SUPPLEMENTARY MATERIALS: Extinctions Caused by Host-Range Expansion*

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SM1. Two-dimensional host-parasite model. The host-range expansion models in the main text are based on a well-known host-parasite model. In this model, the density of the host population (N) has intrinsic growth rate r per unit time, and grows logistically with carrying capacity K. The parasite population, with density P, infects the host with mass-action kinetics at rate α, and with a conversion factor β between hosts and parasites. For example, if the parasite is a lytic virus infecting a bacterium, β represents the burst size. In the absence of the host, the parasite population decays at per capita rate δ. These assumptions yield:

$$\text{(SM1)} \qquad \begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= rN\Big(1 - \frac{N}{K}\Big) - \alpha NP, \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= \beta\alpha NP - \delta P. \end{aligned}$$

This model is equivalent to well-studied predator-prey models with a logistic growth term for the prey species [SM1]. Equilibria and stability results for (SM1) are rederived here for clarity, providing consistency with the models in the main paper.

SM1.1. Equilibria and stability. System (SM1) admits three equilibrium solutions:

$$\text{(SM2)} \qquad \begin{array}{lll} \text{Trivial Equilibrium} & \mathrm{E}_0: & (N,P) = (0,0) \\ \text{Bounded Equilibrium} & \mathrm{E}_1: & (N,P) = (K,0) \\ \text{Positive Equilibrium} & \mathrm{E}_2: & (N,P) = (N_2,P_2), \end{array}$$

where

$$\text{(SM3)} \qquad N_2 = \frac{\delta}{\alpha\beta}, \quad P_2 = \frac{r}{\alpha}\Big(1 - \frac{\delta}{\alpha\beta K}\Big) = \frac{r}{\alpha}\Big(1 - \frac{N_2}{K}\Big).$$

For the existence and stability of the equilibrium, we have the following theorem.

Theorem SM1.1. *The equilibria* E_0 *and* E_1 *exist for positive parameter values, while the equilibrium* E_2 *exists for* $\delta < \alpha\beta K$. E_0 *is always unstable;* E_1 *is GAS for* $\delta > \alpha\beta K$, *and*

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unstable for $\delta < \alpha\beta K$*;* E_2 *exists and is GAS for* $\delta > \alpha\beta K$*. A transcritical bifurcation occurs between the equilibria* E_1 *and* E_2*, at the critical point* $\delta = \alpha\beta K$*.*

Proof. The stability of the equilibria is determined by the Jacobian matrix,

$$J(N,P) = \begin{bmatrix} r\Big(1-\dfrac{2N}{K}\Big) - \alpha P & -\alpha N \\ \beta\alpha P & \beta\alpha N - \delta \end{bmatrix}.$$

Then,

$$J(\mathrm{E}_0) = \begin{bmatrix} r & 0 \\ 0 & -\delta \end{bmatrix}$$

shows that E_0 is always unstable (a saddle).

$$J(\mathrm{E}_1) = \begin{bmatrix} -r & -\dfrac{\alpha}{K} \\ 0 & \alpha\beta K - \delta \end{bmatrix}$$

indicates that E_1 is LAS for $\delta > \alpha\beta K$.

$$J(\mathrm{E}_2) = \begin{bmatrix} -\dfrac{r\delta}{\alpha\beta K} & -\dfrac{\delta}{\beta} \\ r\beta\Big(1-\dfrac{\delta}{\alpha\beta K}\Big) & 0 \end{bmatrix}$$

gives

$$\mathrm{Tr}(J(\mathrm{E}_2)) = -\frac{r\delta}{\alpha\beta K} < 0, \quad \det(J(\mathrm{E}_2)) = r\delta\Big(1-\frac{\delta}{\alpha\beta K}\Big).$$

Thus, E_2 is LAS for $\det(J(\mathrm{E}_2)) > 0$, i.e., $\delta < \alpha\beta K$. This implies that a transcritical bifurcation occurs between E_1 and E_2 at the critical point $\delta = \alpha\beta K$.

Next, we consider the global asymptotic stability (GAS) for E_1 and E_2. First, consider E_1. To achieve this, we construct the Lyapunov function,

$$V_1 = N - K - K\,\ln\Big(\frac{N}{K}\Big) + \frac{1}{\beta}P.$$

Then, differentiating V_1 with respect to time t and evaluating it along the trajectory of system (SM1), we obtain

$$\begin{aligned} \frac{\mathrm{d}V_1}{\mathrm{d}t}\bigg|_{\text{(SM1)}} &= \Big(1-\frac{K}{N}\Big)\frac{\mathrm{d}N}{\mathrm{d}t} + \frac{1}{\beta}\frac{\mathrm{d}P}{\mathrm{d}t} \\ &= (N-K)\Big[r\Big(1-\frac{N}{K}\Big) - \alpha P\Big] + \alpha NP - \frac{\delta}{\beta}P \\ &= -\frac{r}{K}(N-K)^2 - \frac{1}{\beta}(\delta - \alpha\beta K)P. \end{aligned} \tag{SM4}$$

Hence, when $\delta > \alpha\beta K$, $\frac{\mathrm{d}V_1}{\mathrm{d}t}\Big|_{(\mathrm{SM1})} < 0$ as long as $(N,P) \neq (K,0)$, and $\frac{\mathrm{d}V_1}{\mathrm{d}t}\Big|_{(\mathrm{SM1})} = 0$ only if $(N,P) = (K,0)$. This indicates that E_1 is attractive, and thus, together with its LAS, E_1 is GAS for $\delta > \alpha\beta K$.

Similarly, we can show that E_2 is GAS for $\delta < \alpha\beta K$. Let

$$V_2 = N - N_2 - N_2 \ln\left(\frac{N}{N_2}\right) + \frac{1}{\beta}\left(P - P_2 - P_2 \ln\left(\frac{P}{P_2}\right)\right).$$

Then,

$$\begin{aligned} \frac{\mathrm{d}V_2}{\mathrm{d}t}\Big|_{(\mathrm{SM1})} &= \left(1 - \frac{N_2}{N}\right)\frac{\mathrm{d}N}{\mathrm{d}t} + \frac{1}{\beta}\left(1 - \frac{P_2}{P}\right)\frac{\mathrm{d}P}{\mathrm{d}t} \\ &= (N - N_2)\left[r\left(1 - \frac{N}{K}\right) - \alpha P\right] + (P - P_2)\left(\alpha N - \frac{\delta}{\beta}\right) \\ &= -\frac{r}{K}(N - N_2)^2, \end{aligned} \tag{SM5}$$

which implies that $\frac{\mathrm{d}V_2}{\mathrm{d}t}\Big|_{(\mathrm{SM1})} < 0$ as long as $N \neq N_2$, and $\frac{\mathrm{d}V_2}{\mathrm{d}t}\Big|_{(\mathrm{SM1})} = 0$ when $N = N_2$. If $N = N_2$, we have

$$\begin{aligned} 0 &= rN_2\left(1 - \frac{N_2}{K}\right) - \alpha N_2 P, \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= P(\beta\alpha N_2 - \delta) = 0, \end{aligned}$$

which shows that P is a constant and equals $P = \frac{r}{\alpha}\left(1 - \frac{N_2}{K}\right)$, leading to the equilibrium E_2 for $N_2 < K$. Therefore, by the LaSalle's Invariance Principle, we know that E_2 is GAS for $\delta < \alpha\beta K$, i.e., $N_2 < K$. ∎

SM1.2. Boundedness of solutions.

To study the boundedness of solutions to the 2-d model, system (SM1), we construct the Lyapunov function:

$$V_{2\mathrm{d}} = \beta\, N + P. \tag{SM6}$$

Differentiating $V_{2\mathrm{d}}$ with respect to time t and computing it along the trajectory of the 2-d model (SM1) we obtain

$$\frac{\mathrm{d}V_{2\mathrm{d}}}{\mathrm{d}t}\Big|_{(\mathrm{SM1})} = -\frac{\beta r}{K}\, N\,(N - K) - \delta\, P < 0, \quad \text{for} \quad N > K.$$

Thus, we can construct the trapping region in the N-P plane, using the condition $\frac{\mathrm{d}V_{2\mathrm{d}}}{\mathrm{d}t}\Big|_{(\mathrm{SM1})} = 0$, bounded by the N-axis and the parabola:

$$P = -\frac{\beta r}{K\delta}\, N(N - K) = \frac{\beta r}{4\delta} - \frac{\beta r}{K\delta}\left(N - \frac{K}{2}\right)^2.$$

Define the trapping region for the 2-d model as

$$\Omega_{2\mathrm{d}} = \left\{ (N,P) \,\middle|\, 0 \le P \le \frac{\beta r}{4\delta} - \frac{\beta r}{K\delta}\left(N - \frac{K}{2}\right)^2 \right\}. \tag{SM7}$$

Hence, for any positive parameter values, the solutions of the 2-d model are attracted to $\Omega_{2\mathrm{d}}$. Note that the three equilibria E_0, E_1 and E_2 are located on $\Omega_{2\mathrm{d}}$.

SM2. 3-d system.

SM2.1. Nondimensionalisation. We nondimensionalise the system in the following way:

$$t = \bar{t}\bar{\delta}, \quad r_1 = \frac{\bar{r}_1}{\bar{\delta}}, \quad r_2 = \frac{\bar{r}_2}{\bar{\delta}}, \quad K_1 = \frac{\bar{\gamma}_{21}}{\bar{\delta}}\bar{K}_1, \quad K_2 = \frac{\bar{\gamma}_{12}}{\bar{\delta}}\bar{K}_2,$$
$$N_1 = \frac{\bar{\gamma}_{21}}{\bar{\delta}}\bar{N}_1, \quad N_2 = \frac{\bar{\gamma}_{12}}{\bar{\delta}}\bar{N}_2, \quad P = \frac{\bar{\alpha}_1}{\bar{\delta}}\bar{P}, \quad B = \frac{\bar{\beta}_1\bar{\alpha}_1}{\bar{\gamma}_{21}}.$$

The bars signify the parameters and variables that appear in the original system. The nondimensionalised system is:

$$\frac{dN_1}{dt} = r_1 N_1\left(1 - \frac{N_1}{K_1}\right) - N_1N_2 - N_1P,$$
$$\frac{dN_2}{dt} = r_2 N_2\left(1 - \frac{N_2}{K_2}\right) - N_1N_2,$$
$$\frac{dP}{dt} = BN_1P - P.$$

SM2.2. Local stability of 3-d system. Here, we provide the proof of Theorem 2.1 concerning the existence and LAS conditions of equilibria E_0 to E_4.

Proof. The existence condition for E_3 can be easily deduced from $N_{13} > 0$ and $N_{23} > 0$ as $g(K_2 - r_1) > 0$ and $g(K_1 - r_2) > 0$, leading to $(K_2 - r_1)(K_1 - r_2) > 0$. The existence condition for E_4 requires $1 - \frac{1}{BK_1} > 0$, i.e., $K_1 > \frac{1}{B}$.

The stability conditions of the equilibria can be derived from the Jacobian matrix:

$$J(N_1, N_2, P) = \begin{bmatrix} r_1\left(1 - \frac{2N_1}{K_1}\right) - N_2 - P & -N_1 & -N_1 \\ -N_2 & r_2\left(1 - \frac{2N_2}{K_2}\right) - N_1 & 0 \\ BP & 0 & BN_1 - 1 \end{bmatrix}. \tag{SM8}$$

Thus, it is straightforward to obtain the stability conditions for E_0, E_1 and E_2 by evaluating the Jacobian (SM8) at these equilibria as follows: E_0 is always a saddle; E_1 is LAS for $r_2 < K_1 < \frac{1}{B}$; and E_2 is LAS for $r_1 < K_2$. Next, evaluating the Jacobian matrix at E_3 yields $J(E_3)$ whose characteristic polynomial can be written as

$$P(J(E_3)) = \left[\lambda - \frac{1}{g}BK_1r_2(K_2 - r_1) + 1\right](\lambda^2 - \mathrm{Tr}_3\,\lambda + \mathrm{Det}_3),$$

where

$$\mathrm{Tr}_3 = -\frac{1}{g}r_1r_2(K_1 - r_2 + K_2 - r_1), \quad \mathrm{Det}_3 = -\frac{1}{g}r_1r_2(K_1 - r_2)(K_2 - r_1).$$

Noticing that $g = K_1K_2 - r_1r_2$ and that the condition $(K_1 - r_2)(K_2 - r_1) > 0$ must hold for the existence of E_3, we find that $\mathrm{Tr}_3 < 0$. Thus, we only need to consider Det_3 and the linear factor in $P(J(E_3))$. The equilibrium E_3 is LAS if

$$\mathrm{Det}_3 > 0 \quad \text{and} \quad \frac{1}{g}BK_1r_2(K_2 - r_1) - 1 < 0.$$

Clearly, $\mathrm{Det}_3 > 0$ leads to $r_1 > K_2$ and $r_2 > K_1$, which is then combined into the above second condition to yield

$$K_1 < \frac{1}{B} < r_2,\ K_2 < r_1, \quad \text{or} \quad \frac{1}{B} < K_1 < r_2, \quad K_2 < r_1 < r_1^*,$$

where r_1^* is given in (5).

Evaluating the Jacobian matrix at E_4 yields the characteristic polynomial

$$\mathrm{P}(J(\mathrm{E}_4)) = \big(\lambda + \tfrac{1}{B} - r_2\big)\big[\lambda^2 + \tfrac{r_1}{BK_1}\,\lambda + r_1\big(1 - \tfrac{1}{BK_1}\big)\big].$$

Since the existence condition for E_4 requires $K_1 > \frac{1}{B}$, we can conclude that E_4 is LAS if $r_2 < \frac{1}{B}$ $(< K_1)$.

It is easy to see that E_1 and E_3 exchange their stability at $r_2 = K_1$; E_2 and E_3 exchange their stability at $r_1 = K_2$; and E_1 and E_4 exchange their stability at $K_1 = \frac{1}{B}$. Therefore, transcritical bifurcations occur between E_1 and E_3, between E_2 and E_3, as well as between E_1 and E_4. No Hopf bifurcation can occur from these 5 equilibria since none of the 5 corresponding characteristic polynomials can have a pair of purely imaginary eigenvalues; Nor can B-T bifurcation happen since B-T bifurcation appears at the coexistence of Hopf and saddle-node bifurcations. ∎

We next provide the proof of the existence and LAS conditions for E_5.

Proof. The existence conditions for E_5 require $N_{25} > 0$ and $P_5 > 0$. $N_{25} > 0$ gives $r_2 > \frac{1}{B}$, and $P_5 > 0$ yields

$$K_1 > \tfrac{1}{B} \quad \text{and} \quad r_1 - K_2 + \tfrac{g}{K_1 r_2 B} > 0 \quad \Longrightarrow \quad r_1 > r_1^*.$$

Hence, E_5 exists under the conditions: $r_2 > \frac{1}{B}$, $K_1 > \frac{1}{B}$ and $r_1 > r_1^*$.

To find the stability of E_5, we evaluate the Jacobian matrix (SM8) at E_5 to obtain the characteristic polynomial:

$$P_5(\mathrm{E}_5) = \lambda^3 + a_1\,\lambda^2 + a_2\,\lambda + a_3, \tag{SM9}$$

where

$$\begin{aligned} a_1 &= \tfrac{1}{B}\big(Br_2 - 1 + \tfrac{r_1}{K_1}\big),\\ a_2 &= \tfrac{B(BK_1-1)+Br_2-1}{B^2K_1}(r_1 - r_1^{**}), \quad \text{in which} \quad r_1^{**} = \frac{K_1K_2(B+1)(Br_2-1)}{r_2\big[B(BK_1-1)+Br_2-1\big]},\\ a_3 &= \tfrac{(Br_2-1)(BK_1-1)}{B^2K_1}(r_1 - r_1^*). \end{aligned} \tag{SM10}$$

E_5 is LAS if

$$a_k > 0,\ k = 1, 2, 3 \quad \text{and} \quad \Delta_2 = a_1a_2 - a_3 > 0. \tag{SM11}$$

With the existence conditions for E_5, it is clear that $a_1 > 0$ and $a_3 > 0$. $a_2 > 0$ requires $r_1 > r_1^{**}$. $a_3 = 0$ gives two solutions $r_2 = \frac{1}{B}$ and $r_1 = r_1^*$. Therefore, a transcritical bifurcation occurs between E_4 and E_5 at $r_2 = \frac{1}{B}$. Another transcritical bifurcation happens between E_3 and E_5 at $r_1 = r_1^*$.

Now under the conditions: $r_2 > \frac{1}{B}$, $K_1 > \frac{1}{B}$ and $r_1 > \max\{r_1*, r_1^{**}\}$, we consider the stability condition Δ_2, which is obtained as $\Delta_2(r_1) = \frac{1}{B^3 K_1^2 r_2}\,\Delta_{2a}(r_1)$, where

$$\begin{aligned}\Delta_{2a}(r_1) = {} & r_2\big[B(BK_1-1)+Br_2-1\big]r_1^2 \\ & - K_1(Br_2-1)\big[(B+1)K_2 - r_2(Br_2-1)\big]r_1 - K_1^2K_2(Br_2-1)^2.\end{aligned}$$

Since Δ_2 and Δ_{2a} have the same sign, in the following we will consider Δ_{2a}. A simple calculation shows that

$$\begin{aligned}\Delta_{2a}(r_1^*) &= \tfrac{K_1^2K_2(Br_2-1)^2B[(BK_1-1)r_2+K_2]}{r_2(BK_1-1)^2}\,(r_2-K_1),\\ \Delta_{2a}(r_1^{**}) &= \tfrac{K_1^2K_2(Br_2-1)^2B^2}{B(BK_1-1)+Br_2-1}\,(r_2-K_1),\\ r_1^*-r_1^{**} &= \tfrac{K_1K_2(Br_2-1)B}{r_2[B(BK_1-1)+Br_2-1](BK_1-1)}\,(r_2-K_1),\end{aligned}$$

which indicate that

$$\begin{aligned}&\Delta_{2a}(r_1^*) \geq 0, \quad \Delta_{2a}(r_1^{**}) \geq 0, \quad r_1^* \geq r_1^{**} \quad \text{if} \quad r_2 \geq K_1,\\ &\Delta_{2a}(r_1^*) < 0, \quad \Delta_{2a}(r_1^{**}) < 0, \quad r_1^* < r_1^{**} \quad \text{if} \quad r_2 < K_1.\end{aligned}$$

Since $\Delta_{2a}(r_1)$ is a quadratic polynomial in r_1, its graph in the r_1-Δ_{2a} plane is open upwards. Also, note that $\Delta_{2a}(r_1)$ always has a unique positive root, denoted as $r_{1\mathrm{H}}$. Therefore, when $\frac{1}{B} < K_1 \leq r_2$ and $r_1 > r_1^*$, $\Delta_{2a}(r_1) > \Delta_{2a}(r_1^*) \geq 0$ (i.e., $\Delta_2 > 0$), implying that E_5 is LAS. Note that $r_1^* \geq r_1^{**} > r_{1\mathrm{H}}$ when $r_2 \geq K_1$. ■

SM2.3. Example: Hopf bifurcation in the 3-d system. We give an example of Hopf bifurcation from E_5 yielding a stable limit cycle in the 3-d model as follows. Let

$$K_1 = 2, \quad K_2 = 1, \quad r_2 = \frac{3}{2}, \quad B = 1.$$

Then, we have

$$r_1^* = \frac{2}{3}, \quad r_1^{**} = \frac{8}{9}, \quad r_{1\mathrm{H}} = 1,$$

which gives the frequency at the Hopf critical point $r_1 = r_{1\mathrm{H}} = 1$, as $\omega_c = \frac{\sqrt{3}}{6}$. Using the method of normal forms (e.g. [SM2]), we obtain the first focus value, $v_1 = -\frac{45}{208} < 0$, implying that the Hopf bifurcation is supercritical, and the bifurcating limit cycle is stable. The simulation for this example with $r_1 = 0.95$, yielding perturbation $\mu = r_1 - r_{1\mathrm{H}} = -0.05$, is shown in Figure SM1.

SM3. Example: LAS does not necessarily imply GAS. Here we demonstrate that LAS of an equilibrium does not necessarily imply GAS. As described in the main text, in the 3-d model (2) the equilibrium $E_2 = (0, K_2, 0)$ exists for any positive parameter values, and is LAS for $r_2 < K_1$; while the equilibrium $E_4 = \big(\frac{1}{B}, 0, r_2\big(1 - \frac{1}{BK_1}\big)\big)$ exists for $K_1 > \frac{1}{B}$, and is LAS

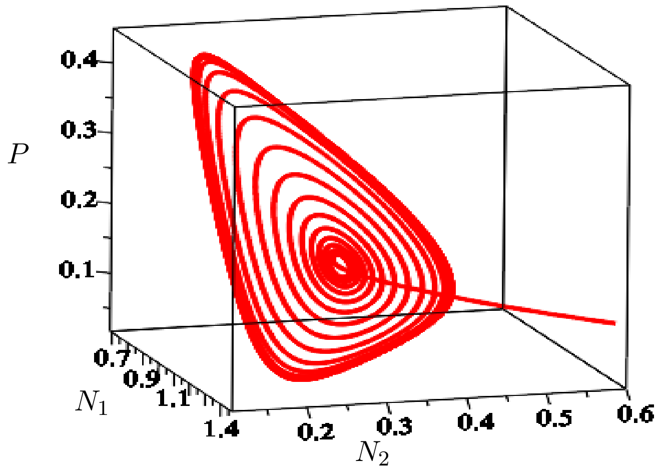


Figure SM1. *Simulated stable limit cycle from the Hopf bifurcation of the 3-d model (10) around* E_5*, with* $B = 1$*,* $K_1 = 2$*,* $K_2 = 1$*,* $r_1 = \frac{19}{20}$ *and* $r_2 = \frac{3}{2}$*, and the initial condition* $(N_1, N_2, P) = (1.4, 0.6, 0.1)$.

for $r_2 < \frac{1}{B}$. Thus, these two equilibria can co-exist for certain parameter values. For example, taking

$$B = \frac{1}{2}, \quad r_1 = 1, \quad K_2 = 2, \quad r_2 = \frac{3}{2}, \quad K_1 = 3,$$

we have bistable E_2 and E_4. The simulation, as shown in Figure SM2(a), indicates the existence of a separatrix between the two tracking areas for E_2 and E_4.

Now, it is easy to show that the GAS of these two equilibria needs the boundedness condition $\mathrm{C}_{\mathrm{bd}}^{\mathrm{3d}}$ (see (14)). For E_4 (a similar proof for E_2), we use the Laypunov function

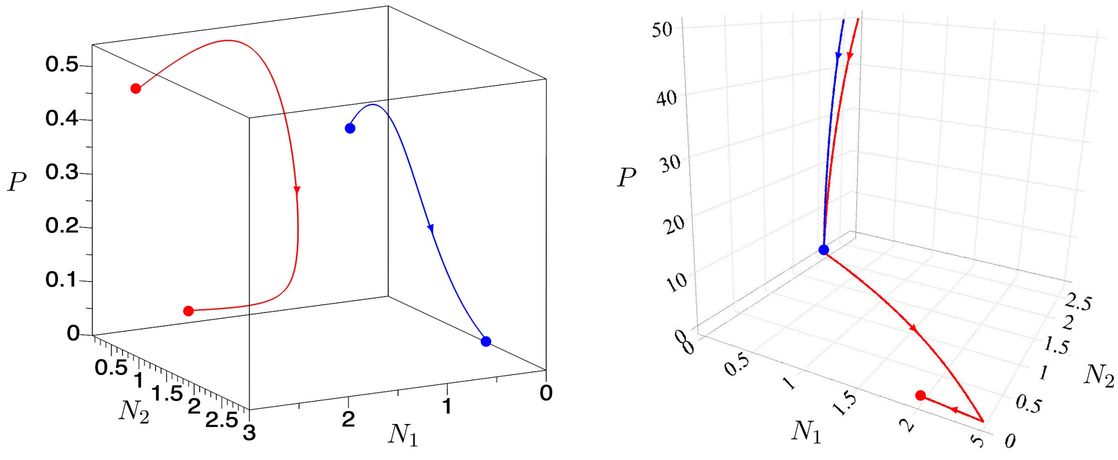
$$V_{\mathrm{3d}}^{\mathrm{E}_4} = N_1 - \frac{1}{B} - \frac{1}{B} \ln\left(\frac{1}{BK_1}\right) + N_2 + \frac{1}{B}\left(P - P_3 - P_3 \ln\left(\frac{P}{P_3}\right)\right),$$

and then use the formulas in (29) and (30) to obtain

$$\left.\frac{\mathrm{d}V_{\mathrm{3d}}^{\mathrm{E}_4}}{\mathrm{d}t}\right|_{(2)} = -\frac{r_1}{K_1}\left[N_1 - \frac{1}{B} + \frac{K_1}{r_1}N_2\right]^2 - \frac{1}{K_2 r_1}\mathrm{C}_{\mathrm{bd}}^{\mathrm{3d}} N_2^2 - \left(\frac{1}{B} - r_2\right)N_2,$$

which clearly shows why the LAS condition is not enough for E_4 to be GAS, and the boundedness condition $\mathrm{C}_{\mathrm{bd}}^{\mathrm{3d}} > 0$ is needed. By adding this condition, it can be seen that there exists a wide range of parameter values such that E_2 or E_4 is GAS, but not both simultaneously (clearly, since a system cannot have more than one GAS equilibrium). We choose the following two parameter sets (satisfying $r_1 r_2 > K_1 K_2$) for simulation to demonstrate that either one of them may be GAS:

$$\text{for } \mathrm{E}_2: \quad B = \frac{1}{2}, \quad r_1 = 1, \quad K_2 = 2, \quad r_2 = 2, \quad K_1 = \frac{3}{2},$$
$$\text{for } \mathrm{E}_4: \quad B = \frac{1}{2}, \quad r_1 = 4, \quad K_2 = 2, \quad r_2 = \frac{3}{2}, \quad K_1 = \frac{5}{2}.$$



(a) (b)

Figure SM2. *Simulated trajectories of the 3-d model* (2) *converging to* E_2 *or* E_4*:* (a) *LAS for* $B = \frac{1}{2}$*,* $r_1 = 1$*,* $K_2 = 2$*,* $r_2 = \frac{3}{2}$*,* $K_1 = 3$*, from the initial point* $(2, 3, 0.5)$ *to* $\mathrm{E}_2 = (0, 2, 0)$ *(the blue curve), and from the initial point* $(3, 1, 0.5)$ *to* $\mathrm{E}_4 = (2, 0, \frac{1}{3})$ *(the red curve); and* (b) *GAS for* $B = \frac{1}{2}$*,* $r_1 = 1$*,* $K_2 = 2$*,* $r_2 = 2$*,* $K_1 = \frac{3}{2}$*, from* $(100, 100, 100)$ *to* $\mathrm{E}_2 = (0, 2, 0)$ *(the blue curve), and for* $B = \frac{1}{2}$*,* $r_1 = 4$*,* $K_2 = 2$*,* $r_2 = \frac{3}{2}$*,* $K_1 = \frac{5}{2}$*, from* $(100, 100, 100)$ *to* $\mathrm{E}_4 = (2, 0, \frac{4}{5})$ *(the red curve).*

The simulation of the 3-d model for the above parameter values is shown in Figure SM2(b), with the initial point $(N_1, N_2, P) = (100, 100, 100)$. It is shown that the blue curve, corresponding to the first set of parameter values, converges to $\mathrm{E}_2 = (0, 2, 0)$, while the red curve converges to E_4.

SM4. 4-d System.

SM4.1. Nondimensionalisation.

We nondimensionalise the system in the following way:

$$t = \bar{t}\bar{\delta}, \quad r_1 = \frac{\bar{r}_1}{\bar{\delta}}, \quad r_2 = \frac{\bar{r}_2}{\bar{\delta}}, \quad K_1 = \frac{\bar{\gamma}_{21}}{\bar{\delta}}\bar{K}_1, \quad K_2 = \frac{\bar{\gamma}_{12}}{\bar{\delta}}\bar{K}_2,$$
$$N_1 = \frac{\bar{\gamma}_{21}}{\bar{\delta}}\bar{N}_1, \quad N_2 = \frac{\bar{\gamma}_{12}}{\bar{\delta}}\bar{N}_2, \quad P = \frac{\bar{\alpha}_1}{\bar{\delta}}\bar{P}, \quad Q = \frac{\bar{\alpha}_1}{\bar{\delta}}\bar{Q},$$
$$A = \frac{\bar{\alpha}_2}{\bar{\alpha}_1}, \quad B = \frac{\bar{\beta}_1\bar{\alpha}_1}{\bar{\gamma}_{21}}, \quad D = \frac{\bar{\beta}_{21}\bar{\alpha}_1}{\bar{\gamma}_{21}}, \quad E = \frac{\bar{\beta}_{22}\bar{\alpha}_2}{\bar{\gamma}_{12}}.$$

The bars signify parameters and variables that appear in the original equations. The nondimensional system is:

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - N_1 P - (1 - c) N_1 Q - N_1 N_2,$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - A N_2 Q - N_1 N_2,$$

$$\frac{dP}{dt} = BN_1P - P$$
$$\frac{dQ}{dt} = D(1-c)N_1Q + EN_2Q - Q.$$

Note, parameter c has not been replaced from the nondimensionalised system. For simplicity of notation in the following sections, we introduce $\tilde{D} = D(1-c)$ and $\tilde{c} = 1 - c$.

SM4.2. Local stability in 4-d system. The stability conditions for the equilibrium solutions of system (10) are obtained from the Jacobian matrix of the 4-d model, given by

(SM12)

$$J(N_1, N_2, P, Q) = \begin{bmatrix} r_1\left(1-\frac{2N_1}{K_1}\right)-N_2-P-\tilde{c}\,Q & -N_1 & -N_1 & -\tilde{c}N_1 \\ -N_2 & r_2\left(1-\frac{2N_2}{K_2}\right)-N_1-AQ & 0 & -AN_2 \\ B\,P & 0 & BN_1-1 & 0 \\ \tilde{D}Q & EQ & 0 & \tilde{D}N_1+EN_2-1 \end{bmatrix}.$$

The proof of Theorem 5.1 for LAS of equilibria $\text{E}_0 - \text{E}_7$ is as follows.

Proof. The proof is similar to that for Theorem 2.1 and Theorem 2.2 in the 3-d model. The only difference is that one additional stability condition may come from the equation $\frac{\mathrm{d}Q}{\mathrm{d}t}$. For E_2, this additional condition is $EK_2 - 1 > 0$. For E_3, this condition is derived from the 4th eigenvalue,

$$\lambda_4 = K_1\tilde{D}r_2(r_1 - K_2) + EK_2r_1(r_2 - K_1) + g < 0,$$

which yields the required conditions given in (24). For E_4, this condition is $B > \tilde{D}$. For E_5, this condition comes from $\lambda_4 = \frac{\tilde{D}}{B} + EK_2\left(1 - \frac{1}{Br_2}\right) - 1 < 0$. For E_6 and E_7, a direct calculation yields the characteristic polynomials:

$$P(J(\text{E}_6)) = \left[\lambda^2 + \tfrac{r_1}{K_1\tilde{D}}\lambda + \tfrac{r_1(K_1\tilde{D}-1)}{K_1\tilde{D}}\right]\left(\lambda + 1 - \tfrac{B}{\tilde{D}}\right)\left[\lambda + \tfrac{Ar_1(K_1\tilde{D}-1)+\tilde{c}\,K_1(1-r_2\tilde{D})}{K_1\tilde{c}\,\tilde{D}}\right],$$
$$P(J(\text{E}_7)) = (\lambda+1)\left[\lambda^2 + \tfrac{r_2}{EK_2}\lambda + \left(1 - \frac{1}{EK_2}\right)r_2\right]\left[\lambda + \tfrac{1}{E} + \tfrac{\tilde{c}r_2}{A}\left(1 - \tfrac{1}{EK_2}\right) - r_1\right],$$

which directly yield the stability conditions for E_6 and E_7, as given in the theorem.

Transcritical bifurcations and Hopf bifurcation can be similarly obtained as those described in Theorem 2.1 and Theorem 2.2. The fact that no B-T bifurcation can occur from E_1-E_5 has been discussed in the proof for Theorem 2.2. For E_6, it can be seen from the characteristic polynomial $P(J(\text{E}_6))$ that three combinations come from $K_1\tilde{D} - 1 = \tilde{D} - B = Ar_1(K_1\tilde{D} - 1) + \tilde{c}\,K_1(1 - r_2\tilde{D}) = 0$; while for E_7, there is only one possibility: $r_1 = K_2 = \frac{1}{E}$. However, similarly, we can verify that the two zero eigenvalues obtained for these four cases are not a double-zero eigenvalue. Therefore, B-T bifurcation is not possible from either E_6 or E_7. ■

The proof of existence and LAS conditions in Theorem 5.3 for E_8 is as follows.

Proof. The existence conditions for E_8 directly follow from the conditions $N_{18} > 0$, $N_{28} > 0$ and $Q_8 > 0$, which are equivalent to that $N_{18\text{n}} > 0$, $N_{28\text{n}} > 0$ and $Q_{8\text{n}} > 0$ since $\text{E}_{8\text{d}} = \tilde{D}N_{18\text{n}} + EN_{28\text{n}}$.

Using a direct computation we obtain the characteristic polynomial for E_8 as follows:

$$P(E_8) = \big(\lambda + 1 - BN_{18}\big)\big(\lambda^3 + a_{18}\lambda^2 + a_{28}\lambda + a_{38}\big),$$

where a_{18}, a_{28} and a_{38} are given in (33). Thus, according to the Routh-Hurwitz criterion, we know that E_8 is LAS if the conditions given in (32) are satisfied. It is easy to see that $a_{18} > 0$ and $a_{38} > 0$ under the existence condition (C_1). Thus, E_8 satisfying the existence condition (C_2) is unstable. The transcritical bifurcation is determined by $a_{38} = 0$, which yields following three transcritical bifurcations:

$$\begin{aligned} &\text{between } (E_8, E_3) \text{ at } Q_8 = 0 \implies \tilde{D} = \tfrac{1}{r_2}\Big[1 + \tfrac{r_1(K_1 - r_2)(1 - EK_2)}{K_1(K_2 - r_1)}\Big], \\ &\text{between } (E_8, E_6) \text{ at } N_{28} = 0 \implies r_2 = \tfrac{1}{\tilde{D}} + \tfrac{Ar_1(\tilde{D}K_1 - 1)}{\tilde{c}\tilde{D}K_1}, \\ &\text{between } (E_8, E_7) \text{ at } N_{18} = 0 \implies r_1 = \tfrac{1}{E} + \tfrac{\tilde{c}r_2(EK_2 - 1)}{AEK_2}. \end{aligned} \tag{SM13}$$

■

Finally, we provide the proof of Theorem 5.4 for existence and LAS of E_9.

Proof. The existence conditions are directly derived from $N_{19} > 0$, $N_{29} > 0$, $P_9 > 0$ and $Q_9 > 0$. The stability of E_9 is determined from its characteristic polynomial

$$P(J(E_9)) = \lambda^4 + a_{19}\lambda^3 + a_{29}\lambda^2 + a_{39}\lambda + a_{49},$$

where the a_{k9} are given in (38). Then by the Routh-Hurwitz criterion, we know that E_9 is LAS under the conditions given in (37).

Transcritical bifurcations occur at the critical point determined by $a_{49} = 0$, which gives two possibilities: one from $Q_9 = 0$, resulting in the critical point,

$$r_2 = \tfrac{EK_2}{BEK_2 - B + \tilde{D}},$$

and the other from $P_9 = 0$, yielding the critical point,

$$r_1 = \tfrac{K_1\big\{AK_2(B - \tilde{D}) + \tilde{c}\big[r_2(BEK_2 - B + \tilde{D}) - EK_2\big]\big\}}{AEK_2(BK_1 - 1)}.$$

Note that $N_{29} = 0$ does not yield critical point. B-T bifurcation might occur at a critical point determined by $P_9 = Q_9 = 0$, leading to the critical point,

$$(r_1, r_2) = \big(\tfrac{K_1(B - \tilde{D})}{E(BK_1 - 1)},\ \tfrac{EK_2}{BEK_2 - B + \tilde{D}}\big),$$

under which the Jacobian matrix of (10) becomes

$$J(E_9) = \begin{bmatrix} \frac{\tilde{D} - B}{BE(BK_2 - 1)} & -\frac{1}{B} & -\frac{1}{B} & -\frac{\tilde{c}}{B} \\ \frac{\tilde{D} - B}{BE} & \frac{\tilde{D} - B}{B(BEK_2 - B + \tilde{D})} & 0 & \frac{A(\tilde{D} - B)}{BE} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which does show two zero eigenvalues, but this is a semi-simple case. Hence, no B-T bifurcation can occur from E_9. ■

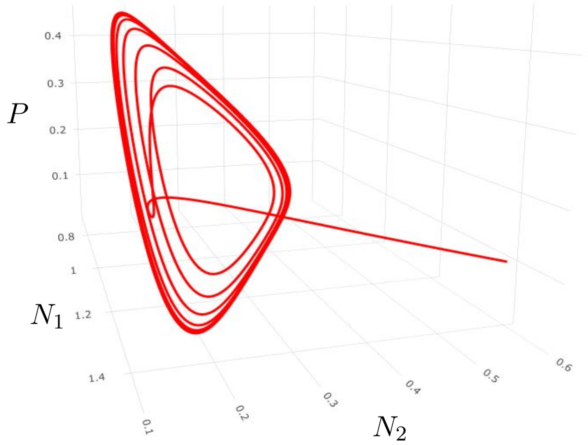


Figure SM3. *Simulated stable limit cycle from the Hopf bifurcation of the 4-d model* (10) *around* E_5*, projected in the* N_1*-*N_2*-*P *space, with* $\tilde{c} = \frac{3}{4}$*,* $A = B = 1$*,* $K_1 = 2$*,* $K_2 = 1$*,* $r_1 = \frac{19}{20}$ *and* $r_2 = \frac{3}{2}$*,* $E = 2$*,* $\tilde{D} = \frac{1}{4}$*, and the initial condition* $(N_1, N_2, P, Q) = (1.4, 0.6, 0.1, 0.5)$*.*

SM4.3. Example: Hopf bifurcation from E_5 in the 4-d system. We give a numerical example of the Hopf bifurcation from E_5 in the 4-d model by choosing the following parameter values:

$$\tilde{c} = \frac{3}{4}, \quad K_1 = 2, \quad B = 1, \quad r_2 = \frac{3}{2}, \quad K_2 = 1, \quad E = 2, \quad \tilde{D} = \frac{1}{4}.$$

Note that the parameter A is not fixed; though it appears in the equations, it does not render into the normal form, and thus does not affect the solution of periodic solutions bifurcating from the Hopf bifurcation and their stability. For the above chosen parameter values, we obtain the first focus value, $v_1 = -\frac{98}{297} < 0$, implying that the Hopf bifurcation is supercritical, and the bifurcating limit cycle is stable. The simulation for this example with $A = 1$ and $r_1 = 0.95$, yielding perturbation $\mu = r_1 - r_{1\mathrm{H}} = -0.05$, as shown in Figure SM3.

SM4.4. Hopf bifurcation from E_8 in the 4-d system. Here we present a numerical example of limit cycles bifurcating from E_8 in the 4-d system.

As an example, we choose $r_2 = \frac{3}{2}$. Then, $\Delta_{28\mathrm{n}} = 0$ gives a solution $r_{1\mathrm{H}} = 1.50864904\cdots$. With these parameter values, we need $B < \frac{1}{N_{18}} = 2.32066172\cdots$. Taking $B = \frac{1}{2}$ yields the following eigenvalues:

$$\lambda_{1,2} = \pm i\,\omega_c, \quad \lambda_3 = -\,0.78454421\cdots, \quad \lambda_4 = -\,1.19884994\cdots,$$

where

$$\omega_c = 0.66328460\cdots.$$

Further, we apply normal form theory and the Maple program [SM2] to find the first focus value, $v_1 = -\,0.22557240\cdots < 0$, implying that the Hopf bifurcation is supercritical and bifurcating limit cycle is stable. Taking $r_1 = 1.4$ which gives the perturbation $\mu = r_1 - r_{1\mathrm{H}} = -\,0.10864904\cdots$, we simulate system (10) to obtain the result, shown in Figure SM4.

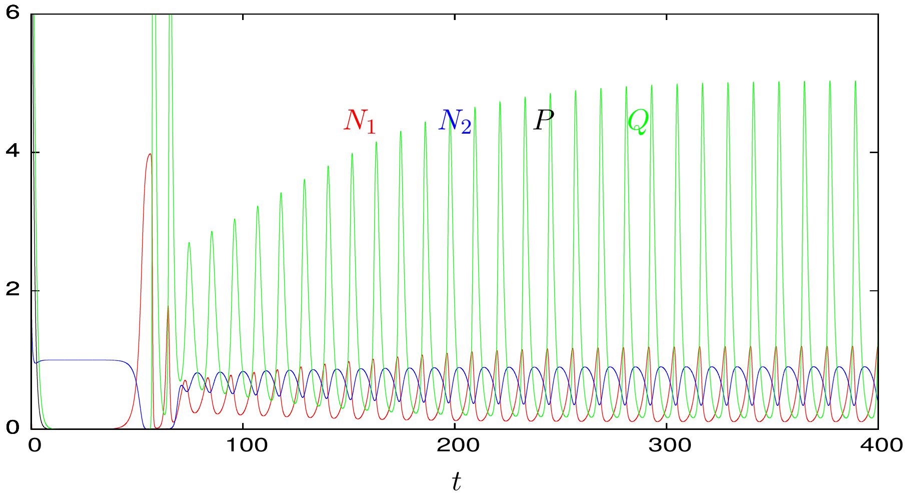


Figure SM4. *Simulated periodic solution bifurcating from the Hopf bifurcation of the 4-d model (10) around* E_8 *with* $\tilde{c} = \frac{2}{5}$, $E = \frac{1}{5}$, $K_2 = 1$, $\tilde{D} = 2$, $K_1 = 4$, $A = \frac{1}{50}$, $B = \frac{1}{2}$, $r_1 = \frac{7}{5}$ *and* $r_2 = \frac{3}{2}$. *The red, blue, black and green curves denote the solutions for* N_1, N_2, P *and* Q, *respectively.*

SM4.5. Hopf bifurcation from E_9 in the 4-d system. Here we provide a numerical example and demonstrate the use of normal form theory to analyse the existence and stability of limit cycles bifurcating from E_9 in the 4-d system.

As an illustrative example, we choose $K_1 = 2$, $K_2 = 3$ with the parameter values given in fig. 1b, and $r_1 = K_2 = 3$, $r_2 = K_1 = 2$ to obtain the follow solutions at the Hopf critical point:

$$\begin{aligned}
&N_{19} = \frac{1}{B} = \frac{235}{413}, \quad N_{29} = \frac{90}{413}, \quad P_9 = \frac{99}{70}, \quad Q_9 = \frac{9}{7}, \\
&a_{19} = \frac{825}{826}, \quad a_{29} = \frac{13581}{5782}, \quad a_{39} = \frac{22275}{40474}, \quad a_{49} = \frac{40095}{40474}, \\
&\Delta_2 = \frac{1225125}{682276}, \quad \Delta_3 = 0.
\end{aligned}$$

With the above parameter values, the Jacobian matrix of system of (10), evaluated at E_9 has a pair of purely imaginary eigenvalues, $\lambda_{1,2} = \pm\frac{3\sqrt{3}}{7}\, i$, and a complex conjugate, $\lambda_{3,4} = -\frac{825}{1652} \pm \frac{3\sqrt{469535}}{1652}\, i$.

We then use normal form theory [SM2] to further analyse the existence and stability of limit cycles. We first we introduce the affine transformation,

$$\begin{pmatrix} N_1 \\ N_2 \\ P \\ Q \end{pmatrix} = \begin{pmatrix} \frac{235}{413} \\ \frac{90}{413} \\ \frac{99}{70} \\ \frac{9}{7} \end{pmatrix} + \begin{bmatrix} 0 & \frac{2350\sqrt{3}}{13629} & -\frac{887125}{3813288} & \frac{3995\sqrt{469535}}{3813288} \\ 0 & -\frac{1175\sqrt{3}}{4543} & -\frac{25715}{317774} & \frac{73\sqrt{469535}}{317774} \\ 1 & 0 & \frac{150337}{107720} & -\frac{3\sqrt{469535}}{21544} \\ -\frac{235}{154} & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

into system (10) to obtain the following new system,

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} =\ & \frac{3\sqrt{3}}{7}\,x_2 - \frac{66409517185}{1698769590623616}\,x_3^2 - \frac{41416598193685}{566256530207872}\,x_4^2 - \frac{4304515\sqrt{3}}{16718163}\,x_1x_2 \\
& - \frac{36313501075}{91993030808}\,x_1x_3 + \frac{142139661\sqrt{469535}}{91993030808}\,x_1x_4 + \frac{32951204945\sqrt{3}}{91993030808}\,x_2x_3 \\
& - \frac{36313501075}{91993030808}\,x_1x_3 + \frac{158904802639\sqrt{469535}}{849384795311808}\,x_3x_4, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} =\ & -\frac{3\sqrt{3}}{7}\,x_1 + \frac{134996846225527\sqrt{3}}{5096308771870848}\,x_3^2 - \frac{265906698812887\sqrt{3}}{5096308771870848}\,x_4^2 + \frac{6926835}{5572721}\,x_1x_2 \\
& + \frac{14858896695\sqrt{3}}{91993030808}\,x_1x_3 - \frac{46145193\sqrt{1408605}}{91993030808}\,x_1x_4 - \frac{55225937195}{68994773106}\,x_2x_3 \\
& - \frac{86729495\sqrt{469535}}{68994773106}\,x_2x_4 - \frac{3051914871181\sqrt{1408605}}{12740771929677120}\,x_3x_4, \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} =\ & -\frac{825}{1652}\,x_3 + \frac{3\sqrt{469535}}{1652}\,x_4 - \frac{9239078245811425}{23782774268730624}\,x_3^2 - \frac{884809143228725}{7927591422910208}\,x_4^2 \\
& + \frac{1973676875\sqrt{3}}{5149194204}\,x_1x_2 - \frac{410130054625}{42500780233296}\,x_1x_3 + \frac{8605231175\sqrt{469535}}{42500780233296}\,x_1x_4 \\
& + \frac{146567022395\sqrt{3}}{3863707293936}\,x_2x_3 - \frac{1083056015\sqrt{1408605}}{42500780233296}\,x_2x_4 + \frac{20190528960079\sqrt{469535}}{11891387134365312}\,x_3x_4, \\
\frac{\mathrm{d}x_4}{\mathrm{d}t} =\ & -\frac{3\sqrt{469535}}{1652}\,x_3 - \frac{825}{1652}\,x_4 + \frac{88182934432388039\sqrt{469535}}{2030335439321533370 88}\,x_3^2 - \frac{39476290370207\sqrt{469535}}{23782774268730624}\,x_4^2 \\
& - \frac{17187499325\sqrt{1408605}}{43958670919548}\,x_1x_2 + \frac{3571562359735\sqrt{469535}}{362829160851647952}\,x_1x_3 - \frac{374687485285}{3863707293936}\,x_1x_4 \\
& - \frac{65386925083295\sqrt{1408605}}{362829160851647952}\,x_2x_3 - \frac{3072800769425\sqrt{3}}{42500780233296}\,x_2x_4 - \frac{1987385800275749}{3963795711455104}\,x_3x_4,
\end{aligned}
$$

whose Jacobian matrix evaluated at the origin is in the Jordan canonical form,

$$
J = \begin{bmatrix} 0 & \frac{3\sqrt{3}}{7} & 0 & 0 \\ -\frac{3\sqrt{3}}{7} & 0 & 0 & 0 \\ 0 & 0 & -\frac{825}{1652} & \frac{3\sqrt{469535}}{1652} \\ 0 & 0 & -\frac{3\sqrt{469535}}{1652} & -\frac{825}{1652} \end{bmatrix}.
$$

In general, one needs to apply center manifold theory first and then apply normal form theory to find the normal form. The method with the Maple program developed in [SM2] combines the two steps in one unified step to obtain the following normal form in polar coordinates up to 3rd-order terms,

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= r\left(\alpha_1\,\mu - \frac{66992230264625}{717665999785812}\,r^2\right), \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{3\sqrt{3}}{7} + \alpha_2\,\mu - \frac{126325590652475\sqrt{3}}{1174362545104056}\,r^2,
\end{aligned}
$$

where μ is the perturbation, defined as $\mu = B - B_{\mathrm{H}}$, and the coefficients α_1 and α_2 are obtained from the linear analysis, given by

$$
\alpha_1 = -\frac{103823}{59780098}, \quad \alpha_2 = \frac{41860550\sqrt{3}}{269010441},
$$

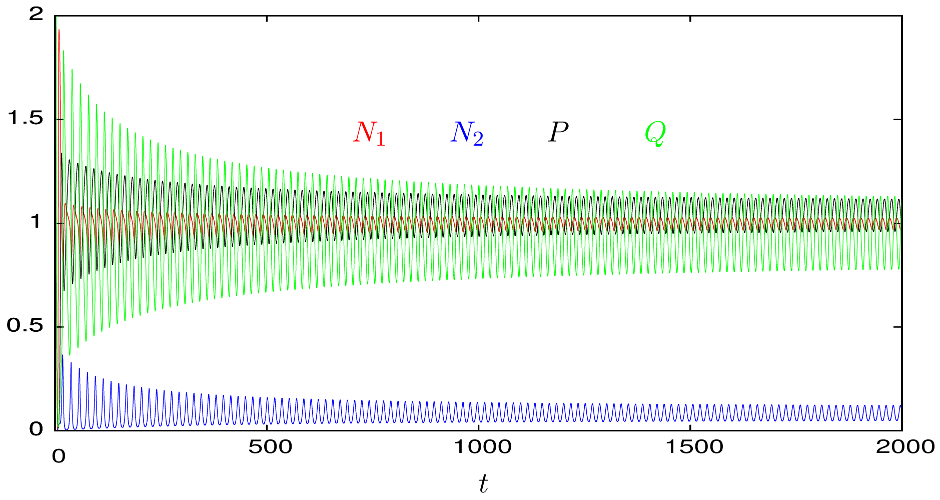


Figure SM5. *Simulated periodic solution bifurcating from the Hopf bifurcation of the 4-d model (10) around* E_9 *with* $\tilde{c} = \frac{2}{5}$, $\tilde{D} = \frac{4}{5}$, $A = 1$, $E = \frac{5}{2}$, $B = 1$, $r_1 = K_2 = 3$ *and* $r_2 = K_1 = 2$. *The red, blue, black and green curves denote the solutions for* N_1, N_2, P *and* Q, *respectively.*

where α_1 is called transversal condition. r and θ represent the amplitude and phase of motion, respectively. Then, the approximations of the amplitude and frequency of the motion are obtained from the normal form as

$$\bar{r} = \sqrt{-\frac{1564636568814}{84096629481125}\,\mu}, \qquad \omega = \frac{3\sqrt{3}}{7} + \frac{311746597115153\sqrt{3}}{1977952725396060}\,\mu.$$

It is seen that since the coefficient of r^2 in the amplitude equation is negative, the Hopf bifurcation is supercritical and bifurcating limit cycle is stable. We choose the perturbation $\mu = 1 - \frac{413}{235} = -\frac{178}{235} \approx 0.75744681$ for simulation, which yields

$$\bar{r} = \sqrt{\frac{5925644877636}{420483147405625}} \approx 0.11871170,$$
$$\omega = \frac{1528906400302039\sqrt{3}}{4944881813490150} \approx 0.53553222.$$

The simulation result is shown in Figure SM5. Comparing it with the simulation for E_8 (see Figure SM4), we note that there the absolute value of the perturbation, $\mu \approx -0.10864904$, is much smaller than that for E_9, while the amplitude of oscillation for E_8 is larger than that for E_9; this implies that the impact of the parameter r_1 (or r_2) is stronger than that of parameter B.

SM4.6. Host-range expansion from E_5. Here we present the proof that it is not possible to find parameter values for which the host, parasite and competitor stably co-exist in the 3-d model, but if the parasite expands its host range, E_5 loses stability and both E_6 and E_7 are stable in the 4-d model.

Theorem SM4.1. *There are no feasible parameter values such that the equilibrium* E_5 *of the* 3*-d model is stable, while it is unstable for the* 4*-d model, simultaneously with both* E_6 *and* E_7 *being LAS.*

Proof. Consider the conditions:

$$\frac{1}{B} < K_1 \le r_2, \quad r_1 > r_1^*,$$

under which E_5 is GAS for the 3-d model, while for the 4-d model, its stability needs two more conditions:

$$E < \frac{Br_2}{K_2(Br_2 - 1)}, \quad \tilde{D} < B - EK_2\Big(B - \frac{1}{r_2}\Big).$$

Note that E_6 exists for $K_1\tilde{D} > 1$, and is LAS for

$$B < \tilde{D}, \quad \text{and} \quad r_2 < \frac{1}{\tilde{D}} + \frac{Ar_1}{\tilde{c}}\Big(1 - \frac{1}{\tilde{D}K_1}\Big).$$

It is easy to see that E_5 (for the 4-d model) is unstable for $B < \tilde{D}$. E_7 exists for $EK_2 > 1$, and is LAS for

$$r_1 < \frac{1}{E} + \frac{\tilde{c}\,r_2}{A}\Big(1 - \frac{1}{EK_2}\Big).$$

Summarizing the above discussions, we have the following conditions required for E_5 being GAS for the 3-d model; unstable for the 4-d model; and E_6 and E_7 both LAS:

$$\begin{aligned} &\frac{1}{B} < K_1 \le r_2, \quad r_1 > r_1^* = \frac{K_1K_2(Br_2-1)}{r_2(BK_1-1)}, \\ &\tilde{D}K_1 > 1, \quad B < \tilde{D}, \quad r_2 < \frac{1}{\tilde{D}} + \frac{Ar_1}{\tilde{c}}\Big(1 - \frac{1}{\tilde{D}K_1}\Big) \triangleq r_{2c}, \\ &EK_2 > 1, \quad r_1 < \frac{1}{E} + \frac{\tilde{c}\,r_2}{A}\Big(1 - \frac{1}{EK_2}\Big) \triangleq r_{1c}. \end{aligned} \tag{SM14}$$

It follows from $K_1 \le r_2 < r_{2c}$ that

$$K_1 < \frac{1}{\tilde{D}} + \frac{Ar_1}{\tilde{c}}\Big(1 - \frac{1}{\tilde{D}K_1}\Big) \implies \Big(K_1 - \frac{1}{\tilde{D}}\Big)\Big(K_1 - \frac{Ar_1}{\tilde{c}}\Big) < 0 \implies K_1 < \frac{Ar_1}{\tilde{c}}. \tag{SM15}$$

We use $r_1 < r_{1c}$ and $r_2 < r_{2c}$ to obtain that

$$\begin{aligned} r_1 &< \frac{1}{E} + \frac{\tilde{c}\,r_2}{A}\Big(1 - \frac{1}{EK_2}\Big) < \frac{1}{E} + \frac{\tilde{c}\,r_{2c}}{A}\Big(1 - \frac{1}{EK_2}\Big) \\ &= \frac{1}{E} + \frac{\tilde{c}}{\tilde{D}A}\Big(1 - \frac{1}{EK_2}\Big) + r_1\Big(1 - \frac{1}{EK_2}\Big)\Big(1 - \frac{1}{\tilde{D}K_1}\Big) \\ \implies r_1 &< \frac{K_1\big[\tilde{D}K_2 + \tilde{c}\,(EK_2 - 1)\big]}{A(\tilde{D}K_1 + EK_2 - 1)} \triangleq r_1^a. \end{aligned} \tag{SM16}$$

Then, using $K_1 < \frac{Ar_1}{\tilde{c}} < \frac{Ar_1^a}{\tilde{c}}$ yields

$$K_1 < \frac{K_1\big[A\tilde{D}K_2 + \tilde{c}\,(EK_2 - 1)\big]}{\tilde{c}\,(\tilde{D}K_1 + EK_2 - 1)} \quad \Longrightarrow \quad AK_2 > \tilde{c}\,K_1. \tag{SM17}$$

Further, it is easy to prove that

$$K_2 - r_1^a = \frac{(EK_2 - 1)(AK_2 - \tilde{c}\,K_1)}{A(\tilde{D}K_2 + (EK_2 - 1)} > 0 \quad \Longrightarrow \quad r_1 < r_1^a < K_2.$$

However, noticing that

$$r_1 > r_1^* = \frac{K_1K_2(Br_2 - 1)}{r_2(BK_1 - 1)} = K_2\,\frac{BK_1 - \frac{K_1}{r_2}}{BK_1 - 1} \geq K_2, \quad \text{due to} \quad r_2 \geq K_1,$$

we know that no feasible parameter values exist for the case that E_5 is GAS for the 3-d model but unstable for the 4-d model, simultaneously with LAS E_6 and E_7.

For the second stability condition for E_5: $\frac{1}{B} < r_2 < K_1$, $r_1 > r_{1\mathrm{H}}$, we should have the same conclusion, since the stability conditions for E_5 (4-d model), E_6 and E_7 are same. ■

SM5. Bistable states. Finally, we consider possible bistable states or co-existence of equilibria, since this is not only an interesting theoretical question, but also an important phenomenon in determining the possible outcomes of host-range expansions. To aid the reader, we provide a visualization of the equilibria of the 3-d and 4-d models in Figure SM6.

It is obvious that the 2-d model (SM1) cannot have coexistence of equilibria. The 3-d model (2) can have bistability only between E_1 and E_2, as well as between E_2 and E_4. The situation becomes much more complex for the 4-d model (10). There are two groups, one of them is an "easier group", which can be identified by directly comparing the stability conditions; while the other is a "harder group", all of which involve the equilibrium E_8, and which require tedious computation such as that demonstrated in the proof of Theorem 4.1. In order to show the existence of bistability, for each case we present a concrete example, without identifying all possible parameter values. More precisely, we have the following result.

Theorem SM5.1. *The* 2*-d model* (SM1) *does not have bistable states. For the* 3*-d model* (2)*, bistable stable states can only exist in the equilibrium pairs* $(\mathrm{E}_1, \mathrm{E}_2)$ *and* $(\mathrm{E}_2, \mathrm{E}_4)$*. For the* 4*-d model* (10)*, there are two groups* (*A*) *and* (*B*)*. Group* (*A*) *contains the bistable pairs:* $(\mathrm{E}_1, \mathrm{E}_2)$, $(\mathrm{E}_1, \mathrm{E}_7)$, $(\mathrm{E}_2, \mathrm{E}_4)$, $(\mathrm{E}_2, \mathrm{E}_6)$, $(\mathrm{E}_2, \mathrm{E}_9)$, $(\mathrm{E}_3, \mathrm{E}_6)$, $(\mathrm{E}_4, \mathrm{E}_7)$, $(\mathrm{E}_5, \mathrm{E}_7)$, *and* $(\mathrm{E}_6, \mathrm{E}_7)$. *Group* (*B*) *contains the bistable pairs:* $(\mathrm{E}_1, \mathrm{E}_8)$, $(\mathrm{E}_2, \mathrm{E}_8)$, $(\mathrm{E}_4, \mathrm{E}_8)$ *and* $(\mathrm{E}_5, \mathrm{E}_8)$. *Typical examples with exact parameter values, exhibiting the bistable states, are listed in Table* SM1 (*for the* 3*-d model*)*, Table* SM2 (*for the group* (*A*) *of the* 4*-d model*) *and Table* SM3 (*for the group* (*B*) *of the* 4*-d model*)*. Characteristic polynomials, denoted by* $P(\lambda)$*, for these examples are also provided in the tables to show stability. In the three tables, E.P. denotes Equilibrium Pair.*

Proof. First note that if there exists a transcritical bifurcation between two equilibria, then they cannot be bistable. For example, there exists a transcritical bifurcation between E_5 and E_9, and so these two equilibria cannot be bistable.

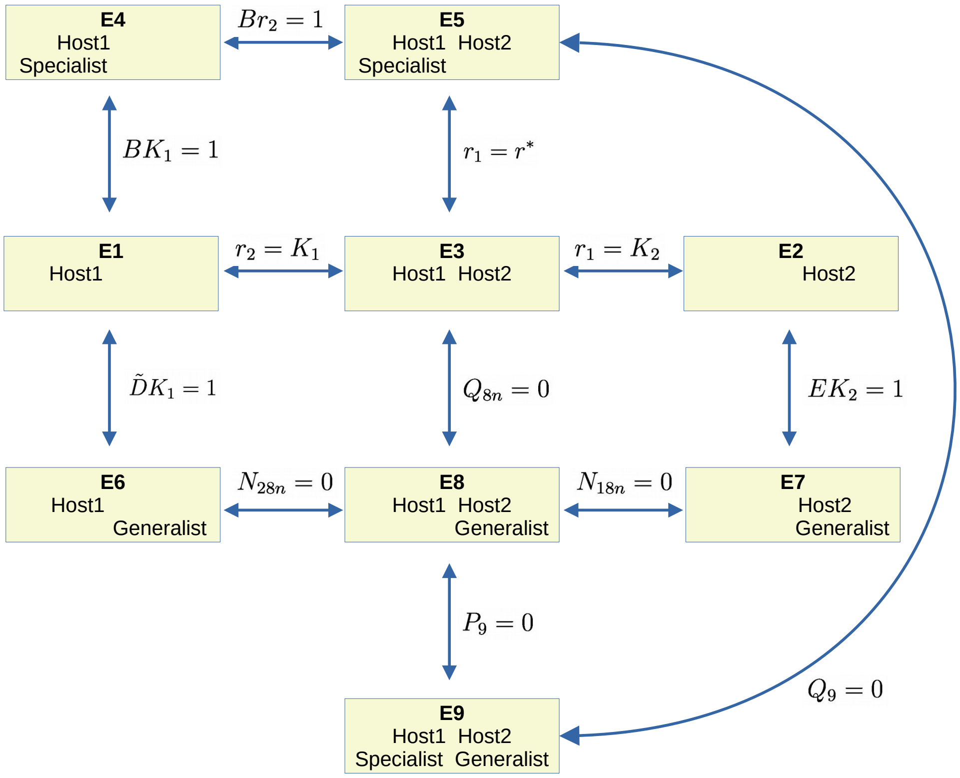


Figure SM6. *A diagram illustrating the equilibrium states of the 4-d model. Transcritical bifurcation conditions label the arrows between equilibria. "Host1(2)" indicates the presence of population $N_{1(2)}$, "Specialist" indicates the presence of population P, while "Generalist" indicates the population Q.*

Table SM1
Bistable Equilibrium Pairs for 3-d Model (2)

E. P.	(B, K_1, K_2, r_1, r_2)	$P(\lambda)$
(E_1, E_2)	$(\frac{1}{2}, \frac{3}{2}, 2, \frac{3}{2}, 1)$	(E_1): $\frac{1}{16}(2\lambda+1)(2\lambda+3)(4\lambda+1)$ (E_2): $\frac{1}{2}(2\lambda+1)(\lambda+1)^2$
(E_2, E_4)	$(\frac{1}{2}, 3, 2, 1, \frac{3}{2})$	(E_2): $\frac{1}{2}(\lambda+1)(2\lambda+3)^2$ (E_4): $\frac{1}{6}(2\lambda+1)(3\lambda^2+2\lambda+1)$

For the 3-d model (2), it is straightforward to use the stability conditions to find the bistable pairs (E_1, E_2) and (E_2, E_4) since the parameter values are overlapping for their existence and stability conditions. For the bistable pairs in Group (A) of the 4-d model, it is not difficulty to find them by carefully inspecting their existence and stability conditions. How-

Table SM2
Bistable Equilibrium Pairs for Group (A) of 4-d Model (10)

E. P.	$(B, K_1, K_2, r_1, r_2, \tilde{D}, E, \tilde{c}, A)$	$P(\lambda)$
$(\mathrm{E}_1, \mathrm{E}_2)$	$(\frac{1}{2},\ \frac{3}{2},\ 2,\ \frac{3}{2},\ 1,\ \frac{1}{3},\ \frac{2}{5},\ \frac{2}{5},\ \frac{1}{2})$	(E_1): $\frac{1}{32}(2\lambda+1)^2(2\lambda+3)(4\lambda+1)$ (E_2): $\frac{1}{10}(\lambda+1)^2(2\lambda+1)(5\lambda+1)$
$(\mathrm{E}_1, \mathrm{E}_7)$	$(\frac{1}{2},\ \frac{3}{2},\ 2,\ \frac{3}{2},\ 1,\ \frac{1}{3},\ \frac{3}{5},\ \frac{2}{5},\ \frac{1}{2})$	(E_1): $\frac{1}{32}(2\lambda+1)^2(2\lambda+3)(4\lambda+1)$ (E_7): $\frac{1}{60}(\lambda+1)(2\lambda+1)(3\lambda+1)(10\lambda+3)$
$(\mathrm{E}_2, \mathrm{E}_4)$	$(\frac{1}{2},\ 3,\ 2,\ 1,\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{5},\ \frac{2}{5},\ \frac{1}{2})$	(E_2): $\frac{1}{10}(\lambda+1)^2(2\lambda+3)(5\lambda+1)$ (E_4): $\frac{1}{6}(2\lambda+1)(3\lambda+1)(3\lambda^2+2\lambda+1)$
$(\mathrm{E}_2, \mathrm{E}_6)$	$(\frac{1}{2},\ 3,\ 2,\ 1,\ \frac{3}{2},\ \frac{3}{5},\ \frac{2}{5},\ \frac{2}{5},\ \frac{1}{2})$	(E_2): $\frac{1}{10}(\lambda+1)^2(2\lambda+3)(5\lambda+1)$ (E_6): $\frac{1}{972}(6\lambda+1)(18\lambda+13)(9\lambda^2+5\lambda+4)$
$(\mathrm{E}_2, \mathrm{E}_9)$	$(\frac{1}{2},\ 3,\ \frac{3}{2},\ \frac{147}{100},\ 3,\ \frac{2}{5},\ \frac{1}{2},\ \frac{2}{5},\ 1)$	(E_2): $\frac{1}{400}(\lambda+1)(\lambda+3)(4\lambda+1)(100\lambda+3)$ (E_9): $\frac{1}{2500}(2500\lambda^4+4450\lambda^3+245\lambda^2+6\lambda+1)$
$(\mathrm{E}_3, \mathrm{E}_6)$	$(\frac{1}{2},\ 1,\ 1,\ 2,\ 3,\ \frac{3}{2},\ \frac{1}{10},\ \frac{2}{5},\ \frac{3}{2})$	(E_3): $\frac{1}{2500}(10\lambda+7)(50\lambda+1)(5\lambda^2+18\lambda+12)$ (E_6): $\frac{1}{54}(3\lambda+2)(6\lambda+1)(3\lambda^2+4\lambda+2)$
$(\mathrm{E}_3, \mathrm{E}_7)$	$(\frac{1}{2},\ 1,\ 1,\ 2,\ 3,\ \frac{1}{4},\ \frac{33}{32},\ \frac{2}{5},\ \frac{1}{45})$	(E_3): $\frac{1}{2000}(40\lambda+1)(10\lambda+)(5\lambda^2+18\lambda+12)$ (E_7): $\frac{1}{363}(\lambda+1)(33\lambda+20)(11\lambda^2+32\lambda+1)$
$(\mathrm{E}_4, \mathrm{E}_7)$	$(\frac{1}{2},\ 3,\ 2,\ 1,\ 1,\ \frac{2}{5},\ \frac{4}{5},\ \frac{2}{5},\ \frac{1}{2})$	(E_4): $\frac{1}{15}(\lambda+1)(5\lambda+1)(3\lambda^2+2\lambda+1)$ (E_7): $\frac{1}{160}(\lambda+1)(20\lambda+11)(8\lambda^2+5\lambda+3)$
$(\mathrm{E}_5, \mathrm{E}_7)$	$(1,\ 2,\ 1,\ 4,\ 3,\ \frac{1}{8},\ \frac{5}{4},\ \frac{2}{5},\ \frac{1}{20})$	(E_5): $\frac{1}{72}(24\lambda+1)(3\lambda^3+12\lambda^2+14\lambda+8)$ (E_7): $\frac{1}{25}(\lambda+1)(5\lambda+8)(5\lambda^2+12\lambda+3)$
$(\mathrm{E}_6, \mathrm{E}_7)$	$(\frac{1}{2},\ \frac{3}{2},\ 1,\ 1,\ \frac{4}{3},\ \frac{3}{4},\ \frac{3}{2},\ \frac{2}{5},\ \frac{1}{2})$	(E_6): $\frac{1}{972}(3\lambda+1)(36\lambda+5)(9\lambda^2+8\lambda+1)$ (E_7): $\frac{1}{405}(\lambda+1)(45\lambda+1)(9\lambda^2+8\lambda+4)$

ever, for Group (B) of the 4-d model, it is quite difficulty to prove whether they are bistable or not. The approach used in the proof of Theorem 4.1 is needed for considering the bistability. Proofs will be given for two examples, one for the pair $(\mathrm{E}_2, \mathrm{E}_9)$ in Group (A), and one for the pair $(\mathrm{E}_5, \mathrm{E}_8)$ in Group (B). Other cases can be similarly proven.

Table SM3
Bistable Equilibrium Pairs for Group (B) of 4-d Model (10)

E. P.	$(B, K_1, K_2, r_1, r_2, \tilde{D}, E, \tilde{c}, A)$	$P(\lambda)$
$(\mathrm{E}_1, \mathrm{E}_8)$	$\left(\frac{1}{2},\ \frac{31}{20},\ \frac{5}{2},\ 2,\ \frac{3}{2},\ \frac{2}{5},\ \frac{4}{5},\ \frac{1}{3},\ \frac{1}{2}\right)$	$(\mathrm{E}_2)\colon\ \frac{1}{40000}(\lambda+2)(20\lambda+1)(40\lambda+9)(50\lambda+19)$ $(\mathrm{E}_8)\colon\ \frac{1}{15583372}(1204\lambda+739)(12943\lambda^2+480\lambda+468)$
$(\mathrm{E}_2, \mathrm{E}_8)$	$\left(\frac{1}{2},\ 1,\ \frac{41}{20},\ 2,\ \frac{4}{5},\ 2,\ \frac{2}{5},\ \frac{2}{5},\ \frac{2}{25}\right)$	$(\mathrm{E}_2)\colon\ \frac{1}{5000}(\lambda+1)(5\lambda+4)(20\lambda+1)(50\lambda+9)$ $(\mathrm{E}_8)\colon\ \frac{1}{9953280}(288\lambda+257)(34560\lambda^3+34080\lambda^2+419\lambda+217)$
$(\mathrm{E}_4, \mathrm{E}_8)$	$\left(\frac{1}{2}, \frac{1001}{500}, \frac{31}{30}, 1, \frac{10}{20}, \frac{2}{5}, \frac{428647}{426405}, \frac{2}{5}, 3\right)$	$(\mathrm{E}_4)\colon\ \frac{1}{100100}(5\lambda+1)(20\lambda+21)(1001\lambda^2+1000\lambda+1)$ $(\mathrm{E}_8)\colon\ \frac{1}{6}(2\lambda+1)(3\lambda+1)(3\lambda^2+2\lambda+1)$

$(\mathrm{E}_5, \mathrm{E}_8)$

$$(B,\ K_1,\ K_2,\ r_1,\ r_2,\ \tilde{D},\ E,\ \tilde{c},\ A) = \left(\tfrac{1}{2},\ \tfrac{40021}{20000},\ 2,\ \tfrac{970471159}{488410000},\ \tfrac{2001}{1000},\ \tfrac{151579}{400210},\ 2,\ \tfrac{2}{5},\ \tfrac{21}{50}\right)$$

$$P(\mathrm{E}_5) = \left(\lambda+\tfrac{96300106}{400410105}\right)\left(\lambda^3+\tfrac{77715624426236610000}{39112859876610000000}\,\lambda^2+\tfrac{1176740478699000}{39112859876610000000}\,\lambda + \tfrac{1686855352339}{39112859876610000000}\right)$$

$$P(\mathrm{E}_8) = \left(\lambda+\tfrac{2309313133421}{26000271920390}\right)\left(\lambda^3+\tfrac{64863514517103697368801043269732340500000}{33017206608527273135328710316100000000000}\lambda^2 + \tfrac{6949883511088218636398660990320959900000}{33017206608527273135328710316100000000000}\lambda + \tfrac{1397220570993614731031667709647277 6551}{33017206608527273135328710316100000000000}\right)$$

For the bistable pair $(\mathrm{E}_2, \mathrm{E}_9)$, the stability for E_2 only needs $r_1 < K_2 < \frac{1}{E}$. The existence condition for E_9 is given in (36). Once the existence condition is satisfied, the stability condition given in (38) will be most likely satisfied, which only needs verifying. Let us start from $K_2 < \frac{1}{E}$ and $E > \frac{B-\tilde{D}}{BK_2}$ to get $1 - \frac{\tilde{D}}{B} < EK_2 < 1$. Then from $r_1 < K_2$ and the condition on r_1 given in (36) we have

$$\frac{K_1\big\{AK_2(B-\tilde{D}) + \tilde{c}\left[r_2(BEK_2 - B + \tilde{D}) - EK_2\right]\big\}}{AEK_2(BK_1 - 1)} < r_1 < K_2, \tag{SM18}$$

which yields

$$AK_2\big[K_1(B-\tilde{D}) - EK_2(BK_1-1)\big] + K_1\tilde{c}\big[r_2(BEK_2 - B + \tilde{D}) - EK_2\big] < 0.$$

Since the term in the second square bracket of the above inequality is positive, the above equality requires

$$K_1(B-\tilde{D}) - EK_2(BK_1-1) < 0 \quad\Longrightarrow\quad EK_2 > 1 - \frac{\tilde{D}K_1 - 1}{BK_1 - 1},$$

which implies $\tilde{D}K_1 > 1$ due to $BK_1 > 1$ (see (36)) and $EK_2 < 1$. It is easy to show that

$$1 - \frac{\tilde{D}}{B} < 1 - \frac{\tilde{D}K_1 - 1}{BK_1 - 1} < EK_2 < 1. \tag{SM19}$$

Then, it follows from (SM18) that

$$A > \frac{K_1\tilde{c}\big[r_2(BEK_2 - B + \tilde{D}) - EK_2\big]}{K_2\big[EK_2(BK_1 - 1) - K_1(B - \tilde{D})\big]}. \tag{SM20}$$

Having established the above relations, we choose $B = \frac{1}{2}$, $\tilde{D} = \frac{2}{5}$, $K_1 = 3$ to satisfy $B > \tilde{D}$ and $BK_1 > 1$. Next, using (SM19) we obtain $\frac{3}{5} < EK_2 < 1$. Choosing $E = \frac{1}{2}$, $K_2 = \frac{3}{2}$ and $\tilde{c} = \frac{2}{5}$ yields $r_2 > \frac{30}{11}$ (see the condition given in (36)). We let $r_2 = 3$. Then, we use (SM20) to obtain $A > \frac{4}{5}$, and choose $A = 1$. With these chosen parameter values, we obtain the characteristic polynomials listed in Table SM2 for (E_2, E_9), showing that both E_2 and E_9 are LAS for this set of parameter values.

Now we turn to the bistable pair (E_4, E_8). To prove this case, we first consider a parameter value at a critical boundary, which yields a zero eigenvalue. Having successfully obtained the result for the critical point, we then give a perturbation to the critical point to remove the zero eigenvalue. To achieve this, we consider the stability of E_4: $r_2 < \frac{1}{B} < \min\{K_1, \frac{1}{\tilde{D}}\}$, and let $K_1 = \frac{1}{B}$, which implies $B > \tilde{D}$, and $\tilde{D}r_2 < Br_2 < 1$. Then, we require that

$$N_{18n} = \frac{1}{B}\big[EK_2(Ar_1 - \tilde{c}\,r_2) - (AK_2 - \tilde{c}\,r_2)\big] > 0,$$

$$N_{28n} = \frac{K_2}{B}\big[Ar_1(B - \tilde{D}) - (1 - \tilde{D}r_2)\tilde{c}\big] > 0,$$

$$Q_{8n} = \frac{1}{B}\big[EK_2(1 - \tilde{D}r_2) - (B - \tilde{D})\,r_1r_2 - EK_2r_1(1 - Br_2)\big] > 0,$$

$$1 - BN_{18} = \frac{(B - \tilde{D})(AK_2 - \tilde{c}\,r_2) - EK_2\,\tilde{c}\,(1 - Br_2)}{EK_2(ABr_2 - \tilde{c}) - \tilde{D}(AK_2 - \tilde{c}\,r_2)} > 0.$$

It is easy to see that $N_{28n} > 0$ gives

$$A > \frac{\tilde{c}\,(1 - \tilde{D}r_2)}{r_1(B - \tilde{D})}, \tag{SM21}$$

which leads to

$$Ar_1 - \tilde{c}\,r_2 > \frac{\tilde{c}\,(1 - \tilde{D}r_2)}{B - \tilde{D}} = \frac{\tilde{c}\,(1 - Br_2)}{B - \tilde{D}} > 0. \tag{SM22}$$

Next, it follows from $N_{18n} > 0$ that

$$E > \frac{AK_2 - \tilde{c}\,r_2}{K_2(Ar_1 - \tilde{c}\,r_2)}. \tag{SM23}$$

Further, we compute

$$\begin{aligned} &EK_2(ABr_2-\tilde{c})-\tilde{D}(AK_2-\tilde{c}\,r_2)-B\tilde{D}N_{18\text{n}} \\ &= EK_2(B-\tilde{D})\Big(Ar_1-\frac{\tilde{c}\,(1-\tilde{D}r_2)}{B-\tilde{D}}\Big)>0, \qquad \text{(due to (SM21))}, \end{aligned}$$

which implies that the denominator of $1-BN_{18}$ is greater than zero provided $N_{18\text{n}}>0$. Thus, letting the numerator of $1-BN_{18}$ be greater than zero leads to

$$E<\frac{(B-\tilde{D})(AK_2-\tilde{c}\,r_2)}{K_2\,\tilde{c}\,(1-Br_2)},$$

which is then combined with (SM23) to yield

$$\frac{AK_2-\tilde{c}\,r_2}{K_2(Ar_1-\tilde{c}\,r_2)}<E<\frac{(B-\tilde{D})(AK_2-\tilde{c}\,r_2)}{K_2\,\tilde{c}\,(1-Br_2)}. \tag{SM24}$$

This provides a feasible interval for E due to $Ar_1-\tilde{c}\,r_2>0$, with the assumption $AK_2-\tilde{c}\,r_2>0$, which will be proven in the next step, see (SM27). Now, from $Q_{8\text{n}}>0$ we have

$$E<\frac{K_2(1-\tilde{D}r_2)-(B-\tilde{D})\,r_1r_2}{K_2r_1(1-Br_1)}, \tag{SM25}$$

which needs

$$K_2>\frac{(B-\tilde{D})\,r_1r_2}{1-\tilde{D}r_2}, \tag{SM26}$$

and then a simple calculation shows that

$$AK_2-\tilde{c}\,r_2>0. \tag{SM27}$$

In addition, it can be shown that

$$\frac{K_2(1-\tilde{D}r_2)-(B-\tilde{D})\,r_1r_2}{K_2r_1(1-Br_1)}<\frac{(B-\tilde{D})(AK_2-\tilde{c}\,r_2)}{K_2\,\tilde{c}\,(1-Br_2)}.$$

Thus, combining (SM24) and the above inequality we have

$$\frac{AK_2-\tilde{c}\,r_2}{K_2(Ar_1-\tilde{c}\,r_2)}<E<\frac{K_2(1-\tilde{D}r_2)-(B-\tilde{D})\,r_1r_2}{K_2r_1(1-Br_1)}, \tag{SM28}$$

which requires that $K_2>r_1$. Comparing this condition $K_2>r_1$ with that given in (SM26) shows that

$$r_1-\frac{(B-\tilde{D})\,r_1r_2}{1-\tilde{D}r_2}=\frac{r_1(1-Br_2)}{1-\tilde{D}r_2}>0,$$

indicating that $K_2>r_1$.

Finally, based on the above derived formulas, we first choose $B = \frac{1}{2}$, and then $K_1 = \frac{1}{B} + \varepsilon = 2 + \varepsilon$, where $0 < \varepsilon \ll 1$. We choose $\varepsilon = \frac{1}{500}$, but set $\varepsilon = 0$ in the following procedure of determining parameter values. We choose $\tilde{D} = \frac{2}{5}$ and $r_2 = \frac{19}{20}$. Further, let $r_1 = 1$ and $\tilde{c} = \frac{2}{5}$. Using (SM21) we have $A > \frac{62}{25}$, and take $A = 3$. Next, we select $K_2 = \frac{31}{30}$ in order to have $K_2 \gtrapprox r_1$. Then, it follows from (SM27) that $\frac{4080}{4061} < E < \frac{3274}{3255}$. We take the middle point of this interval to obtain $E = \frac{428647}{426405}$. Substituting the above chosen parameter values, together with $\varepsilon = \frac{1}{500}$ into system (10) yields the two characteristic polynomials, given in Table SM3, for E_4 and E_8 respectively. ■

Remark SM1. (i) It has been shown that the 2-d model (SM1) does not need a boundedness condition since the solutions of the model are bounded for any positive parameter values, and the two LAS equilibrium solutions are also GAS under their LAS stability conditions. However, the 3-d model (2) and the 4-d model (10) do need the boundedness conditions, in addition to the LAS conditions, to reach GAS. Except the equilibrium E_5 for the 3-d model whose LAS conditions involve the boundedness conditions, all the equilibrium solutions of the 3-d and 4-d models need the boundedness condition to achieve GAS.

(ii) All equilibria of the 3-d and 4-d models are located on the boundary of attracting region Ω_{3d} (for the 3-d model) or Ω_{4d} (for the 4-d model). Without the boundedness condition (C_{bd}^{3d} or C_{bd}^{4d}), when multi-stable equilibria exist, equilibria are LAS and to which equilibria they will converge depends up the initial condition. While when the boundedness condition is satisfied, only one stable equilibrium exists and all trajectories converge to this equilibrium regardless the initial condition.

(iii) In this paper, we only discussed one type of bistable state, that is, when both states are equilibrium solutions. There is another class of bistable states – a stable equilibrium and a stable limit cycle – which can only exist from the bistable equilibrium pairs when one of the equilibria loses its stability and generates a supercritical Hopf bifurcation. Thus, such a bistable phenomenon may only appear in the 4-d model from the equilibrium pairs in Group (B) and the pair (E_2, E_9) in Group (A). Finding the conditions on parameters which produce this type of bistable phenomenon is beyond the scope of this paper, and will be studied in future.

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