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Bifurcation of limit cycles in a cubic-order planar system around a nilpotent critical point

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ABSTRACT

In this paper, bifurcation of limit cycles is considered for planar cubic-order systems with an isolated nilpotent critical point. Normal form theory is applied to compute the generalized Lyapunov constants and to prove the existence of at least 9 small-amplitude limit cycles in the neighborhood of the nilpotent critical point. In addition, the method of double bifurcation of nilpotent focus is used to show that such systems can have 10 small-amplitude limit cycles near the nilpotent critical point. These are new lower bounds on the number of limit cycles in planar cubic-order systems near an isolated nilpotent critical point. Moreover, a set of center conditions is obtained for such cubic systems.

1. Introduction

Dynamical systems can exhibit self-sustained oscillations, called limit cycles, which may appear in almost all fields of science and engineering. Developing limit cycle theory is not only theoretically significant, but also practically important. Limit cycles theory is closely related to the well-known Hilbert's 16th problem, one of the 23 mathematical problems proposed by D. Hilbert in 1900 [25]. A modern version of this problem was included in the 18 most challenging mathematical problems proposed by S. Smale for the 21st century [35].

Consider the following planar differential system:

$$\frac{dx}{dt} = P_n(x, y), \qquad \frac{dy}{dt} = Q_n(x, y), \tag{1.1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are nth-degree polynomials in x and y. The second part of Hilbert's 16th problem is to find an upper bound on the number of limit cycles that system (1.1) can have. This upper bound, denoted as $H(n)$, is called Hilbert number. For general quadratic polynomial systems, four limit

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cycles were found in 1979 [33,14], which were also obtained recently in near-integrable quadratic systems [46]. However, whether $H(2) = 4$ or not is still an open question. For cubic-degree polynomial systems, many results have been obtained on the low bound of the Hilbert number, and the best result so far is $H(3) \geq 13$ [26,27]. In real applications, bifurcation of limit cycles due to Hopf bifurcation is a common phenomenon, but real systems often have dimension higher than two [24,49,50]. In such a case, the system can be first reduced to a 2-dimensional dynamical system by using center manifold theory (e.g., see [24,19]) and then to study the limit cycles bifurcation in the reduced system.

Later, Arnold [7] posed the weak infinitesimal Hilbert's 16th problem, which is closely related to the so-called near-Hamiltonian system [20]:

$$\frac{dx}{dt} = H_y(x,y) + \varepsilon\, p_n(x,\, y), \qquad \frac{dy}{dt} = -\, H_x(x,y) + \varepsilon\, q_n(x,\, y), \tag{1.2}$$

where $H(x,y)$, $p_n(x,y)$ and $q_n(x,y)$ are all polynomial functions in x and y, and $0 < \varepsilon \ll 1$ is a small perturbation parameter. Then, the problem on the study of number of limit cycles is transformed to investigating the zeros of the Abelian integral or the (first-order) Melnikov function:

$$M(h,\delta) = \oint_{H(x,y)=h} q_n(x,y)\, dx - p_n(x,y)\, dy, \tag{1.3}$$

where $H(x,y) = h$ for $h \in (h_1, h_2)$ defines a closed orbit, and δ is a vector parameter, representing the parameters (or coefficients) involved in the polynomials $p_n(x,y)$ and $q_n(x,y)$.

When the study of Hilbert's 16th problem is restricted to the vicinity of an isolated fixed point, which is either an elementary focus or a center, it becomes an investigation on generalized Hopf bifurcations, and the number of bifurcating small-amplitude limit cycles is usually denoted by $M(n)$. It is well known that $M(2) = 3$, obtained by Bautin in 1952 [9]. For $n = 3$, many results have been obtained, divided into two categories. For systems with an elementary focus, the best result obtained so far is 9 limit cycles [44,13, 31]. On the other hand, for systems with a center, there are also a few results obtained in the past two decades. In 1995, Żołądek [52] first proposed a rational Darboux integral, and claimed the existence of 11 small-amplitude limit cycles around a center, which was reinvestigated recently and proved that this system can actually have only 9 limit cycles [45,40]. After more than ten years, another two cubic-order systems were constructed to show 11 limit cycles [15,11]. Recently, the system considered in [15] was used by Yu and Tian to show the existence of 12 small-amplitude limit cycles around a singular point, which is the best result so far for cubic systems.

To consider bifurcation of limit cycles associated with a singularity of focus, Lyapunov constants are needed to solve the center-focus problem and to determine the number and stability of bifurcating limit cycles. There mainly exist three methods for computing Lyapunov constants: the method of normal forms [24, 16,42], the method of Poincaré return map or focus value method [6,30], and the method of Lyapunov function [34,17]. Other approaches can be found, for example, in [24]. To demonstrate the basic idea of these methods, without loss of generality, assume that system (1.1) has a singularity of focus at the origin, and that the Jacobian of the system evaluated at the origin has a purely imaginary pair: $\pm\, i\omega_c$. Then, by using the method of normal forms with the aid of a computer algebra system such as Maple or Mathematica (e.g., see [24,42,38,39]) we compute the normal form to obtain the Lyapunov constants L_k which are used to determine the number of bifurcating limit cycles around the critical point. $v_k\, (k = 0, 1, 2, \cdots)$.

The above mentioned three methods for computing Lyapunov constants have also been used to study the center-focus problem associated with nilpotent critical points, see for example [2,12,32]. But the method of normal forms was only recently applied to compute the so-called *generalized Lyapunov constants* in determining the lower bound of cyclicity [3]. It is well known that it is more difficult to distinguish focus from center when the singular point is degenerate. In [4] Andreev considered the local phase portraits of analytic systems with the origin being a nilpotent singular point, which however does not distinguish focus from center. Later, Takens developed a normal form theory for systems with nilpotent center of foci [36], and Moussu obtained the C^{∞} normal form for analytic nilpotent centers [32]. Further, Berthier and Moussu studied the reversibility of nilpotent centers [10], while Teixeria and Yang applied a convenient normal form to investigate the relationship between reversibility and the center-focus problem, and then studied the reversibility of certain types of polynomial vector fields [37]. Recently, by using Melnikov function method Han and Li [22], and Zhao and Fan [51] considered polynomial Hamiltonian systems with elementary centers to obtain lower bounds on the Hilbert number. Moreover, Han et al. [23] studied polynomial Hamiltonian systems with a nilpotent singular point, and obtained necessary and sufficient conditions for determining the number of limit cycles bifurcating in quadratic and cubic Hamiltonian systems with a nilpotent singular point which may be a center, a cusp or a saddle. However, it should be pointed out that the Melnikov function method used in the above mentioned articles [23,22,51] can not be applied to study the systems considered in this paper, since our systems here are not Hamiltonian, nor even integrable.

The main goal of this paper is to consider bifurcation of limit cycles in cubic polynomial systems and apply our general normal form computation method to obtain new lower bounds on the number of limit cycles. More specifically, we will show that cubic polynomial systems can have at least 9 small-amplitude limit cycles around an isolated nilpotent critical point, and at least 10 small-amplitude limit cycles near an isolated nilpotent critical point. Moreover, a set of center conditions is obtained for such cubic systems. In the next section, we present some basic formulations and preliminary results which are needed in proving our main results in Sections 3, 4 and 5. Conclusion is drawn in Section 6.

2. Mathematical formulation and preliminary results

In this section, we present some basic formulas and preliminary results which will be used in the following sections. Consider the differential system:

$$\begin{aligned} \frac{dx}{dt} &= y + F_1(x,y) = \sum_{j+k}^{\infty} a_{jk} x^j y^k, \\ \frac{dy}{dt} &= F_2(x,y) = \sum_{j+k}^{\infty} b_{jk} x^j y^k, \end{aligned} \tag{2.1}$$

where F_1 and F_2 are analytic in the neighborhood of the origin, with power series beginning from second order. As long as the limit cycles bifurcation is considered near the origin, system (2.1) with a nilpotent center at the origin is more difficult to analyze than the general system (1.1) with an element center or focus at the origin, since the conventional normal form of Hopf bifurcation [24,19] can be directly applied to the latter but not the former. In fact, there exist conventional normal forms for system (2.1) associated with Bogdanov–Takens bifurcation (i.e., the linearized system contains a double-zero eigenvalue at the origin) [24, 19], which is however not able to be directly applied to study bifurcation of limit cycles near the origin. Therefore, a modified normal form of system (2.1) needs to be developed to study bifurcation of limit cycles near the origin. In real applications, many physical systems involve a number of parameters and can thus have higher co-dimensional singularity such as Bogdanov–Takens bifurcation (which is characterized by a double-zero eigenvalue at a critical point, leading to a nilpotent singular point), and thus it is interesting and important to explore the periodic solutions near such a critical point. For example, in the 2-dimensional HIV model [48], a critical point with Bogdanov–Takens bifurcation is identified for certain parameter values and thus the system can be put in the form of system (2.1) in the vicinity of the critical point. Limit cycles due to Hopf bifurcation have been obtained near this critical point and even multiple limit cycles can be found if more parameters are treated as bifurcation parameters. Moreover, homoclinic orbits are identified near this degenerate singular point [48].

To mathematically analyze bifurcation of limit cycles for system (2.1) near the origin, we first present the following result [4,2,3], which can be used to determine the monodromy of the origin of system (2.1).

Lemma 2.1. (*Theorem* 2.1 *in* [3]) *Assume that the origin of system* (2.1) *is an isolated singularity. Define two functions* $f(x)$ *and* $\phi(x)$ *as*

$$\begin{aligned}\phi(x) &= \frac{\partial F_1(x, f(x))}{\partial x} + \frac{\partial F_2(x, f(x))}{\partial y},\\ \psi(x) &= F_2(x, f(x)) = a\, x^{\alpha} + O(x^{\alpha+1}), \quad a \neq 0,\ \alpha \geq 2,\end{aligned}$$

where $y = f(x)$ *is the solution of the equation,* $y + F_1(x, y) = 0$*, passing through the origin* $(0, 0)$*. Write* $\phi(x) = bx^{\beta} + O(x^{\beta+1})$*,* $b \neq 0$ *and* $\beta \geq 1$*, or* $\phi(x) \equiv 0$*. Then, the origin of system* (2.1) *is monodromic if and only if* $a < 0$*,* $\alpha = 2n-1$ $(n \geq 1)$ *being an odd number, and one of the following three conditions holds:*

(i) $\beta > n - 1$;
(ii) $\beta = n - 1$, *and* $b^2 + 4an < 0$;
(iii) $\phi \equiv 0$.

Under the above conditions, we can apply the classical normal form theory, with the following near-identity transformation,

$$x = u + \sum_{i+j=2}^{k} h_{1ij} u^i v^j, \quad y = v + \sum_{i+j=2}^{k} h_{2ij} u^i v^j, \tag{2.2}$$

to obtain the conventional normal form [19,24]:

$$\begin{aligned}\frac{du}{d\tau} &= v + O(\|(u,v)\|^{k+1}),\\ \frac{dv}{d\tau} &= -\,u^{2n-1} + \sum_{j\geq\beta}^{k-1} (\overline{A}_j u^{j+1} + \overline{B}_j u^j v) + O(\|(u,v)\|^{k+1}).\end{aligned} \tag{2.3}$$

This conventional normal form can not be directly used to find the limit cycles bifurcating from the origin. However, if we use the idea of the simplest normal form theorem (or unique normal form theory) (e.g., see [8,43,47,18]) and introduce a time rescaling,

$$\tau = \Big(1 + \sum_{i+j=2}^{k} h_{3ij} u^i v^j\Big)\, t, \tag{2.4}$$

into system (2.3), we obtain

$$\begin{aligned}\frac{du}{d\tau} &= v + O(\|(u,v)\|^{k+1}),\\ \frac{dv}{d\tau} &= -\,u^{2n-1} + v \sum_{j\geq\beta}^{k-1} B_j u^j + O(\|(u,v)\|^{k+1}),\end{aligned} \tag{2.5}$$

where B_j is called the jth-order *generalized Lyapunov constant*. We have developed an algorithm with explicit recursive formulas for computing B_j for the general system (2.1), with a computationally efficient Maple program which can be easily implemented on a computer using Maple. It has been noted that Liu and Li [28] have developed a different method to compute the so-called *quasi Lyapunov constants*, which are equivalent to the generalized Lyapunov constants. However, their method is only applicable for cubic-order systems. Before we particularly consider bifurcation of limit cycles in cubic-order systems with an isolated nilpotent critical point, we present few examples, which have been investigated in [3,5,1], to illustrate the general applicability of our method. The method of normal forms has been used in [3] to study bifurcation of limit cycles, and many examples are presented in this paper. For example, consider the system,

$$\begin{aligned} \frac{dx}{dt} &= -\,y, \\ \frac{dy}{dt} &= x^5 + a\,x^6 + y\,(b\,x^3 + c\,x^4). \end{aligned} \tag{2.6}$$

Note in the first equation of (2.6) that the first term is $-y$ rather than y. But this does not affect the normal forms computation provided we apply a transformation $y \to -y$ if it is necessary for executing a computer program. We used the normal form computation method developed in [3] and coded a Maple program to obtain the following normal form:

$$\begin{aligned} \frac{du}{d\tau} &= -\,v, \\ \frac{dv}{d\tau} &= u^5 + v\big[\,bu^3 + \big(c - \tfrac{5}{7}ab\big)\,u^4 + \big(\tfrac{36}{49}\,a^2b - \tfrac{6}{7}ac\big)\,u^5 \\ &\qquad + \tfrac{13}{294}a^2(21c - 19ab)\,u^6 + \tfrac{80}{1029}a^3(13ab - 14c)\,u^8 + O(u^8)\big], \end{aligned} \tag{2.7}$$

which is exactly the same as that given in [3] except the coefficient $\frac{6}{7}$ which was typed as 6 in [3]. We have used our method and executed our Maple program to obtain the following generalized Lyapunov constants:

$$\begin{aligned} &B_4 = c - \tfrac{5}{7}\,ab, \quad B_6 = \tfrac{13}{294}\,a^2(21c - 19ab), \quad B_8 = -\,\tfrac{729}{19208}\,a^4(33ab - 35c), \\ &B_{10} = -\,\tfrac{5113889}{118590192}\,a^6(47ab - 49c), \;\cdots \end{aligned}$$

It is seen that B_4 and B_6 are exactly the same as that given in (2.7). Further, it is easy to verify that setting $B_4 = B_6 = 0$ leads to $B_{2k} = 0$, $k \geq 4$.

In [5], the authors consider a special case – homogeneous polynomial systems and developed a special approach to calculate the generalized Lyapunov constants. Their methodology is computationally efficient, but can not be applied even to consider a simple cubic polynomial system. The 5th-order homogeneous polynomial system considered in [5] is given by

$$\begin{aligned} \frac{dx}{dt} &= y + Ax^4y + Bx^3y^2 + Cx^2y^3 + Dxy^4 + Ey^5, \\ \frac{dy}{dt} &= -x^5 + Qx^4y + Kx^3y^2 + Lx^2y^3 + Mxy^4 + Ny^5. \end{aligned} \tag{2.8}$$

Using our Maple program, we obtain the following generalized Lyapunov constants:

$$\begin{aligned} &B_4 = Q, \quad B_8 = B + L, \quad B_{12} = \tfrac{1}{7}\big[2L(K + 2A) + 3(D + 5N)\big], \\ &B_{16} = \tfrac{2}{11}\big[(2A + K)(KL + 3N) + L(C + 2M)\big], \\ &B_{20} = \tfrac{14}{81}\big[LM(2A + K) + 3N(C + 2M)\big], \quad B_{24} = \tfrac{20}{741}L^3(2A + K), \end{aligned}$$

where $B_{4(k-1)} = 0$ has been set zero when computing B_{4k} for $k = 2, 3, \ldots, 6$. They are the same as that given in [5], at most different by a positive constant factor.

Another special type of systems called quasi-homogeneous system is considered in [1], which takes the general form:

$$\begin{aligned}\frac{dx}{dt} &= y + \sum_{i=0}^{\infty} P_{q-p+2is}(x,y),\\ \frac{dy}{dt} &= \sum_{i=0}^{\infty} Q_{q-p+2is}(x,y),\end{aligned} \tag{2.9}$$

where $p, q \in \mathbb{N}$, $p \le q$, $s = (n+1)p - q > 0$, $n \in \mathbb{N}$, and P_i and Q_i are quasi-homogeneous polynomials in x and y with $Q_{q-p+2s}(1,0) < 0$. The origin of this system is a nilpotent and monodromic isolated singular point. The authors used their method developed in [1] to obtain the center conditions for the origin of the following system,

$$\begin{aligned}\frac{dx}{dt} &= y + a_1x^5 + a_2x^2y + a_3x^7 + a_4x^4y + a_5xy^2,\\ \frac{dy}{dt} &= -x^7 + b_1x^4y - a_2xy^2 + b_3x^6y + b_4x^3y^2 + b_5y^3.\end{aligned} \tag{2.10}$$

Executing our Maple program, we obtain the following generalized Lyapunov constants for the above system:

$$\begin{aligned}
B_4 &= 5a_1 + b_1, \quad B_6 = 7a_3 + b_3, \quad B_8 = a_5 + 3b_5 - 2a_1(b_4 + 2a_4),\\
B_{10} &= -2(2a_4 + b4)(a_3 - a_1a_2 + 4a_1^3),\\
B_{12} &= -\tfrac{2}{5}(b_4 + 2a_4)\big[a_5 - a_1(4a_4 - b_4 - 50a2a_1^2 + 200a_1^4)\big],\\
B_{14} &= -\tfrac{2}{7}a_1(b_4 + 2a_4)(a_2 - 4a_1^2)(b_4 - a_4 + 62a_1^2 - 268a_1^4),\\
B_{16} &= \tfrac{4}{9}a_1^4(a_2 - 4a_1^2)(3a_4 - 62a_1^2 + 268a_1^4)(5a_4 - 12a_2^2 + 146a_2a_1^2 - 492a_1^4),\\
B_{18} &= -\tfrac{64}{2475}a_1^5(a_2 - 4a_1^2)(9a_2^2 - 187a_2a_1^2 + 704a_1^4)(387a_2^2 - 4681a_2a_1^2 + 13282a_1^4),\\
B_{20} &= \tfrac{32}{975}a_1^5(a_2 - 4a_1^2)(9a_2^2 - 187a_2a_1^2 + 704a_1^4)\\
&\quad \times(1953a_2^3 - 27694a_2^2a_1^2 + 130023a_2a_1^4 - 201730a_1^6),
\end{aligned}$$

where $B_{2(k-1)} = 0$ has been used in computing B_{2k} for $k = 3, 4, \ldots, 10$. Based on these generalized Lyapunov constants, we have the following result.

Proposition 2.1. *The origin of system (2.10) is a center if and only if one of the following conditions is satisfied:*

(i) $5a_1 + b_1 = 7a_3 + b_3 = 2a_4 + b_4 = a_5 + 3b_5 = 0$;
(ii) $a_1 = a_3 = a_5 = b_1 = b_3 = b_5 = 0$; *and*
(iii) $b_1 = -5a_1,\ a_2 = 4a_1^2,\ b_5 = a_1b_4,\ a_5 = a_1(4a_4 - b_4),\ a_3 = b_3 = 0$.

Note that the three center conditions are given in Theorem 3.1 of [1], but the condition $b = -5a_1$ in (iii) was typed as $b_1 = -a_1$ in [1], and in addition, the conditions $a_3 = b_3 = 0$ were missed in (iii). It is easy to verify that under the condition (i) system (2.10) is a Hamiltonian system with the Hamiltonian function:

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{8}x^8 + a_1x^5y + \frac{1}{2}a_2x^2y^2 + a_3x^7y + \frac{1}{2}a_4x^4y^2 - b_5xy^3.$$

For the condition (ii), it is easy to see that system (2.10) is a reversible system since it is invariant under the transformation $(y,t) \to (-y,-t)$.

For the condition (iii), we present a simple proof different from that given in [1]. In fact, for this case, we use the following integrating factor,

$$I_{(\mathrm{iii})} = \frac{4a_4 b_4 (b_4 - 2a_4)(1 + a_4 x^4 + 4a_1 a_4 xy)^{b_4 - 1}}{[2 + b_4 x^4 + 4a_1 b_4 xy + b_4(2a_4 - b_4)y^2]^{1+2a_4}},$$

to obtain the first integral,

$$F(x,y) = \frac{(1 + a_4 x^4 + 4a_1 a_4 xy)^{b_4}}{[2 + b_4 x^4 + 4a_1 b_4 xy + b_4(2a_4 - b_4)y^2]^{2a_4}}.$$

Now, we return to cubic-order systems with an isolated nilpotent critical point and want to find the maximal number of limit cycles which bifurcate in the neighborhood of the critical point. In [28], Liu and Li have considered the following cubic polynomial system,

$$\begin{aligned} \frac{dx}{dt} &= y - 2xy - (a_4 - a_7)x^2 y + a_6 y^2 + a_2 xy^2 + a_5 y^3, \\ \frac{dy}{dt} &= -2x^3 + a_1 x^2 y + y^2 + a_4 xy^2 + a_3 y^3, \end{aligned} \tag{2.11}$$

which contains 7 free parameters. Thus, by adding a linear perturbation, the authors applied their approach to prove the existence of at least 8 small-amplitude limit cycles bifurcating from the origin. In fact, using our method, we can find the generalized Lyapunov constants as follows:

$$\begin{aligned}
B_2 &= a_1, \\
B_4 &= 2\,(a_2 + 3a_3), \\
B_6 &= \tfrac{4\,a_7}{3}\,(3a_3 - 5a_6), \\
B_8 &= \tfrac{4\,a_6 a_7}{105}\,(735 - 105a_4 + 71a_7), \\
B_{10} &= \tfrac{8}{11025}\,a_6 a_7\big(176400 + 18375a_5 + 5460a_7 + 12250a_6^2 - 32a_7^2\big), \\
B_{12} &= \tfrac{32\,a_6 a_7}{573024375}\,\big(30866913000 + 2089303650a_7 - 1188495000a_6^2 \\
&\quad + 29397690a_7^2 - 15232875a_6^2 a_7 - 110996a_7^3\big), \\
B_{14} &= \tfrac{-32\,a_6 a_7}{59727219749071875}\,\big(443894563225159200000 + 21558071645509770000a_7 \\
&\quad - 1647138037233150000a_6^2 - 11437991172477450a_7^2 - 910916029415875a_7^3 \\
&\quad - 22121192499656250a_6^4 + 798220526556a_7^4\big), \\
B_{16} &= \tfrac{32\,a_6 a_7}{18394686513039219979689843750}\,\big(9423379312441682897451542400000 \\
&\quad + 1514298765681319947369112800000a_7 \\
&\quad + 82859324997946429848009339000a_7^2 \\
&\quad + 1864567030459291902188584650a_7^3 \\
&\quad + 14562086011231729200961815a_7^4 - 266619108568395354750 8a_7^5\big),
\end{aligned}$$

where B_{2k} is the kth generalized Lyapunov constant, and $B_{2(k-1)}$ has been set zero in computing B_{2k} for $k = 2, 3, \cdots, 8$. It is noted that the quasi Lyapunov constants, λ_k, $k = 1, 2, \cdots, 8$ obtained in [28], are indeed given by $\lambda_k = \frac{1}{2k+1} B_{2k}$, $k = 1, 2, \cdots, 8$. Then, by applying proper perturbations to show that there exist parameter values satisfying $B_2 = B_3 = \cdots = 0$, but $B_{16} \neq 0$, implying the existence of 7 limit cycles. In addition, the linear perturbation gives one more limit cycle to achieve 8 limit cycles [28]. Later, the same authors considered the following system in [29],

$$\begin{aligned} \frac{dx}{dt} &= y - 2b_{02}xy + a_{02}y^2 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \frac{dy}{dt} &= -2x^3 + b_{02}y^2 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{2.12}$$

which is obviously the same as system (2.11), and can have 8 small-amplitude limit cycles bifurcating from the origin. Moreover, in [29], the authors applied the so called method of double bifurcation of nilpotent focus to get 9 small-amplitude limit cycles, with the distribution of $7 \supset (1 \cup 1)$. That is, there are three singular points, one of which is the origin and other two are near the origin with one limit cycle around each of them, and 7 limit cycles enclose these two limit cycles. The basic idea is to apply perturbation to system (2.12) to obtain a perturbed system as follows:

$$\begin{aligned} \frac{dx}{dt} &= y - 2b_{02}xy + a_{02}y^2 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \frac{dy}{dt} &= 4\delta\varepsilon y + b_{02}y^2 + b_{12}xy^2 + b_{03}y^3 - (x^2 - \varepsilon^2)(2x - b_{21}y), \end{aligned} \tag{2.13}$$

where δ and ε are perturbation parameters, satisfying $0 < |\delta| \ll 1,\ 0 < \varepsilon \ll 1$. It is easy to see that system (2.13) has three fixed points: $(x, y) = (\varepsilon, 0)$, $(-\varepsilon, 0)$ and $(0, 0)$. Thus, at $\delta = \varepsilon = 0$, we have the 8 generalized Lyapunov constants showing the existence of 7 limit cycles around the origin $(0, 0)$. Then, by taking proper perturbation values of δ and ε, we can find two small-amplitude limit cycles inside the 7 limit cycles, each of them encloses one of the two singular points $(\varepsilon, 0)$ and $(-\varepsilon, 0)$. More details about the method of double bifurcation of nilpotent focus can be found in [29]. Although this approach does not give all 9 limit cycles around the origin, it does have one more limit cycle near the origin, compared with the result obtained in [28].

Recently, we have studied bifurcation of near-Hamiltonian systems, described by

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H(x, y, \mu_1)}{\partial y} + \varepsilon\, P(x, y, \mu_2), \\ \frac{dy}{dt} &= -\,\frac{\partial H(x, y, \mu_1)}{\partial x} + \varepsilon\, Q(x, y, \mu_2), \end{aligned} \tag{2.14}$$

where $H(x, y, \mu_1)$ is an nth-degree real polynomial in x and y and P, Q are mth-degree of polynomials in x and y, and μ_1 and μ_2 are vector parameters, and $0 < \varepsilon \ll 1$ is a small perturbation parameter. The function $H(x, y, \mu_1)$ is called the Hamiltonian of system (2.14). When $\varepsilon = 0$, the origin is a nilpotent center of the system.

The monodromy of the origin of system $(2.14)|_{\varepsilon=0}$ has been studied in [21] and detailed classification conditions are given. Very recently, we have applied our new method to consider the following cubic near-Hamiltonian system:

$$\begin{aligned} \frac{dx}{dt} &= y + 2\,xy + 3\,a_1\,y^2 + 2\,a_2\,x^2y + 3\,a_3\,xy^2 + 4\,a_4\,y^3, \\ \frac{dy}{dt} &= -\,4x^3 - y^2 - 2a_2xy^2 - a_3y^3 + \varepsilon\,(\delta x + \delta y + xy + b_1y^2 + b_2x^2y + b_3xy^2 + b_4y^3), \end{aligned} \tag{2.15}$$

which contains 9 free parameters (with δ being the linear perturbation parameter), and shown that there exist at least 9 small-amplitude limit cycles around the origin [41], in which the generalized Lyapunov constants are obtained as follows:

$$\begin{aligned}
B_0 &= \tfrac{1}{2}\delta,\\
B_2 &= b_2 - 1,\\
B_4 &= 12b_4 + 24a_1\, b_1 + 2a_2 - 1,\\
B_6 &= 4(a_3 - 5a_1)\, b_3 + 8\,[2a_1(5-a_2)-5a_3]\, b_1 + 16a_4 + 20a_1(a_3-3a_1) - (2a_2-1)^2,\\
B_8 &= \tfrac{3}{a_3-5a_1}\big\{8\big[(7+2a_2)(2a_2a_1^2 + a_3^2 - 2a_3a_1) + 16(5a_1 - a3)a_1a_4\big]b_1\\
&\qquad + 2a_3 - 8a_2a_3 + 88a_3^2a_1 - 224a_1^2a_3 + 8a_2^2a_3 - 12a_3^3 + 24a_2a_1^2a_3\\
&\qquad - 8a_2a_3^2a_1 + 140a_1^3 - 20a_2^2a_1 + 8a_2^3a_1 - 3a_1 + 14a_2a_1\\
&\qquad - 16(2a_3 - 3a_1 + 2a_2a_1)\, a_4\big\},\\
&\ \ \vdots\\
B_{18} &= \cdots
\end{aligned}$$

where B_0, $B_2, \cdots, B_{2(k-1)}$ have been set zero in computing B_{2k} for $k = 1, 2, \cdots, 9$. Then, by using proper perturbations on the 9 parameters, it has been shown in [41] that there exist at least 9 small-amplitude limit cycles around the origin.

3. 9 limit cycles in a cubic-order system around a nilpotent critical point

In this section, we present our main result of this paper. Consider the cubic polynomial system (2.11) with an additional parameter a_8 and two linear perturbation parameters δ_1 and δ_2:

$$\begin{aligned}
\frac{dx}{dt} &= y + \delta_1\, y + (a_8 - 2)xy - (a_4 - a_7)x^2y + a_6y^2 + a_2xy^2 + a_5y^3,\\
\frac{dy}{dt} &= -\,\delta_1\, x + \delta_2\, y - 2x^3 + a_1x^2y + y^2 + a_4xy^2 + a_3y^3,
\end{aligned} \tag{3.1}$$

where $0 < \delta_1, |\delta_2| \ll 1$. Now system (3.1) can yield 9 limit cycles around the origin, but the computation becomes much more demanding.

In this section, we will consider bifurcation of limit cycles all around the origin of system (3.1), yielding 9 limit cycles, and in the next section, we will apply the method of double bifurcation of nilpotent focus to system (3.1) to obtain 10 limit cycles near the origin.

Theorem 3.1. *For system (3.1) with a nilpotent critical point at the origin, there exist at least* 9 *small-amplitude limit cycles around the origin.*

Proof. First, let the two linear perturbation parameters equal zero, $\delta_1 = \delta_2 = 0$. Then we apply the method of normal forms and our developed Maple program to system (3.1) to obtain

$$B_2 = a_1. \tag{3.2}$$

We set $a_1 = 0$ to have $B_2 = 0$. Then, B_4 is given by

$$B_4 = 2\big(a_2 + 3a_3 - a_6 a_8\big). \tag{3.3}$$

Similarly, letting

$$a_2 = -\,3a_3 + a_6 a_8, \tag{3.4}$$

we obtain $B_4 = 0$. Then, the next two generalized Lyapunov constants B_6 and B_8 become

$$\begin{aligned}
B_6 &= \tfrac{6}{15}(10a_7 - 7a_8^2 + 25a_8)\,a_3 - \tfrac{2}{15}\big[a_8(5a_4 - 14a_8 + 50) - (9a_8 - 50)a_7\big]a_6 \\
&= \tfrac{4}{3}(3a_3 - 5a_6)\,a_7 - \tfrac{2}{15}\big[3(7a_8 - 25)a_3 + (5a_4 - 9a_7 - 14a_8 + 50)a_6\big]a_8, \\
B_8 &= -\tfrac{2}{875}\big[25a_8(23a_8 - 105)a_4 - 5(460a_7 + 1255a_8 - 241a_8^2 - 2100)a_7 \\
&\qquad - 3a_8(62a_8^3 - 1435a_8^2 + 8500a_8 - 14875)\big]a_3 \\
&\quad - \tfrac{4}{875}\big\{\big[875a_4 - 5(42a_8 - 265)a_7 + 62a_8^3 - 1645a_8^2 + 8950a_8 - 14875\big]a_7 \\
&\qquad - (875a_5 - 62a_8^3 + 1435a_8^2 - 8500a_8 + 14875)a_8\big\}a_6 \\
&= \tfrac{4}{35}\big[(46a_3 - 53a_6)a_7 - 35a_4a_6 - 210a_3 + 595a_6\big]a_7 \\
&\quad - \tfrac{2}{875}\big\{25(23a_8 - 105)a_3a_4 + \big[5(241a_8 - 1255)a_3 - 2(210a_7 - 62a_8^2 \\
&\qquad + 1645a_8 - 8950)a_6\big]a_7 - 3(62a_8^3 - 1435a_8^2 + 8500a_8 - 14875)a_3 \\
&\qquad - 2(875a_5 - 62a_8^3 + 1435a_8^2 - 8500a_8 + 14875)a_6\big\}a_8,
\end{aligned} \tag{3.5}$$

where $B_6 = 0$ has been used to compute B_8. It follows from (3.4) and (3.5) that we may classify two cases: (A) $a_6 a_8 = 0$ and (B) $a_6 a_8 \neq 0$.

Case (A) $a_6 a_8 = 0$. In this case, $a_2 = -\,3a_3$. If $a_6 = 0$, then $a_3 = 0$ yields $B_6 = B_8 = 0$, and in fact all $B_{2k} = 0$, $k = 5, 6, \cdots, 10$. This gives a condition

$$\mathrm{C}_1: \quad a_1 = a_2 = a_3 = a_6 = 0, \tag{3.6}$$

under which all the generalized Lyapunov constants, B_{2k}, $k = 1, 2, \ldots, 10$ vanish. Similarly, if $a_8 = 0$, then $a_7 = 0$ yields $B_6 = B_8 = 0$, and this gives another condition,

$$\mathrm{C}_2: \quad a_1 = a_7 = a_8 = 0, \quad a_2 = -\,3\,a_3, \tag{3.7}$$

under which $B_{2k} = 0$, $k = 1, 2, \ldots, 10$.

Next, we want to investigate under the condition $a_6 a_8 = 0$, what is maximal number of limit cycles which can bifurcate from the origin of system (3.1). We first consider $a_6 = 0$, $a_3 \neq 0$ and then $a_8 = 0$, $a_7 \neq 0$. The case $a_6 = a_8 = 0$ is not considered since it yields special cases of C_1.

Case (A_1) $a_6 = 0$, $a_3 \neq 0$. For this case, $B_6 = 0$ yields a solution $a_7 = \frac{1}{10}(7a_8 - 25)a_8$ with $a_8 \neq 0$ since $a_8 = 0$ leads to a special case of C_2. Then, B_8 becomes

$$B_8 = -\tfrac{a_3 a_8}{875}\big[50\,(23a_8 - 105)a_4 - 3(313a_8^3 - 3300a_8^2 + 11225a_8 - 12250)\big].$$

This shows that taking $a_8 = \frac{105}{23}$ yields $B_8 = -\frac{5040}{279841} a_3 \neq 0$, implying that 4 limit cycles can be obtained. Suppose $a_8 \neq \frac{105}{23}$, we solve $B_8 = 0$ for a_4 and then substitute this solution into B_{10} to obtain

$$
\begin{aligned}
B_{10} = \tfrac{2a_3a_8}{65625(23a_8-105)^2} \big[& 1250\,(127a_8-525)(23a_8-105)^2 a_5 \\
& + 3(a_8-5)^2(843883a_8^5 - 17214695a_8^4 + 140051325a_8^3 \\
& - 568007125a_8^2 + 1148437500a_8 - 926100000)\big].
\end{aligned}
$$

Clearly, setting $a_8 = \frac{525}{127}$ gives $B_{10} = \frac{95644595200}{12587618744067} a_3 \neq 0$, yielding 5 limit cycles around the origin. If choosing a root of the second factor in B_{10}, then $a_5 = 0$ (due to $B_{10} = 0$) and B_{12} becomes a function in a_3, giving 6 limit cycles. For example, letting $a_8 = 5$ we have $B_{12} = -\frac{9480}{77} a_3^3 \neq 0$. Now suppose $(127a_8 - 525)(23a_8 - 105) \neq 0$, and the second factor in B_{10} is also nonzero. Then, we solve $B_{10} = 0$ for a_5 and use this solution to simplify B_{12} and B_{14} to obtain

$$
\begin{aligned}
B_{12} &= -\tfrac{a_3a_8}{6015625(23a_8-105)^3(127a_8-525)}\, G_1(a_3^2, a_8), \\
B_{14} &= -\tfrac{a_3a_8}{782031250(23a_8-105)^4(127a_8-525)^2}\, G_2(a_3^2, a_8), \\
B_{16} &= -\tfrac{a_3a_8}{625625000000(23a_8-105)^5(127a_8-525)^2}\, G_3(a_3^2, a_8),
\end{aligned}
$$

where G_i, $i = 1, 2, 3$ are polynomials in a_3^2 and a_8 and linear with respect to a_3^2. In particular,

$$
\begin{aligned}
G_1 = {} & 187500(697a_8 - 2695)(127a_8 - 525)(23a_8 - 105)^3 a_3^2 \\
& - (a_8 - 5)^2 \big(1377405099237a_8^9 - 50096385469230a_8^8 \\
& + 808738100674975a_8^7 - 7606204683786500a_8^6 \\
& + 45929013487571875a_8^5 - 184655576500018750a_8^4 \\
& + 494307270802515625a_8^3 - 849570999768750000a_8^2 \\
& + 850698925256250000a_8 - 378119684250000000\big),
\end{aligned}
$$

which shows that taking $a_8 = \frac{2695}{697}$ yields 6 limit cycles. Next, suppose $a_8 \neq \frac{2695}{697}$. Then, the second factor in G_1 must be nonzero since $a_3 \neq 0$. We solve the equation $G_1 = 0$ for a_3^2 and substitute this solution into B_{14} and B_{16} to obtain two polynomial equations in a_8. It can be shown that there exist 3 real solutions for a_8 such that $a_3^2 > 0$ and $B_{14} = 0$, but $B_{16} \neq 0$, implying that maximal 8 limit cycles can bifurcate from the origin of system (3.1).

The following analysis will be more or less similar to the above discussion.

Case (A_2) $a_8 = 0$, $a_7 \neq 0$. In this case, $B_6 = 0$ yields $a_3 = \frac{5}{3} a_6$ with $a_6 \neq 0$ since $a_6 = 0$ gives a special case of C_1. For this solution, $B_8 = \frac{4a_6a_7}{105}(71a_7 + 735 - 105a_4)$. Letting $a_4 = \frac{71a_7+735}{105}$ yields $B_8 = 0$ and

$$
B_{10} = \tfrac{a_6a_7}{11025}(12250a_6^2 - 32a_7^2 + 5460a_7 + 176400 + 18375a_5),
$$

from which we can solve for a_5 and substitute the solution into B_{12}, B_{14} and B_{16} to obtain their simplified expressions in a_6^2 and a_7, which are linear with respect to a_6^2. In particular, B_{12} is given by

$$
\begin{aligned}
B_{12} = {} & -\tfrac{32a_6a_7}{573024375}\big[18375(829a_7 + 64680)a_6^2 + 110996a_7^3 - 29397690a_7^2 \\
& - 2089303650a_7 - 30866913000\big].
\end{aligned}
$$

It is easy to see that taking $a_7 = -\frac{64680}{829}$ yields $B_{12} = -\frac{2048129405255127 04}{472300192081}\, a_6 \neq 0$, giving 6 limit cycles. Suppose $a_7 \neq -\frac{64680}{829}$. Then, solving $B_{12} = 0$ gives a solution for a_6^2, which is substituted into B_{14} and B_{16} to obtain two polynomial equations in a_7. It can be shown that there exists only one real solution for a_7 such that $a_6^2 > 0$ and $B_{14} = 0$, but $B_{16} \neq 0$, implying that maximal 8 limit cycles can bifurcate from the origin of system (3.1).

Summarizing the above results, we have shown that when $a_6a_8 = 0$, the maximal number of limit cycles can bifurcate from the origin of system (3.1) is 8. So, to find 9 limit cycles, we must consider the case $a_6a_8 \neq 0$.

Case (B) $a_6a_8 \neq 0$. For convenience, define

$$
\begin{aligned}
\mathrm{H}_1 &= 10a_7 - (7a_8 - 25)\, a_8,\\
\mathrm{H}_2 &= -\,(9a_8 - 50)\, a_7 + (5a_4 - 14a_8 + 50)\, a_8,\\
\mathrm{H}_3 &= 5a_4a_8 - (9a_8 - 20)a_7 + (7a_8 - 25)a_8.
\end{aligned}
\tag{3.8}
$$

Then, B_6 can be rewritten as $B_6 = \frac{6}{15}\, a_3\mathrm{H}_1 - \frac{2}{15}\, a_6\mathrm{H}_2$, which shows that if $B_6 = 0$, then $\mathrm{H}_1 = 0$ implies $\mathrm{H}_2 = 0$ due to $a_6a_8 \neq 0$. Hence, in order to have $B_6 = 0$, we need to investigate three cases: $\mathrm{H}_1 = \mathrm{H}_2 = 0$; $\mathrm{H}_1 \neq 0$, $\mathrm{H}_2 = 0$; and $\mathrm{H}_1\mathrm{H}_2 \neq 0$.

Case (B_1) $\mathrm{H}_1\mathrm{H}_2 \neq 0$. First we consider the generic case, $\mathrm{H}_1\mathrm{H}_2 \neq 0$, under which solving $B_6 = 0$ yields a solution for a_3:

$$
a_3 = \frac{\mathrm{H}_2}{3\,\mathrm{H}_1}\, a_6. \tag{3.9}
$$

Next, from the output of our Maple program, we obtain the generalized Lyapunov constant B_8, which is linear in a_5. Thus, solving $B_8 = 0$ for a_5 yields

$$
\begin{aligned}
a_5 = \tfrac{1}{1050 a_8 \mathrm{H}_1} \big\{ & 100\, a_7^2\, a_4(105 - 23a_8) + 10a_4a_7a_8(17a_8^2 - 315a_8 + 1050)\\
& + 25a_4^2a_8^2(23a_8 - 105) + 5a_7^2(4a_7 - a_8^2 + 4a_8)(81a_8 - 355)\\
& - \big[a_4a_8^2 - (4a_7 - a_8^2 + 4a_8)a_7\big]\big(186a_8^3 - 2695a_8^2 + 12400a_8 - 18375\big)\big\}.
\end{aligned}
\tag{3.10}
$$

With the above solutions of a_1, a_2, a_3 and a_5, other higher-order generalized Lyapunov constants are obtained as

$$
\begin{aligned}
B_{10} &= -\,\tfrac{2\, a_6}{826875\, a_8\, \mathrm{H}_1^2}\, F_0\, F_1,\\
B_{12} &= -\,\tfrac{a_6}{136434375\, a_8\, \mathrm{H}_1^3}\, F_0\, F_2,\\
B_{14} &= -\,\tfrac{a_6}{2483105625000\, a_8^2\, \mathrm{H}_1^3}\, F_0\, F_3,\\
B_{16} &= -\,\tfrac{a_6}{223479506250000\, a_8^2\, \mathrm{H}_1^4}\, F_0\, F_4,\\
B_{18} &= -\,\tfrac{a_6}{53188122487500000000\, a_8^3\, \mathrm{H}_1^5}\, F_0\, F_5,
\end{aligned}
\tag{3.11}
$$

where

$$
F_0 = a_4\, a_8^2 - (4a_7 - a_8^2 + 4a_8)a_7, \tag{3.12}
$$

and F_1, F_2, $\cdots$, F_5 are functions in a_4, a_7, a_8 and a_6^2. Note that if $F_0 = 0$, then all the generalized Lyapunov constants B_{2k}, $k \le 10$ vanish. The condition $F_0 = 0$ together with the solutions a_1, a_2, a_3 and a_5 gives the following condition:

$$\mathrm{C}_3:\ a_1 = a_5 = 0,\quad a_2 = a_6\big(a_8 - 2 - \tfrac{2a_7}{a_8}\big),\quad a_3 = \tfrac{2}{3}a_6\big(1 + \tfrac{a_7}{a_8}\big),\quad a_4 = \tfrac{a_7(4a_7 + 4a_8 - a_8^2)}{a_8^2}, \tag{3.13}$$

under which $B_{2k} = 0$, $k = 1, 2, \ldots, 10$.

In order to find the maximal number of limit cycles bifurcating from the origin, we need to use the parameters a_4, a_6, a_7, a_8 to find the solutions such that $F_1 = F_2 = F_3 = F_4 = 0$, but $F_5 \ne 0$ (or $B_{18} \ne 0$). Therefore, in the following, we shall first try to find the solutions from the equations, $F_1 = F_2 = F_3 = F_4 = 0$, and then verify if the condition $B_{18} \ne 0$ is satisfied for these solutions. Since all F_i, $i = 1, 2, 3, 4$ are functions in a_6^2 and in particular, F_1 is linear in a_6^2, given by

$$\begin{aligned} F_1 = {} & 4593750\, a_8\, a_6^2\, \mathrm{H}_3 \\ & + 1250 a_4^2 a_8 \big[315(37a_8 - 175)a_7 - (9619a_8^2 - 80430a_8 + 165375)a_8\big] \\ & - 25a_4\big[50(7513a_8^2 - 11445a_8 - 110250)a_7^2 - 5a_8(38427a_8^3 + 77630a_8^2 \\ & - 2608725a_8 + 6615000)a_7 - a_8^2(201212a_8^4 - 3697415a_8^3 + 24846075a_8^2 \\ & - 72736125a_8 + 78553125)\big] - 500a_7^3(1062a_8^2 - 44885a_8 + 186375) \\ & - 25a_7^2\big(66303a_8^4 - 544520a_8^3 + 3396875a_8^2 - 18947250a_8 + 38587500\big) \\ & - 10a_7a_8\big(137382a_8^5 - 2395130a_8^4 + 13673725a_8^3 - 22645625a_8^2 \\ & - 37261875a_8 + 108871875\big) + a_8^2(7a_8 - 25)\big(11112a_8^5 - 215450a_8^4 \\ & + 1330375a_8^3 - 2339375a_8^2 - 4081875a_8 + 12403125\big). \end{aligned} \tag{3.14}$$

There are two cases: $\mathrm{H}_3 = 0$ and $\mathrm{H}_3 \ne 0$.

First, we consider $\mathrm{H}_3 = 0$ from which we obtain

$$a_4 = \frac{1}{5a_8}\big[(9a_8 - 20)\, a_7 - (7a_8 - 25)\, a_8\big], \tag{3.15}$$

which is substituted into the higher-order generalized Lyapunov constants to yield

$$\begin{aligned} B_{10} &= \tfrac{4a_6}{1378125a_8^2}\,(a_8 - 5)(2a_7 - a_8)\,\overline{F}_1, \\ B_{12} &= \tfrac{2a_6}{227390625a_8^3}\,(a_8 - 5)(2a_7 - a_8)\,\overline{F}_2, \\ B_{14} &= \tfrac{a_6}{2069254687500a_8^4}\,(a_8 - 5)(2a_7 - a_8)\overline{F}_3, \\ B_{16} &= \tfrac{a_6}{186232921875000a_8^5}\,(a_8 - 5)(2a_7 - a_8)\,\overline{F}_4, \end{aligned}$$

where $\overline{F}_1$ is a function in a_7 and a_8, while $\overline{F}_2$, $\overline{F}_3$ and $\overline{F}_4$ are functions in a_7, a_8 and a_6^2. It can be shown that $(a_8 - 5)(2a_7 - a_8) = 0$ yields $B_{2k} = 0$, $k = 1, 2, \ldots, 10$. In fact, $a_8 = 5$ indeed gives a condition,

$$\mathrm{C}_4:\quad a_1 = a_5 = 0,\quad a_8 = 5,\quad a_2 = 2a_6,\quad a_3 = a_6,\quad a_4 = a_7 - 2, \tag{3.16}$$

under which all B_{2k}, $k \le 10$ vanish. However, $2a_7 - a_8 = 0$ yields a special case of C_3.

For other solutions solved from $\overline{F}_1 = \overline{F}_2 = \overline{F}_3 = 0$, it can be shown that maximal 8 limit cycles can be obtained. First it has been noted that the coefficient of a_6^2 in $\overline{F}_2$ is $93a_8^2 - 725a_8 + 1400$. Letting this coefficient equal zero yields polynomials $\overline{F}_1$ and $\overline{F}_2$ in a_7 and it can be shown that there exist four real solutions such that $\overline{F}_1 = 0$ (i.e., $B_{10} = 0$), but $\overline{F}_2 \neq 0$ (i.e., $B_{12} \neq 0$), implying the existence of 6 limit cycles. When $93a_8^2 - 725a_8 + 1400 \neq 0$, we can solve a_6^2 from $\overline{F}_2 = 0$, and then $\overline{F}_3$ and $\overline{F}_4$ also become polynomials in a_7 and a_8. One can show that there exist four real solutions such that $\overline{F}_1 = \overline{F}_3 = 0$, but $\overline{F}_4 \neq 0$, implying that maximal 8 limit cycles can bifurcate from the origin.

Now, suppose $\mathrm{H}_3 \neq 0$. Substituting the solution of $a_6^2 = A_6(a_4, a_7, a_8)$, solved from $F_1 = 0$, into F_2, F_3 and F_4, we obtain

$$F_2 = -\frac{4}{875\,\mathrm{H}_3}\,G_1, \quad F_3 = -\frac{16}{875\,\mathrm{H}_3}\,G_2, \quad F_4 = -\frac{12}{765625\,\mathrm{H}_3}\,G_3, \tag{3.17}$$

where G_1, G_2 and G_3 are respectively, 4th-, 5th- and 7th-degree polynomial functions in a_4. To solve the equations $G_1 = G_2 = G_3 = 0$ for real solutions of the parameters, a_4, a_7 and a_8, we first use the Maple built-in command *eliminate* to eliminate a_4 from the three equations: $G_1 = G_2 = G_3 = 0$, yielding a solution $a_4 = a_4(a_7, a_8)$, and two resultants:

$$\mathrm{R}_{12} = \mathrm{R}_0\,\mathrm{R}_{12\mathrm{a}}, \quad \mathrm{R}_{13} = \mathrm{R}_0\,(93a_8^2 - 725a_8 + 1400)\,\mathrm{R}_{13\mathrm{a}},$$

where the common factor R_0 is given by

$$\begin{aligned}\mathrm{R}_0 = {} & a_8\,\mathrm{H}_1\,(a_8 - 5)(9a_8 - 35) \\ & \times\big[55125(3a_8 - 4)^2 a_7^2 - 10a_8^2\big(18552a_8^2 - 128825a_8 + 223475\big)\,a_7 \\ & -a_8^2\,\big(1852a_8^4 - 194325a_8^3 + 1770175a_8^2 - 5548375a_8 + 5788125\big)\big],\end{aligned} \tag{3.18}$$

and $\mathrm{R}_{12\mathrm{a}}$ and $\mathrm{R}_{13\mathrm{a}}$ are lengthy polynomials in a_7 and a_8 (with 888 terms in $\mathrm{R}_{12\mathrm{a}}$ and 1380 terms in $\mathrm{R}_{13\mathrm{a}}$), which are not listed here for brevity. First, consider R_0. If $\mathrm{R}_0 = 0$, then all generalized Lyapunov constants vanish. Since $a_8\mathrm{H}_1 \neq 0$ and $a_8 = 5$ has been considered in the condition C_4, we only need to consider other two factors. For the big factor, we can show that letting this factor equal zero yields $\mathrm{H}_3 = 0$, violating the assumption. For $a_8 = \frac{35}{9}$, we get one more condition, given below:

$$\mathrm{C}_5: \quad \begin{cases} a_1 = 0, \\ a_8 = \frac{35}{9}, \\ a_2 = \frac{a_6(52488a_7^2 - 161595a_7 + 115150)}{9(81a_7 - 140)(81a_7 - 70)}, \\ a_3 = \frac{a_6(59049a_7^2 - 144585a_7 + 75950)}{9(81a_7 - 140)(81a_7 - 70)}, \\ a_4 = \frac{59049a_7^2 - 119070a_7 + 34300}{945(81a_7 - 140)}, \\ a_5 = \frac{2(27a_7 - 70)(81a_7 - 35)(243a_7 - 350)(162a_7 - 245)}{2679075(81a_7 - 140)^2}, \\ a_6 = \pm\frac{2\,(81a_7 - 70)(243a_7 - 280)}{2835\sqrt{70(81a_7 - 140)}} \qquad \big(a_7 > \frac{140}{81}\big), \end{cases} \tag{3.19}$$

under which $B_{2k} = 0$, $i = 1, 2, \ldots, 10$.

For the remaining parts in R_{12} and R_{13}, we first consider the solution solved from the simple factor of R_{13}, $93a_8^2 - 725a_8 + 1400 = 0$, which gives two real solutions: $a_8^{\pm} = \frac{725 \pm 5\sqrt{193}}{186}$. Substituting the two solutions into the equation $\mathrm{R}_{12\mathrm{a}} = 0$ to solve for a_7, yielding 15 real solutions corresponding to a_8^+ and

11 real solutions corresponding to a_8^-. It can be verified that among the 26 solutions, 2 solutions violate the assumption $\mathrm{H}_3 \neq 0$, 12 ones yield $a_6^2 = A_6(a_4(a_7, a_8), a_7, a_8) < 0$, and other 12 ones lead to $B_{16} \neq 0$, implying that maximal 8 limit cycles can be obtained from the solutions $a_8^{\pm}$. Hence, the feasible solutions for 9 limit cycles must be found from the equations $\mathrm{R}_{12\mathrm{a}} = \mathrm{R}_{13\mathrm{a}} = 0$. Since $\mathrm{R}_{12\mathrm{a}}$ and $\mathrm{R}_{13\mathrm{a}}$ are respectively 23rd- and 29th-degree polynomials in a_7, we apply the Maple built-in command *resultant* to eliminate a_7 from the two equations: $\mathrm{R}_{12\mathrm{a}} = \mathrm{R}_{13\mathrm{a}} = 0$ to obtain a resultant in a_8:

$$\begin{aligned}\mathrm{R}_{123} = {} & C_{123}\, a_8^{326}(9a_8 - 35)^2(a_8 - 5)^5 \\ & \times(3a_8 - 10)^8(697a_8 - 2695)(549a_8 - 1645)\,\mathrm{R}_{123\mathrm{a}}\,\mathrm{R}_{123\mathrm{b}},\end{aligned}$$

where C_{123} is a big integer, and $\mathrm{R}_{123\mathrm{a}}$ (which contains 284 terms) and $\mathrm{R}_{123\mathrm{b}}$ (which includes 1454 terms) are respectively 283th- and 1453th-degree integer polynomials in a_8, each term having a very big integer coefficient. It can be shown that the polynomial $\mathrm{R}_{123\mathrm{b}}$ does not have solutions satisfying $\mathrm{R}_{12\mathrm{a}} = \mathrm{R}_{13\mathrm{a}} = 0$. Thus, we only need to consider the linear factors in R_{123} and the factor $\mathrm{R}_{123\mathrm{a}}$. Since $a_8 \neq 0$, the linear factors have the solutions: $a_8 = \frac{1645}{549}, \frac{10}{3}, \frac{2695}{697}\ \frac{35}{9}$ and 5. $a_8 = 5$ has been considered above in the condition C_4, and a direct computation shows that the solution $a_8 = \frac{10}{3}$ leads to that $\mathrm{R}_{12\mathrm{a}}(a_7)$ and $\mathrm{R}_{13\mathrm{a}}(a_7)$ have no common factors. Moreover, for $a_8 = \frac{2695}{697}$, we have $a_7 = \frac{388080}{485809}$, which yields $\mathrm{H}_1 = 0$ and so is not allowed. Therefore, we only need to consider two values of a_8: $\frac{1645}{549}$ and $\frac{35}{9}$. Each of them yields a unique solution of a_7 satisfying $\mathrm{R}_{12\mathrm{a}}(a_7) = 0$ and $\mathrm{R}_{13\mathrm{a}}(a_7) = 0$. But both them yield a zero divisor for solution $a_4(a_7, a_8)$. Thus, for these two values of a_8, we need reconsider possible bifurcation of limit cycles by investigating the solutions of the equations: $G_1 = G_2 = G_3 = 0$.

(1) $a_8 = \frac{1645}{549}$. For this value, $\mathrm{R}_{12\mathrm{a}}(a_7)$ and $\mathrm{R}_{13\mathrm{a}}(a_7)$ have a common root $a_7 = \frac{101990}{301401}$ under which

$$B_{12} = \tfrac{a_6(301401a_4-49538)}{11163a_4-10744}\, B_{12a}(a_4), \quad B_{14} = \tfrac{a_6(301401a_4-49538)}{11163a_4-10744}\, B_{14a}(a_4),$$

where B_{12a} and B_{14a} are respectively 3rd- and 4th-degree polynomials in a_4. Note that $11163a_4-10744 = 0$ yields $\mathrm{H}_3 = 0$ and so is not allowed, while $301401a_4 - 49538 = 0$ yields a special case of C_3. Moreover, it is easy to show that $B_{12a}(a_4)$ has 3 real roots, and all of them satisfy $a_6^2 = A_6(a_4(a_7, a_8), a_7, a_8) > 0$ and $B_{14a} \neq 0$, implying that there are 6 solutions to yield 7 limit cycles around the origin.

(2) $a_8 = \frac{35}{9}$. For this case we obtain $a_7 = \frac{140}{81}$, and

$$\begin{aligned} B_{12} &= \tfrac{56a_6(81a_4-68)}{345191655699(9a_4-8)}\,(78121827a_4^3 - 206422182a_4^2 + 180139302a_4 - 51948944), \\ B_{14} &= \tfrac{28a_6(81a_4-68)}{363486813451047(9a_4-8)}\,(601147458765a_4^4 - 2125734770745a_4^3 \\ &\quad + 2802717403668a_4^2 - 1634221389834a_4 + 355795637488).\end{aligned}$$

Note that $9a_4 - 8 = 0$ is not allowed since it yields $\mathrm{H}_3 = 0$, and $81a_4 - 68 = 0$ gives a special case of C_3. The 3rd-degree polynomial in B_{12} has one real solution satisfying $a_6^2 = A_6(a_4(a_7, a_8), a_7, a_8) > 0$ and $B_{14} \neq 0$, implying the existence of 7 limit cycles.

Therefore, none of the solutions obtained from the linear factors can give 9 limit cycles.

Next, consider the factor $\mathrm{R}_{123\mathrm{a}}$. It has 53 real roots for a_8, each of them yields a unique solution for a_7 by verifying the common roots of the equations $\mathrm{R}_{12\mathrm{a}}(a_7) = 0$ and $\mathrm{R}_{13\mathrm{a}}(a_7) = 0$, leading to 53 sets of solutions (a_7, a_8). Moreover, all the 53 sets of solutions satisfy $G_1 = G_2 = G_3 = 0$ (i.e., $F_2 = F_3 = F_4 = 0$), but only 24 of them yield $a_6^2 A_6(a_4(a_7, a_8), a_7, a_8) > 0$. These 24 sets of solutions are

$$
\begin{aligned}
(a_7, a_8) = {} & (\ \ 4.398089\cdots, -14.54122\cdots), \ (-66.19700\cdots, -10.81905\cdots), \\
& (\ \ 0.451595\cdots, -0.019793\cdots), \ (-0.545773\cdots, \ \ 0.891421\cdots), \\
& (-9.202151\cdots, \ \ 0.916847\cdots), \ (-0.689736\cdots, \ \ 2.248118\cdots), \\
& (-3.237553\cdots, \ \ 2.525801\cdots), \ (-2.635767\cdots, \ \ 2.674894\cdots), \\
& (\ \ 0.911863\cdots, \ \ 3.171373\cdots), \ (\ \ 0.916134\cdots, \ \ 3.199575\cdots), \\
& (-0.495531\cdots, \ \ 3.233994\cdots), \ (-0.411833\cdots, \ \ 3.357596\cdots), \\
& (\ \ 0.782354\cdots, \ \ 3.464452\cdots), \ (-0.819658\cdots, \ \ 3.530177\cdots), \\
& (\ \ 2.323788\cdots, \ \ 3.574264\cdots), \ (\ \ 2.331817\cdots, \ \ 3.578910\cdots), \\
& (\ \ 0.506545\cdots, \ \ 3.728333\cdots), \ (\ \ 2.897285\cdots, \ \ 4.335131\cdots), \\
& (\ \ 2.989399\cdots, \ \ 4.350327\cdots), \ (\ \ 3.262858\cdots, \ \ 4.519013\cdots), \\
& (\ \ 5.444113\cdots, \ \ 4.999836\cdots), \ (\ \ 5.206692\cdots, \ \ 5.053923\cdots), \\
& (\ \ 5.639909\cdots, \ \ 5.872529\cdots), \ (\ \ 14.51426\cdots, \ \ 8.193654\cdots).
\end{aligned}
$$

Then, for each set of the above solutions, we can find corresponding solutions for $a_4(a_7, a_8)$, $a_6 = \pm\sqrt{A_5(a_4(a_7, a_8), a_7, a_8)}$, a_5, a_3 and a_2. Thus, there are in total 48 solutions, satisfying $B_2 = B_4 = \cdots = B_{16} = 0$, but $B_{18} \neq 0$. For example, taking the fourth solution, we have

$$
\begin{aligned}
a_1 &= 0, & a_2 &= -0.1481082002\cdots, & a_3 &= 0.3895415095\cdots, \\
a_4 &= 0.2161600548\cdots, & a_5 &= 0.3785873532\cdots, & a_6 &= 1.1448190280\cdots, \\
a_7 &= -0.5457733466\cdots, & a_8 &= 0.8914215289\cdots,
\end{aligned}
\tag{3.20}
$$

for which

$$
B_2 = B_4 = \cdots = B_{16} = 0, \quad B_{18} = -\,0.2676264978\cdots \neq 0. \tag{3.21}
$$

Moreover, using the above critical parameter values, we obtain

$$
\det\left[\tfrac{\partial(B_2,B_4,B_6,B_8,B_{10},B_{12},B_{14},B_{16})}{\partial(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8)}\right] = -\,490.0663780732\cdots \neq 0. \tag{3.22}
$$

Therefore, proper perturbations on a_8, a_7, a_4, a_6, a_5, a_3, a_2 and a_1 can be taken to obtain 8 small-amplitude limit cycles around the origin.

Finally, we consider the linear perturbations which yields one more small-amplitude limit cycle. Actually, with the small linear perturbed terms, the origin becomes a focus with eigenvalues $\frac{1}{2}\big[\delta_2 \pm \sqrt{\delta_2^2 - 4\delta_1(1+\delta_1)}\big]$, showing that the zeroth-order focus value is $v_0 = \frac{1}{2}\,\delta_2$. At $\delta_2 = 0$, the origin becomes an elementary center with a purely imaginary pair: $\pm\, i\sqrt{\delta_1(1+\delta_1)}$. Then, by using normal form theory, a simple calculation shows that the first Lyapunov coefficient v_1 is given by

$$
v_1 = \tfrac{1}{8(1+\delta_1)^2}\big[a_1 + (3a_3 + 2a_1 - a_8 a_6 - 2a_6 + a_2)\,\delta_1 + (3a_3 + a_1 + a_2)\,\delta_1^2\big].
$$

Note that $v_1 = \frac{1}{8} a_1$ and $B_2 = a_1$ when $\delta_1 = \delta_2 = 0$, which are in the same order of a_1 (just by difference of a positive constant factor). Thus, we can properly perturb δ_2 such that $v_0 B_2 < 0$ and $|v_0| \ll |B_2|$ to get the 9th small-amplitude limit cycle around the nilpotent critical point (the origin).

Case (B_2) $\mathrm{H}_1 = \mathrm{H}_2 = 0$. Solving $\mathrm{H}_1 = \mathrm{H}_2 = 0$ we have $a_7 = \frac{1}{10}(7a_8 - 25)a_8$ and $a_4 = \frac{3}{50}(3a_8 - 10)(7a_8 - 25)$. There are three cases.

(1) $a_8 = \frac{25}{7}$. Then, $a_4 = a_7 = 0$, and $B_8 = \frac{100}{16807}[12a_3 + (2401a_5 - 8)a_6]$. Letting $a_5 = \frac{8}{2401}$ yields $a_3 = 0$ and

$$\begin{aligned} B_{10} &= \tfrac{-200a_6}{51883209}(12353145a_6^2 - 10712), \\ B_{12} &= \tfrac{-100a_6}{9321683217}(1723910797a_6^2 + 596648), \end{aligned}$$

which clearly indicates that 6 limit cycles can be obtained. If $a_5 \neq \frac{8}{2401}$, then we have $a_3 = -\frac{1}{12}a_6(2401a_5 - 8)$, which is substituted into B_{10} to obtain

$$B_{10} = -\tfrac{25a_6}{86436}\, a_5\big[84035(117649a_6^2 - 80)a_5 + 16470860a_6^2 - 20448\big]. \tag{3.23}$$

It is easy to verify that $a_5 = 0$ gives a special case of C_3. If $a_5 \neq 0$, then $a_6^2 = \frac{80}{117649}$ results in $B_{10} = \frac{57800}{21609}\, a_6 \neq 0$, yielding 5 limit cycles. If $a_6^2 \neq \frac{80}{117649}$, then we solve a_5 from $B_{10} = 0$, which simplifies B_{12} and B_{12} as

$$\begin{aligned} B_{12} = {} & \tfrac{-32a_6(4117715a_6^2-5112)}{2936330213355(117649a_6^2-80)^3}\,(1343441218276120425a_6^6 \\ & - 484017356260489l0a_6^4 + 31519505743936a_6^2 - 6072768000), \\ B_{14} = {} & \tfrac{16a_6(4117715a_6^2-5112)}{2618619284269989(117649a_6^2-80)^3}\,(5021643695607745824 45a_6^6 \\ & - 14623351347578446270a_6^4 + 12842075888066880a_6^2 \\ & - 3507925999104). \end{aligned}$$

Again, one can verify that $4117715a_6^2 - 5112 = 0$ gives a special case of C_3. Otherwise, solving $B_{12} = 0$ gives one real positive solution a_6^2 for which $B_{14} \neq 0$, indicating the existence of 7 limit cycles.

(2) $a_8 = \frac{10}{3}$. Then, we have $a_4 = 0$ and $a_7 = -\frac{5}{9}$, under which $B_8 = \frac{40}{1701}[45a_3 + (567a_5 - 25)a_6]$. If $a_5 = \frac{25}{567}$, then $a_3 = 0$ and

$$B_{10} = -\tfrac{200a_6}{964467}\,(178605a_6^2 - 1774), \quad B_{12} = -\tfrac{400a_6}{31827411}(3880737a_6^2 - 21869),$$

which clearly shows that there exist solutions for the existence of 6 limit cycles. If $a_5 \neq \frac{25}{567}$, then $a_3 = \frac{a_6(25-567a_5)}{45}$, and B_{10} becomes

$$B_{10} = -\frac{8a_6a_5}{567}\big[(189(3969a_6^2 - 61)a_5 + 26460a_6^2 - 83\big].$$

$a_5 = 0$ again yields a special case of C_3. It is easy to see that $3969a_6^2 - 61 = 0$ gives a solution for the existence of 5 limit cycles. If $3969a_6^2 - 61 \neq 0$, then similarly we can prove that there exist 6 solutions for the existence of 7 limit cycles.

(3) $(7a_8 - 25)(3a_8 - 10) \neq 0$. Then $a_4a_7 \neq 0$. Since $a_6 \neq 0$, we solve $B_8 = 0$ to obtain

$$a_5 = \frac{1}{1750a_6}\,(a_8 - 5)(17a_8^2 - 139a_8 + 280)\big[15a_3 - a_6(7a_8 - 15)\big],$$

and then

$$B_{10} = -\tfrac{a_8(15a_3+15a_6-7a_8a_6)}{6890625a_6}\, B_{10a}, \quad B_{12} = -\tfrac{a_8(15a_3+15a_6-7a_8a_6)}{11369531250}\, B_{12a},$$

where B_{10a} and B_{12a} are polynomials in a_3, a_6 and a_8. It is easy to verify that $15a_3 + 15a_6 - 7a_8a_6 = 0$ yields a special case of C_3. Following a similar procedure as used above, we can show that there exist 8 solutions for the existence of 7 limit cycles.

Case (B_3) $\mathrm{H}_1 \neq 0$, $\mathrm{H}_2 = 0$. For this case, $\mathrm{H}_2 = 0$ yields $a_4 = \frac{1}{5a_8}[(14a_8 - 50)a_8 + (9a_8 - 50)a_7]$, and then $B_6 = 0$ requires $a_3 = 0$ due to $\mathrm{H}_1 \neq 0$. Then, we solve $B_8 = 0$ to obtain a solution for a_5, given by

$$\begin{aligned} a_5 = &-\tfrac{1}{875a_8^2}\big[10a_7^2(21a_8^2 - 290a_8 + 875) - a_7a_8(a_8 - 5)(62a_8^2 - 1335a_8 + 4725) \\ &- a_8^2(62a_8^3 - 1435a_8^2 + 8500a_8 - 14875)\big], \end{aligned}$$

under which

$$B_{10} = -\tfrac{4a_6(a_7+a_8)}{1378125a_8^2}\, B_{10a}(a_6^2, a_7, a_8), \qquad B_{12} = -\tfrac{2a_6(a_7+a_8)}{227390625a_8^2}\, B_{12a}(a_6^2, a_7, a_8),$$

where B_{10a} and B_{12a} are polynomials in a_6^2, a_7 and a_8, and in particular,

$$\begin{aligned} B_{10a} = \ & 4593750a_6^2a_8^2 - 250a_7^2(3969a_8^3 - 80750a_8^2 + 526575a_8 - 1102500) \\ & + 5a_7a_8(74208a_8^4 - 2425625a_8^3 + 25885100a_8^2 - 110354125a_8 \\ & + 162618750) + a_8^2(3704a_8^5 + 598890a_8^4 - 16370125a_8^3 \\ & + 135606125a_8^2 - 455039375a_8 + 541603125). \end{aligned}$$

Note that $a_7 + a_8 = 0$ gives a special case of C_3. So solving B_{10a} for a_6^2 and substitute the solution into B_{12} and B_{14} to obtain two polynomial equations in a_7 and a_8. Solving these two polynomial equations, we obtain 10 sets of solutions (a_7, a_8) such that $a_6^2 > 0$ and $B_{16} \neq 0$. This shows that there exist 20 solutions for the existence of 8 limit cycles.

Summarizing the above results obtained for Cases (A) and (B) shows that the maximal number of small-amplitude limit cycles which can bifurcate from the origin is 9.

The proof of Theorem 3.1 is complete. □

4. 10 limit cycles in a cubic-order system near a nilpotent critical point

In this section, we consider system (3.1) again, and will use the method of double bifurcation of nilpotent focus to show that the system can have 10 small-amplitude limit cycles near the origin. To achieve this, we add different perturbations to system (3.1) to obtain the following perturbed system:

$$\begin{aligned} \frac{dx}{dt} &= y + (a_8 - 2)xy - (a_4 - a_7)x^2y + a_6y^2 + a_2xy^2 + a_5y^3, \\ \frac{dy}{dt} &= 4\delta\varepsilon y + y^2 + a_4xy^2 + a_3y^3 - (x^2 - \varepsilon^2)(2x - a_1y), \end{aligned} \tag{4.1}$$

where $0 < |\delta| \ll 1,\ 0 < \varepsilon \ll 1$. Then, for system (4.1) we have the following result.

Theorem 4.1. *For system (4.1) with a nilpotent critical point at the origin, there exist at least* 10 *small-amplitude limit cycles near the origin.*

Proof. The proof has two steps. In the first step, let $\delta = \varepsilon = 0$. Then, as shown in the previous section, we obtain the critical parameter values given in (3.20) such that the conditions in (3.21) and (3.22) are satisfied, and thus we obtain 8 small-amplitude limit cycles around the origin by perturbing the coefficients $a_1, a_2, \ldots, a_8$.

In the second step, by choosing proper values of δ and ε, we can use the method of double bifurcation of nilpotent focus [29] to find two more small-amplitude limit cycles near the origin. In fact, for small δ and ε, the origin of system (4.1) becomes a saddle, having eigenvalues $\varepsilon\big[2\delta \pm \sqrt{4\delta^2+2}\big]$, and two foci arising from the symmetric singular points at $(\pm\varepsilon, 0)$, with eigenvalues $2\varepsilon\big[\delta \pm \sqrt{1+(a_8-2)\varepsilon+(a_7-a_4)\varepsilon^2}\big]$, indicating that the zeroth-order focus values associated with the two foci is given by $v_0 = 2\varepsilon\,\delta$. When $\delta = 0$, the origin is still a saddle (with eigenvalues $\pm\sqrt{2}\,\varepsilon$), while the two foci become elementary centers and Hopf bifurcations occur at the two singular points, with the critical eigenvalues $\pm i\,\omega_c$, where $\omega_c = 2\varepsilon^2\sqrt{1+(a_8-2)\varepsilon+(a_7-a_4)\varepsilon^2} \approx 2\varepsilon^2$. A direct calculation shows that the first Lyapunov constant, associated with the two Hopf critical points, is given by

$$v_1 = \tfrac{\varepsilon^3}{2[1+(a_8-2)\varepsilon+(a_7-a_4)\varepsilon^2]}\,\big\{3a_3a_8 - 2a_6(a_8+a_7) + \big[6a_3a_7 - a_6a_8(a_4+a_7)\big]\varepsilon\big\},$$

where the critical conditions $a_1 = 0$ and $a_2 = -3a_3 + a_6a_8$ (see (3.2) and (3.4)) have been used. With the critical solution (3.20), $v_1 \approx 0.50065566\,\varepsilon^3 > 0$. Thus, we can perturb $\delta = 0$ to $\delta < 0$ such that $|\delta| \ll \varepsilon^3$, leading to bifurcations of two small-amplitude limit cycles around the two symmetric singular points $(\pm\varepsilon, 0)$. Then, by proper perturbations on other parameters to get $B_2 < 0$ and $v_1 \ll |B_2|$, and so on higher-order generalized Lyapunov constants. These two additional limit cycles are enclosed by the 8 small-amplitude limit cycles, giving rise to 10 small-amplitude limit cycles with the distribution of $8 \supset (1 \cup 1)$. □

5. Center conditions for the nilpotent critical point

In this section, we will present a set of center conditions for system (3.1) under which the nilpotent critical point – the origin, becomes a center. First of all, it requires $\delta_1 = \delta_2 = 0$. Then, based on the generalized Lyapunov constants, we can find the conditions under which the origin of system (3.1) is a center. As a matter of fact, the critical conditions $\mathrm{C_i}$, $i = 1, 2, 3, 4, 5$ have been shown in the proof of Theorem 3.1 to be the candidates for the center conditions of the origin since they yield all the generalized Lyapunov constants to vanish.

Theorem 5.1. *When $\delta_1 = \delta_2 = 0$, the origin of system (3.1) is a center if and only if one of the following conditions is satisfied:*

C_1: $a_1 = a_2 = a_3 = a_6 = 0$;
C_2: $a_1 = a_7 = a_8 = a_2 + 3a_3 = 0$;
C_3: $a_1 = a_5 = a_2 - a_6\big(a_8 - 2 - \tfrac{2a_7}{a_8}\big) = 3a_3 - 2a_6\big(1 + \tfrac{a_7}{a_8}\big) = a_4 + \tfrac{a_7(a_8^2-4a_8-4a_7)}{a_8^2} = 0$ $(a_8 \neq 0)$;
C_4: $a_1 = a_5 = a_8 - 5 = a_2 - 2a_6 = a_3 - a_6 = a_4 - a_7 + 2 = 0$.

Proof. The *necessity* of the conditions $\mathrm{C_i}$, $i = 1, 2, 3, 4$ has been proved in Theorem 3.1 since all these conditions and C_5 yield the generalized Lyapunov constants B_{2k}, $k = 1, 2, \ldots, 10$ to vanish. No other possible center conditions have been found from the proof of Theorem 3.1. So we only need to prove the *sufficiency* of these conditions.

First, consider the condition C_1. Under C_1 system (3.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= y + (a_8 - 2)\,xy - (a_4 - a_7)\,x^2 y + a_5\,y^3, \\ \frac{dy}{dt} &= -\,2x^3 + y^2 + a_4\,xy^2. \end{aligned} \tag{5.1}$$

It is easy to see that system (5.1) is a reversible system since the system is invariant under the transformation: $(y,t) \to (-y,-t)$. Hence, the origin of system (5.1) is a center.

For condition C_2, system (3.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= y - 2xy - a_4\,x^2 y + a_6\,y^2 - 3a_3\,xy^2 + a_5\,y^3, \\ \frac{dy}{dt} &= -\,2x^3 + y^2 + a_4\,xy^2 + a_3\,y^3, \end{aligned} \tag{5.2}$$

which is a Hamiltonian system with the Hamiltonian function,

$$H(x,y) = \frac{1}{2}\,y^2 + \frac{1}{2}\,x^4 - xy^2 - \frac{a_4}{2}\,x^2y^2 + \frac{a_6}{3}\,y^3 - a_3xy^3 + \frac{a_5}{4}\,y^4.$$

Next, consider the condition C_3 under which system (3.1) can be written as

$$\begin{aligned} \frac{dx}{dt} &= y + (a_8-2)xy + \tfrac{2a_7(a_8^2-2a_8-2a_7)}{a_8^2}\,x^2y + a_6y^2 + a_6\big(a_8 - 2 - \tfrac{2a_7}{a_8}\big)\,xy^2, \\ \frac{dy}{dt} &= -\,2x^3 + y^2 - \tfrac{a_7(a_8^2-4a_8-4a_7)}{a_8^2}\,xy^2 + \tfrac{2}{3}\,a_6\big(1 + \tfrac{a_7}{a_8}\big)\,y^3. \end{aligned} \tag{5.3}$$

It can be shown that there exist integrating factors under different conditions, given by

$$\begin{aligned} I_1 &= \big[(a_8^2 - 2a_8 - 2a_7)x + a_8\big]^{\frac{-\,a_8^2}{a_8^2-2a_8-2a_7}}, \\ &\quad \text{if } \ (a_8^2-2a_8-2a_7)(3a_8^2-8a_8-8a_7) \\ &\qquad \times (a_8^2-3a_8-3a_7)(a_8^2-4a_8-4a_7)(a_8+a_7) \neq 0, \\ I_2 &= \tfrac{6a_8^4}{a_8^4y^2\{2a_6y+3[1+(a_8-2)x]\}-12\,[6+6a_8x+3a_8^2x^2+a_8^3x^3]}, \\ &\qquad\qquad \text{if } \ a_8^2 - 2a_8 - 2a_7 = 0, \\ I_3 &= \frac{1}{(a_8x+4)^4}, \quad \text{if } \ 3a_8^2 - 8a_8 - 8a_7 = 0, \\ I_4 &= \frac{1}{(a_8x+3)^3}, \quad \text{if } \ a_8^2 - 3a_8 - 3a_7 = 0, \\ I_5 &= \frac{1}{(a_8x+2)^2}, \quad \text{if } \ a_8^2 - 4a_8 - 4a_7 = 0, \\ I_6 &= \frac{1}{(a_8x+1)}, \quad \text{if } \ a_8 + a_7 = 0, \end{aligned} \tag{5.4}$$

such that system (5.3) has the following corresponding first integrals:

$$
\begin{aligned}
F_1(x,y) =\;& I_1\Big\{\tfrac{1}{2}y^2+\tfrac{a_8-2}{2}xy^2+\tfrac{a_6}{3}y^3+\tfrac{a_7(a_8^2-2a_8-2a_7)}{a_8^2}\,x^2y^2+\tfrac{a_6}{3}\big(a_8-2-\tfrac{2a_7}{a_8}\big)xy^3 \\
&+\tfrac{(a_8^2-2a_8-2a_7)x+a_8}{(3a_8^2-8a_8-8a_7)(a_8^2-3a_8-3a_7)(a_8^2-4a_8-4a_7)(a_8+a_7)} \\
&\times\Big[\,2(a_8^2-3a_8-3a_7)(a_8^2-4a_8-4a_7)(a_8+a_7)\,x^3 \\
&\quad -3a_8(a_8+a_7)(a_8^2-4a_8-4a_7)\,x^2+6a_8^2(a_8+a_7)\,x+3a_8^3\Big]\Big\}, \\
F_2(x,y) =\;& 2\ln a_8^2-\ln I_2-a_8x, \qquad (5.5)\\
F_3(x,y) =\;& \tfrac{y^2[8a_6y+3(3a_8x-8x+4)]}{96(a_8x+4)^3}+\tfrac{8[9a_8^2x^2+54a_8x+88]}{3a_8^4(a_8x+4)^3}+\tfrac{1}{a_8^4}\ln(a_8x+4)^2, \\
F_4(x,y) =\;& \tfrac{y^2[2a_6y+2(a8-3)x+3]}{18(a_8x+3)^2}+\tfrac{2a_8^3x^3+12a_8^2x^2-36a_8x-135}{a_8^4(a_8x+3)^2}-\tfrac{9}{a_8^4}\ln(a_8x+3)^2, \\
F_5(x,y) =\;& \tfrac{y^2[4a_6y+3x(a8-4)+6]}{24(a_8x+2)}+\tfrac{a_8^3x^3-6a_8^2x^2-16a_8x+16}{a_8^4(a_8x+2)}+\tfrac{12}{a_8^4}\ln(a_8x+2)^2, \\
F_6(x,y) =\;& \tfrac{1}{6}\,y^2(2a_6y-6x+3)+\tfrac{(2a_8^2x^2-3a_8x+6)x}{3a_8^3}-\tfrac{1}{a_8^4}\ln(a_8x+1)^2.
\end{aligned}
$$

Now, for the condition C_4, system (3.1) becomes

$$
\begin{aligned}
\frac{dx}{dt} &= y+3xy+2x^2y+a_6y^2+2a_6\,xy^2, \\
\frac{dy}{dt} &= -\,2x^3+y^2+(a_7-2)\,xy^2+a_6\,y^3.
\end{aligned} \tag{5.6}
$$

We apply the following transformation and time rescaling

$$
u=\frac{x}{1+x},\quad v=\frac{y}{1+x},\quad t=(1-u)^2\tau \quad\Longrightarrow\quad x=\frac{u}{1-u},\quad y=\frac{v}{1-u}, \tag{5.7}
$$

to system (5.6) to obtain

$$
\begin{aligned}
\frac{du}{d\tau} &= v\big[1-u^2+a_6\,v(1-u^2)\big], \\
\frac{dv}{d\tau} &= u\big[-2u^2+(a_7-4)v^2-a_6\,v^3\big].
\end{aligned} \tag{5.8}
$$

This is a reversible system since it is invariant under the transformation $(u,\tau)\to(-u,-\tau)$. Hence, the origin of system (5.8) is a center, implying that the origin of the original system (3.1) is a center since the origin is invariant under the transformation (5.7). □

Finally, consider the condition C_5 (see Eqn. (3.19)). This condition is necessary for the origin of system (3.1) being a center has been proved in Theorem 3.1. For *sufficiency* of this condition, we have the following conjecture.

Conjecture 5.1. *The condition* C_5 *is also sufficient for the origin of system* (3.1) *being a center.*

6. Conclusion

In this paper, we have shown that planar cubic polynomial vector fields with an isolated nilpotent critical point can have at least 9 small-amplitude limit cycles around the critical point and at least 10 small-amplitude limit cycles near the critical point with the distribution of $8\supset(1\cup1)$. Normal form theory has been applied to compute the generalized Lyapunov constants, and then to determine the number of bifurcating limit cycles near the critical point. Moreover, a set of center conditions for the nilpotent point has been obtained for such cubic polynomial systems. It has demonstrated the general applicability of our method and program to solve different types of polynomial systems with a nilpotent singular point.

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