Forced oscillations, bifurcations and stability of a molecular system

Part 2: Resonances

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The bifurcation and stability analysis of the non-autonomous system studied in a companion paper is extended to cases in which external resonance occurs. In particular, three representative resonance cases are studied. They are 1:1 primary resonance, 1:2 superharmonic resonance, and 2:1 subharmonic resonance cases. Again, the intrinsic harmonic balancing technique is used as the main method of analysis which is facilitated by MAPLE, a symbolic computer language.

1. Introduction
In a companion paper, a three-dimensional model of a molecular system is considered. The model represents a non-autonomous system subjected to a harmonic excitation with a frequency \( \Omega \). The stability and bifurcation behaviour of this system in the vicinity of a critical point, at which the associated autonomous system exhibits Hopf bifurcation, is analysed under the assumption that the internal frequency \( \omega_c \) and the excitation frequency \( \Omega \) are not rationally linked; i.e.

\[
m_1 \Omega + m_2 \omega_c \neq 0,
\]

where \( m_1 \) and \( m_2 \) are integers. This non-resonance condition may not hold in a number of cases, thus resulting in resonance.

In this paper three representative resonance cases are studied: These are 1:1 (\( \omega_c / \Omega \)) primary resonance, 1:2 superharmonic resonance, and 2:1 subharmonic resonance cases. The analysis will be carried out with the aid of intrinsic harmonic balancing technique and MAPLE.

2. Resonance cases
2.1. Primary resonance 1:1
With reference to equations (8)–(10) obtained in Part 1 (Yu et al. 1996), the following scaling is introduced:

\[
\omega_c - \Omega \rightarrow \varepsilon \delta, \quad w_1 \rightarrow \varepsilon w_1, \quad \mu \rightarrow \varepsilon \mu \quad \text{and} \quad E \rightarrow \varepsilon^2 E,
\]

which helps to facilitate perturbation analysis. It is noted that \( E \) is scaled to \( \varepsilon^2 E \) rather than \( \varepsilon E \) as in the non-resonant case (Yu et al. 1996). From the viewpoint of physics, this is because of the fact that 1:1 resonant case needs much less energy from the external force to excite the dynamic motions. In fact, if the nonlinear terms are ignored, this system produces an unbounded response, as expected.

The possible periodic solutions in the vicinity of the critical point can be described in the parameter form:

\[
w_i = w_i(\tau; \varepsilon), \quad \mu = \mu(\varepsilon) \quad \text{and} \quad \tau = \omega(\varepsilon)t,
\]

and furthermore, \( w_i \) can be assumed to be represented by a Fourier series of the form

\[
w_i(\tau; \varepsilon) = \sum_{m=0}^{M} p_{im}(\varepsilon) \cos m \tau + r_{im}(\varepsilon) \sin m \tau,
\]

where \( m \) and \( M \) are integers and \( M \) is open-ended.

Substituting the assumed solution (2) into (8)–(10)
given by Yu et al. (1996) yields the identity
\[ \omega(e) \frac{\partial \omega(e)}{\partial \epsilon} = W[\omega|\omega(e); \mu(e); \Omega] \quad i = 1, 2, 3. \quad (4) \]

A sequence of perturbation equations is then generated by differentiating the above identity with respect to the independent perturbation parameter \( \epsilon \) successively. Furthermore, evaluating the resulting equations at the critical point yields an ordered set of equations.

\( \epsilon^0 \) (zeroth-order) equations:
\[
\begin{align*}
\Omega \frac{\partial \omega_0^0}{\partial \tau} &= \Omega \omega_2^0, \\
\Omega \frac{\partial \omega_0^1}{\partial \tau} &= -\Omega \omega_1^0, \\
\Omega \frac{\partial \omega_0^2}{\partial \tau} &= \alpha_1 \omega_3^0,
\end{align*}
\]

\( \epsilon^1 \) (first-order) equations:
\[
\begin{align*}
\Omega \frac{\partial \omega_1^0}{\partial \tau} + \omega \frac{\partial \omega_1^0}{\partial \tau} &= \Omega \omega_2^0 + \delta \omega_2^0 - \frac{3}{1610} \sqrt{230} \cos \Omega t \\
&\quad + \frac{4}{4125625} \left[ \frac{1}{23} \left( 9938 \sqrt{230} \omega_1^0 \right) \\
&\quad - 424409 \omega_2^2 - 148159 \omega_3^2 \right] \mu^0 \\
&\quad + \frac{72}{175} \sqrt{14123} \left[ -124 (\omega_2^0)^2 + (\omega_1^0)^2 + \sqrt{230} \omega_1^0 \omega_2^0 \\
&\quad + \sqrt{230} \omega_1^0 \omega_3^2 - 123 \omega_2^0 \omega_3^2 \right],
\end{align*}
\]

\[
\begin{align*}
\Omega \frac{\partial \omega_1^1}{\partial \tau} + \omega \frac{\partial \omega_1^1}{\partial \tau} &= -\Omega \omega_1^1 - \delta \omega_1^1 + \frac{1}{35} \cos \Omega t \\
&\quad + \frac{2}{20628125} \sqrt{410} (3527 \sqrt{230} \omega_1^0 \\
&\quad + 89548 \omega_2^2 + 108428 \omega_1^0) \mu^0 \\
&\quad + \frac{72}{875} \sqrt{410} [4 (\omega_2^0)^2 + 9 (\omega_1^0)^2 + 2 \sqrt{230} \omega_2^0 \omega_1^0 \\
&\quad + 2 \sqrt{230} \omega_2^0 \omega_3^2 + 13 \omega_2^0 \omega_3^2].
\end{align*}
\]

\[
\begin{align*}
\Omega \frac{\partial \omega_1^2}{\partial \tau} + \omega \frac{\partial \omega_1^2}{\partial \tau} &= \alpha_1 \omega_3^2 - \frac{1}{35} \cos \Omega t \\
&\quad - \frac{72}{20628125} \sqrt{410} (657 \sqrt{230} \omega_1^0 \\
&\quad - 6457 \omega_2^2 + 5248 \omega_3^2) \mu^0
\end{align*}
\]

\[
- \frac{72}{875} \sqrt{410} [4 (\omega_2^0)^2 + 9 (\omega_1^0)^2 + 2 \sqrt{230} \omega_1^0 \omega_2^0 \\
&\quad + 2 \sqrt{230} \omega_2^0 \omega_3^2 + 13 \omega_2^0 \omega_3^2],
\]

ctc., where the superscripts indicate evaluations at \( \epsilon = 0 \), and the primes denote derivatives with respect to \( \epsilon \).

Now, substituting the Fourier series (2) into the zeroth-order equations (5), and balancing the harmonics, produces the following algebraic equations.

For \( m = 0 \) (constant):
\[
\begin{align*}
0 & \quad 0 \quad \{ p_{10}^0 \} \\
-\Omega & \quad 0 \quad \{ p_{20}^0 \} = 0 \Rightarrow p_{10}^0 = 0, \quad i = 1, 2, 3. \quad (9)
\end{align*}
\]

For \( m = 1 \):
\[
\begin{align*}
\begin{bmatrix}
  r_{11}^0 = -p_{11}^0, \\
  p_{21}^0 = r_{11}^0,
\end{bmatrix}
\end{align*}
\]

and
\[
\begin{align*}
p_{31}^0 = r_{31}^0 = 0.
\end{align*}
\]

For \( m > 1 \):
\[
\begin{align*}
\begin{bmatrix}
  m\Omega & \Omega & 0 & 0 & 0 & 0 & \{ p_{1m}^0 \} \\
  m\Omega & m\Omega & 0 & 0 & 0 & 0 & \{ p_{2m}^0 \} \\
  0 & 0 & m\Omega & -\Omega & 0 & 0 & \{ r_{1m}^0 \} \\
  0 & 0 & -\Omega & m\Omega & 0 & 0 & \{ p_{2m}^0 \} \\
  0 & 0 & 0 & 0 & m\Omega & \alpha_1 & \{ p_{3m}^0 \} \\
  0 & 0 & 0 & 0 & -\alpha_1 & m\Omega & \{ r_{3m}^0 \}
\end{bmatrix}
&= p_{1m}^0 = r_{1m}^0 = 0, \quad i = 1, 2, 3. \quad (12)
\end{align*}
\]

Unlike in the non-resonant case, here one cannot assume that \( r_{1,0i}(\epsilon) = 0 \) because now phase is involved in the system owing to resonance. Therefore, \( w_i^0 \) can be written as
\[
\begin{align*}
w_i^0 &= p_{1i}^0 \cos \Omega t + r_{1i}^0 \cos \Omega t = a_{1i}^0 \cos (\Omega t + \phi), \\
w_i^0 &= r_{1i}^0 \cos \Omega t - p_{1i}^0 \sin \Omega t = -a_{1i}^0 \sin (\Omega t + \phi), \\
w_i^0 &= 0
\end{align*}
\]

where
\[
a_{1i}^0 = [(p_{1i}^0)^2 + (r_{1i}^0)^2]^{1/2} \quad \text{and} \quad \phi = -\tan^{-1}\left(\frac{p_{1i}^0}{r_{1i}^0}\right).
\]

(14)
The above procedure that deals with the zeroth-order equations can be employed to consider the first-order equations (7)–(9) as well, and the details of analysis are omitted for brevity. The non-zero first-order coefficients \( p_{01}, p_{21}, p_{11}, r_{11}, r_{12}, r_{21} \) are given in the Appendix. \( \mu^0 \) and \( \omega' \) can be determined from (7) and (8) when \( m = 1 \), which yields the following relations.

\[
\cos \tau \text{ of (7)}:
\]
\[
\Omega \omega' r_{11}^0 - \Omega' r_{11} - \Omega p_{21}^0
= -\frac{4}{4125625} \sqrt{\frac{41}{23}} \left[ 9938 \sqrt{230} p_{11}^0 - 42449 r_{11}^0 \right] \mu^0
+ \delta r_{11}^0 - \frac{3}{1610} \sqrt{230} E.
\]
\[
(15)
\]

\[
\sin \tau \text{ of (7)}:
\]
\[
-\Omega \omega' r_{11}^0 - \Omega p_{11}^0 - \Omega r_{21}^0
= -\frac{4}{4125625} \sqrt{\frac{41}{23}} \left[ 9938 \sqrt{230} r_{11}^0 + 42449 p_{11}^0 \right] \mu^0
- \delta p_{11}^0.
\]
\[
(16)
\]

\[
\cos \tau \text{ of (8)}:
\]
\[
-\Omega \omega' r_{11}^0 + \Omega r_{21}^0 + \Omega p_{11}^0
= \frac{2}{20628125} \sqrt{\frac{410}{230}} \left[ 3527 \sqrt{230} p_{11}^0 + 89548 r_{11}^0 \right] \mu^0
- \delta p_{11}^0 + \frac{1}{35} E.
\]
\[
(17)
\]

\[
\sin \tau \text{ of (8)}:
\]
\[
-\Omega \omega' r_{11}^0 - \Omega p_{11}^0 - \Omega r_{11}^0
= \frac{2}{20628125} \sqrt{\frac{410}{230}} \left[ 3527 \sqrt{230} r_{11}^0 - 89548 p_{11}^0 \right] \mu^0
- \delta r_{11}^0.
\]
\[
(18)
\]

A simple manipulation of the above equations, in the form of
\[
p_{11}^0 \times [(15) - (18)] + r_{11}^0 \times [(16) + (17)],
\]
results in

\[
\mu^0 = \frac{503125}{4608} \sqrt{\frac{410}{230}} \left[ \frac{1}{(p_{11}^0)^2 + (r_{11}^0)^2} \right] \times \left[ \frac{3}{3220} \sqrt{230} p_{11}^0 - \frac{1}{70} r_{11}^0 \right] E,
\]
\[
(19)
\]

and
\[
r_{11}^0 \times [(15) - (18)] - r_{11}^0 \times [(16) + (17)],
\]
yields

\[
\begin{align*}
\omega' &= \delta - \frac{4932}{20125} \sqrt{\frac{41}{23}} \mu^0 - \left[ \frac{1}{(p_{11}^0)^2 + (r_{11}^0)^2} \right] \\
&\quad \times \left[ \frac{3}{3220} \sqrt{230} p_{11}^0 + \frac{1}{70} r_{11}^0 \right] E.
\end{align*}
\]
\[
(20)
\]

By using (14), (19) and (20) can be rewritten as

\[
\begin{align*}
\mu^0 &= \frac{E(a_3 \cos \phi + a_4 \sin \phi)}{a_1 a_{11}^0} , \\
\omega' &= \delta - a_2 \mu^0 + \frac{E(a_3 \sin \phi - a_4 \cos \phi)}{a_{11}^0} ,
\end{align*}
\]
\[
(21)
\]

where

\[
\begin{align*}
a_1 &= \frac{4608}{503125} \sqrt{\frac{410}{23}}, \\
a_2 &= \frac{4932}{20125} \sqrt{\frac{41}{23}}, \\
a_3 &= \frac{3}{3220} \sqrt{230}, \\
a_4 &= \frac{1}{70}.
\end{align*}
\]
\[
(22)
\]

Then the \( w'_i \) are obtained as

\[
\begin{align*}
w'_i &= p_{10}^0 + p_{11}^0 \cos \Omega t + r_{11}^0 \sin \Omega t \\
+ p_{12}^0 \cos 2\omega t + r_{12}^0 \sin 2\omega t,
\end{align*}
\]
\[
(23)
\]

where frequency \( \omega \) is given by

\[
\omega = \omega_0 + \omega \epsilon
= \Omega + \epsilon \left[ \delta - a_2 \mu^0 + \frac{E}{a_{11}^0} (a_3 \sin \phi - a_4 \cos \phi) \right],
\]
\[
(24)
\]

and the first-order asymptotic solutions are in the form of

\[
w_i = w_0^0 + w'_i \epsilon.
\]
\[
(25)
\]

2.2. 1:2 superharmonic resonance

The procedure applied in the case of 1:1 resonance can be used again. Here the system is scaled by

\[
\omega_0 - 2\Omega \rightarrow \epsilon \omega, \quad w_i \rightarrow \epsilon w_i, \quad \mu \rightarrow \epsilon \mu, \quad E \rightarrow \epsilon E.
\]
\[
(26)
\]

Note that the scaling factor for \( E \) is \( \epsilon \) rather than \( \epsilon^2 \) which was used in the case of 1:1 resonance. Equations (2)–(4) can be used again here provided \( \tau = \omega(\epsilon)t \) is modified as

\[
\tau = \frac{1}{2} \omega(\epsilon)t, \quad \text{so that} \quad \omega_0 = \omega(0) = 2\Omega.
\]
\[
(27)
\]

Then the zeroth-order, first-order, etc., perturbation equations can be obtained from (3) by differentiation with respect to \( \epsilon \) successively as follows.
\[ e^0 \text{ (zeroth-order) equations:} \]
\[
\begin{align*}
\Omega \frac{\partial w^0_1}{\partial t} &= 2\Omega w^0_2 - \frac{3}{1610} \sqrt{230} E \cos \Omega t, \\
\Omega \frac{\partial w^0_2}{\partial t} &= -2\Omega w^0_1 + \frac{1}{35} E \cos \Omega t, \\
\Omega \frac{\partial w^0_3}{\partial t} &= \alpha_1 w^0_3 - \frac{1}{35} E \cos \Omega t.
\end{align*}
\]
\[ (28) \]

\[ e^1 \text{ (first-order) equations:} \]
\[
\begin{align*}
\Omega \frac{\partial w^1_1}{\partial t} + \frac{1}{2} \omega^* \frac{\partial w^0_1}{\partial t} &= 2\Omega w^2_1 + \delta w^0_2 + \frac{4}{4125 625} \sqrt{23} \\
&\times (9938\sqrt{230} w^0_2 - 424 409 w^0_3 - 148 159 w^0_4) \mu^0 \\
&+ \frac{72}{175} \sqrt{23} w^0_2 + \sqrt{230} w^0_1 w^0_2 \\
&+ \frac{64}{7175} \sqrt{23} E \cos \Omega t(-\sqrt{230} w^0_1 + 16 w^0_2 - 4 w^0_3), \\
&= -2\Omega w^1_1 - \delta w^0_1 + \frac{2}{20628 125} \sqrt{410} \\
&\times (3527\sqrt{230} w^0_1 + 98 428 w^0_2 + 108 428 w^0_3) \mu^0 \\
&+ \frac{72}{875} \sqrt{410} w^0_1^2 [4(w^0_2)^2 + 9(w^0_3)^2 + 2\sqrt{230} w^0_1 w^0_2 \\
&+ 2\sqrt{230} w^0_1 w^0_3 + 13 w^0_2 w^0_3], \\
&\frac{2}{35 875} \sqrt{23} E \cos \Omega t(-\sqrt{230} w^0_1 + 4 w^0_2 - w^0_3), \\
&= \alpha_1 w^1_3 - \frac{72}{20 628 125} \sqrt{410} \\
&\times (657\sqrt{230} w^0_1 - 6457 w^0_2 + 5248 w^0_3) \mu^0 \\
&- \frac{72}{875} \sqrt{410} [4(w^0_2)^2 + 9(w^0_3)^2 + 2\sqrt{230} w^0_1 w^0_2 \\
&+ 2\sqrt{230} w^0_1 w^0_3 + 13 w^0_2 w^0_3], \\
&\frac{72}{35 875} \sqrt{23} E \cos \Omega t(-\sqrt{230} w^0_1 + 4 w^0_2 - w^0_3), \\
&\ldots
\end{align*}
\]
\[ (29) \]

Now, substituting the Fourier series (2) into the zeroth-order equations (28) and balancing the harmonics results in the non-zero zeroth-order coefficients \( p^0_{11} \) and \( r^0_{11} \), \( i = 1, 2, 3 \), which are given in the Appendix, and
\[
r^0_{12} = -r^0_{12}, \quad p^0_{12} = r^0_{12}.
\]

\[ (32) \]

The \( p^0_{12} \) and \( r^0_{12} \) play the same role, such as \( p^0_{11} \) and \( r^0_{11} \) in the 1:1 resonant case, as the amplitude of periodic motions.

\[
\begin{align*}
w^0_1 &= p^0_{11} \cos \omega t + r^0_{11} \sin \omega t + a^0_{12} \cos (2\omega t + \phi), \\
&= p^0_{21} \cos \omega t + r^0_{21} \sin \omega t - a^0_{12} \sin (2\omega t + \phi), \\
w^0_2 &= p^0_{31} \cos \omega t + r^0_{31} \sin \omega t,
\end{align*}
\]
\[ (33) \]

where
\[
a^0_{12} = [(p^0_{12})^2 + (r^0_{12})^2]^{1/2} \quad \text{and} \quad \phi = -\tan^{-1} \left( \frac{r^0_{12}}{p^0_{12}} \right).
\]
\[ (34) \]

Next, the above procedure of harmonic balancing can be applied to the first-order equations (29)–(31) to obtain the non-zero coefficients \( p^1_{0}, p^1_{11}, r^1_{11}, p^1_{12}, r^1_{12}, p^1_{13}, p^1_{14} \) and \( r^1_{14} \), which are given in the Appendix. Furthermore, a similar manipulation that has led to (21) and (22) results in
\[
\begin{align*}
\mu^0 &= \frac{E^2(b_3 \cos \phi + b_4 \sin \phi)}{b^0_{12}}, \\
\omega^* &= \delta - b_2 \mu^0 + \frac{E^2(b_3 \sin \phi - b_4 \cos \phi)}{a^0_{12}}.
\end{align*}
\]
\[ (35) \]

where
\[
\begin{align*}
b_1 &= \frac{4608}{503 125} \sqrt{410}, \\
b_2 &= \frac{4932}{20 125} \sqrt{23}, \\
b_3 &= \frac{139487 381 250(\Omega^2 + 100)^2}{\sqrt{410} \left[ 139487 381 250(\Omega^2 + 100)^2 \right]^{1/2}} \\
&\times (6093850^5 - 3867876\sqrt{230} \Omega^4 - 115 267 720 \Omega^2 \\
&- 264 451 800 \sqrt{230} \Omega^2 - 59 784 038 000 \Omega \\
&+ 13 252 020 000 \sqrt{230}), \\
b_4 &= \frac{\sqrt{410}}{139487 381 250(\Omega^2 + 100)^2} \\
&\times (243 110 \sqrt{230} \Omega^4 + 128 908 008 \Omega^4 \\
&- 2 799 560 \sqrt{230} \Omega^3 + 25 518 325 200 \Omega^2 \\
&- 2 812 900 000 \sqrt{230} \Omega + 841 118 280 000),
\end{align*}
\]
\[ (36) \]
and therefore
\[ w'_1 = p'_{10} + \sum_{j=1}^{4} p'_{ij} \cos(j\omega t) + r'_{ij} \sin(j\omega t), \quad (37) \]
where
\[ \omega = \omega_e + \omega' \varepsilon \]
\[ = 2\Omega + \varepsilon \left( \delta - b_2 \mu^0 + \frac{E}{a_{12}} (b_3 \sin \phi - b_4 \cos \phi) \right) E, \quad (38) \]
and the first-order asymptotic solutions are given by (25).

2.3. 2 : 1 subharmonic resonance

Following the procedure of the previous subsections, we can obtain the results for this case. First, a scaling given by
\[ \omega_e - \frac{1}{2} \Omega = \varepsilon \delta, \quad w_1 \rightarrow \varepsilon w_1, \quad \mu \rightarrow \varepsilon \mu, \quad E \rightarrow \varepsilon E, \quad (39) \]
is used, with
\[ \tau = \omega_e t, \quad \omega_e = \omega_e(0) = \frac{1}{2} \Omega. \quad (40) \]
The zeroth-order and first-order, etc. perturbation equations are given as follows.

\[ \varepsilon^0 (zeroth-order) \text{ equations:} \]
\[ \frac{1}{2} \Omega \frac{\partial w^0_1}{\partial \tau} + \omega \frac{\partial w^0_2}{\partial \tau} = \frac{1}{2} \Omega w^0_2 - \frac{3}{1610} \sqrt{230} E \cos \Omega t, \]
\[ \frac{1}{2} \Omega \frac{\partial w^0_2}{\partial \tau} = -\frac{1}{2} \Omega w^0_1 + \frac{E}{35} \cos \Omega t, \]
\[ \frac{1}{2} \Omega \frac{\partial w^0_3}{\partial \tau} = a_1 w^0_3 - \frac{1}{35} E \cos \Omega t. \quad (41) \]

\[ \varepsilon^1 (first-order) \text{ equations:} \]
\[ \frac{1}{2} \Omega \frac{\partial w^1_1}{\partial \tau} + \omega \frac{\partial w^1_1}{\partial \tau} = \frac{1}{2} \Omega w^1_2 + \frac{4}{4125625} \sqrt{23} \]
\[ \times (9938 \sqrt{230} w^0_2 - 424409 w^0_2 - 148159 w^0_3) \mu^0 \]
\[ + \frac{72}{175} \sqrt{\frac{41}{23}} \left( -124(w^0_2)^2 + (w^0_3)^2 + \sqrt{230} w^0_2 w^0_3 \right) \]
\[ + \sqrt{230} w^0_2 w^0_3 - 123 w^0_2 w^0_3 \}
\[ - \frac{64}{7175} \sqrt{\frac{41}{23}} E \cos \Omega t \left( -\sqrt{230} w^0_1 + 16 w^0_2 - 4 w^0_3 \right), \quad (42) \]

\[ \frac{1}{2} \Omega \frac{\partial w^0_4}{\partial \tau} + \omega \frac{\partial w^0_2}{\partial \tau} = -\frac{1}{2} \Omega w^0_4 + \frac{2}{20628125} \sqrt{410} \]
\[ \times (3527 \sqrt{230} w^0_1 + 89548 w^0_2 + 108428 w^0_3) \mu^0 \]
\[ + \frac{72}{875} \sqrt{410} \left[ (w^0_2)^2 + 9(w^0_3)^2 + 2 \sqrt{230} w^0_2 w^0_3 \right] \]
\[ + 2 \sqrt{230} w^0_2 w^0_3 + 13 w^0_2 w^0_3 \}
\[ - \frac{2}{35} \frac{\sqrt{410}}{E \cos \Omega t (\sqrt{230} w^0_1 + 4 w^0_2 - w^0_3),} \quad (43) \]

\[ \frac{1}{2} \Omega \frac{\partial w^1_4}{\partial \tau} + \omega \frac{\partial w^0_3}{\partial \tau} = a_1 w^0_3 - \frac{72}{20628125} \sqrt{410} \]
\[ \times (657 \sqrt{230} w^0_1 - 6457 w^0_2 + 5248 w^0_3) \mu^0 \]
\[ - \frac{72}{875} \sqrt{410} \left[ (w^0_2)^2 + 9(w^0_3)^2 + 2 \sqrt{230} w^0_2 w^0_3 \right] \]
\[ + 2 \sqrt{230} w^0_2 w^0_3 + 13 w^0_2 w^0_3 \}
\[ + \frac{72}{35} \frac{\sqrt{410}}{E \cos \Omega t (\sqrt{230} w^0_1 + 4 w^0_2 - w^0_3)}, \quad (44) \]

cetc.

By balancing the harmonics of the zeroth-order perturbation equations with the aid of Fourier series (2), one obtains the non-zero coefficients \( p^0_{12} \) and \( r^0_{12} \), \( i = 1, 2, 3 \), which are given in the Appendix, and
\[ r^0_{11} = -p^0_{11}, \quad p^0_{21} = r^0_{11}. \quad (45) \]

Therefore
\[ w^0_1 = p^0_{12} \cos 2\omega t + r^0_{12} \sin 2\omega t + a^0_{11} \cos (\omega t + \phi), \]
\[ w^0_2 = p^0_{22} \cos 2\omega t + r^0_{22} \sin 2\omega t - a^0_{11} \sin (\omega t + \phi), \]
\[ w^0_3 = p^0_{32} \cos 2\omega t + r^0_{32} \sin 2\omega t, \]
\[ (46) \]

where \( a^0_{11} \) and \( \phi \) are given by (14). The first-order perturbation equations produce the non-zero coefficients \( p^0_{10}, p^0_{11}, r^0_{11}, p^0_{12}, r^0_{12}, p^0_{22}, r^0_{33}, p^0_{14}, r^0_{14} \) and \( r^0_{14} \), which are listed in the Appendix. \( \mu^0 \) and \( \omega' \) are given by
\[ \mu^0 = \frac{E(c_3 \cos \phi + c_4 \sin \phi)}{c_1 a^0_{11}} \]
\[ \omega' = \delta - 2\mu^0 + \frac{E(c_3 \sin \phi - c_4 \cos \phi)}{a^0_{11}}, \quad (47) \]
where
\begin{align*}
c_1 &= \frac{4608}{503\,125} \sqrt{410}, \\
&\quad \frac{4932}{20\,125} \sqrt{23}, \\
c_3 &= -\frac{\sqrt{410}}{28\,879\,375(\Omega^2 + 100)\Omega} \times (70\,840\,\Omega^3 + 97\,539\sqrt{230} \Omega^2 + 8\,441\,920\Omega \\
&\quad + 10\,897\,800\sqrt{230}), \\
c_4 &= -\frac{\sqrt{410}}{28\,879\,375(\Omega^2 + 100)\Omega} \times \left(\frac{8155}{2}\sqrt{230}\Omega^3 - 1\,265\,916\Omega^2 \\
&\quad + 293\,360\sqrt{230}\Omega - 113\,012\,400\right),
\end{align*}

and
\begin{equation}
w_i = p_{i0} + \sum_{j=1}^{4} p_{ij} \cos \, (j\omega t) + r_{ij} \sin \, (j\omega t),
\end{equation}

where
\begin{equation}
\omega = \omega + \omega',
\end{equation}

\begin{equation}
= \frac{1}{2} \Omega + \varepsilon \left[\delta - c_2\mu^0 + \frac{E}{\alpha^0} (c_3 \sin \phi - c_4 \cos \phi)\right],
\end{equation}

and the first-order asymptotic solutions are given by (25).

3. Verification of solutions
To verify a given order of approximate solution, one direct way is to substitute this solution back into the original differential equations ((8)–(10) of Yu et al. 1996). If the substitution of the Nth-order solution, $w_i = \cdots + O(\varepsilon^{N+1})$, yields a result of the order $O(\varepsilon^N)$, then the Nth-order solution contains all possible contributions to this order, and it is a consistent approximation. Otherwise the approximation is inconsistent.

Consider, for example, the 1:1 resonance; the first-order asymptotic solutions given by (23), with the aid of (24) and the approximation $\mu = \mu^0$, has been verified by the symbolic computation program developed with the aid of MAPLE. It has been shown that substituting these solutions into original differential equations indeed results in a second-order residual order $O(\varepsilon^2)$. The solutions of the other two resonant cases (i.e. 1:2 and 2:1 resonances) have also been verified. In fact, all the results and formulae given in the preceding section are obtained by using a symbolic computation program written in the MAPLE language.

4. Stability analysis
The approach of the stability analysis used in the first part of this paper for the non-resonant case will be applied here for the resonance studies. Rate equations can be obtained by simple algebraic manipulations. This approach has been used to study an internal resonance case of autonomous systems (Yu and Huseyn, 1993). The rate equations can be also derived by using the MTHB method following the procedure given in the Appendix of the companion paper.

4.1. 1:1 primary resonance
Manipulating (15)–(18) by constructing
\begin{equation}
\frac{\varepsilon}{\alpha^0} \{ p_{11} \times [(15) - (18)] + r_{11} \times [(16) + (17)]\},
\end{equation}

leads to
\begin{equation}
\frac{d\rho}{dt} = \varepsilon [a_1\mu \rho - E(a_3 \cos \phi + a_4 \sin \phi)],
\end{equation}

and
\begin{equation}
\frac{d\phi}{dt} = \varepsilon [\rho (\delta - a_2\mu) + E(a_3 \sin \phi - a_4 \cos \phi)],
\end{equation}

yields
\begin{equation}
\rho \frac{d\phi}{dt} = \varepsilon (\rho (\delta - a_2\mu) + E(a_3 \sin \phi - a_4 \cos \phi)),
\end{equation}

where the constant variables $p_{11}$ and $r_{11}$ have been replaced by time variables $\rho(t)$ and $\phi(t)$ in polar coordinates as $\rho \cos \phi$ and $\rho \sin \phi$; $\phi$ is the phase of motions.

Now, by using the first approximation $\mu = \mu^0$ and back scaling (see the forward scaling (1))
\begin{equation}
e_{w_t} \rightarrow w_t \quad \text{(and so $\varepsilon \rho \rightarrow \rho$)} \quad \varepsilon \mu \rightarrow \mu \quad \text{and} \quad \varepsilon^2 E \rightarrow E,
\end{equation}

one transfers (51) and (52) to unscaled equations:
\begin{equation}
\frac{d\rho}{dt} = a_1\mu \rho - E(a_3 \cos \phi + a_4 \sin \phi),
\end{equation}

\begin{equation}
\frac{d\phi}{dt} = \delta - a_2\mu + \frac{E}{\rho} (a_3 \sin \phi - a_4 \cos \phi).
\end{equation}

Periodic solutions can be easily derived from (54) by setting $\frac{d\rho}{dt} = \frac{d\phi}{dt} = 0$ as follows:
\begin{equation}
\rho^2 = \frac{E^2 (a_3^2 + a_4^2)}{(a_1 \mu)^2 + (\delta - a_2 \mu)^2},
\end{equation}

(55)
which represents the amplitude of a family of periodic motions and
\[
\begin{align*}
\cos \phi &= \frac{\rho}{E(a_3^2 + a_4^2)} \left[ a_3 a_1 \mu + a_4 (\delta - a_2 \mu) \right], \\
\sin \phi &= \frac{\rho}{E(a_3^2 + a_4^2)} \left[ a_4 a_1 \mu - a_3 (\delta - a_2 \mu) \right],
\end{align*}
\]
(56)
which determines the corresponding phases as
\[
\phi = \tan^{-1} \left[ \frac{a_4 a_1 \mu - a_3 (\delta - a_2 \mu)}{a_3 a_1 \mu + a_4 (\delta - a_2 \mu)} \right].
\]
(57)

The stability conditions for the above periodic solutions can be determined from the Jacobian of (54), given by
\[
J = \begin{bmatrix}
a_1 \mu & E(a_3 \sin \phi - a_4 \cos \phi) \\
-\frac{E}{\rho^2} (a_3 \sin \phi - a_4 \cos \phi) & \frac{E}{\rho} (a_3 \cos \phi + a_4 \sin \phi)
\end{bmatrix},
\]
(58)

which, in turn, yields the characteristic polynomial:
\[
\lambda^2 + tr \lambda + \Delta = 0,
\]
(59)

where
\[
tr = 2a_1 \mu \quad \text{and} \quad \Delta = (a_1 \mu)^2 + (\delta - a_2 \mu)^2.
\]
(60)

If the trace \( tr > 0 \) and the determinant \( \Delta > 0 \), the periodic solution is stable. It is noted that coefficients \( a_3 \) and \( a_4 \) do not affect the stability of the periodic solutions which indeed contain these two coefficients. As \( \Delta > 0 \) is always satisfied, the only critical value is \( \mu = 0 \), i.e.

\[
\begin{cases}
\text{stable,} & \text{if } \mu < 0, \\
\text{unstable,} & \text{if } \mu > 0.
\end{cases}
\]

The above analysis is only up to first-order perturbations. If the second-order perturbation is included, then the rate equation (54) becomes
\[
\frac{d\rho}{dt} = a_1 \mu^0 \rho - E(a_3 \cos \phi + a_4 \sin \phi) \\
+ a_5 \rho^3 + \frac{E^2}{\rho} (a_6 + a_7 \cos 2\phi + a_8 \sin 2\phi),
\]
(61)
and
\[
\frac{d\phi}{dt} = (\delta - a_2 \mu^0) + \frac{E}{\rho} (a_3 \sin \phi - a_4 \cos \phi) \\
+ a_9 \rho^2 + \frac{E^2}{\rho^2} (a_{10} + a_{11} \cos 2\phi + a_{12} \sin 2\phi),
\]
(62)

where the coefficients \( a_i, i = 5, 6, \ldots, 10 \) are given in the Appendix. Now, finding the periodic solutions from (61) and (62) becomes very involved. However, it can be shown that these solutions yield first-order approximations given by (55) and (57).

### 4.2. \( 1:2 \) superharmonic resonance

Using a similar algebraic manipulation and back scaling
\[
e w_i \to w_i \quad \text{(and so } \varepsilon \rho \to \rho) \quad \varepsilon \mu \to \mu \quad \text{and } \varepsilon E \to E,
\]
(63)

one obtains the unscaled rate equations:
\[
\begin{align*}
\frac{d\rho}{dt} &= b_1 \mu - E^2 (b_3 \cos \phi + b_4 \sin \phi), \\
\frac{d\phi}{dt} &= \frac{\rho}{E^2} (b_3 \sin \phi - b_4 \cos \phi),
\end{align*}
\]
(64)

where \( b_i \) are given by (36). The periodic solutions are then found as
\[
\rho^2 = \frac{E^4 (b_3^2 + b_4^2)}{(b_1 \mu)^2 + (\delta - b_2 \mu)^2},
\]
(65)
\[
\cos \phi = \frac{\rho}{E^2 (b_3^2 + b_4^2)} [b_3 b_1 \mu + b_4 (\delta - b_2 \mu)],
\]
\[
\sin \phi = \frac{\rho}{E^2 (b_3^2 + b_4^2)} [b_4 b_1 \mu - b_3 (\delta - b_2 \mu)],
\]

and the stability conditions are determined from the characteristic polynomial \( \lambda^2 + tr \lambda + \Delta = 0 \) which is identical to (59) but \( tr \) and \( \Delta \) are now given by
\[
tr = 2b_1 \mu \quad \text{and} \quad \Delta = (b_1 \mu)^2 + (\delta - b_2 \mu)^2,
\]
(66)

and therefore

\[
\begin{cases}
\text{stable,} & \text{if } \mu < 0, \\
\text{unstable,} & \text{if } \mu > 0.
\end{cases}
\]

It is noted that the stability conditions for the \( 1:2 \) resonance are identical to those of \( 1:1 \) resonance. This is not surprising because the rate equations (64) will be identical to (54) if \( b_1 \) and \( E^2 \) are replaced by \( a_1 \) and \( E \), respectively. Furthermore, note that \( a_3 = b_1 \) and \( a_4 = b_2 \), and stability conditions are determined from these two parameters only (\( a_3, a_4 \) and \( b_3, b_4 \) do not influence the stability analysis).

### 4.3. \( 2:1 \) subharmonic resonance

Using the same approach as in the cases of \( 1:1 \) and \( 1:2 \) resonances, one may obtain the unscaled rate equations for this case, as
\[
\begin{align*}
\frac{d\rho}{dt} &= \rho [c_1 \mu - E (c_3 \cos 2\phi + c_4 \sin 2\phi)], \\
\frac{d\phi}{dt} &= \delta - c_2 \mu + E (c_3 \sin 2\phi - c_4 \cos 2\phi)
\end{align*}
\]
(67)
where $c_i$ are given by (48). By setting $d\rho/dt = d\phi/dt = 0$, two families of periodic solutions are obtained.

(a) 
\[
\begin{align*}
\rho &= 0, \\
c_3 \sin 2\phi &= c_4 \cos 2\phi = -\frac{\delta - c_2 \mu}{E}
\end{align*}
\]  
(68)

(b) 
\[
\begin{align*}
\rho &\neq 0, \\
\cos 2\phi &= \frac{1}{E(c_3^2 + c_4^2)} \left[c_3 c_1 \mu + c_4 (\delta - c_2 \mu)\right], \\
\sin 2\phi &= \frac{1}{E(c_3^2 + c_4^2)} \left[c_4 c_1 \mu - c_3 (\delta - c_2 \mu)\right]
\end{align*}
\]  
(69)

It should be noted that the periodic solutions (a) and (b) have the same form (see the asymptotic solution formula given by (25), (46) and (49)). Also it is noted that solution (b) must satisfy $\cos^2 2\phi + \sin^2 2\phi = 1$, which in turn results in
\[
(c_1 \mu)^2 + (\delta - c_2 \mu)^2 = E^2(c_3^2 + c_4^2). 
\]  
(70)

The second equation of solution (a), which determines the phase of the periodic solutions, can be rewritten as
\[
E(c_3^2 + c_4^2)^{1/2} \sin (2\phi - \beta) = -(\delta - c_2 \mu), 
\]  
(71)

where $\beta$ is determined from
\[
\frac{\cos \beta}{(c_3^2 + c_4^2)^{1/2}} = \frac{a_3}{(c_3^2 + c_4^2)^{1/2}} \\
\frac{\sin \beta}{(c_3^2 + c_4^2)^{1/2}} = \frac{a_4}{(c_3^2 + c_4^2)^{1/2}}. 
\]  
(72)

The phase $\phi$ of the periodic solution is then given by
\[
\phi = \frac{1}{2} (k\pi + \beta + \gamma) \quad \left(-\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}\right), 
\]  
(73)

The asymptotic solution for solution (a) ($\rho = c_{1,4}^0$) is described by
\[
\begin{align*}
w_1 &= p_{10} + p_{12} \cos 2\omega t + r_{12}^0 \sin 2\omega t + p_{12} \cos 2\omega t + r_{12}^0 \sin 2\omega t \\
&+ p_{14} \cos 4\omega t + r_{14}^0 \sin 4\omega t, 
\end{align*}
\]  
(74)

where the coefficients are given in the Appendix.

The stability conditions of the above periodic solutions can be determined from the Jacobian:
\[
J = \begin{bmatrix}
(c_1 \mu - E(c_3 \cos 2\phi + c_4 \sin 2\phi)) \\
0 \\
2E(c_3 \sin 2\phi - c_4 \cos 2\phi) \\
2E(c_3 \cos 2\phi + c_4 \sin 2\phi)
\end{bmatrix}.
\]  
(75)

as
\[
c_1 \mu - E(c_3 \cos 2\phi + c_4 \sin 2\phi) < 0,
\]  
and
\[
E(c_3 \cos 2\phi + c_4 \sin 2\phi) < 0, \quad E > 0,
\]  
(76)

which are applicable to both solutions (a) and (b). It can be seen from the above equation that $\mu$ must be $< 0$ (because $c_i > 0$). It is a necessary condition but not sufficient.

Now consider solution (a); because
\[
E(c_3 \cos 2\phi + c_4 \sin 2\phi) = E(c_3^2 + c_4^2)^{1/2} \cos (2\phi - \beta),
\]  
(77)

where
\[
2\phi - \beta = k\pi + \gamma, \quad -\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}, \quad \text{for } k = 0, 1, 2, \ldots,
\]  
(78)

the following relations
\[
E(c_3 \cos 2\phi + c_4 \sin 2\phi) = \begin{cases} 
> 0, & k = 0, 2, 4, \ldots, \\
< 0, & k = 1, 3, 5, \ldots,
\end{cases}
\]  
(79)

are obtained and therefore, the solution
\[
\phi = \frac{1}{2} (k\pi + \beta + \gamma), \quad \text{for } k = 0, 2, 4, \ldots,
\]  

is unstable.

So that the solution
\[
\phi = \frac{1}{2} (k\pi + \beta + \gamma), \quad \text{for } k = 1, 3, 5, \ldots,
\]  

to be stable, an additional condition
\[
c_1 \mu - E(c_3 \cos 2\phi + c_4 \sin 2\phi) < 0,
\]  
(80)

should be satisfied, which can be written as
\[
(c_1 \mu)^2 + (\delta - c_2 \mu)^2 > E^2(c_3^2 + c_4^2).
\]  
(81)

The above relation gives the critical curve
\[
(c_1 \mu)^2 + (\delta - c_2 \mu)^2 = E^2(c_3^2 + c_4^2),
\]  
(82)

from which the periodic solution (b) bifurcates from the periodic solution (a). Actually, this condition must be satisfied to obtain the solution (b).

Next, consider the solution (b): the second stability condition is
\[
E(c_3 \cos 2\phi + c_4 \sin 2\phi) = c_1 \mu < 0,
\]  
(83)

and thus the first stability condition becomes zero. Therefore, the stability of the solution (b) is not
determined. To determine the amplitude $\rho$ of the solution $(b)$ and its stability needs certain cubic terms in the original differential equations.

5. Conclusions
A systematic perturbation procedure, based on the intrinsic harmonic balancing technique, has been developed to study non-autonomous systems. Ordered asymptotic solutions, together with the construction of rate equations for stability analysis, are derived in each step of perturbations. It has been found that the resonance cases have some significant differences from the non-resonance case, which is considered in Part 1.

Amplitudes of motions are coupled with phases in the cases of resonance, while they do not involve phases in the non-resonant case.

The system considered in this paper exhibits (phase-locked) periodic solutions only in resonant cases. However, periodic as well as quasi-periodic solutions exist in the non-resonant case.

Bifurcation sequences are different in that, for the non-resonant case, two distinct families of periodic solutions may bifurcate from the initial equilibrium solution, and then these periodic motions may lose stability and further bifurcate into the same family of quasi-periodic motions on a torus. However, for the $1:1$ and $1:2$ resonant cases, no bifurcations are exhibited and the only possible dynamic solution is a family of periodic motions. The $2:1$ resonance, on the other hand, has two families of periodic solutions, one of which bifurcates from the other.

The stability conditions of resonant cases depend on both amplitudes and phases of motions, while they rely on amplitudes only in the non-resonant case.

Appendix
The coefficients given in the first-order approximation (23) are given as follows:

\[
\begin{align*}
  \rho'_{21} + r'_{11} &= \frac{1}{1520870040 \Omega} \\
  &\times (a \cos 2\phi - b \sin 2\phi + 52187 \sqrt{230})E, \\
  r'_{21} + p'_{11} &= -\frac{1}{1520870040 \Omega} \\
  &\times (a \sin 2\phi + b \cos 2\phi - 941876)E,
\end{align*}
\]

where $a = 82135 \sqrt{230}$ and $b = 1344864$. It is noted in the above equation that two of the four coefficients, say, $p'_{11}$ and $r'_{11}$, can be set to zero because two equations have four variables.

\[
\begin{align*}
  p'_{21} &= \frac{1}{16898560(\Omega^2 + 100)} \left[ a \cos 2\phi - b \sin 2\phi \\
  &\quad (10851 \sqrt{230} \Omega + 12331680) \right]E, \\
  r'_{21} &= -\frac{1}{16898560(\Omega^2 + 100)} \left[ a \sin 2\phi + b \cos 2\phi \\
  &\quad (12331680 - 10851 \sqrt{230}) \right]E, \\
  a &= 49593 \sqrt{230} \Omega - 1563080, \\
  b &= 1563080 + 495930 \sqrt{230}, \\
  p'_{10} &= \frac{144}{875 \Omega} (a'_{10})^2, \\
  p'_{20} &= \frac{4464}{175 \sqrt{230}} (a'_{11})^2, \\
  p'_{30} &= -\frac{72 \sqrt{410}}{4375} (a'_{11})^2, \\
  p'_{12} &= \frac{24 \sqrt{410}}{20125 \Omega} (a \cos 2\phi + b \sin 2\phi)(a'_{10})^2, \\
  r'_{12} &= -\frac{24 \sqrt{410}}{20125 \Omega} (a \sin 2\phi - b \cos 2\phi)(a'_{11})^2, \\
  a &= 161, b = 85 \sqrt{230}, \\
  p'_{22} &= \frac{12 \sqrt{410}}{20125 \Omega} (a \cos 2\phi - b \sin 2\phi)(a'_{11})^2, \\
  r'_{22} &= -\frac{12 \sqrt{410}}{20125 \Omega} (a \sin 2\phi + b \cos 2\phi)(a'_{11})^2, \\
  a &= 154 \sqrt{230}, b = 299, \\
  p'_{52} &= -\frac{36 \sqrt{410}}{875(\Omega^2 + 25)} (a \cos 2\phi - b \sin 2\phi)(a'_{11})^2, \\
  r'_{52} &= \frac{36 \sqrt{410}}{875(\Omega^2 + 25)} (a \sin 2\phi + b \cos 2\phi)(a'_{11})^2, \\
  a &= \sqrt{230} \Omega - 10, b = 2 \Omega + 5 \sqrt{230}, \\
  a_5 &= -\frac{53136}{17609375(\Omega^2 + 25) \Omega} (1656 \Omega^3 - 6025 \sqrt{230} \Omega^2 \\
  &\quad - 38525 \Omega - 123750 \sqrt{230}), \\
  a_6 &= \frac{\sqrt{230}}{734300563046400(\Omega^2 + 100) \Omega} \\
  &\times (15536548 \sqrt{230})^3 + 28827463450 \sqrt{230} \Omega + 617376819200),
\end{align*}
\]
\[ a_{7} = \frac{\sqrt{230}}{734300563460(\Omega^2 + 100)\Omega} \times (4447382517\Omega^2 - 646108270\sqrt{230}\Omega \\
- 6173768192000), \]
\[ a_{8} = \frac{1}{2} \frac{958}{1576969\Omega^2} \times 4751183090\sqrt{230}\Omega - 60458534400) \]
\[ a_{9} = \frac{53136}{3521875(\Omega^2 + 25)\Omega} (86\sqrt{230}\Omega^2 - 147806\Omega^2 \]
\[ + 3225\sqrt{230}\Omega - 3615225), \]
\[ a_{10} = \frac{1}{367150281523200(\Omega^2 + 100)\Omega} \times (338507597477\Omega^2 - 468813900685\sqrt{230}\Omega \]
\[ + 699176780200), \]
\[ a_{11} = \frac{1}{367150281523200(\Omega^2 + 100)\Omega} \times (1979502157\Omega^2 - 53105753545\sqrt{230}\Omega \]
\[ + 272205269200), \]
\[ a_{12} = \frac{1}{183575140761600(\Omega^2 + 100)\Omega} \times (111412580435\Omega^2 - 1562743013\sqrt{230}\Omega \]
\[ + 21353963800), \]
\[ p_{n}^{0} = \frac{2}{105\Omega} E, \quad r_{n}^{0} = \frac{\sqrt{230}}{1610\Omega} E, \]
\[ p_{21}^{0} = \frac{\sqrt{230}}{80\Omega} E, \quad r_{21}^{0} = -\frac{1}{105\Omega} E, \]
\[ p_{31}^{0} = -\frac{2}{7(\Omega^2 + 100)} E, \quad r_{31}^{0} = -\frac{\Omega}{35(\Omega^2 + 100)} E \]
\[ p_{10}^{0} = -\frac{\sqrt{410}}{3032334375(\Omega^2 + 100)\Omega^3} \times (35\sqrt{230}\Omega^3 - 741852\Omega^2 + 970280\sqrt{230}\Omega \]
\[ - 62041200)E^2 \]
\[ + \frac{72\sqrt{410}}{875\Omega} (a_{12}^{0})^2, \]
\[ p_{20}^{0} = \frac{4\sqrt{410}}{69743690625(\Omega^2 + 100)\Omega^3} \times (32200\Omega^3 + 878679\sqrt{230}\Omega^2 - 6531252\Omega \]
\[ + 39904275\sqrt{230})E^2 + \frac{2232}{175\Omega} \frac{\sqrt{410}}{23} (a_{12}^{0})^2, \]
\[ p_{50}^{0} = \frac{4\sqrt{410}}{5053890625(\Omega^2 + 100)\Omega^2} \times (105\sqrt{230}\Omega^3 + 79054\Omega^2 + 91065\sqrt{230}\Omega \]
\[ - 51701000E^2 - \frac{72\sqrt{410}}{4375} (a_{12}^{0})^2, \]
\[ p_{11}^{1} = -\frac{1}{105\Omega} \delta E \times (a \cos \phi - b \sin \phi), \]
\[ - \frac{\frac{4\sqrt{410}}{345884069538000000(\Omega^2 + 100)^2} \left( \frac{E}{a_{12}^{0}} \right)^2 E^2 \]
\[ \times \left[ \left( \frac{1586818345}{2} \right)^{\sqrt{230}\Omega^3} + 6789255881392\Omega^6 \]
\[ - 1389613360345\sqrt{230}\Omega^5 \]
\[ + 11447334454532000000 \]
\[ + 144428159770000\sqrt{230}\Omega^3 \]
\[ + 26320415546230000\Omega^2 \]
\[ + 228934968476350000\sqrt{230}\Omega \]
\[ + 2973533197497000000 \cos \phi \]
\[ - 10(28799520059\Omega^7 + 449134674340\sqrt{230}\Omega^6 \]
\[ - 5511213213170\Omega^5 \]
\[ + 16174318985120\sqrt{230}\Omega^4 \]
\[ - 18395357655298000\Omega^3 \]
\[ + 14101168666070000\sqrt{230}\Omega^2 \]
\[ - 76806220382140000 \Omega \]
\[ + 30419924420100000\sqrt{230}) \sin \phi \]
\[ r_{11}^{1} = \frac{-\sqrt{230}}{3220\Omega^2} \delta E \times \frac{2\sqrt{410}}{86638125(\Omega^2 + 100)\Omega^2} \times (a \sin \phi + b \cos \phi) \]
\[ - \frac{\sqrt{410}}{63642668794392000000(\Omega^2 + 100)^3} \Omega^4 \]
\[ p'_{10} = \frac{2 \sqrt{410}}{3 \times 0.102} \times 3.34 \times 3.379 (\Omega^2 + 100) \Omega^3 \times (35 \sqrt{230} \Omega^3 - 21.978 \Omega^2 + 970.280 \sqrt{230} \Omega^2 - 158.620 \Omega^4) E^2 + \frac{288 \sqrt{410}}{875 \Omega} (a_{11}^0)^2, \]

\[ p'_{20} = -\frac{4 \sqrt{410}}{697439625} \times (1288000 \Omega^3 - 3557151 \sqrt{230} \Omega^2 - 261250000 \Omega - 749463600 \sqrt{230} E^2 + \frac{8928}{175 \Omega^2} (a_{11}^0)^2, \]

\[ p'_{30} = -\frac{2 \sqrt{410} E^2}{5053890625 \Omega^2 + 100 \Omega^2} \times (210 \sqrt{230} \Omega^2 - 285413 \Omega^2 + 182130 \sqrt{230} \Omega + 264368000 \Omega - \frac{72 \sqrt{410}}{4375} (a_{11}^0)^2, \]

\[ p'_{21} + r'_{11} = -\frac{4 \sqrt{410}}{156863256000 \Omega^2 + 100 \Omega^2} \times (a \cos \phi + b \sin \phi + c \cos 3 \phi + d \sin 3 \phi) E a_{11}^0, \]

\[ r'_{21} + p'_{11} = -\frac{4 \sqrt{410}}{156863256000 \Omega^2 + 100 \Omega^2} \times (a \sin \phi - b \cos \phi - c \sin 3 \phi + d \cos 3 \phi) E a_{11}^0. \]

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References
