parts of $\lambda_{1,2}(\mu)$, respectively, moves to cross the imaginary axis at $\mu = \mu^*$ such that

$$\alpha(\mu^*) = 0 \quad \text{and} \quad \frac{d\alpha(\mu^*)}{d\mu} \neq 0. \tag{9}$$

The second condition of Eq. (9) is usually called the transversality condition, implying that the crossing of the complex conjugate pair at the imaginary axis is not tangent to the imaginary axis. Without loss of generality, one may assume that when μ is varied from $\mu < \mu^*$ to $\mu > \mu^*$, the $\lambda_{1,2}(\mu)$ moves from the left-half of complex plane to the right, and the remaining eigenvalues have negative real parts in the vicinity of the critical point $\mu = \mu^*$. According to Hopf theory [Hopf, 1942], a family of limit cycles will bifurcate from the equilibrium solution $\mathbf{x}^*$ at the critical point μ^*, where the equilibrium solution $\mathbf{x}^*$ changes its stability.

The goals of Hopf bifurcation control are:

(i) to move the critical point $(\mathbf{x}^*, \mu^*)$ to a designated position $(\mathbf{x}^*, \tilde{\mu})$;
(ii) to stabilize all possible Hopf bifurcations.

Goal (i) only requires linear analysis, while goal (ii) must apply nonlinear systems theory. In general, if the purpose of the control is to avoid bifurcations, one should employ linear analysis to maximize the stability interval for the equilibrium solution under consideration. The best is to completely eliminate possible bifurcations using a feedback control. If this is not feasible, then one may have to consider stabilizing the bifurcating limit cycles using a nonlinear state feedback [Chen *et al.*, 2000]. In certain circumstances, one may wish to create a Hopf bifurcation, which can be easily achieved using the above two steps in a reverse way [Chen *et al.*, 2001].

At the designed position, $\mathbf{x}^*, \mathbf{f}(\mathbf{x}^*, \mu) = \mathbf{0}$ for all $\mu \in R$. To achieve objective (i), calculate the Jacobian of system (5) at $\mathbf{x}^*$ to obtain

$$J(\mu) = \left[\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}^*} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}^*}. \tag{10}$$

Thus, by Hopf theory, $J(\mu)$ contains a complex conjugate pair of eigenvalues $\tilde{\lambda}_{1,2}(\mu) = \tilde{\alpha}(\mu) + i\tilde{\omega}(\mu)$ satisfying

$$\tilde{\alpha}(\tilde{\mu}) = 0 \quad \text{and} \quad \frac{d\tilde{\alpha}(\tilde{\mu})}{d\mu} \neq 0, \tag{11}$$

and the remaining eigenvalues of $J(\mu)$ have negative real part at the critical point $\mu = \tilde{\mu}$.

Once the first step discussed above is done, one may decide if it is necessary to continue toward the next step. If the aim of the control is to eliminate an existing Hopf bifurcation but the linear analysis does not reach the goal, then one must use the nonlinear part of the control to stabilize the Hopf bifurcation. This can be achieved using normal form theory. The main task in applying normal form theory is to compute the leading nonzero coefficient in the normal form, which determines whether the Hopf bifurcation is supercritical or subcritical. For Hopf bifurcation, this coefficient is also called the first *Laypunov coefficient* or the first-order *focus value*. This coefficient can be explicitly expressed in terms of the second- and third-order derivatives of the vector field of (11) evaluated at the critical point. The first-order focus value, v_1, of a general n-dimensional nonlinear system can be computed using, for example, the Maple program developed in [Yu, 1998].

3. The Lorenz System

The Lorenz system is described by [Lorenz, 1963]

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= dx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{12}$$

where a, b and d are real parameters, usually taking positive values. The typical Lorenz chaotic attractor is depicted in Fig. 1(a).

A simpler form of Lorenz system has been considered by Wang and Abed [1995] and Chen *et al.* [2000] using a washout filter to control bifurcation. The advantage of this method keeps the equilibrium solutions unchanged (without solving the equilibrium solutions of the system). The disadvantage of this method is not only increasing the dimension of the system by one, but also destroying the symmetry of the original system. This simple Lorenz system was reconsidered by Yu and Chen [2004] using the polynomial formula (7). It was shown in [Yu & Chen, 2004] that a simple cubic-order controller can be applied to control the symmetric equilibrium solutions as well as the limit cycles bifurcating from a Hopf critical point. However, the control law adopted in [Yu & Chen, 2004] does not keep all the three equilibrium solutions of the original system unchanged, but only the two symmetric equilibrium solutions. Although this is enough for controlling Hopf bifurcation emerging from the two symmetric equilibrium solutions, it does not satisfy the requirement that all equilibrium solutions should not be changed.

In this paper, we shall apply a different, but still simple, control law to keep all the three equilibrium solutions unchanged. First, it is easy to show that system (12) has three equilibrium solutions, C_0, C_+ and C_-, given below:

$$\begin{aligned} &C_0 : x_e^0 = y_e^0 = z_e^0 = 0, \\ &C_\pm : x_e^\pm = y_e^\pm = \pm\sqrt{b(d-1)}, \\ &z_e^\pm = d-1, \quad (d>1). \end{aligned} \tag{13}$$

Suppose all the parameters a, b and d are positive. Then C_0 is globally stable for $0 < d < 1$, and pitchfork bifurcation occurs at the critical point, $d = 1$, where the equilibrium C_0 loses its stability and bifurcates into either C_+ or C_-. The two equilibria C_+ and C_- are stable for $1 < d < d_H$, where

$$d_H = \frac{a(a+b+3)}{a-b-1} \quad (a > b+1), \tag{14}$$

and at this critical point C_+ and C_- lose their stability, giving rise to Hopf bifurcation. Is is easy to see that when $0 < a < b+1, d_H < 0$, implying that there is no Hopf bifurcation and the two equilibria C_+ and C_- are always stable as long as $0 < a < b+1$. Note that $a > b+1$ implies $a > 1$ for $b > 0$.

3.1. *Without control*

When no control is applied to system (12), the critical point at which Hopf bifurcation occurs is defined by Eq. (14). At this critical point, the Jacobian of system (12) evaluated at C_+ and C_- has a real negative eigenvalue $-(a+b+1)$ and a purely imaginary pair $\pm i\omega_c$, $(i^2 = -1)$ where

$$\omega_c = \sqrt{\frac{2ab(a+1)}{a-b-1}}, \quad (a > b+1, b > 0). \tag{15}$$

Applying the following transformation,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pm\sqrt{b(d-1)} \\ \pm\sqrt{b(d-1)} \\ d-1 \end{pmatrix} + T \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}, \tag{16}$$

where

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & \dfrac{\omega_c}{a} & -\dfrac{b+1}{a} \\ \dfrac{\omega_c^2}{a\sqrt{b(d_H-1)}} & -\dfrac{(a+1)\omega_c}{a\sqrt{b(d_H-1)}} & -\dfrac{b(a+b+1)}{a\sqrt{b(d-1)}} \end{bmatrix} \tag{17}$$

to system (12) yields

$$\begin{aligned} \dot{\tilde{x}} &= \omega_c\tilde{y} + \frac{(a-b-1)[2ab\tilde{x} + b\omega_c\tilde{y} + b(a-b-1)\tilde{z}]}{a^3 - (b+1)^3 + a[(a-b-1)(1+3b)+4b]}\mu + \cdots \\ \dot{\tilde{y}} &= -\omega_c\tilde{x} - \frac{(a+b+1)(a-b-1)[2ab\tilde{x} + b\omega_c\tilde{y} + 2b(a-b-1)\tilde{z}]}{\omega_c\{a^3 - (b+1)^3 + a[(a-b-1)(1+3b)+4b]\}}\mu + \cdots \\ \dot{\tilde{z}} &= -(a+b+1)\tilde{z} - \frac{(a-b-1)[2ab\tilde{x} + b, \omega_c\tilde{y} + b(a-b-1)\tilde{z}]}{a^3 - (b+1)^3 + a[(a-b-1)(1+3b)+4b]}\mu + \cdots \end{aligned} \tag{18}$$

where $\cdots$ denotes quadratic terms, and $\mu = d - d_H$, represents a bifurcation parameter.

Employing the Maple programs developed in [Yu, 1998] for computing the normal forms of Hopf and generalized Hopf bifurcations yields the following normal form:

$$\begin{aligned} \dot{\rho} &= \rho(v_0\mu + v_1\rho^2) + \cdots, \\ \dot{\theta} &= \omega_c(1 + \tau_0\mu + \tau_1\rho^2) + \cdots, \end{aligned} \tag{19}$$

where v_0 and τ_0 are obtained from linear analysis (e.g. using the formula given in [Yu & Huseyin, 1988]), while v_1 and τ_1 must depend on the nonlinear analysis via normal form computation,

$$v_0 = \frac{b(a-b-1)^2}{2\{a^3-(b+1)^3+a[(a-b-1)(1+3b)+4b]\}},$$

$$v_1 = \frac{b(a-b-1)}{4(a+b+1)\{2a[(a-b-1)(3b+2)+b(b+2)]+(a-b-1)^3\}} \times \frac{4a[2a(3a+b^2+1)+(a^2+5)(a-b-1)]+[(a-b-1)(5a+b+3)+12a](a-b-1)^2}{4a[(a-b-1)(3b+1)+2b(b+2)]+(a-b-1)^3},$$

$$\tau_0 = \frac{(a-b-1)[(a^2-1)+b(a-b-1)]}{2(a+1)\{a^3-(b+1)^3+a[(a-b-1)(1+3b)+4b]\}},$$

$$\begin{aligned}\tau_1 = &-\frac{(a-b-1)}{12ab(a+1)(a+b+1)\{2a[(a-b-1)(3b+2)+b(b+2)]+(a-b-1)^3\}}\\ &\times \frac{1}{4a[(a-b-1)(3b+1)+2b(b+2)]+(a-b-1)^3}[48(2a-1)a^2(a^2-1)^2\\ &-4a^2(a^2-1)(5a^2+61a-34)(a-b-1)-a(240a^4-172a^3-364a^2+196a+4)(a-b-1)^2\\ &+a(145a^3+235a^2-209a-11)(a-b-1)^3+a(74a^2-146a-4)(a-b-1)^4\\ &-(61a^2-10a+1)(a-b-1)^5+(8a-1)(a-b-1)^6].\end{aligned}$$

Here, ρ and θ in Eq. (19) represent the amplitude and phase of motion, respectively. The first equation of (19) can be used for bifurcation and stability analysis. It is obvious that $v_0 > 0$ and $v_1 > 0$ for $b > 0, a > b+1$. Thus, when $\mu < 0$ (i.e. $d < d_H$), the two equilibrium solutions C_+ and C_- are stable; when $\mu > 0$, these two equilibrium solutions lose stability and Hopf bifurcation occurs at the critical point $\mu = 0$ (i.e. $d = d_H$), and the Hopf bifurcation is subcritical (i.e. the bifurcating limit cycles are unstable) due to $v_1 > 0$.

3.2. *With feedback control*

Now, we study how to apply feedback controls to stabilize system (12). By using formula (7), noticing the symmetry of system (12) with respect to C_+ and C_-, we may have many different control laws. For an illustration, in this paper we apply the following simple quadratic nonlinear, state feedback control law:

$$u_2 = ky(z-d+1), \tag{20}$$

to the second equation of system (12), and then the closed-loop system is given by

$$\begin{aligned}\dot{x} &= a(y-x),\\ \dot{y} &= dx - y - xz - ky(z-d+1),\\ \dot{z} &= -bz + xy.\end{aligned} \tag{21}$$

It is easy to see that this control (20) does not change the equilibrium solutions C_0 and $C_\pm$ of the original system (12). Similarly, Hopf bifurcation may occur from the equilibria $C_\pm$. The main results for Hopf bifurcation control of system (21) are summarized below, followed by a detailed analysis. Hopf bifurcation emerging from the equilibria $C_\pm$ can be controlled as being supercritical if the feedback control gain coefficient k is chosen as

$$k \in (-1, k_-) \cup (k_+, \infty), \quad \text{with} \quad k_\pm = \frac{-B \pm \sqrt{B^2-4AC}}{2A}, \tag{22}$$

where

$$\begin{aligned}A = &-16a^2(a+1)^2(a-1) - 8a(a+1)\\ &\times(a^3+2a^2-5a-1)(a-b-1)\\ &-4a(a^3-9a^2-7a+5)(a-b-1)^2\\ &+(4a^3-2a^2-26a-4)(a-b-1)^3\\ &-2(6a+1)(a-b-1)^4+2(a-b-1)^5,\\ B = &-(1+k)[-16a^3(a+1)^2(a-1)\\ &-8a^2(a+1)(a^2-6a-2)(a-b-1)\\ &+2a(24a^3+14a^2-4a+10)(a-b-1)^2\end{aligned}$$

$$
\begin{aligned}
&- 2a(4a^2 + 9a - 5)(a - b - 1)^3 \\
&- 14a^2(a - b - 1)^4 + (2a - 1)(a - b - 1)^5], \\
C &= (1 + k)^2 a(a - b - 1)\{(a - b - 1)[(5a + b + 3) \\
&\times (a - b - 1)^2 + 4a(a + 3)(a - b - 1) + 20a] \\
&+ 4a^2(b^2 + ab + 7a + 1)\}.
\end{aligned} \tag{23}
$$

First, note that the stability conditions of these equilibrium solution are changed, due to the control. In other words, the critical points have been changed due to the control. In fact, the characteristic polynomial associated with C_0 is

$$
\begin{aligned}
P_0(\lambda) &= \lambda^3 + [a + b + 1 + (1 - d)k]\lambda^2 \\
&+ [b(a + d) + (a + b)(1 - d)(1 + k)]\lambda \\
&+ ab(1 - d)(1 + k).
\end{aligned} \tag{24}
$$

To have C_0 stable, it requires that all the coefficients of P_0 are positive, and the Huiwitz quantity

$$
\begin{aligned}
\Delta_0 &= [a + 1 + (1 - d)k]\{b[a + b + 1 + (1 - d)k] \\
&+ a(1 - d)(1 + k)\} > 0.
\end{aligned}
$$

is also satisfied. This clearly shows that C_0 is stable if $(1 - d)(1 + k) > 0$, in addition to $a > 0, b > 0$. Thus, if $0 < d < 1$, C_0 is stable for any value of $k > -1$. $k = 0$ makes the controlled system (21) return to the uncontrolled system (12). One may choose $k < -1$ to increase the stability interval of d to $d \in (1, \infty)$.

Note that under the control (20), the two equilibrium solutions of the controlled system (21) are not only kept symmetric but also have the same stability condition. As a matter of fact, the characteristic polynomial for the two equilibrium solutions are

$$
\begin{aligned}
P_\pm(\lambda) &= \lambda^3 + (a + b + 1)\lambda^2 + b[a + d + (d - 1)k]\lambda \\
&+ 2ab(d - 1)(1 + k).
\end{aligned} \tag{25}
$$

Similarly, in order for $C_\pm$ to be stable, besides requiring all the coefficients of $P_\pm$ to be positive, we need

$$
\Delta_\pm = b\{a(a+b+3) - (a-b-1)[1+(d-1)(1+k)]\} > 0.
$$

Therefore, $C_\pm$ are stable when

$$
(d - 1)(1 + k) > 0 \quad \text{and} \quad \Delta_\pm > 0.
$$

It is clear that in addition to $(d - 1)(1 + k) > 0$, if $0 < a < b + 1$, then $C_\pm$ are stable. Only if $a > b + 1$ $(b > 0)$, then $C_\pm$ becomes unstable and Hopf bifurcation emerges from these two symmetric equilibrium solutions.

Next, suppose $b > 0$ and $a > b + 1$, we perform a nonlinear analysis to determine the stability of Hopf bifurcation. The Hopf critical point can be still expressed in terms of d, given by

$$
d_H = 1 + \frac{(a + 1)(a + b + 1)}{(a - b - 1)(1 + k)}, \quad (a > b + 1, b > 0). \tag{26}
$$

At this critical point, the eigenvalues of the Jacobian are still the same as that of uncontrolled system (12): $-(a + b + 1)$ and $\pm i\omega_c$, where ω_c is given in (15). Again, let $d = d_H + \mu$. Then applying the transformation (16), with

$$
T = \begin{bmatrix}
1 & 0 & 1 \\
1 & \dfrac{\omega_c}{a} & -\dfrac{b + 1}{a} \\
\dfrac{\omega_c^2}{a(1 + k)\sqrt{b(d_H - 1)}} & -\dfrac{(a + 1)\omega_c}{a(1 + k)\sqrt{b(d_H - 1)}} & -\dfrac{b(a + b + 1)}{a(1 + k)\sqrt{b(d_H - 1)}}
\end{bmatrix} \tag{27}
$$

to system (12) results in

$$
\begin{aligned}
\dot{\tilde{x}} &= \omega_c\tilde{y} + \frac{(1 + k)(a - b - 1)[2ab\tilde{x} + b\omega_c\tilde{y} + b(a - b - 1)\tilde{z}]}{a^3 - (b + 1)^3 + a[(a - b - 1)(1 + 3b) + 4b]}\mu + \cdots \\
\dot{\tilde{y}} &= -\omega_c\tilde{x} - \frac{(1 + k)(a + b + 1)(a - b - 1)[2ab\tilde{x} + b\omega_c\tilde{y} + 2b(a - b - 1)\tilde{z}]}{\omega_c\{a^3 - (b + 1)^3 + a[(a - b - 1)(1 + 3b) + 4b]\}}\mu + \cdots \\
\dot{\tilde{z}} &= -(a + b + 1)\tilde{z} - \frac{(1 + k)(a - b - 1)[2ab\tilde{x} + b, \omega_c\tilde{y} + b(a - b - 1)\tilde{z}]}{a^3 - (b + 1)^3 + a[(a - b - 1)(1 + 3b) + 4b]}\mu + \cdots.
\end{aligned} \tag{28}
$$

Employing the Maple programs [Yu, 1998] to system (28) yields the normal form (19) with

$$v_0 = \frac{(1+k)b(a-b-1)^2}{2\{a^3-(b+1)^3+a[(a-b-1)(1+3b)+4b]\}},$$

$$v_1 = \frac{bg(k)}{4a(1+k)(a+b+1)\{2a(a+1)+(a-b-1)(a+b+1)^2\}[8ab(a+1)+(a-b-1)(a+b+1)^2]},$$

where $g(k) = (Ak^2 + Bk + C)$, with the coefficients A, B and C given in (23).

To consider the sign of v_1, first note that the sign of v_1 is the same as that of $g(k)$ for $1 + k > 0$. It is easy to see that $C > 0$ for $b > 0, a > b + 1$. Next, we wish to prove that $A < 0$ for $b > 0, a > b + 1$. To achieve this, we first have

$$A(a = b + 1) = -16b(b + 2)^2(b + 1)^2 < 0, \quad \forall\, b > 0.$$

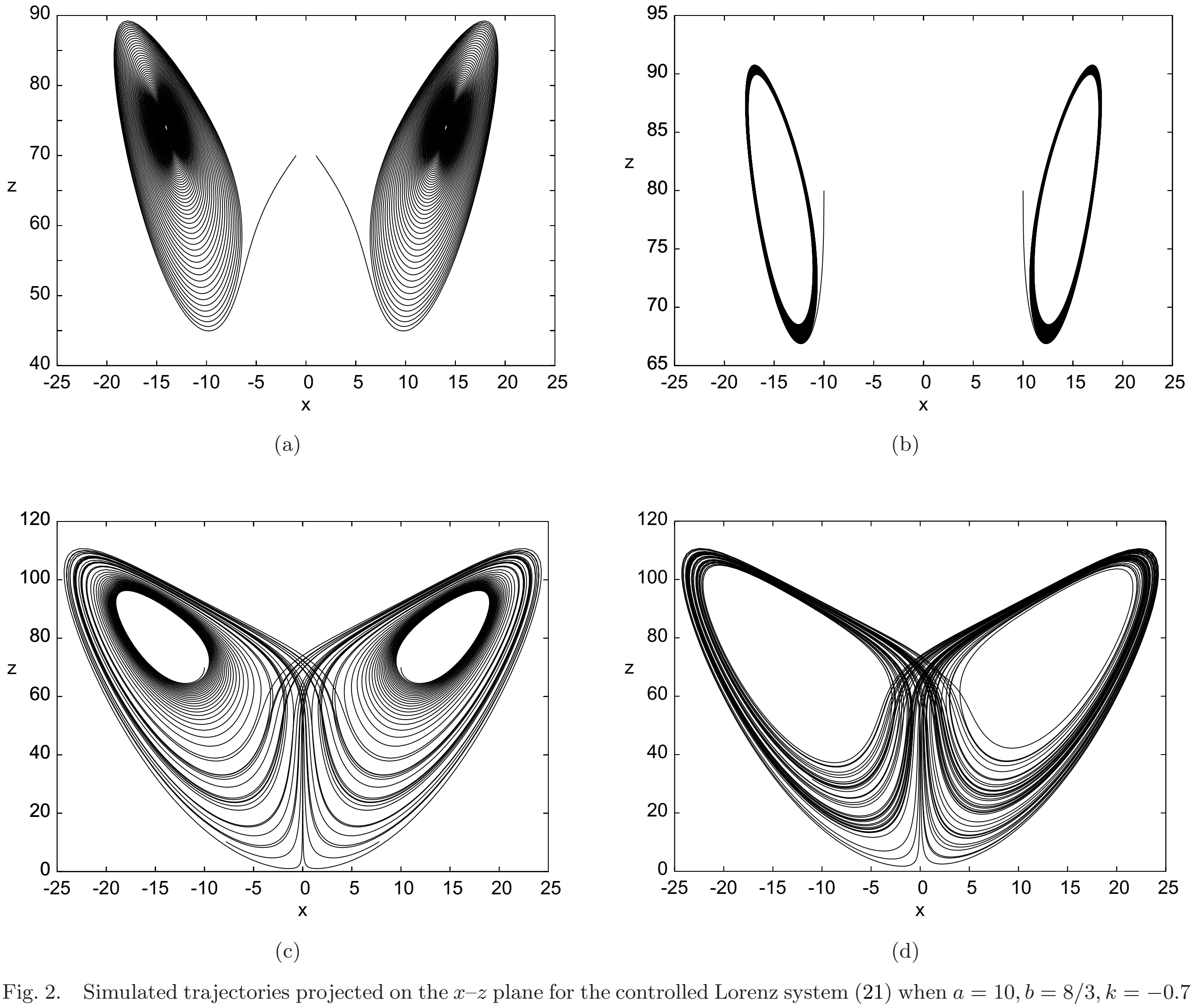

Fig. 2. Simulated trajectories projected on the x–z plane for the controlled Lorenz system (21) when $a = 10, b = 8/3, k = -0.7$ for (a) $d = 75$, converging to $C_\pm$ from the initial conditions $x(0) = \pm 1, y(0) = \pm 10, z(0) = 70$; (b) $d = 82$, converging to limit cycles from the initial conditions $x(0) = \pm 10, y(0) = \pm 10, z(0) = 80$; (c) $d = 85$, leading to co-existence of stable limit cycles and chaos from the initial condition $x(0) = \pm 10, y(0) = \pm 10, z(0) = 70$; and (d) $d = 85$, leading to chaos from the initial condition $x(0) = 1, y(0) = 10, z(0) = 70$.

Then, differentiating A with respect to a, and let $A_1 = \partial A/\partial a$. Then we obtain

$$A_1(a = b+1) = -8(b+1)(b+2)(b^3 + 15b^2 + 20b + 1) < 0, \quad \forall\, b > 0.$$

Further, let $A_2 = \partial^2 A/\partial a^2$, for which we have

$$A_2(a = b+1) = -88b^4 - 792b^3 - 1840b^2 - 1408b - 224 < 0, \quad \forall\, b > 0.$$

Finally, let $A_3 = \partial^3 A/\partial a^3$, yielding

$$\begin{aligned} A_3 = -12\{&2a^2(23a+30) + (39a+20)(a-1) \\ &+ [30a(a-1) + 12a(b+1) + 4a^2 + 3] \\ &\times (a-b-1) + [2(b+1) + 11](a-b-1)^2\} \\ < 0,& \quad \forall\, b > 0 \quad \text{and} \quad a > b+1. \end{aligned}$$

Therefore, we know that

$$A < 0, \quad \forall\, b > 0 \quad \text{and} \quad a > b+1.$$

Hence, the roots $k_\pm$ of the quadratic function $g(k)$, given in (22), satisfy $k_- < 0$ and $k_+ > 0$.

To end this section, we present some numerical simulation results to illustrate the theoretical

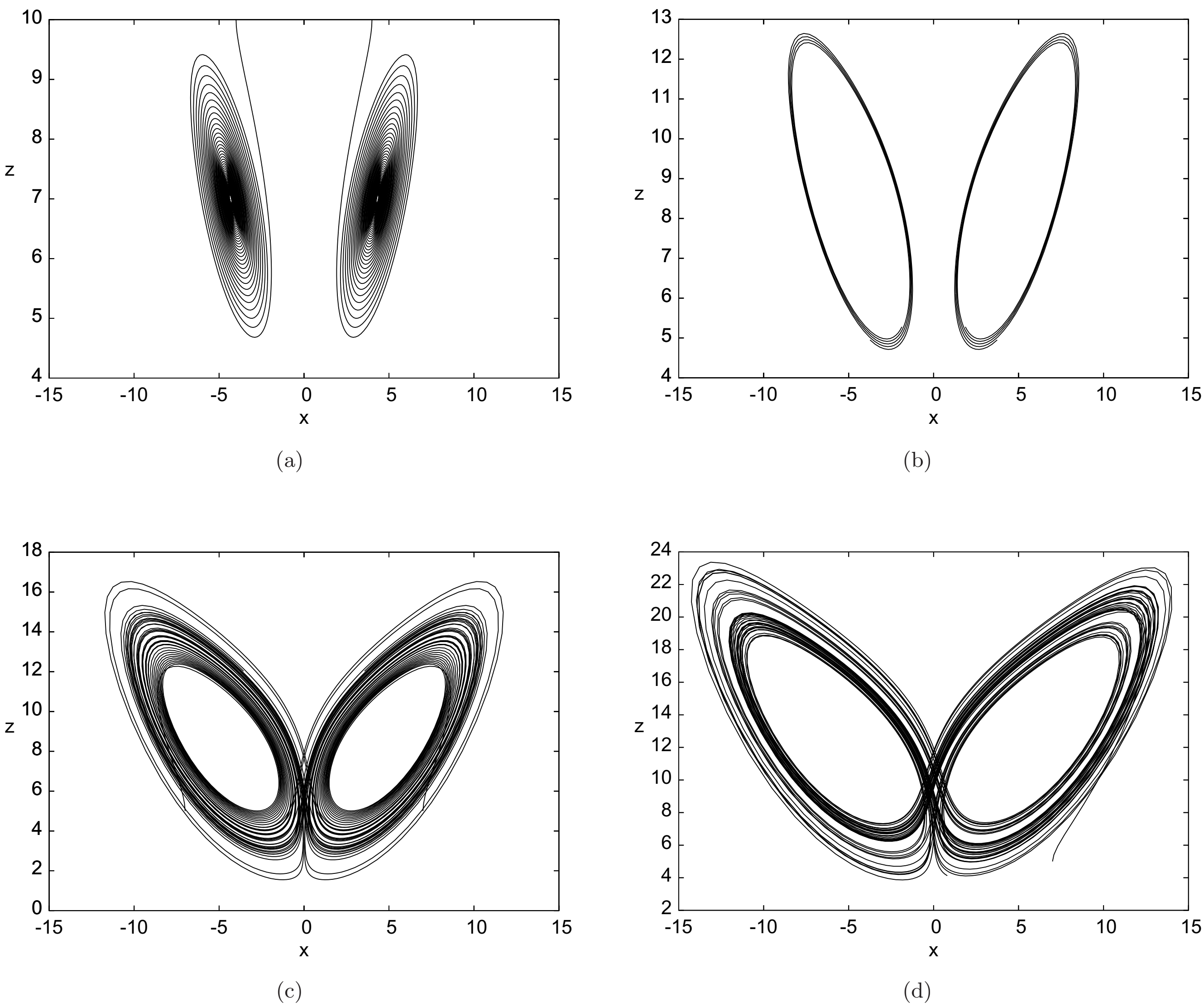

Fig. 3. Simulated trajectories projected on the x–z plane for the controlled Lorenz system (21) when $a = 10, b = 8/3, k = 2.0$ for (a) $d = 8$, converging to $C_\pm$ from the initial conditions $x(0) = \pm 4, y(0) = \pm 4, z(0) = 10$; (b) $d = 9.5$, converging to limit cycles from the initial conditions $x(0) = \pm 6, y(0) = \pm 6, z(0) = 8.5$; (c) $d = 9.5$, leading to co-existence of stable limit cycles and chaos from the initial condition $x(0) = \pm 7, y(0) = \pm 7, z(0) = 5$ and (d) $d = 13.5$, leading to chaos from the initial condition $x(0) = 7, y(0) = 7, z(0) = 5$.

predictions. We choose the typical values: $a = 10, b = 8/3$. For this case, $a > b + 1$. It is well known that the equilibrium C_0 of the uncontrolled system is stable for $0 < d < 1$, and the two symmetric equilibria $C_\pm$ are stable for $1 < d < 470/19$, and a subcritical Hopf bifurcation occurs at the critical point $d_H = 470/19 \approx 24.7368$. For the controlled system,

$$d_H = 1 + \frac{451}{19(1+k)}, \quad \text{with } \omega_c = 4\sqrt{\frac{110}{19}}, \tag{29}$$

and

$$k_\pm = \frac{902138819 \pm \sqrt{9561317949938982921}}{3635254304}$$
$$\Rightarrow k_- \approx -0.6024, \quad k_+ \approx 1.0988. \tag{30}$$

Thus, for this case, we may choose $k \in (-1, -0.6024) \cup (1.0988, \infty)$ such that the Hopf bifurcation is supercritical. The marginal values of d_H at k_- and k_+ are given by

$$d_H^- \approx 60.7053, \quad d_H^+ \approx 12.3099,$$

respectively. Hence, we may choose the value of $k < k_-$ so that the stability interval of $C_\pm$ is increased from $d_H = 24.7368$ for the uncontrolled system to a value greater than $d_H = 60.7053$. For example, if $k = -0.7$, then $d_H = 80.1228$, much larger than that of the uncontrolled system. Moreover, we can always choose k such that the bifurcating limit cycles are stable.

Numerical simulation results are shown in Figs. 2 and 3, corresponding to $k = -0.7$ and $k = 2.0$, respectively. The results indicate that the predictions given by the theoretical analysis are correct. Several cases are presented. Figure 2 corresponds to the value of $k = -0.7 < k_-$ with different values of d. For this case, $d_H \approx 80.1228$. Thus, for $d = 75$, the system trajectories converge to the two equilibria $C_\pm$ if the initial conditions are not far away from these equilibria, as shown in Fig. 2(a). When $d = 82$, $C_\pm$ becomes unstable and Hopf bifurcation occurs and the bifurcating limit cycles are stable, see Fig. 2(b). When d is increased to $d = 85$, the system may exhibit co-existence of stable limit cycles and chaos [see Fig. 2(c)] or just chaos [see Fig. 2(d)], depending upon initial conditions.

When $k = 2.0$, $d_H \approx 8.9123$. When $d = 8$, the solution trajectories converge to $C_\pm$ from the initial conditions: $x(0) = y(0) = \pm 4, z(0) = 10$, as shown in Fig. 3(a). When $d = 9.5$, $C_\pm$ become unstable and bifurcating limit cycles are stable from the initial conditions: $x(0) = y(0) = \pm 6, z(0) = 8.5$, see Fig. 3(b). Again, this case also shows co-existence of stable limit cycles and chaos. For example, for the same value of $d = 9.5$, the system exhibits co-existence of limit cycles and chaos if the initial conditions are chosen as $x(0) = y(0) = \pm 7, z(0) = 5$ [see Fig. 3(c)]. When d is increased to $d = 13.5$, the system becomes chaotic for the same initial conditions [see Fig. 3(d)].

4. The Chen System

The Chen system is given by [Chen & Lü, 2003]

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x + cy - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{31}$$

where a, b and c are real parameters, usually taking positive values. The typical Chen's chaotic attractor is shown in Fig. 1(b).

Similarly, Chen system (31) also has three equilibrium solutions, C_0, C_+ and C_-, given by

$$\begin{aligned} &C_0 : x_e^0 = y_e^0 = z_e^0 = 0, \\ &C_\pm : x_e^\pm = y_e^\pm = \pm\sqrt{b(2c - a)}, \quad z_e^\pm = 2c - a, \\ &\qquad\qquad (2c > a > 0). \end{aligned} \tag{32}$$

Suppose all the parameters a, b and c are positive. Then a simple linear analysis shows that C_0 is stable for $a > 2c$. Pitchfork bifurcation occurs at the critical point, $a = 2c$, where the equilibrium C_0 loses its stability and bifurcates into either C_+ or C_-. The characteristic polynomial associated with the two equilibria C_+ and C_- is

$$P_\pm(\lambda) = \lambda^3 + (a + b - c)\lambda^2 + bc\lambda + 2abc(2c - a). \tag{33}$$

Thus, $C_\pm$ are stable if

$$a + b - c > 0, \quad 2c - a > 0 \quad \text{and}$$
$$b(2a^2 - c(3a + c - b) > 0,$$

which are equivalent to

$$0 < a < 2c \quad \text{and} \quad b > 3a + c - \frac{2a^2}{c}. \tag{34}$$

These conditions imply that when $3a + c - 2a^2/c < 0$, i.e. when

$$\frac{1}{4}(\sqrt{17} + 3)c < a < 2c,$$

the two equilibria $C_\pm$ are always stable. If $0 < a < 1/4(\sqrt{17}+3)c$, then there exists a critical point

$$b_H = 3a + c - \frac{2a^2}{c}, \quad \left(0 < a < \frac{1}{4}(\sqrt{17}+3)c\right), \tag{35}$$

at which C_+ and C_- lose stability, giving rise to Hopf bifurcation. It should be noted here that when $b > b_H, C_\pm$ are stable (unstable if $b < b_H$).

4.1. *Without control*

The uncontrolled system (31) has a Hopf critical point, defined by Eq. (35). At this critical point, the Jacobian of system (31) evaluated at C_+ and C_- has a real negative eigenvalue $-(2a(2c-a)/c)$ and a purely imaginary pair $\pm i\omega_c$, where

$$\omega_c = \sqrt{c^2 + 3ac - 2a^2}, \quad \left(0 < a < \frac{1}{4}(\sqrt{17}+3)c\right). \tag{36}$$

Note that $a < 1/4(\sqrt{17}+3)c < 2c$ implying that the real eigenvalue is indeed negative.

By applying the transformation,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pm\sqrt{b(2c-a)} \\ \pm\sqrt{b(2c-a)} \\ 2c-a \end{pmatrix} + T \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}, \tag{37}$$

where

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & \dfrac{\omega_c}{a} & \dfrac{2a-3c}{c} \\ \dfrac{\omega_c^2}{a\sqrt{b(2c-a)}} & \dfrac{(c-a)\omega_c}{a\sqrt{b(2c-a)}} & \dfrac{2(a-2c)\omega_c^2}{c^2\sqrt{b(2c-a)}} \end{bmatrix} \tag{38}$$

to system (31) we obtain

$$\begin{aligned} \dot{\tilde{x}} &= \omega_c \tilde{y} + \frac{(a-c)c^3\tilde{x} + c^3\omega_c\tilde{y} + 4a^2(a-2c)^2\tilde{z}}{c^2\omega_c^2 + 4a^2(a-2c)^2}\mu + \cdots \\ \dot{\tilde{y}} &= -\omega_c\tilde{x} - \frac{2a(a-2c)[c^3(a-c)\tilde{x} + c^3\omega_c\tilde{y} + 4a^2(a-2c)^2\tilde{z}]}{c\omega_c[c^2\omega_c^2 + 4a^2(a-2c)^2]}\mu + \cdots \\ \dot{\tilde{z}} &= -\frac{2a(2c-a)}{c}\tilde{z} - \frac{(a-c)c^3\tilde{x} + c^3\omega_c\tilde{y} + 4a^2(a-2c)^2\tilde{z}}{c^2\omega_c^2 + 4a^2(a-2c)^2}\mu + \cdots \end{aligned} \tag{39}$$

where $\mu = d - d_H$ has been used.

Applying the formula for v_0 [Yu & Huseyin, 1988] and the Maple programs [Yu, 1998] to system (39) yields the following focus value for the normal form (19):

$$v_0 = \frac{-c^2(c^2 - 5ac + 2a^2)}{2[c^2\omega_c^2 + 4a^2(a-2c)^2]},$$

$$v_1 = \frac{c(4c-a)\omega_c^2(c-a)(c^3 + 2ac^2 + 2ca^2 - 2a^3)}{8(2c-a)[c^2\omega_c^2 + a^2(a-2c)^2][c^2\omega_c^2 + 4a^2(a-2c)^2]}.$$

To consider the sign of v_1, we can first show that the factor $c^3 + 2ac^2 + 2ca^2 - 2a^3$ is greater than 0 for $c > 0.5748a$. Further, note that the condition $a < (1/4)(\sqrt{17}+3)c$ means $c > 0.5616a$. Therefore, we obtain

$$v_1 < 0 \quad \text{when } 0.5748a < c < a,$$

$$v_1 > 0 \quad \text{when } 0.5616a < c < 0.5748a \ \text{ or } \ c > a.$$

Also, note that $v_0 > 0$ if $c < ((5+\sqrt{17})/2)a \approx 4.5615a$.

Let us consider the typical parameter values for the typical Chen attractor: $a = 35, b = 3, c = 28$. According to the above formula, for this case, we have $v_0 > 0$, and $b_H = 45.5, 0.5748a < c < a$, implying that $v_1 < 0$. Thus, the two equilibria $C_\pm$ are unstable and the solution trajectory is chaotic [see Fig. 1(b)]. If we choose b close to b_H, then we may obtain stable $C_\pm$ and stable limit cycles. For example, taking $b = 48$ gives stable $C_\pm$ and $b = 43$ leads to stable limit cycles. The simulation results are shown in Figs. 4(a) and 4(b). It should be noted that the convergence of trajectory is quite robust even for very large initial conditions.

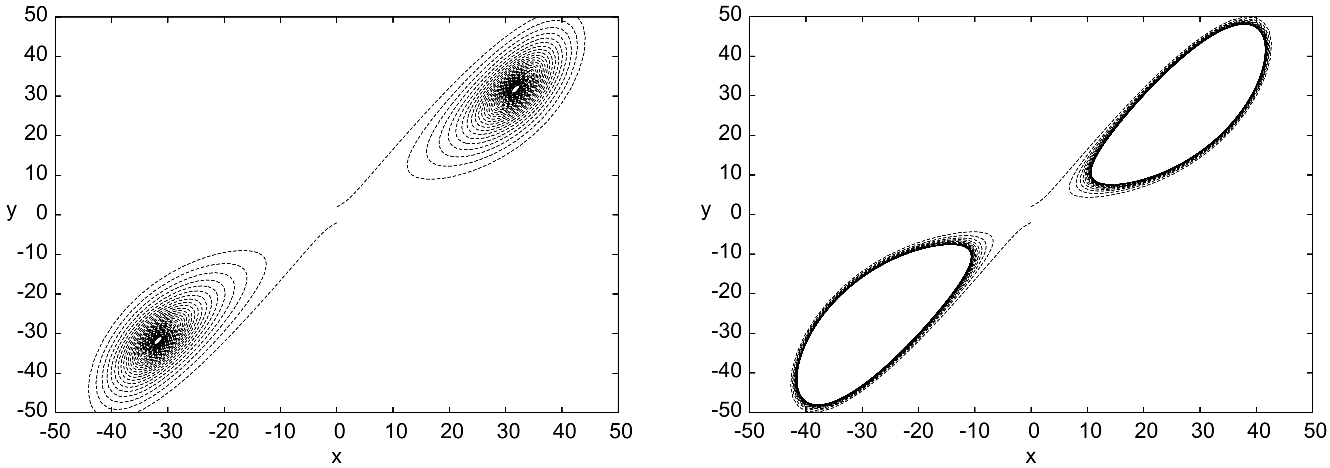

Fig. 4. Simulated trajectories projected on the x–y plane for the uncontrolled Chen system (31) when $a = 35, c = 28$, with initial conditions $x(0) = 0, y(0) = \pm 2, z(0) = 28$, for (a) $b = 48$, converging to $C_\pm$ and (b) $b = 43$, converging to stable limit cycles.

4.2. *With feedback control*

To control Hopf bifurcation in the Chen system (31), we use a slightly different control law from that for the Lorenz system, given by

$$u_2 = kx(z - 2c + a), \tag{40}$$

and thus the closed-loop Chen system is

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x + cy - xz - kx(z - 2c + a), \\ \dot{z} &= -bz + xy, \end{aligned} \tag{41}$$

whose equilibria C_0 and $C_\pm$ are the same as that of the uncontrolled system (31). Similarly, we have the following result for Hopf bifurcation control of system (41): Hopf bifurcation emerging from the equilibria $C_\pm$ is supercritical if the feedback control gain coefficient k is chosen such that $\tilde{v}_1 < 0$ when $(a - c)(2c - a) > 0$, where

$$\begin{aligned} \tilde{v}_1 = {} & (2c - a)^4 k^4 - (c + 3a)(2c - a)^3 k^3 \\ & - (9c^2 + 15ac - 4a^2)(2c - a)^2 k^2 \\ & - (11c^3 + 19ac^2 - 6a^3)(2c - a)k \\ & - (4c - a)(c^3 + 2c^2 a + 2ca^2 - 2a^3). \end{aligned} \tag{42}$$

Linear analysis shows that the equilibrium C_0 is stable if $a - c > 0$ and $(a - 2c)(1 + k) > 0$. For the two symmetric equilibria $C_\pm$, the characteristic polynomial is given by

$$\begin{aligned} P_\pm(\lambda) = {} & \lambda^3 + (a + b - c)\lambda^2 + b[c + (2c - a)k]\lambda \\ & + 2ab(2c - a)(1 + k). \end{aligned} \tag{43}$$

Hence, $C_\pm$ are stable if

$$\begin{aligned} & (2c - a)(1 + k) > 0, \quad c + (2c - a)k > 0 \quad \text{and} \\ & (a + b - c)[c + (2c - a)k] > 2a(2c - a)(1 + k). \end{aligned} \tag{44}$$

Next, consider possible Hopf bifurcation from $C_\pm$. The critical point can be still expressed in terms of b as

$$b_H = \frac{(a + c)(2c - a)(1 + k) - (a - c)^2}{c + (2c - a)k}. \tag{45}$$

So, $b_H > 0$ implies that $(a + c)(2c - a)(1 + k) - (a - c)^2 > 0$. Again, here it should be noted that $C_\pm$ are stable when $b > b_H$.

At the critical point b_H, the eigenvalues of the Jacobian are changed to

$$\begin{aligned} \lambda_{1,2} &= \pm i\sqrt{(a + c)(2c - a)(1 + k) - (a - c)^2}, \\ \lambda_3 &= -\frac{2a(2c - a)(1 + k)}{c + (2c - a)k} < 0, \end{aligned} \tag{46}$$

assuming that $(2c - a)(1 + k) > 0, c + (2c - a)k > 0$ and $(a + c)(2c - a)(1 + k) - (a - c)^2 > 0$.

Next, let $d = d_H + \mu$. Then applying the transformation (37), with

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & \dfrac{\omega_c}{a} & \dfrac{2a-3c-(2c-a)k}{c+(2c-a)k} \\ \dfrac{\omega_c^2}{a(1+k)\sqrt{b(2c-a)}} & \dfrac{(c-a)\omega_c}{a(1+k)\sqrt{b(2c-a)}} & -\dfrac{2(2c-a)\omega_c^2}{[c+(2c-a)k]^2\sqrt{b(2c-a)}} \end{bmatrix} \tag{47}$$

to system (41) yields

$$\dot{\tilde{x}} = \omega_c\tilde{y} + \frac{(a-c)[c+(2c-a)k]^3\tilde{x} + [c+(2c-a)k]^3\omega_c\tilde{y} + 4(1+k)^2a^2(2c-a)^2\tilde{z}}{4(1+k)^2a^2(2c-a)^2 + \omega_c^2[c+(2c-a)k]^2}\mu + \cdots$$

$$\dot{\tilde{y}} = -\omega_c\tilde{x} - \frac{2(1+k)a(2c-a)\{(a-c)[c+(2c-a)k]^3\tilde{x} + \omega_c[c+(2c-a)k]^3\tilde{y} + 4(1+k)^2a^2(2c-a)^2\tilde{z}\}}{\omega_c[c+(2c-a)k]\{4(1+k)^2a^2(2c-a)^2 + \omega_c^2[c+(2c-a)k]^2\}}\mu + \cdots$$

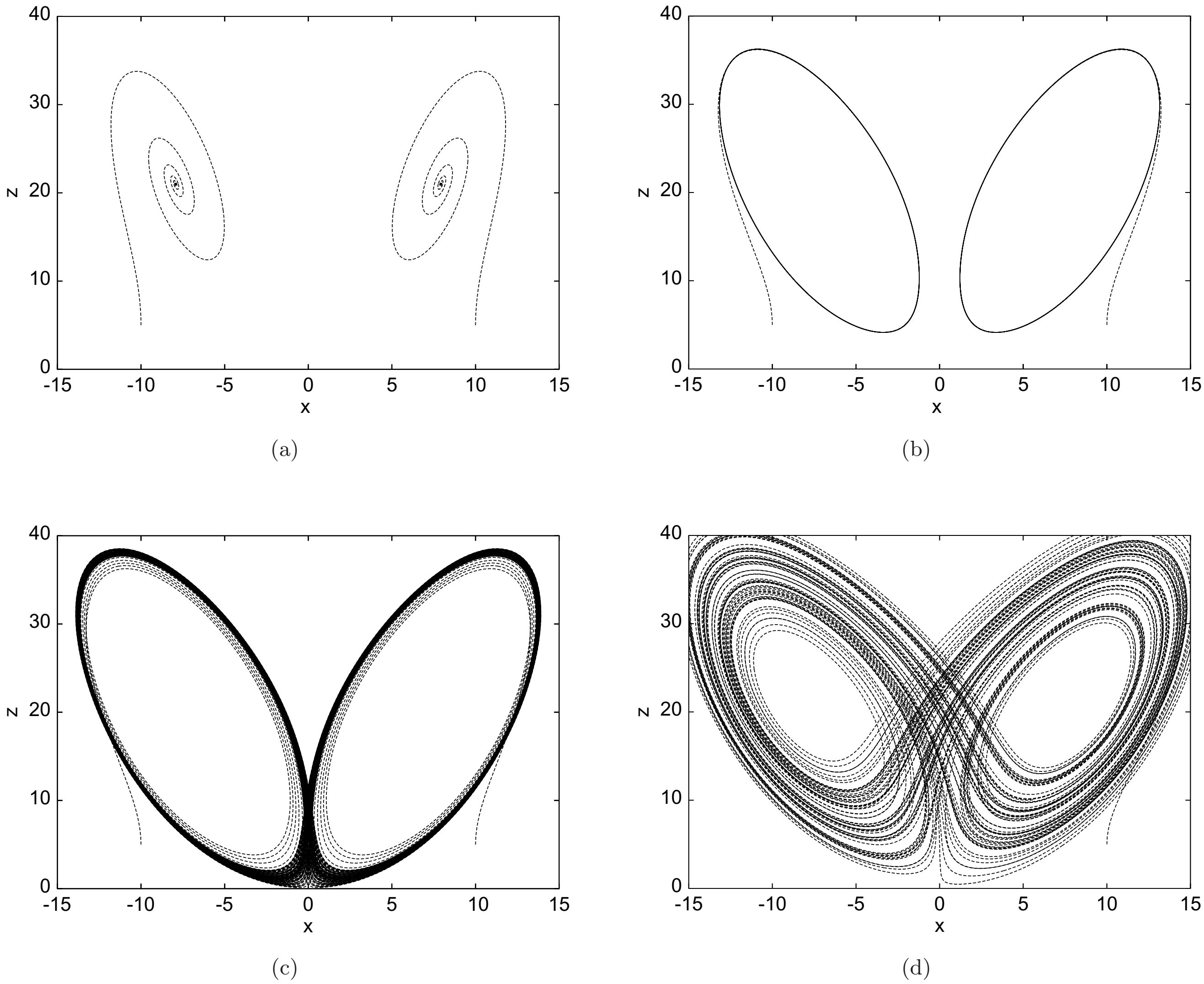

Fig. 5. Simulated trajectories projected on the x–z plane for the controlled Chen system (51), with initial conditions $x(0) = y(0) = \pm 10, z(0) = 5$, when $a = 35, b = 3, c = 28$ for (a) $k = -35/36$, converging to $C_\pm$; (b) $k = -0.9435$, converging to stable limit cycles; (c) $k = -0.941$, co-existence of stable limit cycles and chaos and (d) $k = -0.94$, chaos.

$$\dot{\tilde{z}} = -\frac{2a(2c-a)(1+k)}{c+(2c-a)k}\tilde{z} - \frac{(a-c)[c+(2c-a)k]^3\tilde{x} + [(2c-a)k]^3\omega_c\tilde{y} + 4(1+k)^2a^2(2c-a)^2\tilde{z}}{4(1+k)^2a^2(2c-a)^2 + \omega_c^2[c+(2c-a)k]^2}\mu + \cdots \tag{48}$$

Now, applying the formula [Yu & Huseyin, 1988] to system (41) yields

$$v_0 = \frac{-\omega_c^2[c+(2c-a)k]^2}{2\{4(1+k)^2a^2(2c-a)^2 + \omega_c^2[c+(2c-a)k]^2\}} < 0,$$

and the Maple programs [Yu, 1998] gives v_1 being a fourth-degree polynomial of k,

$$v_1 = \frac{\omega_c^2[c+(2c-a)k](a-c)\tilde{v}_1}{8(2c-a)\{a^2(2c-a)^2(1+k)^2 + \omega_c^2[c+(2c-a)k]^2\}[4a^2(2c-a)^2(1+k)^2 + \omega_c^2[c+(2c-a)k]^2},$$

where $\tilde{v}_1$ is given in (42). Therefore, the Hopf bifurcation is supercritical (resp. subcritical) if $v_1 < 0 (v_1 > 0)$; or equivalently if $\tilde{v}_1 < 0 (\tilde{v}_1 > 0)$ when $(a-c)(2c-a) > 0$.

To illustrate the application of the above analytical results, let us choose the parameter values for the Chen chaotic attractor: $a = 35, c = 28$. For this case, $c < a < 2c$, and

$$b_H = \frac{7(26+27k)}{4+3k}, \quad \omega_c = 7\sqrt{26+27k}. \tag{49}$$

In order to have $\omega_c > 0$, k must be chosen as

$$k > -\frac{26}{27} \approx -0.962962963, \tag{50}$$

which guarantees $b_H > 0$. For this case, $\tilde{v}_1$ becomes

$$\tilde{v}_1 = 7203(27k^4 - 171k^3 - 1032k^2 - 1474k - 638),$$

which has four real roots:

$$k = -2.059664313, \quad -1.142249478, \quad -0.9572404427, \quad 10.49248757.$$

Combined with condition (50), we obtain that $v_1 < 0$ if

$$k \in (-0.9572404427, 10.49248757).$$

For $b = 3$, the equilibria $C_\pm$ of the uncontrolled system are unstable. To stabilize $C_\pm$, it is seen from (44) that $k > -1$ since $2c - a > 0$, as well as $k < -(17/18)$. So the value of k for stabilizing $C_\pm$ is located in a very narrow interval $(-1, -(17/18))$. For example, we may choose $k = -(35/36)$. The simulation result is shown in Fig. 5(a). It is noted that this control is quite robust, i.e. the initial condition can be chosen far away from the equilibria.

If we choose $k = -0.941$, which yields $b_H = 3.52676 > 3$ and close to 3 (implying that the bifurcation parameter μ is small). For this control value, $C_\pm$ are unstable, giving rise to bifurcation of stable limit cycles, see Fig. 5(b). When k slightly increases, the system may exhibit co-existence of stable limit cycles and chaos, as shown in Fig. 5(c), or just chaotic motion [see Fig. 5(d)].

5. The Lü System

The Lü system is given by [Chen & Lü, 2003]

$$\dot{x} = a(y-x), \quad \dot{y} = cy - xz, \quad \dot{z} = -bz + xy, \tag{51}$$

where a, b and c are real parameters, usually taking positive values. The typical Lü's chaotic attractor is shown in Fig. 1(c).

The Lü system (51) also has three equilibrium solutions, C_0, C_+ and C_-, given by

$$\begin{aligned} &C_0 : x_e^0 = y_e^0 = z_e^0 = 0, \\ &C_\pm : x_e^\pm = y_e^\pm = \pm\sqrt{bc}, \quad z_e^\pm = c. \end{aligned} \tag{52}$$

Suppose a, b and c be positive. Then the characteristic polynomial associated with the equilibrium C_0 is $P_0(\lambda) = (\lambda+1)(\lambda+b)(\lambda-c)$, showing that C_0 is unstable when $c > 0$. The characteristic polynomial associated with the equilibria $C_\pm$ is given by

$$P_\pm(\lambda) = \lambda^3 + (a+b-c)\lambda^2 + ab\lambda + 2abc. \tag{53}$$

It can be shown that when $a + b - 3c > 0, C_\pm$ are stable, and they lose stability at the critical point:

$$b_H = 3c - a, \tag{54}$$

from which Hopf bifurcation occurs. When $b > b_H, C_\pm$ are stable.

5.1. *Without control*

At the critical point defined in (54), the uncontrolled Lü system (51) emerges to a Hopf bifurcation. At this critical point, the Jacobian of system (51) evaluated at C_+ and C_- has a real negative eigenvalue $-2c$ and a purely imaginary pair $\pm i\sqrt{a(3c-a)}$. With the transformation,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pm\sqrt{bc} \\ \pm\sqrt{bc} \\ c \end{pmatrix} + T \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}, \tag{55}$$

where

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & \dfrac{\omega_c}{a} & \dfrac{a-2c}{a} \\ \dfrac{\omega_c^2}{a\sqrt{bc}} & \dfrac{(c-a)\omega_c}{\sqrt{bc}} & \dfrac{2c(a-3c)}{a\sqrt{bc}} \end{bmatrix}, \tag{56}$$

we may transform (31) to

$$\begin{aligned} \dot{\tilde{x}} &= \omega_c\tilde{y} - \frac{a(a-c)\tilde{x} + a\omega_c\tilde{y} + 4c^2\tilde{z}}{(a+c)(a-4c)}\mu + \cdots \\ \dot{\tilde{y}} &= -\omega_c\tilde{x} + \frac{2c[a(a-c)\tilde{x} + a\omega_c\tilde{y} + 4c^2\tilde{z}]}{\omega_c(a+c)(a-4c)}\mu + \cdots \\ \dot{\tilde{z}} &= -2c\tilde{z} + \frac{a(a-c)\tilde{x} + a\omega_c\tilde{y} + 4c^2\tilde{z}}{(a+c)(a-4c)}\mu + \cdots \end{aligned} \tag{57}$$

where $\mu = d - d_H$ has been used.

Similarly, applying the formula and the Maple programs to system (51) yields the following focus values:

$$v_0 = \frac{\omega_c^2}{2(a+c)(a-4c)},$$
$$v_1 = \frac{3\omega_c^2(a-c)(2a-5c)}{8c(a+c)(4c-a)(c^2+\omega_c^2)}.$$

It is easy to see that $v_1 < 0$ if

$$c < a < \frac{5}{2}c.$$

For the parameter values of the typical Lü system: $a = 30, b = 44/15, c = 111/5$, we have $b_H = 183/5 > 44/15$. Hence, the two equilibria $C_\pm$ are unstable, and the trajectories are chaotic, as shown in Fig. 1(c). However, if we choose b close to b_H, then we may obtain stable limit cycles since $c < a < (5/2)c$ is satisfied for this case. For example, simulation results for $b = 35$ and $b = 40$ are depicted in Figs. 6(a) and 6(b), respectively, confirming the analytical predictions. These results are similar to the Chen system [see Fig. 4], not sensitive to the initial conditions.

5.2. *With feedback control*

For the consistency, we use a similar control law as that used for the Chen system, given by

$$u_2 = kx(z-c), \tag{58}$$

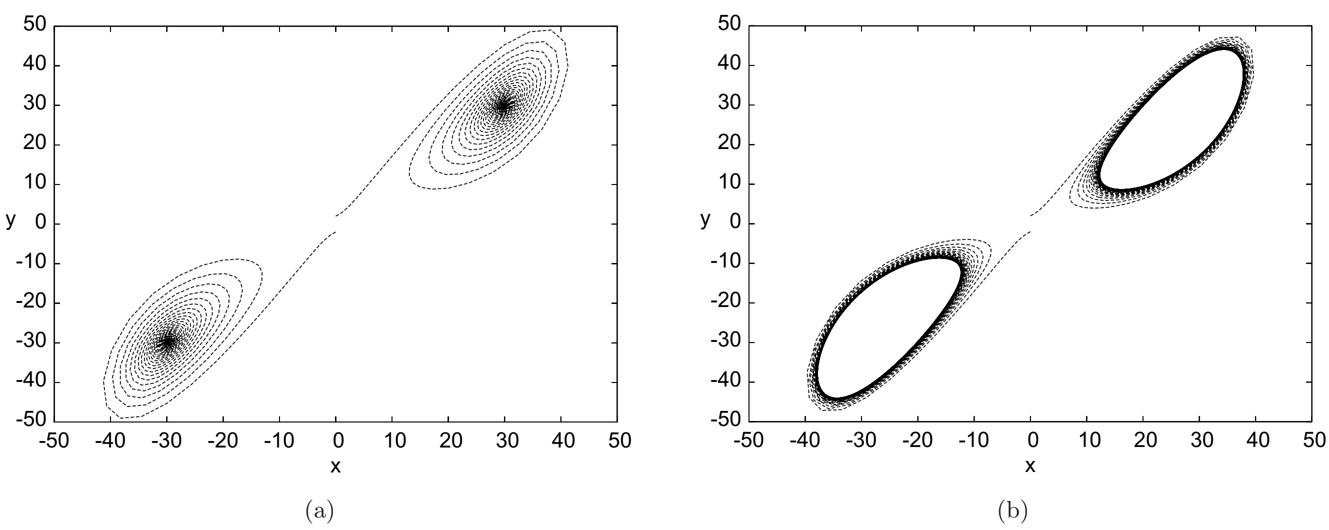

Fig. 6. Simulated trajectories projected on the x–y plane for the uncontrolled Lü system (51) when $a = 30, c = 111/5$, with initial conditions $x(0) = 0, y(0) = \pm 2, z(0) = 15$, for (a) $b = 40$, converging to $C_\pm$ and (b) $b = 35$, converging to stable limit cycles.

which, in turn, yields the closed-loop Lü system as

$$\begin{aligned}\dot{x} &= a(y-x),\\ \dot{y} &= cy - xz - kx(z-c),\\ \dot{z} &= -bz + xy,\end{aligned} \tag{59}$$

whose equilibria C_0 and $C_\pm$ are not changed. For the controlled system (59), we have the following result for Hopf bifurcation control: Hopf bifurcation emerging from the equilibrium $C_\pm$ is supercritical if the feedback control gain coefficient k is chosen such that $v_1 < 0$, where

$$\begin{aligned}v_1 &= \frac{(a-c)(a+ck)}{8c[a^2c^2(1+k)^2+\omega_c^2(c+ck)^2][4a^2c^2(1+k)^2+\omega_c^2(c+ck)^2]}\tilde{v}_1,\\ \tilde{v}_1 &= c^4k^4 + c^3(a-5c)k^3 + c^2a(a-21c)k^2 + ca(9a^2-35ca+2c^2)k + 3a^3(2a-5c).\end{aligned} \tag{60}$$

We give a detailed analysis below. First of all, a linear analysis shows that the equilibrium C_0 is stable if $(1+k) < 0$ and $a-c > 0$. Similarly, one can show that the two symmetric equilibria $C_\pm$ are stable if

$$1+k>0, \quad a+ck>0 \quad \text{and} \quad (a+b-c)(a+ck) > 2ac(1+k). \tag{61}$$

Hopf bifurcation for Lü system may occur at the critical point:

$$b_H = \frac{c(a+c)(1+k)-(a-c)^2}{a+ck}. \tag{62}$$

The eigenvalues evaluated at $b = b_H$ are

$$\lambda_{1,2} = \pm i\sqrt{c(a+c)(1+k)-(a-c)^2}, \quad \lambda_3 = -\frac{2ac(1+k)}{a+ck} < 0, \tag{63}$$

under the assumption: $1+k>0, a+ck>0$ and $c(a+c)(1+k)-(a-c)^2>0$.

Let $d = d_H + \mu$. Then applying the transformation (37), with

$$T = \begin{bmatrix} 1 & 0 & 1\\ 1 & \dfrac{\omega_c}{a} & \dfrac{a-2c-ck}{a+ck}\\ \dfrac{\omega_c^2}{a(1+k)\sqrt{bc}} & \dfrac{(c-a)\omega_c}{a(1+k)\sqrt{bc}} & -\dfrac{2c\omega_c^2}{(a+ck)^2\sqrt{bc}}\end{bmatrix} \tag{64}$$

to system (59) yields

$$\begin{aligned}\dot{\tilde{x}} &= \omega_c\tilde{y} + \frac{(a-c)(a+ck)^3\tilde{x} + (a+ck)^3\omega_c\tilde{y} + 4(1+k)^2a^2c^2\tilde{z}}{4(1+k)^2a^2c^2+\omega_c^2(a+ck)^2}\mu + \cdots\\ \dot{\tilde{y}} &= -\omega_c\tilde{x} - \frac{2ac(1+k)\{(a-c)(a+ck)^3\tilde{x} + (a+ck)^3\omega_c\tilde{y} + 4a^2c^2(1+k)^2\tilde{z}\}}{\omega_c(a+ck)[4(1+k)^2a^2c^2+\omega_c^2(a+ck)^2]}\mu + \cdots\\ \dot{\tilde{z}} &= -\frac{2ac(1+k)}{a+ck}\tilde{z} - \frac{(a-c)(a+ck)^3\tilde{x} + (a+ck)^3\omega_c\tilde{y} + 4(1+k)^2a^2c^2\tilde{z}}{4(1+k)^2a^2c^2+\omega_c^2(a+ck)^2}\mu + \cdots.\end{aligned} \tag{65}$$

The formula [Yu & Huseyin, 1988] and the Maple programs [Yu, 1998] that are employed to system (59) result in

$$v_0 = \frac{-\omega_c^2(a+ck)^2}{2[4(1+k)^2a^2c^2+\omega_c^2(a+ck)^2]} < 0,$$

and v_1 being a fourth-degree polynomial of k, given in (60). Thus, when $v_1 < 0$ (>0) (or equivalently, $\tilde{v}_1 < 0$ (>0) if $a > c$), the Hopf bifurcation is supercritical (resp. subcritical).

To end this section, we present a couple of numerical simulation results to illustrate the application. For the typical parameter values of the Lü chaotic attractor: $a = 30, c = 111/5$, we have

$$b_H = \frac{3(3050+3219k)}{5(50+37k)} \quad \text{and} \quad \omega_c = \frac{3}{5}\sqrt{3050+3219k}. \tag{66}$$

$\omega_c > 0$ requires

$$k > -\frac{3050}{3219} \approx -0.9474992234, \qquad (67)$$

which guarantees $b_H > 0$, and v_1 then becomes

$$\tilde{v}_1 = \frac{81}{625}(1874161k^4 - 6838155k^3 - 49763150k^2 - 73097200k - 31875000),$$

which has fourth real roots:

$$k \approx -2.002559540,\ -1.167523751,\ -0.9378275909,\ 7.756559531.$$

Combined with the condition (67), we obtain that $v_1 < 0$ if

$$k \in (-0.9378275909, 7.756559531).$$

For $b = 44/15$, the equilibria $C_\pm$ of the uncontrolled system are unstable. To stabilize $C_\pm$, it is seen from (61) that $-1 < k < -(25250/27343) \approx -0.9234539004$. For example, we may choose $k = -0.96$. The simulation result is shown in Fig. 7(a). Again, it has been noted that under this control the basins of the stable equilibria $C_\pm$ are quite large.

If we choose $k = -0.923$, this yields $b_H = 2.985538 > 44/15$ and close to 44/15. For this control value, $C_\pm$ are unstable, giving rise to

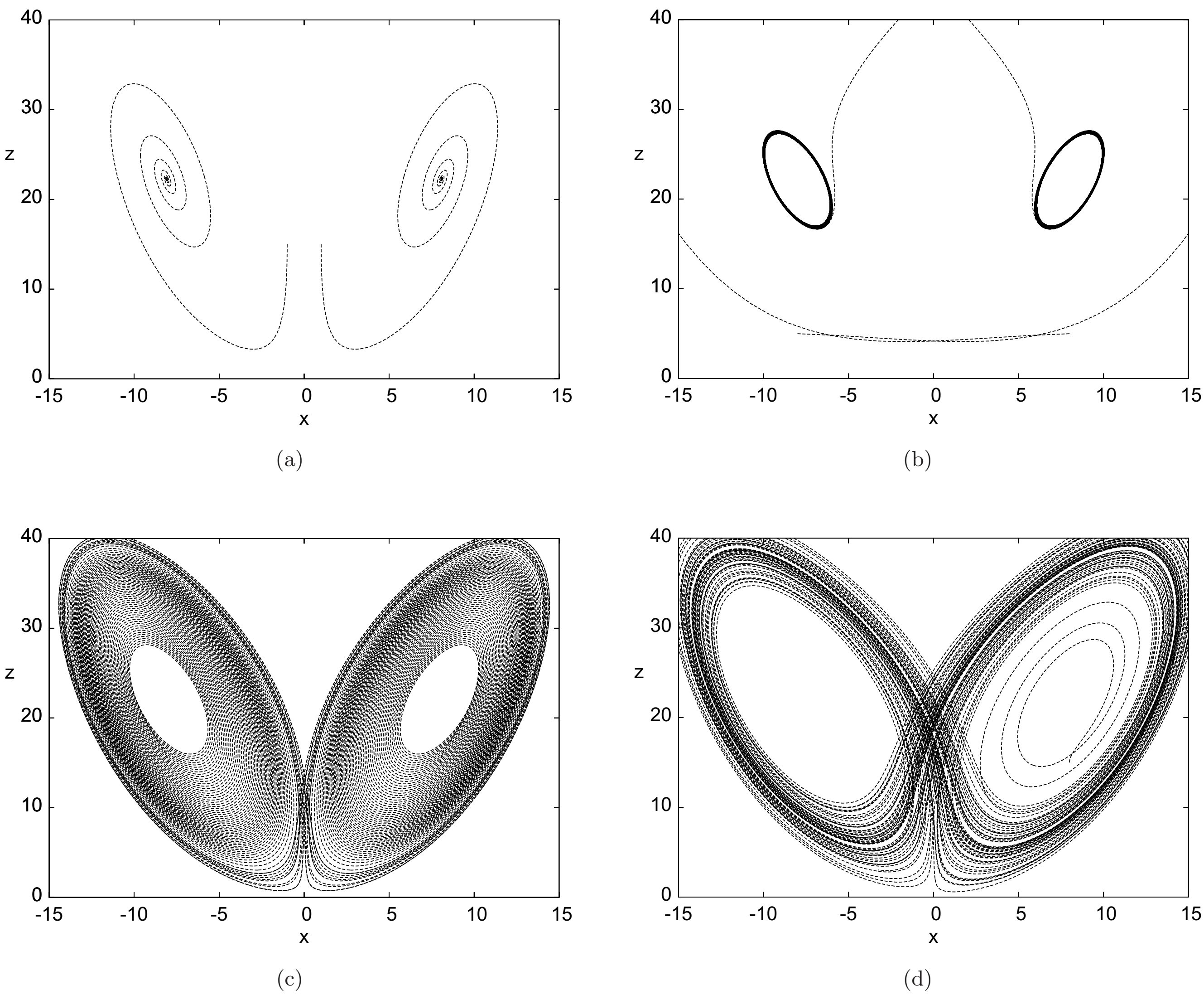

Fig. 7. Simulated trajectories projected on the x–z plane for the controlled Lü system (51) when $a = 30, b = 44/15, c = 111/5$ for (a) $k = -0.96$, converging to $C_\pm$ with initial conditions $x(0) = y(0) = \pm 1, z(0) = 15$; (b) $k = -0.923$, converging to stable limit cycles with initial conditions $x(0) = \pm 8, y(0) = 0, z(0) = 5$; (c) $k = -0.92$, co-existence of stable limit cycles and chaos with initial conditions $x(0) = y(0) = \pm 8, z(0) = 15$ and (d) $k = -0.90$, chaos with initial conditions $x(0) = y(0) = \pm 8, z(0) = 15$.

bifurcation of stable limit cycles, as shown in Fig. 7(b). When k is increased a little bit, the system can still have stable limit cycles but with chaos co-existing [see Fig. 7(c)] or just chaotic motion [see Fig. 7(c)].

6. Conclusion

In this paper, an early developed control formula is used for controlling Hopf bifurcations in a class of Lorenz-like systems. It has been shown that simple control laws in a single quadratic term can be applied, which not only leaves unchanged the equilibrium solutions of the original system, but can also stabilize equilibrium solutions or periodic motions bifurcating from a Hopf critical point. This approach does not guarantee the global stability, but does not require ultimate boundedness of trajectories, which is usually needed when applying Lyapunov function method. In certain cases, it may be possible to suppress chaotic motions via Hopf bifurcation control, in particular, by stabilizing equilibrium solutions. The method proposed in this paper can be extended to consider other singular cases, associated with double Hopf, Hopf-zero and double zero.

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