Bifurcation of limit cycles in a cubic-order planar system around a nilpotent critical point

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In this paper, bifurcation of limit cycles is considered for planar cubic-order systems with an isolated nilpotent critical point. Normal form theory is applied to compute the generalized Lyapunov constants and to prove the existence of at least 9 small-amplitude limit cycles in the neighborhood of the nilpotent critical point. In addition, the method of double bifurcation of nilpotent focus is used to show that such systems can have 10 small-amplitude limit cycles near the nilpotent critical point. These are new lower bounds on the number of limit cycles in planar cubic-order systems near an isolated nilpotent critical point. Moreover, a set of center conditions is obtained for such cubic systems.

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1. Introduction

Dynamical systems can exhibit self-sustained oscillations, called limit cycles, which may appear in almost all fields of science and engineering. Developing limit cycle theory is not only theoretically significant, but also practically important. Limit cycles theory is closely related to the well-known Hilbert’s 16th problem, one of the 23 mathematical problems proposed by D. Hilbert in 1900 [25]. A modern version of this problem was included in the 18 most challenging mathematical problems proposed by S. Smale for the 21st century [35].

Consider the following planar differential system:

\[
\frac{dx}{dt} = P_n(x, y), \quad \frac{dy}{dt} = Q_n(x, y),
\]

(1.1)

where \(P_n(x, y)\) and \(Q_n(x, y)\) are \(n\)th-degree polynomials in \(x\) and \(y\). The second part of Hilbert’s 16th problem is to find an upper bound on the number of limit cycles that system (1.1) can have. This upper bound, denoted as \(H(n)\), is called Hilbert number. For general quadratic polynomial systems, four limit

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cycles were found in 1979 [33,14], which were also obtained recently in near-integrable quadratic systems [46]. However, whether $H(2) = 4$ or not is still an open question. For cubic-degree polynomial systems, many results have been obtained on the low bound of the Hilbert number, and the best result so far is $H(3) \geq 13$ [26,27]. In real applications, bifurcation of limit cycles due to Hopf bifurcation is a common phenomenon, but real systems often have dimension higher than two [24,49,50]. In such a case, the system can be first reduced to a 2-dimensional dynamical system by using center manifold theory (e.g., see [24,19]) and then to study the limit cycles bifurcation in the reduced system.

Later, Arnold [7] posed the weak infinitesimal Hilbert’s 16th problem, which is closely related to the so-called near-Hamiltonian system [20]:

$$\frac{dx}{dt} = H_y(x, y) + \varepsilon p_n(x, y), \quad \frac{dy}{dt} = -H_x(x, y) + \varepsilon q_n(x, y),$$

(1.2)

where $H(x, y)$, $p_n(x, y)$ and $q_n(x, y)$ are all polynomial functions in $x$ and $y$, and $0 < \varepsilon \ll 1$ is a small perturbation parameter. Then, the problem on the study of number of limit cycles is transformed to investigating the zeros of the Abelian integral or the (first-order) Melnikov function:

$$M(h, \delta) = \int_{H(x,y)=h} q_n(x, y) \, dx - p_n(x, y) \, dy,$$

(1.3)

where $H(x, y) = h$ for $h \in (h_1, h_2)$ defines a closed orbit, and $\delta$ is a vector parameter, representing the parameters (or coefficients) involved in the polynomials $p_n(x, y)$ and $q_n(x, y)$.

When the study of Hilbert’s 16th problem is restricted to the vicinity of an isolated fixed point, which is either an elementary focus or a center, it becomes an investigation on generalized Hopf bifurcations, and the number of bifurcating small-amplitude limit cycles is usually denoted by $M(n)$. It is well known that $M(2) = 3$, obtained by Bautin in 1952 [9]. For $n = 3$, many results have been obtained, divided into two categories. For systems with an elementary focus, the best result obtained so far is 9 limit cycles [44,13,31]. On the other hand, for systems with a center, there are also a few results obtained in the past two decades. In 1995, Žoładek [52] first proposed a rational Darboux integral, and claimed the existence of 11 small-amplitude limit cycles around a center, which was reinvestigated recently and proved that this system can actually have only 9 limit cycles [45,40]. After more than ten years, another two cubic-order systems were constructed to show 11 limit cycles [15,11]. Recently, the system considered in [15] was used by Yu and Tian to show the existence of 12 small-amplitude limit cycles around a singular point, which is the best result so far for cubic systems.

To consider bifurcation of limit cycles associated with a singularity of focus, Lyapunov constants are needed to solve the center-focus problem and to determine the number and stability of bifurcating limit cycles. There mainly exist three methods for computing Lyapunov constants: the method of normal forms [24,16,42], the method of Poincaré return map or focus value method [6,30], and the method of Lyapunov function [34,17]. Other approaches can be found, for example, in [24]. To demonstrate the basic idea of these methods, without loss of generality, assume that system (1.1) has a singularity of focus at the origin, and that the Jacobian of the system evaluated at the origin has a purely imaginary pair: $\pm i\omega_0$. Then, by using the method of normal forms with the aid of a computer algebra system such as Maple or Mathematica (e.g., see [24,42,38,39]) we compute the normal form to obtain the Lyapunov constants $L_k$ which are used to determine the number of bifurcating limit cycles around the critical point. $v_k$ ($k = 0, 1, 2, \cdots$).

The above mentioned three methods for computing Lyapunov constants have also been used to study the center-focus problem associated with nilpotent critical points, see for example [2,12,32]. But the method of normal forms was only recently applied to compute the so-called generalized Lyapunov constants in determining the lower bound of cyclicity [3]. It is well known that it is more difficult to distinguish focus from center when the singular point is degenerate. In [4] Andreev considered the local phase portraits of
analytic systems with the origin being a nilpotent singular point, which however does not distinguish focus from center. Later, Takens developed a normal form theory for systems with nilpotent center of foci [36], and Moussu obtained the $C^\infty$ normal form for analytic nilpotent centers [32]. Further, Berthier and Moussu studied the reversibility of nilpotent centers [10], while Teixera and Yang applied a convenient normal form to investigate the relationship between reversibility and the center-focus problem, and then studied the reversibility of certain types of polynomial vector fields [37]. Recently, by using Melnikov function method Han and Li [22], and Zhao and Fan [51] considered polynomial Hamiltonian systems with elementary centers to obtain lower bounds on the Hilbert number. Moreover, Han et al. [23] studied polynomial Hamiltonian systems with a nilpotent singular point, and obtained necessary and sufficient conditions for determining the number of limit cycles bifurcating in quadratic and cubic Hamiltonian systems with a nilpotent singular point which may be a center, a cusp or a saddle. However, it should be pointed out that the Melnikov function method used in the above mentioned articles [23,22,51] can not be applied to study the systems considered in this paper, since our systems here are not Hamiltonian, nor even integrable.

The main goal of this paper is to consider bifurcation of limit cycles in cubic polynomial systems and apply our general normal form computation method to obtain new lower bounds on the number of limit cycles. More specifically, we will show that cubic polynomial systems can have at least 9 small-amplitude limit cycles around an isolated nilpotent critical point, and at least 10 small-amplitude limit cycles near an isolated nilpotent critical point. Moreover, a set of center conditions is obtained for such cubic systems. In the next section, we present some basic formulations and preliminary results which are needed in proving our main results in Sections 3, 4 and 5. Conclusion is drawn in Section 6.

2. Mathematical formulation and preliminary results

In this section, we present some basic formulas and preliminary results which will be used in the following sections. Consider the differential system:

$$\frac{dx}{dt} = y + F_1(x, y) = \sum_{j+k} a_{jk} x^j y^k,$$

$$\frac{dy}{dt} = F_2(x, y) = \sum_{j+k} b_{jk} x^j y^k,$$  \hspace{1cm} (2.1)

where $F_1$ and $F_2$ are analytic in the neighborhood of the origin, with power series beginning from second order. As long as the limit cycles bifurcation is considered near the origin, system (2.1) with a nilpotent center at the origin is more difficult to analyze than the general system (1.1) with an element center or focus at the origin, since the conventional normal form of Hopf bifurcation [24,19] can be directly applied to the latter but not the former. In fact, there exist conventional normal forms for system (2.1) associated with Bogdanov–Takens bifurcation (i.e., the linearized system contains a double-zero eigenvalue at the origin) [24, 19], which is however not able to be directly applied to study bifurcation of limit cycles near the origin. Therefore, a modified normal form of system (2.1) needs to be developed to study bifurcation of limit cycles near the origin. In real applications, many physical systems involve a number of parameters and can thus have higher co-dimensional singularity such as Bogdanov–Takens bifurcation (which is characterized by a double-zero eigenvalue at a critical point, leading to a nilpotent singular point), and thus it is interesting and important to explore the periodic solutions near such a critical point. For example, in the 2-dimensional HIV model [48], a critical point with Bogdanov–Takens bifurcation is identified for certain parameter values and thus the system can be put in the form of system (2.1) in the vicinity of the critical point. Limit cycles due to Hopf bifurcation have been obtained near this critical point and even multiple limit cycles can be found if more parameters are treated as bifurcation parameters. Moreover, homoclinic orbits are identified near this degenerate singular point [48].
To mathematically analyze bifurcation of limit cycles for system (2.1) near the origin, we first present the following result [4,2,3], which can be used to determine the monodromy of the origin of system (2.1).

Lemma 2.1. (Theorem 2.1 in [3]) Assume that the origin of system (2.1) is an isolated singularity. Define two functions \( f(x) \) and \( \phi(x) \) as

\[
\phi(x) = \frac{\partial F_1(x, f(x))}{\partial x} + \frac{\partial F_2(x, f(x))}{\partial y},
\]

\[
\psi(x) = F_2(x, f(x)) = ax^\alpha + O(x^{\alpha+1}), \quad a \neq 0, \alpha \geq 2,
\]

where \( y = f(x) \) is the solution of the equation, \( y + F_1(x, y) = 0 \), passing through the origin \((0,0)\). Write \( \phi(x) = bx^\beta + O(x^{\beta+1}) \), \( b \neq 0 \) and \( \beta \geq 1 \), or \( \phi(x) \equiv 0 \). Then, the origin of system (2.1) is monodromic if and only if \( a < 0 \), \( \alpha = 2n - 1 \) (\( n \geq 1 \)) being an odd number, and one of the following three conditions holds:

(i) \( \beta > n - 1 \);
(ii) \( \beta = n - 1 \), and \( b^2 + 4an < 0 \);
(iii) \( \phi \equiv 0 \).

Under the above conditions, we can apply the classical normal form theory, with the following near-identity transformation,

\[
x = u + \sum_{i+j=2}^{k} h_{1ij}u^iv^j, \quad y = v + \sum_{i+j=2}^{k} h_{2ij}u^iv^j,
\]

(2.2)

to obtain the conventional normal form [19,24]:

\[
\frac{du}{d\tau} = v + O(||(u,v)||^{k+1}),
\]

\[
\frac{dv}{d\tau} = -u^{2n-1} + \sum_{j \geq \beta}^{k-1} (A_ju^{j+1} + B_ju^jv) + O(||(u,v)||^{k+1}).
\]

(2.3)

This conventional normal form can not be directly used to find the limit cycles bifurcating from the origin. However, if we use the idea of the simplest normal form theorem (or unique normal form theory) (e.g., see [8,43,47,18]) and introduce a time rescaling,

\[
\tau = \left(1 + \sum_{i+j=2}^{k} h_{3ij}u^iv^j\right)t,
\]

(2.4)

into system (2.3), we obtain

\[
\frac{du}{d\tau} = v + O(||(u,v)||^{k+1}),
\]

\[
\frac{dv}{d\tau} = -u^{2n-1} + v \sum_{j \geq \beta}^{k-1} B_ju^j + O(||(u,v)||^{k+1}),
\]

(2.5)

where \( B_j \) is called the \( j \)th-order generalized Lyapunov constant. We have developed an algorithm with explicit recursive formulas for computing \( B_j \) for the general system (2.1), with a computationally efficient Maple program which can be easily implemented on a computer using Maple. It has been noted that Liu and Li [28] have developed a different method to compute the so-called quasi Lyapunov constants, which are
equivalent to the generalized Lyapunov constants. However, their method is only applicable for cubic-order systems. Before we particularly consider bifurcation of limit cycles in cubic-order systems with an isolated nilpotent critical point, we present few examples, which have been investigated in [3,5,1], to illustrate the general applicability of our method. The method of normal forms has been used in [3] to study bifurcation of limit cycles, and many examples are presented in this paper. For example, consider the system,

\[
\begin{align*}
\frac{dx}{dt} &= -y, \\
\frac{dy}{dt} &= x^5 + ax^6 + y(bx^3 + cx^4).
\end{align*}
\]  

(2.6)

Note in the first equation of (2.6) that the first term is \(-y\) rather than \(y\). But this does not affect the normal forms computation provided we apply a transformation \(y \rightarrow -y\) if it is necessary for executing a computer program. We used the normal form computation method developed in [3] and coded a Maple program to obtain the following normal form:

\[
\begin{align*}
\frac{du}{d\tau} &= -v, \\
\frac{dv}{d\tau} &= u^5 + v\left[ bu^3 + (c - \frac{5}{7}ab) u^4 + \left( \frac{36}{125} a^2 b - \frac{6}{7} ac \right) u^5 \\
&\quad + \frac{12}{294} a^2 (21c - 19ab) u^6 + \frac{80}{1029} a^3 (13ab - 14c) u^8 + O(u^9) \right],
\end{align*}
\]  

(2.7)

which is exactly the same as that given in [3] except the coefficient \(\frac{9}{7}\) which was typed as 6 in [3]. We have used our method and executed our Maple program to obtain the following generalized Lyapunov constants:

\[
\begin{align*}
B_4 &= c - \frac{5}{7}ab, & B_6 &= \frac{13}{294} a^2 (21c - 19ab), & B_8 &= -\frac{729}{1728} a^4 (33ab - 35c), \\
B_{10} &= -\frac{5113889}{118590192} a^6 (47ab - 49c), & \cdots
\end{align*}
\]

It is seen that \(B_4\) and \(B_6\) are exactly the same as that given in (2.7). Further, it is easy to verify that setting \(B_4 = B_6 = 0\) leads to \(B_{2k} = 0, k \geq 4\).

In [5], the authors consider a special case – homogeneous polynomial systems and developed a special approach to calculate the generalized Lyapunov constants. Their methodology is computationally efficient, but can not be applied even to consider a simple cubic polynomial system. The 5th-order homogeneous polynomial system considered in [5] is given by

\[
\begin{align*}
\frac{dx}{dt} &= y + Ax^4 y + Bx^3 y^2 + Cx^2 y^3 + Dxy^4 + Ey^5, \\
\frac{dy}{dt} &= -x^5 + Qx^4 y + Kx^3 y^2 + Lx^2 y^3 + Mxy^4 + Ny^5.
\end{align*}
\]  

(2.8)

Using our Maple program, we obtain the following generalized Lyapunov constants:

\[
\begin{align*}
B_4 &= Q, & B_8 &= B + L, & B_{12} &= \frac{1}{6} \left[ 2L(K + 2A) + 3(D + 5N) \right], \\
B_{16} &= \frac{2}{11} \left[ (2A + K)(KL + 3N) + L(C + 2M) \right], \\
B_{20} &= \frac{14}{11} \left[ LM(2A + K) + 3N(C + 2M) \right], & B_{24} &= \frac{36}{11L} L^3 (2A + K),
\end{align*}
\]

where \(B_{4k} = 0\) has been set zero when computing \(B_{4k}\) for \(k = 2, 3, \ldots, 6\). They are the same as that given in [5], at most different by a positive constant factor.
Another special type of systems called quasi-homogeneous system is considered in [1], which takes the general form:

\[
\begin{align*}
\frac{dx}{dt} &= y + \sum_{i=0}^{\infty} P_{q-p+2is}(x, y), \\
\frac{dy}{dt} &= \sum_{i=0}^{\infty} Q_{q-p+2is}(x, y),
\end{align*}
\]

(2.9)

where \(p, q \in \mathbb{N}, p \leq q, s = (n+1)p - q > 0, n \in \mathbb{N},\) and \(P_i\) and \(Q_i\) are quasi-homogeneous polynomials in \(x\) and \(y\) with \(Q_{q-p+2s}(1, 0) < 0.\) The origin of this system is a nilpotent and monodromic isolated singular point. The authors used their method developed in [1] to obtain the center conditions for the origin of the following system,

\[
\begin{align*}
\frac{dx}{dt} &= y + a_1 x^5 + a_2 x^2 y + a_3 x^7 + a_4 x^4 y + a_5 xy^2, \\
\frac{dy}{dt} &= -x^7 + b_1 x^4 y - a_2 xy^2 + b_3 x^6 y + b_4 x^3 y^2 + b_5 y^3.
\end{align*}
\]

(2.10)

Executing our Maple program, we obtain the following generalized Lyapunov constants for the above system:

\[
\begin{align*}
B_4 &= 5a_1 + b_1, \quad B_6 = 7a_3 + b_3, \quad B_8 = a_5 + 3b_5 - 2a_1(b_4 + 2a_4), \\
B_{10} &= -2(2a_4 + b_4)(a_3 - a_1 a_2 + 4a_1^3), \\
B_{12} &= -\frac{2}{3}(b_4 + 2a_4)(a_5 - a_1(4a_4 - b_4 - 50a_2a_1^2 + 200a_1^4)], \\
B_{14} &= -\frac{4}{3}a_1(b_4 + 2a_4)(a_2 - 4a_1^2)(b_4 - a_4 + 62a_1^2 - 268a_1^4), \\
B_{16} &= \frac{4}{3}a_1^2(a_2 - 4a_1^2)(3a_4 - 62a_1^2 + 268a_1^4)(5a_4 - 12a_1^2 + 146a_2a_1^2 - 492a_1^4), \\
B_{18} &= -\frac{64}{27}a_1^5(a_2 - 4a_1^2)(9a_4^2 - 187a_2a_1^2 + 704a_1^4)(387a_2^2 - 4681a_2a_1^2 + 13282a_1^4), \\
B_{20} &= \frac{32}{975}a_1^5(a_2 - 4a_1^2)(9a_4^2 - 187a_2a_1^2 + 704a_1^4) \\
&\quad \times (1953a_2^4 - 27694a_2a_1^2 + 130023a_2a_1^4 - 201730a_1^6),
\end{align*}
\]

where \(B_{2(k-1)} = 0\) has been used in computing \(B_{2k}\) for \(k = 3, 4, \ldots, 10.\) Based on these generalized Lyapunov constants, we have the following result.

**Proposition 2.1.** The origin of system (2.10) is a center if and only if one of the following conditions is satisfied:

(i) \(5a_1 + b_1 = 7a_3 + b_3 = 2a_4 + b_4 = a_5 + 3b_5 = 0;\)

(ii) \(a_1 = a_3 = a_5 = b_1 = b_3 = b_5 = 0;\) and

(iii) \(b_1 = -5a_1, \quad a_2 = 4a_1^2, \quad b_5 = a_1b_4, \quad a_5 = a_1(4a_4 - b_4), \quad a_3 = b_3 = 0.\)

Note that the three center conditions are given in Theorem 3.1 of [1], but the condition \(b = -5a_1\) in (iii) was typied as \(b_1 = -a_1\) in [1], and in addition, the conditions \(a_3 = b_3 = 0\) were missed in (iii). It is easy to verify that under the condition (i) system (2.10) is a Hamiltonian system with the Hamiltonian function:

\[
H(x, y) = \frac{1}{2}y^2 + \frac{1}{8}x^8 + a_1 x^5 y + \frac{1}{2}a_2 x^2 y^2 + a_3 x^7 y + \frac{1}{2}a_4 x^4 y^2 - b_5 xy^3.
\]
For the condition (ii), it is easy to see that system (2.10) is a reversible system since it is invariant under the transformation \((y, t) \rightarrow (-y, -t)\).

For the condition (iii), we present a simple proof different from that given in [1]. In fact, for this case, we use the following integrating factor,

\[
I_{(iii)} = \frac{4a_4 b_4 (b_4 - 2a_4) (1 + a_4 x^4 + 4a_1 a_4 y)^{b_4 - 1}}{[2 + b_4 x^4 + 4a_1 b_4 y + b_4 (2a_4 - b_4) y^2]^{1 + 2a_4}},
\]

to obtain the first integral,

\[
F(x, y) = \frac{(1 + a_4 x^4 + 4a_1 a_4 y)^{b_4}}{[2 + b_4 x^4 + 4a_1 b_4 y + b_4 (2a_4 - b_4) y^2]^{2a_4}}.
\]

Now, we return to cubic-order systems with an isolated nilpotent critical point and want to find the maximal number of limit cycles which bifurcate in the neighborhood of the critical point. In [28], Liu and Li have considered the following cubic polynomial system,

\[
\begin{align*}
\frac{dx}{dt} &= y - 2xy - (a_4 - a_7)x^2 y + a_6 y^2 + a_2 x y^2 + a_5 y^3, \\
\frac{dy}{dt} &= -2x^3 + a_1 x^2 y + y^2 + a_4 x y^2 + a_3 y^3,
\end{align*}
\]

(2.11)

which contains 7 free parameters. Thus, by adding a linear perturbation, the authors applied their approach to prove the existence of at least 8 small-amplitude limit cycles bifurcating from the origin. In fact, using our method, we can find the generalized Lyapunov constants as follows:

\[
\begin{align*}
B_2 &= a_1, \\
B_4 &= 2(a_2 + 3a_3), \\
B_6 &= \frac{4a_2}{3} (3a_3 - 5a_6), \\
B_8 &= \frac{4a_2 a_7}{105} (735 - 105a_4 + 71a_7), \\
B_{10} &= \frac{8}{1025} a_6 a_7 \{176400 + 18375a_5 + 5460a_7 + 12250a_6^2 - 32a_7^2\}, \\
B_{12} &= \frac{32a_6 a_7}{579024375} \{30866913000 + 2089303650a_7 - 1188495000a_6^2 \\
&\quad + 29397690a_7^2 - 15232875a_6^2 a_7 - 110996a_7^3\}, \\
B_{14} &= \frac{-32a_6 a_7}{59727219749071875} \{44389456322515920000 + 2155807164550977000a_7 \\
&\quad - 1647138037233150000a_6^2 - 11437991172477450a_7^2 - 910916029415875a_7^3 \\
&\quad - 2212192499656250a_6^4 + 798220526556a_7^4\}, \\
B_{16} &= \frac{32a_6 a_7}{183948685130392199786884375} \{9423379312441682897451542400000 \\
&\quad + 151429876581319947369112800000a_7 \\
&\quad + 82859324997946429848009339000a_7^2 \\
&\quad + 1864567030459291902188584650a_7^3 \\
&\quad + 14562086011213729200961815a_7^4 - 2666191085683953547508a_7^5\},
\end{align*}
\]
where $B_{2k}$ is the $k$th generalized Lyapunov constant, and $B_{2(k-1)}$ has been set zero in computing $B_{2k}$ for $k = 2, 3, \cdots, 8$. It is noted that the quasi Lyapunov constants, $\lambda_k$, $k = 1, 2, \cdots, 8$, obtained in [28], are indeed given by $\lambda_k = \frac{1}{2k+1} B_{2k}$, $k = 1, 2, \cdots, 8$. Then, by applying proper perturbations to show that there exist parameter values satisfying $B_2 = B_3 = \cdots = 0$, but $B_{16} \neq 0$, implying the existence of 7 limit cycles. In addition, the linear perturbation gives one more limit cycle to achieve 8 limit cycles [28]. Later, the same authors considered the following system in [29],

$$
\begin{align*}
\frac{dx}{dt} &= y - 2b_{02}xy + a_{02}y^2 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\frac{dy}{dt} &= -2x^3 + b_{02}y^2 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3,
\end{align*}
$$

(2.12)

which is obviously the same as system (2.11), and can have 8 small-amplitude limit cycles bifurcating from the origin. Moreover, in [29], the authors applied the so called method of double bifurcation of nilpotent focus to get 9 small-amplitude limit cycles, with the distribution of $7 \supset (1 \cup 1)$. That is, there are three singular points, one of which is the origin and other two are near the origin with one limit cycle around each of them, and 7 limit cycles enclose these two limit cycles. The basic idea is to apply perturbation to system (2.12) to obtain a perturbed system as follows:

$$
\begin{align*}
\frac{dx}{dt} &= y - 2b_{02}xy + a_{02}y^2 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\frac{dy}{dt} &= 4\delta \varepsilon y + b_{02}y^2 + b_{12}xy^2 + b_{03}y^3 - (x^2 - \varepsilon^2)(2x - b_{21}y),
\end{align*}
$$

(2.13)

where $\delta$ and $\varepsilon$ are perturbation parameters, satisfying $0 < \vert \delta \vert \ll 1$, $0 < \varepsilon \ll 1$. It is easy to see that system (2.13) has three fixed points: $(x, y) = (\varepsilon, 0), (-\varepsilon, 0)$ and $(0, 0)$. Thus, at $\delta = \varepsilon = 0$, we have the 8 generalized Lyapunov constants showing the existence of 7 limit cycles around the origin $(0, 0)$. Then, by taking proper perturbation values of $\delta$ and $\varepsilon$, we can find two small-amplitude limit cycles inside the 7 limit cycles, each of them encloses one of the two singular points $(\varepsilon, 0)$ and $(-\varepsilon, 0)$. More details about the method of double bifurcation of nilpotent focus can be found in [29]. Although this approach does not give all 9 limit cycles around the origin, it does have one more limit cycle near the origin, compared with the result obtained in [28].

Recently, we have studied bifurcation of near-Hamiltonian systems, described by

$$
\begin{align*}
\frac{dx}{dt} &= \frac{\partial H(x, y, \mu_1)}{\partial y} + \varepsilon P(x, y, \mu_2), \\
\frac{dy}{dt} &= -\frac{\partial H(x, y, \mu_1)}{\partial x} + \varepsilon Q(x, y, \mu_2),
\end{align*}
$$

(2.14)

where $H(x, y, \mu_1)$ is an $n$th-degree real polynomial in $x$ and $y$ and $P, Q$ are $m$th-degree polynomials in $x$ and $y$, and $\mu_1$ and $\mu_2$ are vector parameters, and $0 < \varepsilon \ll 1$ is a small perturbation parameter. The function $H(x, y, \mu_1)$ is called the Hamiltonian of system (2.14). When $\varepsilon = 0$, the origin is a nilpotent center of the system.

The monodromy of the origin of system (2.14)$|_{\varepsilon=0}$ has been studied in [21] and detailed classification conditions are given. Very recently, we have applied our new method to consider the following cubic near-Hamiltonian system:

$$
\begin{align*}
\frac{dx}{dt} &= y + 2xy + 3a_1y^2 + 2a_2x^2y + 3a_3xy^2 + 4a_4y^3, \\
\frac{dy}{dt} &= -4x^3 - y^2 - 2a_2xy^2 - a_3y^3 + \varepsilon (\delta x + \delta y + xy + b_1y^2 + b_2x^2y + b_3xy^2 + b_4y^3),
\end{align*}
$$

(2.15)
which contains 9 free parameters (with \( \delta \) being the linear perturbation parameter), and shown that there exist at least 9 small-amplitude limit cycles around the origin [41], in which the generalized Lyapunov constants are obtained as follows:

\[
B_0 = \frac{1}{2} \delta, \\
B_2 = b_2 - 1, \\
B_4 = 12b_4 + 24a_1 b_1 + 2a_2 - 1, \\
B_6 = 4(a_3 - 5a_1) b_3 + 8[2a_1(5-a_2)-5a_3]b_1 + 16a_4 + 20a_1(a_3-3a_1)-(2a_2-1)^2, \\
B_8 = \frac{3}{a_3-b_1} \{8[(7+2a_2)(2a_2a_1^2 + a_3^2 - 2a_3a_1) + 16(5a_1-a_3)a_1a_4]b_1 \\
+ 2a_3 - 8a_2a_3 + 88a_3^2a_1 - 224a_1a_3 + 8a_2^2a_3 - 12a_3^2 + 24a_2a_1a_3 \\
- 8a_2a_3^2a_1 + 140a_3^2 - 20a_2^2a_1 + 8a_3^2a_1 - 3a_1 + 14a_2a_1 \\
- 16(2a_3 - 3a_1 + 2a_2a_1)a_4 \}, \\
\vdots \\
B_{18} = \cdots
\]

where \( B_0, B_2, \cdots, B_{2(k-1)} \) have been set zero in computing \( B_{2k} \) for \( k = 1, 2, \cdots, 9 \). Then, by using proper perturbations on the 9 parameters, it has been shown in [41] that there exist at least 9 small-amplitude limit cycles around the origin.

3. 9 limit cycles in a cubic-order system around a nilpotent critical point

In this section, we present our main result of this paper. Consider the cubic polynomial system (2.11) with an additional parameter \( a_8 \) and two linear perturbation parameters \( \delta_1 \) and \( \delta_2 \):

\[
\begin{align*}
\frac{dx}{dt} &= y + \delta_1 y + (a_8 - 2)xy - (a_4 - a_7)x^2y + a_6y^2 + a_2xy^2 + a_5y^3, \\
\frac{dy}{dt} &= -\delta_1 x + \delta_2 y - 2x^3 + a_1x^2y + y^2 + a_4xy^2 + a_3y^3,
\end{align*}
\]

(3.1)

where \( 0 < \delta_1, |\delta_2| \ll 1 \). Now system (3.1) can yield 9 limit cycles around the origin, but the computation becomes much more demanding.

In this section, we will consider bifurcation of limit cycles all around the origin of system (3.1), yielding 9 limit cycles, and in the next section, we will apply the method of double bifurcation of nilpotent focus to system (3.1) to obtain 10 limit cycles near the origin.

**Theorem 3.1.** For system (3.1) with a nilpotent critical point at the origin, there exist at least 9 small-amplitude limit cycles around the origin.

**Proof.** First, let the two linear perturbation parameters equal zero, \( \delta_1 = \delta_2 = 0 \). Then we apply the method of normal forms and our developed Maple program to system (3.1) to obtain

\[
B_2 = a_1.
\]

(3.2)
We set $a_1 = 0$ to have $B_2 = 0$. Then, $B_4$ is given by

$$B_4 = 2(a_2 + 3a_3 - a_6a_8).$$

(3.3)

Similarly, letting

$$a_2 = -3a_3 + a_6a_8,$$

we obtain $B_4 = 0$. Then, the next two generalized Lyapunov constants $B_6$ and $B_8$ become

$$B_6 = \frac{6}{15}(10a_7 - 7a_8^2 + 25a_8)a_3 - \frac{2}{15}[a_8(5a_4 - 14a_8 + 50) - (9a_8 - 50)a_7]a_6$$

$$= \frac{4}{3}(3a_3 - 5a_6)a_7 - \frac{2}{15}[3(7a_8 - 25)a_3 + (5a_4 - 9a_7 - 14a_8 + 50)a_6]a_8,$$

$$B_8 = -\frac{2}{875}[25a_8(23a_8 - 105)a_4 - 5(460a_7 + 1255a_8 - 241a_8^2 - 2100)a_7$$

$$- 3a_8(62a_8^2 - 1435a_8^2 + 8500a_8 - 14875)]a_3$$

$$- \frac{4}{875}\{[875a_4 - 5(42a_8 - 265)a_7 + 62a_8^3 - 1645a_8^2 + 8950a_8 - 14875]a_7$$

$$- (875a_5 - 62a_3^3 + 1435a_3^2 - 8500a_8 + 14875)a_8\}a_6$$

$$= \frac{4}{35}[(46a_3 - 53a_6)a_7 - 35a_4a_6 - 210a_8 + 595a_8]a_7$$

$$- \frac{2}{875}\{25(23a_8 - 105)a_3a_4 + [5(241a_8 - 1255)a_3 - 2(210a_7 - 62a_8^2$$

$$+ 1645a_8 - 8950)a_6]a_7 - 3(62a_8^3 - 1435a_8^2 + 8500a_8 - 14875)a_3$$

$$- 2(875a_5 - 62a_3^3 + 1435a_3^2 - 8500a_8 + 14875)a_6\}a_8,$$

where $B_6 = 0$ has been used to compute $B_8$. It follows from (3.4) and (3.5) that we may classify two cases:

(A) $a_6a_8 = 0$ and (B) $a_6a_8 \neq 0$.

**Case (A)** $a_6a_8 = 0$. In this case, $a_2 = -3a_3$. If $a_6 = 0$, then $a_3 = 0$ yields $B_6 = B_8 = 0$, and in fact all $B_{2k} = 0$, $k = 5, 6, \cdots , 10$. This gives a condition

$$C_1 : \ a_1 = a_2 = a_3 = a_6 = 0,$$

(3.6)

under which all the generalized Lyapunov constants, $B_{2k}$, $k = 1, 2, \ldots , 10$ vanish. Similarly, if $a_8 = 0$, then $a_7 = 0$ yields $B_6 = B_8 = 0$, and this gives another condition,

$$C_2 : \ a_1 = a_7 = a_8 = 0, \quad a_2 = -3a_3,$$

(3.7)

under which $B_{2k} = 0$, $k = 1, 2, \ldots , 10$.

Next, we want to investigate under the condition $a_6a_8 = 0$, what is maximal number of limit cycles which can bifurcate from the origin of system (3.1). We first consider $a_6 = 0, a_3 \neq 0$ and then $a_8 = 0, a_7 \neq 0$. The case $a_6 = a_8 = 0$ is not considered since it yields special cases of $C_1$.

**Case (A)** $a_6 = 0, a_3 \neq 0$. For this case, $B_6 = 0$ yields a solution $a_7 = \frac{1}{15}(7a_8 - 25)a_8$ with $a_8 \neq 0$ since $a_8 = 0$ leads to a special case of $C_2$. Then, $B_8$ becomes

$$B_8 = -\frac{a_3a_8}{875}[50(23a_8 - 105)a_4 - 3(313a_8^3 - 3300a_8^2 + 11225a_8 - 12250)].$$
This shows that taking \( a_8 = \frac{105}{23} \) yields \( B_8 = -\frac{5040}{298641} a_3 \neq 0 \), implying that 4 limit cycles can be obtained. Suppose \( a_8 \neq \frac{105}{23} \), we solve \( B_8 = 0 \) for \( a_4 \) and then substitute this solution into \( B_{10} \) to obtain

\[
B_{10} = \frac{2a_8 a_6}{65625(23a_8 - 105)^7} \left[ 1250(127a_8 - 525)(23a_8 - 105)^2 a_5 
+ 3(a_8 - 5)^2(843883a_8^2 - 17214695a_8^4 + 140051325a_8^6
- 568007125a_8^7 + 1148437500a_8 - 926100000) \right].
\]

Clearly, setting \( a_8 = \frac{525}{127} \) gives \( B_{10} = \frac{956445952000}{12587618744967} a_3 \neq 0 \), yielding 5 limit cycles around the origin. If choosing a root of the second factor in \( B_{10} \), then \( a_5 = 0 \) (due to \( B_{10} = 0 \)) and \( B_{12} \) becomes a function in \( a_3 \), giving 6 limit cycles. For example, letting \( a_8 = 5 \) we have \( B_{12} = -\frac{2480}{77} a_3^3 \neq 0 \). Now suppose \((127a_8 - 525)(23a_8 - 105) \neq 0 \), and the second factor in \( B_{10} \) is also nonzero. Then, we solve \( B_{10} = 0 \) for \( a_5 \) and use this solution to simplify \( B_{12} \) and \( B_{14} \) to obtain

\[
\begin{align*}
B_{12} &= -\frac{6015625(23a_8 - 105)^7}{65625(23a_8 - 105)^7} G_1(a_3, a_8), \\
B_{14} &= -\frac{782031250(23a_8 - 105)^7}{65625(23a_8 - 105)^7} G_2(a_3, a_8), \\
B_{16} &= -\frac{625625000000(23a_8 - 105)^7}{65625(23a_8 - 105)^7} G_3(a_3, a_8),
\end{align*}
\]

where \( G_i, i = 1, 2, 3 \) are polynomials in \( a_3^2 \) and \( a_8 \) and linear with respect to \( a_3^3 \). In particular,

\[
G_1 = 187500(697a_8 - 2695)(127a_8 - 525)(23a_8 - 105)^3 a_3^2
- (a_8 - 5)^2(1377405099237a_8^9 - 50096385469230a_8^7
+ 80873810674975a_8^7 - 7606204683786500a_8^6
+ 4592901348751875a_8^5 - 184655576500018750a_8^4
+ 494307270802515625a_8^3 - 849570999768750000a_8^2
+ 850698925256250000a_8 - 378119684250000000),
\]

which shows that taking \( a_8 = \frac{2695}{697} \) yields 6 limit cycles. Next, suppose \( a_8 \neq \frac{2695}{697} \). Then, the second factor in \( G_1 \) must be nonzero since \( a_3 \neq 0 \). We solve the equation \( G_1 = 0 \) for \( a_3^2 \) and substitute this solution into \( B_{14} \) and \( B_{16} \) to obtain two polynomial equations in \( a_8 \). It can be shown that there exist 3 real solutions for \( a_8 \) such that \( a_3^2 > 0 \) and \( B_{14} = 0 \), but \( B_{16} \neq 0 \), implying that maximal 8 limit cycles can bifurcate from the origin of system \((3.1)\).

The following analysis will be more or less similar to the above discussion.

**Case** \((A_2) \) \( a_8 = 0, a_7 \neq 0 \). In this case, \( B_6 = 0 \) yields \( a_3 = \frac{5}{3} a_6 \) with \( a_6 \neq 0 \) since \( a_6 = 0 \) gives a special case of \( C_1 \). For this solution, \( B_8 = \frac{4a_8 a_6}{105} (71a_7 + 735 - 105a_4) \). Letting \( a_4 = \frac{71a_7 + 735}{105} \) yields \( B_8 = 0 \) and

\[
B_{10} = \frac{a_6 a_7}{11025}(12250a_7^2 - 32a_7^2 + 5460a_7 + 176400 + 18375a_7),
\]

from which we can solve for \( a_6 \) and substitute the solution into \( B_{12}, B_{14} \) and \( B_{16} \) to obtain their simplified expressions in \( a_7^2 \) and \( a_7 \), which are linear with respect to \( a_6^2 \). In particular, \( B_{12} \) is given by

\[
B_{12} = -\frac{32a_6 a_7}{573024375} \left[ 18375(829a_7 + 64680)a_7^2 + 110996a_7^2 - 29397690a_7^2
- 2089303650a_7 - 30866913000 \right].
\]
It is easy to see that taking \( a_7 = -\frac{64680}{829} \) yields \( B_{12} = -\frac{204812940525512704}{472809192381} \) \( a_6 \neq 0 \), giving 6 limit cycles. Suppose \( a_7 \neq -\frac{64680}{829} \). Then, solving \( B_{12} = 0 \) gives a solution for \( a_6^2 \), which is substituted into \( B_{14} \) and \( B_{16} \) to obtain two polynomial equations in \( a_7 \). It can be shown that there exists only one real solution for \( a_7 \) such that \( a_6^2 > 0 \) and \( B_{14} = 0 \), but \( B_{16} \neq 0 \), implying that maximal 8 limit cycles can bifurcate from the origin of system (3.1).

Summarizing the above results, we have shown that when \( a_6 a_8 = 0 \), the maximal number of limit cycles can bifurcate from the origin of system (3.1) is 8. So, to find 9 limit cycles, we must consider the case \( a_6 a_8 \neq 0 \).

**Case (B)** \( a_6 a_8 \neq 0 \). For convenience, define

\[
H_1 = 10a_7 - (7a_8 - 25)a_8,
H_2 = -(9a_8 - 50)a_7 + (5a_4 - 14a_8 + 50)a_8, \\
H_3 = 5a_4 a_8 - (9a_8 - 20)a_7 + (7a_8 - 25)a_8.
\]

Then, \( B_6 \) can be rewritten as \( B_6 = \frac{6}{15} a_3 H_1 - \frac{2}{15} a_6 H_2 \), which shows that if \( B_6 = 0 \), then \( H_1 = 0 \) implies \( H_2 = 0 \) due to \( a_6 a_8 \neq 0 \). Hence, in order to have \( B_6 = 0 \), we need to investigate three cases: \( H_1 = H_2 = 0 \); \( H_1 \neq 0, H_2 = 0 \); and \( H_1 H_2 \neq 0 \).

**Case (B1)** \( H_1 H_2 \neq 0 \). First we consider the generic case, \( H_1 H_2 \neq 0 \), under which solving \( B_6 = 0 \) yields a solution for \( a_3 \):

\[
a_3 = \frac{H_2}{3H_1} a_6. \tag{3.9}
\]

Next, from the output of our Maple program, we obtain the generalized Lyapunov constant \( B_8 \), which is linear in \( a_5 \). Thus, solving \( B_8 = 0 \) for \( a_5 \) yields

\[
a_5 = \frac{1}{1050a_8 H_1} \left\{ 100a_5^2 a_4 (105 - 23a_8) + 10a_4 a_7 a_8 (17a_8^2 - 315a_8 + 105) \\
+ 25a_4^2 a_8^2 (23a_8 - 105) + 5a_4^2 (4a_7 - a_8^2 + 4a_8) (81a_8 - 355) \\
- [a_4 a_5^2 - (4a_7 - a_8^2 + 4a_8) a_7] (18a_8^3 - 2695a_8^2 + 12400a_8 - 18375) \right\}. \tag{3.10}
\]

With the above solutions of \( a_1, a_2, a_3 \) and \( a_5 \), other higher-order generalized Lyapunov constants are obtained as

\[
B_{10} = -\frac{2a_5}{826875 a_8 H_1} F_0 F_1, \\
B_{12} = -\frac{a_5}{136434375 a_8 H_1} F_0 F_2, \\
B_{14} = -\frac{a_5}{2483105625000 a_8^3 H_1} F_0 F_3, \\
B_{16} = -\frac{a_5}{223479506250000 a_8^4 H_1} F_0 F_4, \\
B_{18} = -\frac{a_5}{5318812248750000000 a_8^5 H_1} F_0 F_5,
\]

where

\[
F_0 = a_4 a_5^2 - (4a_7 - a_8^2 + 4a_8) a_7. \tag{3.12}
\]
and $F_1$, $F_2$, $\ldots$, $F_5$ are functions in $a_4, a_7, a_8$ and $a_6^2$. Note that if $F_0 = 0$, then all the generalized Lyapunov constants $B_{2k}$, $k \leq 10$ vanish. The condition $F_0 = 0$ together with the solutions $a_1$, $a_2$, $a_3$ and $a_5$ gives the following condition:

$$C_3 : a_1 = a_5 = 0, \quad a_2 = a_6(a_8 - 2 - \frac{2\alpha_7}{a_8}), \quad a_3 = \frac{2}{3}a_6(1 + \frac{\alpha_7}{a_8}), \quad a_4 = \frac{a_7(4a_7 + 4a_8 - a_6^2)}{a_6^2},$$

under which $B_{2k} = 0$, $k = 1, 2, \ldots, 10$.

In order to find the maximal number of limit cycles bifurcating from the origin, we need to use the parameters $a_4, a_6, a_7, a_8$ to find the solutions such that $F_1 = F_2 = F_3 = F_4 = 0$, but $F_5 \neq 0$ (or $B_{18} \neq 0$). Therefore, in the following, we shall first try to find the solutions from the equations, $F_1 = F_2 = F_3 = F_4 = 0$, and then verify if the condition $B_{18} \neq 0$ is satisfied for these solutions. Since all $F_i$, $i = 1, 2, 3, 4$ are functions in $a_6^2$ and in particular, $F_1$ is linear in $a_6^2$, given by

$$F_1 = \begin{array}{c}
4593750 a_8 a_6^2 H_3 \\
+ 1250a_2^2 a_6 [315(37a_8 - 175)a_7 - (9619a_8^2 - 80430a_8 + 165375)a_8] \\
- 25a_4[50(7513a_8^2 - 11445a_8 - 110250)a_8^2 - 5a_8(38427a_8^2 + 77630a_8^2) \\
- 2608725a_8 + 6615000)a_7 - a_6^2(201212a_8^4 - 3697415a_8^3 + 24846075a_8^2) \\
- 72736125a_8 + 78553125)] - 500a_7^2(1062a_8^2 - 44885a_8 + 186375) \\
- 25a_7^2(66303a_8^4 - 544520a_8^2 + 3396875a_8^2 - 18947250a_8 + 38587500) \\
- 10a_7a_8(137382a_8^3 - 2395130a_8^2 + 13673725a_8^3 - 22645625a_8^2) \\
- 37261875a_8 + 108871875] + a_6^2(7a_8 - 25)(11112a_8^3 - 215450a_8^4 \\
+ 1330375a_8^2 - 2339375a_8^2 - 4081875a_8 + 12403125). \end{array}$$

(3.14)

There are two cases: $H_3 = 0$ and $H_3 \neq 0$.

First, we consider $H_3 = 0$ from which we obtain

$$a_4 = \frac{1}{5a_8} \left[ (9a_8 - 20)a_7 - (7a_8 - 25)a_8 \right],$$

(3.15)

which is substituted into the higher-order generalized Lyapunov constants to yield

$$B_{10} = \frac{4a_6}{1378125a_8} (a_8 - 5)(2a_7 - a_8) \overline{F}_1,$$

$$B_{12} = \frac{2a_6}{227390625a_8} (a_8 - 5)(2a_7 - a_8) \overline{F}_2,$$

$$B_{14} = \frac{a_6}{20692546875a_8^3} (a_8 - 5)(2a_7 - a_8) \overline{F}_3,$$

$$B_{16} = \frac{a_6}{18623921875000a_8^5} (a_8 - 5)(2a_7 - a_8) \overline{F}_4,$$

(3.16)

where $\overline{F}_1$ is a function in $a_7$ and $a_8$, while $\overline{F}_2$, $\overline{F}_3$ and $\overline{F}_4$ are functions in $a_7$, $a_8$ and $a_6^2$. It can be shown that $(a_8 - 5)(2a_7 - a_8) = 0$ yields $B_{2k} = 0$, $k = 1, 2, \ldots, 10$. In fact, $a_8 = 5$ indeed gives a condition,

$$C_4 : a_1 = a_5 = 0, \quad a_8 = 5, \quad a_2 = 2a_6, \quad a_3 = a_6, \quad a_4 = a_7 - 2,$$

under which all $B_{2k}, k \leq 10$ vanish. However, $2a_7 - a_8 = 0$ yields a special case of $C_3$. 

---

For other solutions solved from $F_1 = F_2 = F_3 = 0$, it can be shown that maximal 8 limit cycles can be obtained. First it has been noted that the coefficient of $a_6^2$ in $F_2$ is $93a_8^2 - 725a_8 + 1400$. Letting this coefficient equal zero yields polynomials $F_1$ and $F_2$ in $a_7$ and it can be shown that there exist four real solutions such that $F_1 = 0$ (i.e., $B_{10} = 0$), but $F_2 \neq 0$ (i.e., $B_{12} \neq 0$), implying the existence of 6 limit cycles. When $93a_8^2 - 725a_8 + 1400 \neq 0$, we can solve $a_6^2$ from $F_2 = 0$, and then $F_3$ and $F_4$ also become polynomials in $a_7$ and $a_8$. One can show that there exist four real solutions such that $F_1 = F_3 = 0$, but $F_4 \neq 0$, implying that maximal 8 limit cycles can bifurcate from the origin.

Now, suppose $H_3 \neq 0$. Substituting the solution of $a_6^2 = A_6(a_4, a_7, a_8)$, solved from $F_1 = 0$, into $F_2$, $F_3$ and $F_4$, we obtain

$$F_2 = -\frac{4}{875 H_3} G_1, \quad F_3 = -\frac{16}{875 H_3} G_2, \quad F_4 = -\frac{12}{765625 H_3} G_3,$$

where $G_1$, $G_2$ and $G_3$ are respectively, 4th-, 5th- and 7th-degree polynomial functions in $a_4$. To solve the equations $G_1 = G_2 = G_3 = 0$ for real solutions of the parameters, $a_4$, $a_7$ and $a_8$, we first use the Maple built-in command eliminate to eliminate $a_4$ from the three equations: $G_1 = G_2 = G_3 = 0$, yielding a solution $a_4 = a_4(a_7, a_8)$, and two resultants:

$$R_{12} = R_0 R_{12a}, \quad R_{13} = R_0 (93a_8^2 - 725a_8 + 1400) R_{13a},$$

where the common factor $R_0$ is given by

$$R_0 = a_8 H_1 (a_8 - 5)(9a_8 - 35) \times \left[55125(3a_8 - 4)^2a_7^2 - 10a_8^2(18552a_8^2 - 128825a_8 + 223475) a_7 - a_8^2 (1852a_8^4 - 194325a_8^3 + 1770175a_8^2 - 5548375a_8 + 5788125)\right],$$

and $R_{12a}$ and $R_{13a}$ are lengthy polynomials in $a_7$ and $a_8$ (with 888 terms in $R_{12a}$ and 1380 terms in $R_{13a}$), which are not listed here for brevity. First, consider $R_0$. If $R_0 = 0$, then all generalized Lyapunov constants vanish. Since $a_8 H_1 \neq 0$ and $a_8 = 5$ has been considered in the condition $C_4$, we only need to consider other two factors. For the big factor, we can show that letting this factor equal zero yields $H_3 = 0$, violating the assumption. For $a_8 = \frac{35}{9}$, we get one more condition, given below:

$$C_5 : \begin{cases} a_1 = 0, \\
    a_8 = \frac{35}{9}, \\
    a_2 = \frac{a_6(52488a_7^2 - 161595a_7 + 115150)}{9(81a_7 - 140)(81a_7 - 70)}, \\
    a_3 = \frac{a_6(59049a_7^2 - 144585a_7 + 75950)}{9(81a_7 - 140)(81a_7 - 70)}, \\
    a_4 = \frac{59049a_7^2 - 119070a_7 + 34300}{945(81a_7 - 140)}, \\
    a_5 = \frac{2(27a_7 - 70)(81a_7 - 35)(243a_7 - 350)(162a_7 - 245)}{2679075(81a_7 - 140)^2}, \\
    a_6 = \frac{2(81a_7 - 70)(243a_7 - 280)}{2835\sqrt{70}(81a_7 - 140)} \left( a_7 > \frac{140}{81} \right), \end{cases}$$

under which $B_{2k} = 0$, $i = 1, 2, \ldots, 10$.

For the remaining parts in $R_{12}$ and $R_{13}$, we first consider the solution solved from the simple factor of $R_{13}$, $93a_8^2 - 725a_8 + 1400 = 0$, which gives two real solutions: $a_8^\pm = \frac{725 \pm 5\sqrt{193}}{186}$. Substituting the two solutions into the equation $R_{12a} = 0$ to solve for $a_7$, yielding 15 real solutions corresponding to $a_8^+$ and

24 of verifying \(a_8\) factors imply that maximal 8 limit cycles can be obtained from the solutions \(a_8^\pm\). Hence, the feasible solutions for 9 limit cycles must be found from the equations \(R_{12a} = R_{13a} = 0\). Since \(R_{12a}\) and \(R_{13a}\) are respectively 23rd- and 29th-degree polynomials in \(a_7\), we apply the Maple built-in command resultant to eliminate \(a_7\) from the two equations: \(R_{12a} = R_{13a} = 0\) to obtain a resultant in \(a_8\):

\[
R_{123} = C_{123} a_8^{326} (9 a_8 - 35)^2 (a_8 - 5)^5 \times (3 a_8 - 10)^8 (697 a_8 - 2695) (549 a_8 - 1645) R_{123a} R_{123b},
\]

where \(C_{123}\) is a big integer, and \(R_{123a}\) (which contains 284 terms) and \(R_{123b}\) (which includes 1454 terms) are respectively 283rd- and 1453th-degree integer polynomials in \(a_8\), each term having a very big integer coefficient. It can be shown that the polynomial \(R_{123}\), does not have solutions satisfying \(R_{12a} = R_{13a} = 0\). Thus, we only need to consider the linear factors in \(R_{123}\) and the factor \(R_{123a}\). Since \(a_8 \neq 0\), the linear factors have the solutions: \(a_8 = \frac{1645}{549}, \frac{10}{3}, \frac{2695}{369}, \frac{35}{9}\) and \(5\). \(a_8 = 5\) has been considered above in the condition \(C_4\), and a direct computation shows that the solution \(a_8 = \frac{10}{3}\) leads to that \(R_{12a}(a_7)\) and \(R_{13a}(a_7)\) have no common factors. Moreover, for \(a_8 = \frac{2695}{369}\), we have \(a_7 = \frac{38809}{485809}\), which yields \(H_1 = 0\) and so is not allowed. Therefore, we only need to consider two values of \(a_8: \frac{1645}{549}\) and \(\frac{35}{9}\). Each of them yields a unique solution of \(a_7\) satisfying \(R_{12a}(a_7) = 0\) and \(R_{13a}(a_7) = 0\). But both them yield a zero divisor for solution \(a_4(a_7, a_8)\). Thus, for these two values of \(a_8\), we need reconsider possible bifurcation of limit cycles by investigating the solutions of the equations: \(G_1 = G_2 = G_3 = 0\).

1. \(a_8 = \frac{1645}{549}\). For this value, \(R_{12a}(a_7)\) and \(R_{13a}(a_7)\) have a common root \(a_7 = \frac{101996}{301401}\) under which

\[
B_{12} = \frac{a_6 (301401 a_4 - 49538)}{11163 a_4 - 10744} B_{12a}(a_4), \quad B_{14} = \frac{a_6 (301401 a_4 - 49538)}{11163 a_4 - 10744} B_{14a}(a_4),
\]

where \(B_{12a}\) and \(B_{14a}\) are respectively 3rd- and 4th-degree polynomials in \(a_4\). Note that \(11163 a_4 - 10744 = 0\) yields \(H_3 = 0\) and so is not allowed, while \(301401 a_4 - 49538 = 0\) yields a special case of \(C_3\). Moreover, it is easy to show that \(B_{12a}(a_4)\) has 3 real roots, and all of them satisfy \(a_6^2 = A_6(a_4(a_7, a_8), a_7, a_8) > 0\) and \(B_{14a} \neq 0\), implying that there are 6 solutions to yield 7 limit cycles around the origin.

2. \(a_8 = \frac{35}{9}\). For this case we obtain \(a_7 = \frac{140}{81}\), and

\[
B_{12} = \frac{56 a_6 (81 a_4 - 68)}{3519163699 (9 a_4 - 8)} (78121827 a_3^2 - 206422182 a_3 + 180139302 a_4 - 51948944),
\]

\[
B_{14} = \frac{28 a_6 (81 a_4 - 68)}{363486813451047 (9 a_4 - 8)} (601147458765 a_4 - 2125734770745 a_4^3 + 280271743668 a_4^2 - 1634221389834 a_4 + 355795637488).
\]

Note that \(9 a_4 - 8 = 0\) is not allowed since it yields \(H_3 = 0\), and \(81 a_4 - 68 = 0\) gives a special case of \(C_3\). The 3rd-degree polynomial in \(B_{12}\) has one real solution satisfying \(a_6^2 = A_6(a_4(a_7, a_8), a_7, a_8) > 0\) and \(B_{14} \neq 0\), implying the existence of 7 limit cycles.

Therefore, none of the solutions obtained from the linear factors can give 9 limit cycles.

Next, consider the factor \(R_{123a}\). It has 53 real roots for \(a_8\), each of them yields a unique solution for \(a_7\) by verifying the common roots of the equations \(R_{12a}(a_7) = 0\) and \(R_{13a}(a_7) = 0\), leading to 53 sets of solutions \((a_7, a_8)\). Moreover, all the 53 sets of solutions satisfy \(G_1 = G_2 = G_3 = 0\) (i.e., \(F_2 = F_3 = F_4 = 0\)), but only 24 of them yield \(a_6^2 A_6(a_4(a_7, a_8), a_7, a_8) > 0\). These 24 sets of solutions are
Then, for each set of the above solutions, we can find corresponding solutions for $a_4(a_7, a_8), a_6 = \pm \sqrt{A(a_7, a_8, a_7, a_8), a_5, a_3}$ and $a_2$. Thus, there are in total 48 solutions, satisfying $B_2 = B_4 = \cdots = B_{16} = 0$, but $B_{18} \neq 0$. For example, taking the fourth solution, we have

$$
\begin{align*}
(a_7, a_8) &= (4.398089\cdots, -14.54122\cdots), (-66.19700\cdots, -10.81905\cdots), \\
&\quad (0.451595\cdots, -0.019793\cdots), (-0.545773\cdots, 0.891421\cdots), \\
&\quad (-9.202151\cdots, 0.916847\cdots), (-0.689736\cdots, 2.248118\cdots), \\
&\quad (-3.237553\cdots, 2.525801\cdots), (-2.635767\cdots, 2.674894\cdots), \\
&\quad (0.911863\cdots, 3.17137\cdots), (0.916134\cdots, 3.199575\cdots), \\
&\quad (-0.495531\cdots, 3.233994\cdots), (-0.411833\cdots, 3.357596\cdots), \\
&\quad (0.782354\cdots, 3.464452\cdots), (-0.819658\cdots, 3.530177\cdots), \\
&\quad (2.323788\cdots, 3.574264\cdots), (2.331817\cdots, 3.578910\cdots), \\
&\quad (0.506545\cdots, 3.728333\cdots), (2.897285\cdots, 4.335131\cdots), \\
&\quad (2.989399\cdots, 4.350327\cdots), (3.262858\cdots, 4.519013\cdots), \\
&\quad (5.444113\cdots, 4.999836\cdots), (5.206692\cdots, 5.053923\cdots), \\
&\quad (5.639909\cdots, 5.872529\cdots), (14.51426\cdots, 8.193654\cdots).
\end{align*}
$$

for which

$$
B_2 = B_4 = \cdots = B_{16} = 0, \quad B_{18} = -0.2676264978\cdots \neq 0. \quad (3.21)
$$

Moreover, using the above critical parameter values, we obtain

$$
\det \left[ \frac{\partial(B_2, B_4, B_6, B_8, B_{10}, B_{12}, B_{14}, B_{16})}{\partial(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)} \right] = -490.0663780732\cdots \neq 0. \quad (3.22)
$$

Therefore, proper perturbations on $a_8, a_7, a_4, a_6, a_5, a_3, a_2$ and $a_1$ can be taken to obtain 8 small-amplitude limit cycles around the origin.

Finally, we consider the linear perturbations which yields one more small-amplitude limit cycle. Actually, with the small linear perturbed terms, the origin becomes a focus with eigenvalues $\frac{1}{2} \left[ \delta_2 \pm \sqrt{\delta_2^2 - 4\delta_1(1+\delta_1)} \right]$, showing that the zeroth-order focus value is $v_0 = \frac{1}{2} \delta_2$. At $\delta_2 = 0$, the origin becomes an elementary center with a purely imaginary pair: $\pm i\sqrt{\delta_1(1+\delta_1)}$. Then, by using normal form theory, a simple calculation shows that the first Lyapunov coefficient $v_1$ is given by

$$
v_1 = \frac{1}{8(1+\delta_1)^2} \left[ a_1 + (3a_3 + 2a_4 - a_8a_6 - 2a_6 + a_2) \delta_1 + (3a_3 + a_1 + a_2) \delta_1^2 \right].
$$
Note that $v_1 = \frac{1}{8}a_1$ and $B_2 = a_1$ when $\delta_1 = \delta_2 = 0$, which are in the same order of $a_1$ (just by difference of a positive constant factor). Thus, we can properly perturb $\delta_2$ such that $v_0B_2 < 0$ and $|v_0| \ll |B_2|$ to get the 9th small-amplitude limit cycle around the nilpotent critical point (the origin).

Case (B2) $H_1 = H_2 = 0$. Solving $H_1 = H_2 = 0$ we have $a_7 = \frac{1}{10}(7a_8 - 25)a_8$ and $a_4 = \frac{3}{90}(3a_8 - 10)(7a_8 - 25)$. There are three cases.

(1) $a_8 = \frac{25}{6}$. Then, $a_4 = a_7 = 0$, and $B_8 = \frac{100}{100000}[12a_3 + (2401a_5 - 8)a_6]$. Letting $a_5 = \frac{8}{2401}$ yields $a_3 = 0$ and

\begin{align*}
B_{10} &= -\frac{200a_6}{51883209}(12353145a_6^2 - 10712), \\
B_{12} &= -\frac{100a_6}{9321683217}(1723910797a_6^2 + 596648),
\end{align*}

which clearly indicates that 6 limit cycles can be obtained. If $a_5 \neq \frac{8}{2401}$, then we have $a_3 = -\frac{1}{12}a_6(2401a_5 - 8)$, which is substituted into $B_{10}$ to obtain

\begin{equation}
B_{10} = -\frac{25a_6}{86436}a_5\left[84035(117649a_6^2 - 80)a_5 + 16470860a_6^2 - 20448\right].
\end{equation}

It is easy to verify that $a_5 = 0$ gives a special case of $C_3$. If $a_5 \neq 0$, then $a_6^2 = \frac{80}{117649}$ results in $B_{10} = \frac{57800}{2169}a_6 \neq 0$, yielding 5 limit cycles. If $a_6^2 \neq \frac{80}{117649}$, then we solve $a_5$ from $B_{10} = 0$, which simplifies $B_{12}$ and $B_{14}$ as

\begin{align*}
B_{12} &= -\frac{-32a_6(4117715a_6^2 - 5112)}{293633012355(17649a_6^2 - 80)^3}(1343441218276120425a_6^6 \\
&\quad - 48401735626048910a_6^8 + 31519505743936a_6^2 - 6072768000), \\
B_{14} &= -\frac{16a_6(4117715a_6^2 - 5112)}{2618619284260998(17649a_6^2 - 80)^6}(502164369560774582445a_6^6 \\
&\quad - 14623351347578446270a_6^8 + 12842075888066880a_6^2 \\
&\quad - 3507925999104).
\end{align*}

Again, one can verify that $4117715a_6^2 - 5112 = 0$ gives a special case of $C_3$. Otherwise, solving $B_{12} = 0$ gives one real positive solution $a_6^2$ for which $B_{14} \neq 0$, indicating the existence of 7 limit cycles.

(2) $a_8 = \frac{10}{3}$. Then, we have $a_4 = 0$ and $a_7 = -\frac{8}{9}$, under which $B_8 = \frac{40}{1001}[45a_3 + (567a_5 - 25)a_6]$. If $a_5 = \frac{25}{507}$, then $a_3 = 0$ and

\begin{align*}
B_{10} &= -\frac{200a_6}{964467}(178605a_6^2 - 1774), \\
B_{12} &= -\frac{400a_6}{31827411}(3880737a_6^2 - 21869),
\end{align*}

which clearly shows that there exist solutions for the existence of 6 limit cycles. If $a_5 \neq \frac{25}{507}$, then $a_3 = \frac{a_6(25 - 567a_5)}{45}$, and $B_{10}$ becomes

\begin{align*}
B_{10} &= -\frac{8a_6a_5}{567}\left[(189(3969a_6^2 - 61)a_5 + 26460a_6^2 - 83)\right].
\end{align*}

$a_5 = 0$ again yields a special case of $C_3$. It is easy to see that $3969a_6^2 - 61 = 0$ gives a solution for the existence of 5 limit cycles. If $3969a_6^2 - 61 \neq 0$, then similarly we can prove that there exist 6 solutions for the existence of 7 limit cycles.

(3) $(7a_8 - 25)(3a_8 - 10) \neq 0$. Then $a_4a_7 \neq 0$. Since $a_6 \neq 0$, we solve $B_8 = 0$ to obtain

\begin{equation}
a_5 = \frac{1}{1750a_6}(a_8 - 5)(17a_8^2 - 139a_8 + 280)[15a_3 - a_6(7a_8 - 15)],
\end{equation}
and then
\[ B_{10} = \frac{-a_8(15a_3+15a_6-7a_8a_6)}{6890625a_8} B_{10a}, \quad B_{12} = \frac{-a_8(15a_3+15a_6-7a_8a_6)}{11369531250} B_{12a}, \]

where \( B_{10a} \) and \( B_{12a} \) are polynomials in \( a_3, a_6 \) and \( a_8 \). It is easy to verify that \( 15a_3 + 15a_6 - 7a_8a_6 = 0 \) yields a special case of C3. Following a similar procedure as used above, we can show that there exist 8 solutions for the existence of 7 limit cycles.

**Case (B3) \( H_1 \neq 0, H_2 = 0.** For this case, \( H_2 = 0 \) yields \( a_4 = \frac{1}{5a_8}[(14a_8 - 50)a_8 + (9a_8 - 50)a_7] \), and then \( B_6 = 0 \) requires \( a_3 = 0 \) due to \( H_1 \neq 0 \). Then, we solve \( B_8 = 0 \) to obtain a solution for \( a_5 \), given by
\[
\begin{align*}
a_5 &= -\frac{1}{875a_8^2} \left[ 100a_7^2(21a_8^2 - 290a_8 + 875) - a_7a_8(a_8 - 5)(62a_8^2 - 1335a_8 + 4725) \\
&\quad - a_8^3(62a_8^2 - 1435a_8^2 + 8500a_8 - 14875) \right],
\end{align*}
\]
under which
\[
B_{10} = -\frac{4a_8(a_7+a_8)}{1378125a_8^2} B_{10a}(a_6^2, a_7, a_8), \quad B_{12} = -\frac{2a_8(a_7+a_8)}{227390625a_8^2} B_{12a}(a_6^2, a_7, a_8),
\]
where \( B_{10a} \) and \( B_{12a} \) are polynomials in \( a_6^2, a_7 \) and \( a_8 \), and in particular,
\[
B_{10a} = 4593750a_6^2a_8^2 - 250a_8^2(3969a_8^3 - 80750a_8^2 + 526575a_8 - 1102500) \\
+ 5a_7a_8(74208a_8^4 - 2425625a_8^3 + 55885100a_8^2 - 110354125a_8) \\
+ 162618750) + a_8^3(3704a_8^5 + 598890a_8^4 - 16370125a_8^3) \\
+ 135606125a_8^2 - 455039375a_8 + 541603125).
\]

Note that \( a_7 + a_8 = 0 \) gives a special case of C3. So solving \( B_{10a} \) for \( a_6^2 \) and substitute the solution into \( B_{12} \) and \( B_{14} \) to obtain two polynomial equations in \( a_7 \) and \( a_8 \). Solving these two polynomial equations, we obtain 10 sets of solutions \( (a_7, a_8) \) such that \( a_6^2 > 0 \) and \( B_{16} \neq 0 \). This shows that there exist 20 solutions for the existence of 8 limit cycles.

Summarizing the above results obtained for Cases (A) and (B) shows that the maximal number of small-amplitude limit cycles which can bifurcate from the origin is 9.

The proof of Theorem 3.1 is complete. □

4. 10 limit cycles in a cubic-order system near a nilpotent critical point

In this section, we consider system (3.1) again, and will use the method of double bifurcation of nilpotent focus to show that the system can have 10 small-amplitude limit cycles near the origin. To achieve this, we add different perturbations to system (3.1) to obtain the following perturbed system:
\[
\begin{align*}
\frac{dx}{dt} &= y + (a_8 - 2)xy - (a_4 - a_7)x^2y + a_6y^2 + a_2xy^2 + a_5y^3, \\
\frac{dy}{dt} &= 4\delta x + y^2 + a_4xy^2 + a_3y^3 - (x^2 - \varepsilon^2)(2x - a_1y),
\end{align*}
\] (4.1)

where \( 0 < |\delta| \ll 1, \ 0 < \varepsilon \ll 1 \). Then, for system (4.1) we have the following result.

**Theorem 4.1.** For system (4.1) with a nilpotent critical point at the origin, there exist at least 10 small-amplitude limit cycles near the origin.
Proof. The proof has two steps. In the first step, let $\delta = \varepsilon = 0$. Then, as shown in the previous section, we obtain the critical parameter values given in (3.20) such that the conditions in (3.21) and (3.22) are satisfied, and thus we obtain 8 small-amplitude limit cycles around the origin by perturbing the coefficients $a_1, a_2, \ldots, a_8$.

In the second step, by choosing proper values of $\delta$ and $\varepsilon$, we can use the method of double bifurcation of nilpotent focus [29] to find two more small-amplitude limit cycles near the origin. In fact, for small $\delta$ and $\varepsilon$, the origin of system (4.1) becomes a saddle, having eigenvalues $\varepsilon \left[2\delta \pm \sqrt{\delta^2 + 2}\right]$, and two foci arising from the symmetric singular points at $(\pm \varepsilon, 0)$, with eigenvalues $2\varepsilon \left[\delta \pm \sqrt{1 + (a_8 - 2)\varepsilon + (a_7 - a_4)\varepsilon^2}\right]$, indicating that the zeroth-order focus values associated with the two foci is given by $v_0 = 2\varepsilon \delta$. When $\delta = 0$, the origin is still a saddle (with eigenvalues $\pm \sqrt{2} \varepsilon$), while the two foci become elementary centers and Hopf bifurcations occur at the two singular points, with the critical eigenvalues $\pm i \omega_c$, where $\omega_c = 2\varepsilon^2 \sqrt{1 + (a_8 - 2)\varepsilon + (a_7 - a_4)\varepsilon^2} \approx 2\varepsilon^2$. A direct calculation shows that the first Lyapunov constant, associated with the two Hopf critical points, is given by

$$v_1 = \frac{\varepsilon^3}{2[1+(a_8-2)\varepsilon+(a_7-a_4)\varepsilon^2]} \left\{ 3a_3a_8 - 2a_6(a_8 + a_7) + [6a_3a_7 - a_6a_8(a_4 + a_7)]\varepsilon \right\},$$

where the critical conditions $a_1 = 0$ and $a_2 = -3a_3 + a_6a_8$ (see (3.2) and (3.4)) have been used. With the critical solution (3.20), $v_1 \approx 0.50065566 \varepsilon^3 > 0$. Thus, we can perturb $\delta = 0$ to $\delta < 0$ such that $|\delta| \ll \varepsilon^3$, leading to bifurcations of two small-amplitude limit cycles around the two symmetric singular points $(\pm \varepsilon, 0)$. Then, by proper perturbations on other parameters to get $B_2 < 0$ and $v_1 \ll |B_2|$, and so on higher-order generalized Lyapunov constants. These two additional limit cycles are enclosed by the 8 small-amplitude limit cycles, giving rise to 10 small-amplitude limit cycles with the distribution of $8 \supset (1 \cup 1)$. \[\Box\]

5. Center conditions for the nilpotent critical point

In this section, we will present a set of center conditions for system (3.1) under which the nilpotent critical point – the origin, becomes a center. First of all, it requires $\delta_1 = \delta_2 = 0$. Then, based on the generalized Lyapunov constants, we can find the conditions under which the origin of system (3.1) is a center. As a matter of fact, the critical conditions $C_i, i = 1, 2, 3, 4, 5$ have been shown in the proof of Theorem 3.1 to be the candidates for the center conditions of the origin since they yield all the generalized Lyapunov constants to vanish.

Theorem 5.1. When $\delta_1 = \delta_2 = 0$, the origin of system (3.1) is a center if and only if one of the following conditions is satisfied:

$C_1$: $a_1 = a_2 = a_3 = a_6 = 0$;

$C_2$: $a_1 = a_7 = a_8 = a_2 + 3a_3 = 0$;

$C_3$: $a_1 = a_5 = a_2 - a_6(a_8 - 2 - \frac{2a_2}{a_8}) = 3a_3 - 2a_6(1 + \frac{a_7}{a_8}) = a_4 + \frac{a_2(a_7^2 - 4a_7 - 4a_8)}{a_8} = 0$ ($a_8 \neq 0$);

$C_4$: $a_1 = a_5 = a_8 - 5 = a_2 - 2a_6 = a_3 - a_6 = a_4 - a_7 + 2 = 0$.

Proof. The necessity of the conditions $C_i, i = 1, 2, 3, 4$ has been proved in Theorem 3.1 since all these conditions and $C_5$ yield the generalized Lyapunov constants $B_{2k}, k = 1, 2, \ldots, 10$ to vanish. No other possible center conditions have been found from the proof of Theorem 3.1. So we only need to prove the sufficiency of these conditions.
First, consider the condition $C_1$. Under $C_1$ system (3.1) becomes

$$
\frac{dx}{dt} = y + (a_8 - 2)xy - (a_4 - a_7)x^2y + a_5y^3, \\
\frac{dy}{dt} = -2x^3 + y^2 + a_4xy^2.
$$

(5.1)

It is easy to see that system (5.1) is a reversible system since the system is invariant under the transformation: $(y, t) \rightarrow (-y, -t)$. Hence, the origin of system (5.1) is a center.

For condition $C_2$, system (3.1) becomes

$$
\frac{dx}{dt} = y - 2xy - a_4x^2y + a_6y^2 - 3a_3xy^2 + a_5y^3, \\
\frac{dy}{dt} = -2x^3 + y^2 + a_4xy^2 + a_3y^3,
$$

(5.2)

which is a Hamiltonian system with the Hamiltonian function,

$$
H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^4 - xy^2 - \frac{a_4}{2}x^2y^2 + \frac{a_6}{3}y^3 - a_3xy^2 + \frac{a_5}{4}y^4.
$$

Next, consider the condition $C_3$ under which system (3.1) can be written as

$$
\frac{dx}{dt} = y + (a_8 - 2)xy + \frac{2a_7(a_8^2 - 2a_8 - 2a_7)}{a_8^2}x^2y + a_6y^2 + a_6(a_8 - 2 - \frac{2a_7}{a_8})xy^2, \\
\frac{dy}{dt} = -2x^3 + y^2 - \frac{a_7(a_8^2 - 4a_8 - 4a_7)}{a_8}xy^2 + \frac{2}{3}a_6(1 + \frac{a_7}{a_8})y^3.
$$

(5.3)

It can be shown that there exist integrating factors under different conditions, given by

$$
I_1 = \left[ (a_8^2 - 2a_8 - 2a_7)x + a_8 \right] \frac{-x^2}{a_8^2 - 2a_8 - 2a_7}, \\
\text{if } (a_8^2 - 2a_8 - 2a_7)(3a_8^2 - 8a_8 - 8a_7) \\
\times (a_8^2 - 3a_8 - 3a_7)(a_8^2 - 4a_8 - 4a_7)(a_8 + a_7) \neq 0,
$$

$$
I_2 = \frac{6a_8^4}{a_8^2y^2(2a_8y + 1 + (a_8 - 2)x)x^2(12[6 + 6a_8x + 3a_8^2x^2 + a_8^4x^3])}, \\
\text{if } a_8^2 - 2a_8 - 2a_7 = 0,
$$

(5.4)

$$
I_3 = \frac{1}{(a_8x + 4)^4}, \quad \text{if } 3a_8^2 - 8a_8 - 8a_7 = 0,
$$

$$
I_4 = \frac{1}{(a_8x + 3)^3}, \quad \text{if } a_8^2 - 3a_8 - 3a_7 = 0,
$$

$$
I_5 = \frac{1}{(a_8x + 2)^2}, \quad \text{if } a_8^2 - 4a_8 - 4a_7 = 0,
$$

$$
I_6 = \frac{1}{(a_8x + 1)}, \quad \text{if } a_8 + a_7 = 0,
$$

such that system (5.3) has the following corresponding first integrals:

\[ F_1(x, y) = I_1 \left\{ \frac{1}{2} y^2 + \frac{a_6}{2} x^2 y + \frac{a_8}{3} y^3 + \frac{a_6 (a_8^2 - 2a_8 - 2a_7)}{a_8^3} x^2 y^2 + \frac{a_6}{3} (a_8 - 2 - \frac{2a_7}{a_8}) x y^3 \\
+ \frac{(a_8^2 - 2a_8 - 2a_7) x + a_8}{(3a_8^2 - 8a_8 - 2a_7)(a_8^2 - 3a_8 - 3a_7)(a_8^2 - 4a_8 - 4a_7)(a_8 + a_7)} \right\} \times \left[ 2(a_8^2 - 3a_8 - 3a_7)(a_8^2 - 4a_8 - 4a_7)(a_8 + a_7)x^3 \right.
- 3a_8(a_8 + a_7)(a_8^2 - 4a_8 - 4a_7)x^2 + 6a_8^2(a_8 + a_7)x + 3a_8^3 \right), \]
\[ F_2(x, y) = 2 \ln a_8^2 - \ln I_2 - a_8 x, \quad (5.5) \]
\[ F_3(x, y) = \frac{y^2 [8a_8 y + (3a_6 - 8x + 4)]}{96(a_8 x + 4)^3} + \frac{8[9a_6^2 x^2 + 54a_8 x + 88]}{3a_8^3 (a_8 x + 4)^3} + \frac{1}{a_8^3} \ln(a_8 x + 4)^2, \]
\[ F_4(x, y) = \frac{y^2 [2a_6 y + (2a_8 - 3x + 3)]}{18(a_8 x + 3)^3} + \frac{2a_6^2 x^2 + 12a_6^2 x^2 - 36a_8 x - 135}{a_8^3 (a_8 x + 3)^3} - \frac{9}{a_8^3} \ln(a_8 x + 3)^2, \]
\[ F_5(x, y) = \frac{y^2 [4a_6 y + 3(a_8 - 4) + 6]}{24(a_8 x + 2)^3} + \frac{a_8^3 x^3 - 6a_8^2 x^2 - 16a_8 x + 16}{a_8^3 (a_8 x + 2)^3} + \frac{12}{a_8^3} \ln(a_8 x + 2)^2, \]
\[ F_6(x, y) = \frac{y^2 (2a_6 y - 6x + 3) + (2a_6^2 x^2 - 3a_8 x + 6)}{3a_8^3} - \frac{1}{a_8^3} \ln(a_8 x + 1)^2. \]

Now, for the condition C4, system (3.1) becomes
\[
\begin{align*}
\frac{dx}{dt} &= y + 3xy + 2x^2y + a_6 y^2 + 2a_6 xy^2, \\
\frac{dy}{dt} &= -2x^3 + y^2 + (a_7 - 2)xy^2 + a_6 y^3.
\end{align*}
\]  
(5.6)

We apply the following transformation and time rescaling
\[
\begin{align*}
u &= \frac{x}{1 + x}, \\
v &= \frac{y}{1 + x}, \\
t &= (1 - u)^2 \tau \\
\implies x &= \frac{u}{1 - u}, \\
y &= \frac{v}{1 - u},
\end{align*}
\]  
(5.7)

to system (5.6) to obtain
\[
\begin{align*}
\frac{du}{d\tau} &= v[1 - u^2 + a_6 v(1 - u^2)], \\
\frac{dv}{d\tau} &= u[-2u^2 + (a_7 - 4)u^2 - a_6 v^3].
\end{align*}
\]  
(5.8)

This is a reversible system since it is invariant under the transformation \((u, \tau) \to (-u, -\tau)\). Hence, the origin of system (5.8) is a center, implying that the origin of the original system (3.1) is a center since the origin is invariant under the transformation (5.7).

Finally, consider the condition C5 (see Eqn. (3.19)). This condition is necessary for the origin of system (3.1) being a center has been proved in Theorem 3.1. For sufficiency of this condition, we have the following conjecture.

**Conjecture 5.1.** *The condition C5 is also sufficient for the origin of system (3.1) being a center.*

6. Conclusion

In this paper, we have shown that planar cubic polynomial vector fields with an isolated nilpotent critical point can have at least 9 small-amplitude limit cycles around the critical point and at least 10 small-amplitude limit cycles near the critical point with the distribution of 8 \(\circ (1 \cup 1)\). Normal form theory has been applied to compute the generalized Lyapunov constants, and then to determine the number of
bifurcating limit cycles near the critical point. Moreover, a set of center conditions for the nilpotent point has been obtained for such cubic polynomial systems. It has demonstrated the general applicability of our method and program to solve different types of polynomial systems with a nilpotent singular point.

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