Twelve limit cycles around a singular point in a planar cubic-degree polynomial system

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ABSTRACT

In this paper, we prove the existence of 12 small-amplitude limit cycles around a singular point in a planar cubic-degree polynomial system. Based on two previously developed cubic systems in the literature, which have been proved to exhibit 11 small-amplitude limit cycles, we applied a different method to show 11 limit cycles. Moreover, we show that one of the systems can actually have 12 small-amplitude limit cycles around a singular point. This is the best result so far obtained in cubic planar vector fields around a singular point.

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1. Introduction

Studying bifurcation of limit cycles in planar polynomial systems is related to the second part of the well-known Hilbert’s 16th problem [1]. The progress in the solution of the problem is very slow. It has not even solved the simplest quadratic systems after more than one century since the problem was posed by Hilbert at the Paris conference of the International Congress of Mathematicians in 1900. More precisely, the second part of Hilbert’s 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that the following system,

\[ \dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \quad (1) \]

can have, where $P_n(x, y)$, and $Q_n(x, y)$, represent $n$th-degree polynomials of $x$, and $y$. In early 1990’s, Ilyashenko and Yakovenko [2], and Écalle [3] independently proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, the best result is $H(2) \geq 4$, obtained more than 30 years ago [4,5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$, is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [7,8]. Note that the 13 limit cycles obtained in [7,8] are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems. A comprehensive review on the study of Hilbert’s 16th problem can be found in a survey article [9].

In order to help understand and attack Hilbert’s 16th problem the so called weak Hilbert’s 16th problem was posed by Arnold [10], which is closely related to the so-called near-Hamiltonian system [11]:

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\[ \dot{x} = H_y(x, y) + ep_n(x, y), \quad \dot{y} = -H_x(x, y) + e q_n(x, y), \] where \( H(x, y), \ p_n(x, y), \) and \( q_n(x, y) \) are all polynomial functions of \( x, y, \) and 0 < \( \varepsilon \ll 1, \) represents a small perturbation. Investigating the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the (first-order) Melnikov function, given as an integral

\[ M(h, \delta) = \int_{H(x, y) = h} q_n(x, y) dx - p_n(x, y) dy, \] along closed orbits \( H(x, y) = h \) for \( h \in (h_1, h_2), \) where \( \delta \) denotes the parameters (or coefficients) involved in the polynomial functions \( q_n \) and \( p_n. \)

When we focus on the maximum number of small-amplitude limit cycles, \( M(n), \) bifurcating from an elementary center or an elementary focus in system (1), one of the best-known results is \( M(2) = 3, \) which was solved by Bautin in 1952 [12]. For \( n = 3, \) a number of results have been obtained. Around an elemental focus, James and Lloyd [13] considered a particular class of cubic systems to obtain 8 limit cycles in 1991, and the systems were reinvestigated couple of years later by Ning et al. [14] to find another solution of 8 limit cycles. Yu and Corless [15] constructed a cubic system and combined symbolic and numerical computations to show 9 limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [16]. Another cubic system was also recently constructed by Lloyd and Pearson [17] to show 9 limit cycles with purely symbolic computation.

On the other hand, around a center, there are also a few results obtained in the past two decades. Žołądek studied classification of cubic centers and listed 17 cases for reversible centers and 35 cases for Darboux centers [18,19]. In 1995, Žołądek [20] first proposed a rational Darboux integral,

\[ H_0 = \frac{f_1}{f_2} = \frac{(x^4 + 4x^2 + 4y^2)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^5}, \]

and used it to prove the existence of 11 small-amplitude limit cycles around a center. This result was extensively cited by many researchers in this area. After more than ten years, another two cubic systems are constructed to show 11 limit cycles [21,22]. Recently, the system defined by (4) was reinvestigated by Yu and Han with the method of focus value computation, who only obtained 9 limit cycles [23]. This obvious difference motivated a further investigation on this problem. Very recently, Tian and Yu [24] has proved that the 11 limit cycles obtained by Žołądek [20] are not correct, and the mistakes leading to the erroneous result have been identified.

In this paper, we will consider the two cubic systems proposed by Christopher [21], and Bondar and Sadovskii [22]. The first system discussed in [21] is determined by a Darboux first integral,

\[ H_1 = \frac{(xy^2 + x + 1)^5}{x^3(xy^2 + \frac{1}{2}xy^2 + \frac{25}{12}y^3 + \frac{5}{4}xy + \frac{15}{4} + a)^2}, \]

where \( a \) is a parameter, from which we obtain the following dynamical system,

\[ \dot{x} = (32a^2 - 75)^2 x(-6 - 9x - 3x^2 + 8xy - 12y^2), \]
\[ \dot{y} = (32a^2 - 75)^2 (24a - 16ax + 90y + 15xy - 16ax^2 + 60y^3), (32a^2 - 75 \neq 0), \]

System (6) has an equilibrium point, given by

\[ x_c = \frac{6(8a^2 + 25)}{32a^2 - 75}, \quad y_c = \frac{70a}{32a^2 - 75}. \]

Shifting the equilibrium point \((x_c, y_c)\) to the origin and setting \( a = 2 \) yields the system:

\[ \dot{x} = -10(342 + 53x)(289x - 2112y + 159x^2 - 848xy + 636y^2), \]
\[ \dot{y} = -605788x + 988380y - 432745xy + 755568y^2 - 89888xy^2 + 168540y^3, \]

which has been studied in [21] to show 11 small-amplitude limit cycles around the origin (i.e., around the equilibrium point \((x_c, y_c)\)) of system (6).

The second system given in [22] is described by

\[ \dot{x} = y(1 - 2r(3r^2 + 5)x + (r^2 + 3)(3r^2 + 1)^2 x^2) \equiv f_1 (x, y), \]
\[ \dot{y} = -x(1 - 8r|x|)[1 - 3r(r^2 + 3|x|) + 2(2/3r^2 - 1) - r(r^2 + 3)(3r^2 - 7)|x|xy - [r(r^2 + 11) - (r^2 + 3)(3r^4 + 22r^2 - 1)x]y^2 + 2r(r^2 + 3)(r^2 - 1)y^3 \equiv f_2 (x, y), \]

where \( r \) is a parameter. It can be shown that the origin of system (9) is a center [22].

To find the small-amplitude limit cycles bifurcating from the origin of the systems (8) and (9), in general we may apply perturbations to the systems and then compute the Melnikov functions around the loops defined by the first integral.
$H(x, y) = h$. For system (8), we may use $H_1$, while for system (9), we need to find the first integral, which is not an easy job. Even we have these $H$ functions, it is difficult to compute the Melnikov functions. Therefore, we turn to using focus value computation to analyze bifurcation of limit cycles around the origin. Suppose the focus values around the origin of the system (8) or (9) are given in the following form:

$$V = \sum_{i=0}^{\infty} \varepsilon^i V_i,$$

where $\varepsilon$ is a small perturbation parameter. We call $\nu_j$ the j-th $\varepsilon^i$-order focus value of the system, and note that $\nu_j = 0$, $j = 0, 1, 2, \ldots$ since the origin is a center of these two systems. We use $M(n)$ to denote the number of limit cycles bifurcating from a singular point, where $n$ is the order of the system.

The rest of the paper is organized as follows. In the next section, we use the method of focus value computation to show that there are 11 small-amplitude limit cycles around the origin of the systems (8) and (9). In Section 3, we use system (6) with the free coefficient $a$ to prove that there exist 12 small-amplitude limit cycles around the origin. Conclusion is drawn in Section 4.

### 2. 11 Limit cycles in systems (8) and (9)

In this section, we will use the method of focus value computation to show that the systems (8) and (9) can have 11 small-amplitude limit cycles bifurcating from the origin, i.e., $M(3) \geq 11$. First, we consider system (8) and have the following result.

**Theorem 1.** System (8) can have 11 small-amplitude limit cycles bifurcating from the origin by proper cubic perturbations.

**Proof.** Adding perturbations to the non-perturbed system, we have two choices: either to the original system (6) (with $a = 2$) or after the shifting the equilibrium (7) to the origin plus a linear transformation applied such that the Jacobian of the resulting system is in the Jordan canonical form. These two choices are equivalent, giving the same result on the number of limit cycles, but the latter is simpler. Therefore, we take the second choice. We first apply a linear transformation and a time scaling, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 \begin{array}{cc} 286 & 0 \\ 12 & 2 \end{array} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \rightarrow \frac{t}{12\sqrt{23602332855}};$$

into system (8) to obtain

$$\begin{align*}
\dot{x} &= \frac{-38483312\sqrt{23602332855}x^2 + 5022227x^4 + 5312\sqrt{23602332855}y^2 - 2039614912\sqrt{23602332855}x^3 y^3}{696872478744 + 23602332855} \\
\dot{y} &= \frac{-15317y^2 + 15481312\sqrt{23602332855}y^3}{23602332855 - \frac{757589944320}{35312640} - \frac{774425276416}{4490071331}} - \frac{165731}{47083520}y^3 + \frac{280912\sqrt{23602332855}y^2}{4490071331} - \frac{2383681000808448}{869538447360} - \varepsilon a_3(x,y) \equiv f_2(x,y) + \varepsilon q_3(x,y),
\end{align*}$$

where the linear part of the unperturbed system is now in the Jordan canonical form, and the cubic polynomial perturbations have been added, given in the general form:

$$\begin{align*}
p_3(x,y) &= a_{10}x + a_{10}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{12}x^2 y + a_{01}y^3, \\
q_3(x,y) &= b_{10}x + b_{10}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{12}x^2 y + b_{01}y^3.
\end{align*}$$

To make the origin of the system be an elementary center, it requires that $a_{10} + b_{10} = 0$, or $b_{10} = -a_{10}$. To further simplify the analysis, introducing another linear transformation and a time scaling, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a_{10} & 1 + a_{10} \\ -1 + b_{10} \\ -a_{10} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \rightarrow \frac{t}{\omega_e},$$

where $\omega_e = \sqrt{1 + \varepsilon (a_{10} - b_{10}) - \varepsilon^2 (a_{10} + b_{10})}$, into system (11) yields

$$\begin{align*}
\dot{x} &= f_1(x,y) + \varepsilon p_1(x,y), \\
\dot{y} &= f_2(x,y) + \varepsilon q_3(x,y),
\end{align*}$$

where higher-order $\varepsilon$ terms have been dropped since system (11) only has $\varepsilon$-order terms added, $f_1$ and $f_2$ are given in (11), and $p_1$ and $q_3$ are given below.
\[ p_1(x, y) = \begin{cases} 0 & \text{if } x = 0, \quad y = 0; \\ \frac{1}{2} (a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + a_5 x^6) & \text{otherwise}, \end{cases} \]

\[ q_1(x, y) = \begin{cases} 0 & \text{if } x = 0, \quad y = 0; \\ \frac{1}{2} (b_0 y + b_1 x y) & \text{otherwise}. \end{cases} \]

Now, based on system (13), we use focus value computation (e.g., using the Maple program developed in [25]) to obtain the \( \epsilon \)-order focus values \( \nu_{11}, \nu_{12}, \ldots, \nu_{1n} \), all of them are linear functions of \( a_0 \) and \( b_0 \). For example, \( \nu_{11} \) is given by

\[
\nu_{11} = \frac{8776116145175287}{1132248475384012800} a_{10} - \frac{215539887437}{226496950768025600} a_{10} - \frac{440628167}{13242672226713600} b_{10} - \frac{4187}{1710058144} a_{20} + \frac{24857}{60192} a_{19} - \frac{57611}{60192} a_{20} - \frac{2597}{500179520} b_{20} + \frac{159}{12160} b_{11} - \frac{371}{2044631952} b_{20} + \frac{1}{8} (a_{12} + b_{21} + 3a_{30} + 3b_{30}).
\]

Thus, we may use any one of the parameters, say \( b_{30} \), to solve \( \nu_{11} = 0 \) to obtain \( b_{30} \) expressed in terms of other parameters.

Similarly, we can solve the equation \( \nu_{12} = 0 \) for \( b_{21} \), \( \nu_{13} = 0 \) for \( b_{21} \), \( \nu_{14} = 0 \) for \( b_{20} \), \( \nu_{15} = 0 \) for \( b_{20} \), \( \nu_{16} = 0 \) for \( b_{11} \), \( \nu_{17} = 0 \) for \( b_{20} \), \( \nu_{18} = 0 \) for \( b_{10} \), \( \nu_{19} = 0 \) for \( b_{10} \), \( \nu_{20} = 0 \) for \( b_{21} \), respectively. Finally, under theses solutions, the 11th- and 12th-order focus values are given by \( \nu_{111} = c_{11} f_{10} \) and \( \nu_{112} = c_{12} f_{10} \), where \( c_{11} \) and \( c_{12} \) are some constants, and the common factor \( f_{10} \) is a function of \( a_{10}, a_{20}, a_{11}, a_{20}, a_{12}, a_{21}, a_{30}, b_{10} \), given by

\[
f_{10} = 691234068956115 \times 23602332855a_{10} + 77703544185425357945a_{10} - 2095858834724574 \]

This shows that the best result we can obtain is \( \nu_{ij} = 0, j = 0, 1, 2, \ldots, 10 \) but \( \nu_{111} \neq 0 \), implying that system (11) can have at most 11 small-amplitude limit cycles bifurcating from the origin. The above function \( f_{10} \) also implies that besides the \( b_j \) parameters, only two \( a_k \) parameters are used to solve the focus values. In other words, we may leave one free parameter, say \( a_{10} \), which can be used to scale the focus values, and set all other parameters zero, \( a_{10} = a_{20} = a_{11} = a_{20} = a_{12} = a_{30} = 0 \). Certainly, one can choose other possible combinations of the parameters \( a_k \) and \( b_k \) to show the same result. Thus, without loss of generality, we may assume \( 0 < \nu_{111} \leq 1 \).

Finally, taking small perturbations in backward order on \( a_{21} \) for \( \nu_{110} \), on \( a_{30} \) for \( \nu_{109} \), on \( b_{10} \) for \( \nu_{108} \), on \( b_{20} \) for \( \nu_{107} \), on \( b_{11} \) for \( \nu_{106} \), on \( b_{20} \) for \( \nu_{105} \), on \( b_{30} \) for \( \nu_{104} \), on \( b_{21} \) for \( \nu_{103} \), on \( b_{12} \) for \( \nu_{102} \), on \( b_{20} \) for \( \nu_{101} \), and on \( b_{11} \) for \( \nu_{100} \) so that

\[
\nu_{ij} \nu_{ij+1} < 0, \quad |\nu_{ij+1}| < |\nu_{ij}|, \quad \text{for } j = 0, 1, 2, \ldots, 10.
\]

This shows that there exist 11 small-amplitude limit cycles around the origin of system (8). \( \square \)

Next, we consider system (9) and have the following result.

**Theorem 2.** System (9) can have 11 small-amplitude limit cycles bifurcating from the origin by proper cubic perturbations.

**Proof.** According to [22], the perturbations added to system (9) are given by

\[
\hat{p}_1(x, y) = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + a_5 x^6, \\
\hat{q}_1(x, y) = a_0 y + a_1 x y + a_2 x^2 y + a_3 x^3 y + a_4 x^4 y.
\]
and so the perturbed system (9) becomes

\[
\begin{align*}
\dot{x} &= f_1(x, y) + \varepsilon p_3(x, y), \\
\dot{y} &= f_2(x, y) + \varepsilon q_3(x, y),
\end{align*}
\]

(17)

where \( f_1 \) and \( f_2 \) are given in (9). First note that the zero-order \( \varepsilon \)-order focus value \( \nu_{i0} = a_{i0} \). Letting \( a_{i0} = 0 \) and then executing the Maple program for computing the focus values [25] results in

\[
\nu_{i1} = \frac{1}{8} \left[ 4r(3r^2 + 11)a_1 + 8r(r^2 + 4)a_3 + 3a_4 + 4(3r^2 - 1)(a_7 + a_9) + 2r(r^2 + 3)a_8 + a_{11} \right],
\]

\[
\nu_{i2} = -\frac{1}{96} \left[ 4r(729r^8 + 7816r^6 + 26110r^4 + 27656r^2 + 89)a_4 - 24r^2(3r^2 + 1)(r^2 + 3)^2 a_2 \\
+ 8r(245r^6 + 2773r^4 + 9669r^2 + 10643r^2 - 2)a_3 + (709r^6 + 5107r^4 + 7227r^2 + 45)a_4 - 4r(3r^4 + 58r^2 - 13)a_5 \\
+ (215r^6 + 1809r^4 + 3177r^2 + 15)a_6 + 4(759r^8 + 4548r^6 - 2596r^2 + 5022r^4 + 11)a_7 \\
+ 2r(r^2 + 3)(263r^6 + 1601r^4 + 2585r^2 - 1)a_8 + 4(735r^8 + 4744r^6 + 5778r^4 - 2744r^2 - 1)a_9 \\
+ 20r(3r^2 - 22r^2 + 3)a_{10} + (236r^6 + 1553r^4 + 2649r^2 - 17)a_{11} \right],
\]

Similarly, we can linearly solve the equations \( \nu_{ii} = 0 \) for \( a_i \), \( i = 1, 2, \ldots, 10 \) one by one. Then, the 11th and 12th-order focus values are obtained as

\[
\nu_{11} = 2340 \frac{f_{201}(r)}{f_{201}(r)} F_{21}(r) \text{ and } \nu_{12} = -120 \frac{f_{202}(r)}{f_{202}(r)} F_{22}(r),
\]

where \( F_{20N}, F_{20D}, F_{21} \) and \( F_{22} \) are all polynomials of \( r \), given by

\[
F_{20N} = a_{11} r^{11}(r^2 - 1)^{11}(r^2 + 3)^8(3r^2 + 1)^9(3r^6 - 31r^4 - 19r^2 - 1),
\]

\[
F_{20D} = 132337775714^4 + 26996002137r^{12} + 37965987792r^{30} - 3268655299296r^{38} - 16760339520156r^{26} + 12963886884900r^{24} + 38642548810176r^{22} + 3147430151568r^{20} + 6604961423494r^{18} + 3872384663134r^{16} + 6683800535760r^{14} - 3528405585600r^{12} - 58333647916r^{10} + 237931831540r^8 + 32335049504r^6 + 6091835792r^4 + 715493989r^2 + 26239665,
\]

\[
F_{21} = 63r^{12} + 868r^{10} + 1407r^8 - 1232r^6 - 803r^4 + 588r^2 + 133,
\]

\[
F_{22} = 355761r^{18} + 9537262r^{16} + 77877345r^{14} + 179927041r^{12} + 41987927r^{10} - 169741431r^8 - 36416005r^6 + 63798403r^4 + 14431452r^2 + 3831737.
\]

In order to obtain maximal number of limit cycles, we may solve \( F_{21} = 0 \) for \( r \). However, unfortunately, \( F_{21} = 0 \) has no real solutions for \( r \). Thus, the best result we can get is \( \nu_{i1} = 0 \), \( i = 0, 1, 2, \ldots, 10 \), but \( \nu_{11} \neq 0 \), implying that system (17) can have at most 11 small-amplitude limit cycles bifurcating from the origin. Again, without loss of generality, we may use the parameter \( a_{i1} \) to scale the focus values such that \( 0 < \nu_{i1} \ll 1 \). Further, by perturbing the parameters, in backward order, on \( a_i \) for \( \nu_{ii} = 10.9, \ldots, 0 \), such that the relations given in (15) hold, which implies that system (17) exhibits 11 small-amplitude limit cycles around the origin. □

3. 12 Limit cycles in system (6)

Now we return to system (6) and let the parameter \( a \) be free to vary. In this case, we have the following result.

**Theorem 3.** System (6) can have 12 small-amplitude limit cycles bifurcating from the origin by using proper cubic perturbations with a properly chosen value of \( a \), i.e., \( M(3) \geq 12 \).

In order to prove Theorem 3, we need a lemma. Consider the following generally perturbed system:

\[
\begin{align*}
\dot{x} &= P(x, y, \delta_1) + \varepsilon p(x, y, \delta_2), \\
\dot{y} &= Q(x, y, \delta_1) + \varepsilon q(x, y, \delta_2),
\end{align*}
\]

(18)

where \( P, Q, p \) and \( q \) are polynomials of \( x \) and \( y \). Suppose that the vector parameter \( \delta_1 \) involved in \( P \) and \( Q \) is \( m \) dimension, while the vector parameter \( \delta_2 \) involved in \( p \) and \( q \) is \( l \) dimension. It is further assumed that when \( \varepsilon = 0 \) system (18) is
integrable. In Hamiltonian case, \( P = H_c, \ Q = -H_c, \) where \( H \) is a Hamiltonian. If it is not a Hamiltonian system one needs to multiply the integrating factor in order to use Melnikov function. This increases complexity of computation. While with the method of focus value computation, it does not need to find the integrating factor and thus greatly simplifies the computation. For system (18), we have the following lemma.

**Lemma 1.** By properly choosing the parameters \( \delta_1 \) and \( \delta_2 \) in system (18), \( k \) small-amplitude limit cycles exist around the origin of the system, satisfying \( k < m - l \). The exact number of the limit cycles depends upon how many parameters can be chosen independently to solve the focus value equations (or to determine the zeros of Melnikov functions).

The proof can follow the proof for Theorem 3 in [26].

We first consider \( \varepsilon \)-order focus values (equivalent to first-order Melnikov function), and then consider \( \varepsilon^2 \)-order focus values. We will show that using \( \varepsilon^2 \)-order focus values does not increase the number of limit cycles.

3.1. Based on \( \varepsilon \)-order focus values

**Proof.** Following the procedure in the proof for Theorem 1, for system (6), we first shift the equilibrium point defined in (7) to the origin, and then apply a liner transformation with a proper time scaling such that the Jacobian of the resulting system evaluated at the origin is in the form of \(
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\). The time scaling is taken as \( t \rightarrow \frac{t}{\omega_e} \), where \( \omega_e \) is given by

\[
\omega_e = \sqrt{5(8a^2 + 25)(32a^2 - 75)(16384a^6 - 14400a^4 + 165000a^2 + 84375)}.
\]

Then, adding the \( \varepsilon \)-order perturbation terms \( p_3(x, y) \) and \( q_3(x, y) \), given by

\[
p_3(x, y) = a_{101}x + a_{011}y + a_{201}x^2 + a_{111}xy + a_{021}y^2 + a_{301}x^3 + a_{211}x^2y + a_{121}xy^2 + a_{031}y^3,
\]

\[
q_3(x, y) = b_{101}x + b_{011}y + b_{201}x^2 + b_{111}xy + b_{021}y^2 + b_{301}x^3 + b_{211}x^2y + b_{121}xy^2 + b_{031}y^3,
\]

into system (6), resulting a new system. Here, note that we add one more sub-index “1” to explicitly indicate the \( \varepsilon \)-order perturbations in order to distinguish from the \( \varepsilon^2 \)-order perturbations considered in the next subsection. Further, under the condition \( b_{011} = -a_{011} \), we apply another linear transformation and a second time scaling, given below,

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\frac{1}{1+a_{011}} & 0 \\
\frac{a_{011}}{1+a_{011}} & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}, \quad t \rightarrow \frac{t}{\omega_e'}
\]

where \( \omega_{e'} = \sqrt{1 + \varepsilon(a_{011} - b_{011}) + \varepsilon^2(a_{011}^2 + a_{011}b_{011})} \), and obtain the final system, given by

\[
\ddot{x} = -\frac{\sqrt{5\alpha}}{768a^2(4a^2 - 5)^2\omega_e} x^2 + \frac{(32a^2 - 75)A_0}{384B_0} x^2 y + \frac{\sqrt{5(32a^2 - 75)}\omega_e}{3840B_0}\alpha x^3 y^2 - \frac{\sqrt{5(32a^2 - 75)}C_0}{4608B_0}\omega_e x^3 y^2 + \varepsilon \left\{ a_{201} - \frac{(32a^2 - 75)A_0}{384B_0} a_{101} + \frac{\sqrt{5(8a^2 + 25)}C_0}{1536B_0}\omega_e (a_{011} - b_{011}) \right\} x^2 + \frac{a_{111} - \frac{(32a^2 - 75)A_0}{384B_0} a_{011} + \frac{\sqrt{5(32a^2 - 75)}\omega_e}{1920B_0}}{a_{011} + \frac{\sqrt{5(32a^2 - 75)}\omega_e}{7680B_0}} x y + \left\{ a_{201} + \frac{\sqrt{5(32a^2 - 75)}\omega_e}{9216B_0}\alpha (a_{011} - b_{011}) \right\} y^2 + \left\{ a_{301} - \frac{(32a^2 - 75)(128a^4 - 176a^2 - 225)}{2304B_0} a_{101} + \frac{\sqrt{5(32a^2 - 75)}C_0}{9216B_0}\omega_e (a_{011} - b_{011}) \right\} x^3 + \left\{ a_{211} - \frac{(32a^2 - 75)(128a^4 - 176a^2 - 225)}{2304B_0} a_{011} + \frac{\sqrt{5(32a^2 - 75)}\omega_e}{11520B_0}\alpha a_{101} \right\} x^2 y + \left\{ a_{121} + \frac{\sqrt{5(32a^2 - 75)}\omega_e}{46080(8a^2 + 25)} (3a_{011} + b_{011}) \right\} x y^2 + a_{031} y^3 \right\},
\]

Similarly, we linearly solve the polynomial equations one by one for
\(a\) and \(b\) using \(\omega_k\) in [19]. Now applying the method of focus value computation (e.g., executing the Maple program in [25]) yields \(v_{1j} \equiv 1 \ldots 4\). For example,

\[
 \begin{align*}
 v_{11} &= \frac{1}{8} [a_{11} + b_{211} + 3(a_{101} + b_{011})] + \left[ \frac{(32a^2 - 75)^3(-55188750000a^4 - 33328125000a^2 - 83056640625)}{(460800a^2(4a^2 - 5)^2(6a^2 + 25)\omega_k^2)} \right] a_{101} \\
 &\quad + \frac{(32a^2 - 75)^3(1111616a^2 + 270400a^2 - 232500a^2 + 421875)}{921600a^2(4a^2 - 5)^2\omega_k} b_{011} - \frac{21\sqrt{5}(32a^2 - 75)(1024a^4 + 100a^2 + 5625)}{921600a^2(4a^2 - 5)^2\omega_k} b_{111} \\
 &\quad - \frac{(32a^2 - 75)(32a^2 - 75)\sqrt{5}(32a^2 - 75)(128a^2 + 225)}{49(6a^2 + 25)(32a^2 - 75)^3320(4a^2 - 5)\omega_k} b_{011} - \frac{7\sqrt{5}(32a^2 - 75)^3(128a^2 + 225)}{1920(4a^2 - 5)(8a^2 + 25)} a_{111} \\
 &\quad - \frac{(32a^2 - 75)(32a^2 - 75)^3}{48(4a^2 - 5)\omega_k} b_{211}.
\end{align*}
\]

Similarly, we linearly solve the polynomial equations one by one for each \(v_{11} = 0\) using \(b_{011}\), for \(v_{12} = 0\) using \(b_{211}\), for \(v_{13} = 0\) using \(b_{111}\), for \(v_{14} = 0\) using \(b_{011}\), for \(v_{15} = 0\) using \(b_{211}\), for \(v_{16} = 0\) using \(b_{011}\), for \(v_{17} = 0\) using \(b_{211}\), for \(v_{18} = 0\) using \(b_{011}\), for \(v_{19} = 0\) using \(a_{101}\), for \(v_{20} = 0\) using \(a_{211}\), and then obtain

\[
 \begin{align*}
 v_{111} &= 12103 F_{301}^{\text{CNH}} F_{312}^{\text{CNH}}, \\
 v_{112} &= -931 F_{301}^{\text{CNH}} F_{322}^{\text{CNH}},
\end{align*}
\]

where \(F_{301}^{\text{CNH}}\) is a 6th-degree polynomial of \(a\), and \(F_{301}^{\text{CNH}}\) is given by

\[
 F_{301}^{\text{CNH}} = \frac{15(32a^2 - 75)^3(2048a^6 - 3200a^4 - 13500a^2 + 5625)}{a_{101}^2(4a^2 - 5)^{11}(8a^2 + 25)^4\omega_k^2} + \frac{\sqrt{5}(16a^2 + 15)(32a^2 - 75)^3(2048a^6 + 7360a^4 - 55300a^2 + 16875)}{a_{011}^2(4a^2 - 5)^{11}(8a^2 + 25)^4\omega_k^2} - \frac{6(8a^2 + 25)(32a^2 - 75)^3(17408a^6 - 40640a^4 - 67500a^2 + 28125)}{a_{101}^2(4a^2 - 5)^{11}(8a^2 + 25)^4\omega_k^2}.
\]
The lengthy expressions for $F_{31}$ and $F_{32}$ are given in Appendix A. It can be shown by using the Groebner basis reduction procedure that $F_{21}/F_{20} 
eq 0$ and $F_{30}/F_{31} 
eq 0$. The polynomial equation $F_{31} = 0$ has three real solutions for $a^2$ (up to 1000 digit points, but only list 50 digit points here):

\[
a^2 = 4.08009735271177103610297484395201964354626904458021\ldots,
\]

\[
55.4186330411036260819951662137654320243144364768321\ldots.
\]

and all of them satisfy $\omega_k > 0$ (see the expression of $\omega_k$ given in (19)). So there are in total six solutions. Taking the positive value of the second solution for $a^2$:

\[
a = 7.4443692171401381002446239854627395063347650296272\ldots
\]

and setting the non-used parameters $a_{01} = a_{20} = a_{11} = a_{21} = a_{02} = a_{12} = a_{03} = 0$, and $a_{10} = 1$, we obtain the following critical parameter values:

\[
b_{031} = -0.19166152145498089355202548357797946751132495935848\ldots,
\]

\[
b_{121} = 0.1147084514014593144202950698087209236746686969414\ldots,
\]

\[
b_{211} = -1.21440862253395164484193434261717547030904451575014\ldots,
\]

\[
b_{301} = 0.13315882740016516186295687278609694132824695861367\ldots.
\]

\[
b_{021} = 0.39631189749427819043808615679104642347912703580286\ldots
\]

\[
b_{111} = -5.32984926540348883870841754242686045572842701602586\ldots
\]

\[
b_{201} = -1.24805818157971430865735712066201836587770395955\ldots
\]

\[
b_{101} = -19.01617439016444745664609221188921421020650255893027\ldots
\]

\[
a_{031} = -0.3403910326924869344157930981604851752629690721232\ldots.
\]

\[
a_{211} = -0.15448521013159245023264811458682922516993711282715\ldots
\]

under which the focus values become

\[
\nu_{11} = -0.6 \times 10^{-999}, \quad \nu_{12} = 0.4 \times 10^{-999}, \quad \nu_{13} = -0.6 \times 10^{-999},
\]

\[
\nu_{14} = 0.12 \times 10^{-998}, \quad \nu_{15} = -0.13 \times 10^{-997}, \quad \nu_{16} = 0.8 \times 10^{-997},
\]

\[
\nu_{17} = 0.4 \times 10^{-996}, \quad \nu_{18} = 0.8 \times 10^{-995}, \quad \nu_{19} = -0.2 \times 10^{-994},
\]

\[
\nu_{10} = 0.16 \times 10^{-992}, \quad \nu_{11} = -0.5 \times 10^{-991},
\]

\[
\nu_{12} = 0.34448628111762061550983080164\ldots \times 10^{-18}.
\]

Therefore, we can take perturbations in backward order on $a$ for $\nu_{111}$, on $a_{21}$ for $\nu_{110}$, on $a_{30}$ for $\nu_{19}$, on $b_{101}$ for $\nu_{18}$, on $b_{201}$ for $\nu_{17}$, on $b_{311}$ for $\nu_{16}$, on $b_{021}$ for $\nu_{15}$, on $b_{301}$ for $\nu_{14}$, on $b_{121}$ for $\nu_{13}$, on $b_{121}$ for $\nu_{12}$, on $b_{301}$ for $\nu_{11}$, on $b_{011}$ for $\nu_{10}$, to obtain 12 small-amplitude limit cycles bifurcating from the origin. \(\square\)

3.2. Based on $\varepsilon^2$-order focus value

**Proof.** In order to show that higher-order Melnikov functions will not generate more limit cycles, equivalently, here we use the $\varepsilon^2$-order focus values to prove this. To achieve this, we change the perturbations to include the $\varepsilon^2$-order perturbations, given in the form of
\[
p_3(x, y) = a_{101}x + a_{011}y + a_{201}x^2 + a_{111}xy + a_{021}y^2 + a_{301}x^3 + a_{211}x^2y + a_{121}xy^2 + a_{031}y^3 + \epsilon [a_{102}x + a_{012}y + a_{202}x^2 + a_{112}xy + a_{022}y^2 + a_{302}x^3 + a_{212}x^2y + a_{122}xy^2 + a_{032}y^3].
\]
\[
q_3(x, y) = b_{101}x + b_{011}y + b_{201}x^2 + b_{111}xy + b_{021}y^2 + b_{301}x^3 + b_{211}x^2y + b_{121}xy^2 + b_{031}y^3 + \epsilon [b_{102}x + b_{012}y + b_{202}x^2 + b_{112}xy + b_{032}y^3].
\]

(23)

In order to make the origin an elementary center, we have \( b_{011} = -a_{101} \) and \( b_{012} = -a_{102} \). Also, in order to have all \( \epsilon \)-order focus values to vanish, we may solve \( a_{112} \) from the equation \( F_{40v} = 0 \). Then, all the solutions for the \( \epsilon \)-order perturbation parameters are obtained, as given in Appendix B. Now, we use these parameter expressions to simplify the \( \epsilon^2 \)-order focus values \( v_{2j} \), \( j = 1, 2, \ldots \) and then linearly solve the polynomial equations one by one for \( v_{21} = 0 \) using \( b_{021} \), for \( v_{22} = 0 \) using \( b_{122} \), for \( v_{23} = 0 \) using \( b_{123} \), for \( v_{24} = 0 \) using \( b_{132} \), for \( v_{25} = 0 \) using \( b_{202} \), for \( v_{26} = 0 \) using \( b_{122} \), for \( v_{27} = 0 \) using \( b_{202} \), for \( v_{28} = 0 \) using \( b_{122} \), for \( v_{29} = 0 \) using \( a_{021} \), and for \( v_{210} = 0 \) using \( a_{121} \). Finally, we obtain

\[
v_{211} = -146381281540702208 \frac{F_{40v}}{5F_{30d}} F_{31},
\]
\[
v_{212} = 180161577280864256 \frac{F_{40v}}{5F_{30d}} F_{32}
\]

where the common factor \( F_{40v} \) is a function of the unused \( \epsilon \)-order perturbation parameters, \( a_{101}, a_{011}, a_{201}, a_{111}, a_{021}, a_{301} \) and \( \epsilon^2 \)-order perturbation parameters, \( a_{102}, a_{012}, a_{202}, a_{112}, a_{022}, a_{302} \) while the polynomial functions, \( F_{30d}, F_{31} \) and \( F_{32} \) are exactly the same as that obtained from using the \( \epsilon \)-order focus values, see Eq. (22). Therefore, the best result we can have is \( v_{2j} = 0, j = 0, 1, 2, \ldots, 10 \), but \( v_{211} \neq 0 \), indicating that using \( \epsilon^2 \)-order perturbations still gives 12 small-amplitude limit cycles bifurcating from the origin. This suggests that using higher-order perturbations may not increase the number of limit cycles. \( \square \)

4. Conclusion

In this paper, we have applied the method of focus value computation to confirm the results of 11 small-amplitude limit cycles around a singular point in two existing systems in the literature. Further, we used one of the two systems with a free parameter to obtain 12 small-amplitude limit cycles. This is the best result so far obtained in cubic planar vector fields around a singular point.

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Appendix A

In this appendix, we list the expressions for \( F_{31}, F_{32} \) in Eq. (22).

\[
F_{31} = \frac{100002671659204408191270590349312}{124556484375} \frac{1}{a^{28}} + \frac{81968692409139232369633976548694256}{124556484375} \frac{1}{a^{48}} - \frac{144491654217902126140674110413441335296}{199290375} \frac{1}{a^{46}}
\]
\[
+ \frac{32150625}{23958075} \frac{1}{a^{44}} - \frac{673280660493096644103991475399527715840}{531441} \frac{1}{a^{42}} - \frac{86756528992167810184858626362595239526400}{177147} \frac{1}{a^{40}}
\]
\[
+ \frac{163554126127002314837288798161362485248000}{59049} \frac{1}{a^{38}} - \frac{7023522799360115593776510850258960384000000}{39049} \frac{1}{a^{36}}
\]
\[
+ \frac{3768999767982326590470055765207433216000000000}{6561} \frac{1}{a^{34}} - \frac{1716840349613790147437200519764208000000000000}{243} \frac{1}{a^{32}}
\]
\[
- \frac{25969044713153543051947369123644524500000000000}{2187} \frac{1}{a^{30}} + \frac{2298402431789470657144226366532002092285156250}{243} \frac{1}{a^{28}}
\]
\[
F_{32} = \frac{5801794936047356598957858042355286489324032}{84075626953125} \cdot a^{66}
\]
\[
+ \frac{4871211304076929540039730660578079195954937856}{84075626953125} \cdot a^{64}
\]
\[
+ \frac{98115617017701347629012133223277101678229848064}{560504796875} \cdot a^{62}
\]
\[
+ \frac{19624689786663604325437326676781184026010216103936}{3363025078125} \cdot a^{60}
\]
\[
+ \frac{21256591905213042405396221854632976200299114921984}{134521003125} \cdot a^{58}
\]
\[
+ \frac{195263677424322371088615241747992055543599795936}{1793613375} \cdot a^{56}
\]
\[
+ \frac{70119459709914198500295736221723652294678493003776}{1076168025} \cdot a^{54}
\]
\[
+ \frac{57721050631981993916086425126111691780038902520176}{43046721} \cdot a^{52}
\]
\[
+ \frac{709920387371447570583565409396663171263951096564940800}{4782969} \cdot a^{50}
\]
\[
+ \frac{104296349050750214912771728398207142493292218300416000}{14348907} \cdot a^{48}
\]
\[
+ \frac{8385315027121568675222026121264946195301195055104000000}{4782969} \cdot a^{46}
\]
\[
+ \frac{87437962180467579496457634960502924312130761523200000000}{582678688158772734164733109697595433857419827200000000000}{531441} \cdot a^{44}
\]
\[
+ \frac{21292488896396114083846506545169895680528187655000000000000}{531441} \cdot a^{42}
\]
\[
+ \frac{4724009367317348656168824748752235171538304308593750000000}{59049} \cdot a^{40}
\]
Appendix B

In this appendix, we list the solutions for the $\epsilon$-order perturbation parameters under which all the $\epsilon$-order focus values vanish.

\[ b_{031} = \frac{-\left(32\alpha^2 - 75\right)(2048\alpha^6 - 3200\alpha^4 - 13500\alpha^2 + 5625)}{3072(8\alpha^2 + 25)(16\alpha^2 + 15)B_4D_s} a_{011} \frac{\sqrt{5}(32\alpha^2 - 75)^2 \omega_c}{46080(8\alpha^2 + 25)B_6} a_{101} \]

\[ + \frac{(17408\alpha^6 - 40640\alpha^4 - 67500\alpha^2 + 28125)}{7680(8\alpha^2 + 25)(16\alpha^2 + 15)B_4D_s} a_{201} \]

\[ + \frac{\sqrt{5}(32\alpha^2 - 75)(16384\alpha^6 + 109056\alpha^4 - 55800\alpha^2 - 52500\alpha^2 + 84375)}{7680(16\alpha^2 + 15)B_4D_s} a_{111} \]

\[ \frac{32\alpha^2 - 75}{1536(16\alpha^2 + 15)} \left[ \frac{67108864\alpha^{14} - 6291456\alpha^{12} - 2059927552\alpha^{10} + 5214044160\alpha^8}{B_4D_s} \right] a_{021} \]

\[ + \frac{80940800\alpha^6 + 7945500000\alpha^4 - 1307812500\alpha^2 - 474609375}{B_4D_s} a_{021} \]
\[
\sqrt{5} \left\{ 60397776a^4 + 304087040a^2 + 19492044800a^0 + 53643008000a^6 \right\} - \\
\frac{78456000000a^6 + 155209500000a^4 + 35859375000a^2 - 21357421875}{(16a^2 + 15)|D_x\omega_c|} \right\} a_{031},
\]

\[
b_{121} = \frac{\sqrt{5}\omega_c}{15360} \left[ 14680064a^{12} + 195297280a^{10} - 106700800a^8 - 1193728000a^6 \right] \]

\[
\left\{ 32a^2 - 75 \right\} \left( 8a^2 + 25 \right) \left( 16a^2 + 15 \right) |D_x\omega_c| \]

\[
+ \frac{2031300000a^4 + 1746562500a^2 - 791015625}{(32a^2 - 75)(8a^2 + 25)(16a^2 + 15)|D_x\omega_c|} \right\} a_{011},
\]

\[
\sqrt{5}\omega_c \left[ 22020996a^{12} + 985661440a^{10} - 81203200a^8 - 441408000a^6 \right] \]

\[
\left\{ 8a^2 + 25 \right\} \left( 16a^2 + 15 \right) |D_x\omega_c| \]

\[
+ \frac{10975500000a^4 + 8732812500a^2 - 3955078125}{(8a^2 + 25)(16a^2 + 15)|D_x\omega_c|} \right\} a_{201},
\]

\[
\frac{(32a^2 - 75)(8a^2 + 25)}{16777216a^{12} + 17301504a^{10} - 299581440a^8 + 495603200a^6}{7680(16a^2 + 15)|D_x\omega_c|}
\]

\[
\frac{1821720000a^4 + 761062500a^2 - 474609375}{B_x|D_x\omega_c|} \right\} a_{111},
\]

\[
- \frac{\sqrt{5}(8a^2 + 25)(32a^2 - 75)^2}{22020996a^{12} + 985661440a^{10} - 81203200a^8 - 441408000a^6}{16a^2 + 15}
\]

\[
+ \frac{10975500000a^4 + 8732812500a^2 - 3955078125}{(8a^2 + 25)(16a^2 + 15)|D_x\omega_c|} \right\} a_{021},
\]

\[
\frac{(32a^2 - 75)}{19058917376a^{16} - 78454456320a^{14} - 480496844800a^{12} + 4597039104000a^{10}}{5(16a^2 + 15)|D_x\omega_c|}
\]

\[
+ \frac{3557792000000a^8 - 45933048000000a^6 + 40921650000000a^4 + 5299804687500a^2 + 4805419921875}{D_x|D_x\omega_c|} \right\} a_{031},
\]

\[
b_{211} = -\frac{35(32a^2 - 75)}{46080(8a^2 + 25)} \left[ 10173741824a^{16} + 10536091648a^{14} - 14659092480a^{12} - 13002997760a^{10} \right]
\]

\[
+ \frac{206808064000a^8 + 250732800000a^6 - 557212500000a^4 - 176554687500a^2 + 10678719075}{(16a^2 + 15)|D_x\omega_c|} \right\} a_{101},
\]

\[
+ \frac{7\sqrt{(16a^2 + 15)(16a^2 + 225)(32a^2 - 75)^2}{46080a^2(4a^2 - 5)^2|D_x\omega_c|} \right\} a_{011} + \frac{536870912a^{16} + 17012097024a^{14}}{7680(16a^2 + 15)|D_x\omega_c|}
\]

\[
+ \frac{536870912a^{16} + 17012097024a^{14} + 13662945280a^{12} - 259778150400a^{10} + 308864000000a^8}{7680(16a^2 + 15)|D_x\omega_c|}
\]

\[
+ \frac{317592000000a^6 - 914512500000a^4 - 294257812500a^2 + 177978515625}{7680(16a^2 + 15)|D_x\omega_c|}
\]

\[
- \frac{\sqrt{5}(8a^2 + 25)(32a^2 - 75)^2}{4194304a^{12} - 52297728a^{10} - 169410560a^8 - 1519603200a^6}{884160000a^4 + 420187500a^2 - 284765625}{B_x|D_x\omega_c|}
\]

\[
+ \frac{7(8a^2 + 25)(32a^2 - 75)^2}{1536(16a^2 + 15)} \left[ 32078036992a^{14} - 48473571328a^{12} + 4065460224a^{10} \right]
\]

\[
+ \frac{151542988800a^8 - 581091840000a^6 - 411075000000a^4 - 7973437500a^2 + 64072265625}{B_x|D_x\omega_c|} \right\} a_{021},
\]

\[
\frac{\sqrt{5}(8a^2 + 25)(32a^2 - 75)^2}{3(16a^2 + 15)} \left[ 3848290697216a^{10} + 12685185908736a^{18} - 98385621155840a^{16} 
\]

\[
+ 46400222032000a^4 + 87475427736000a^{12} - 488871183360000a^{10} - 2960110272000000a^8}{D_x|D_x\omega_c|} \right\} a_{031},
\]

\[
+ \frac{183037554000000000a^6 + 10748143125000000a^4 - 2100858394873500a^2 - 1081219482421875}{D_x|D_x\omega_c|} \right\} a_{031},
\]

\[
\]
\[
\begin{align*}
b_{301} &= \frac{\sqrt{3}(32a^2 - 75)}{3072(16a^2 + 15)} \left[ 240518168576a^{18} + 1523102777344a^{16} - 12233610362880a^{14} - 12036918476800a^{12} \right. \\
&\quad + 151498719232000a^{10} - 20093644800000a^8 - 9932664000000a^6 \\
&\quad + 2744161875000000a^4 + 2705273475000a^2 - 24027099609375 \bigg] a_{101} \\
&\quad - \frac{(8a^2 + 25)(32a^2 - 75)^2(16a^2 + 225)(3328a^4 - 8320a^2 - 3375)}{3072a^2(4a^2 - 5)^5a_5^2} a_{111} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)}{2560(16a^2 + 15)} \left[ 22338299392a^{18} + 4358586499072a^{16} - 16686451261440a^{14} \\
&\quad - 8007718010880a^{12} - 39044669440000a^{10} + 41013638400000a^8 + 22383480000000a^6 \\
&\quad + 5743321875000000a^4 - 25866210937500a^2 - 40045166015625 \bigg] a_{201} \\
&\quad - \frac{(8a^2 + 25)^2(32a^2 - 75)^2}{512(16a^2 + 15)} \left[ 21646802944a^{12} - 72642068480a^{10} + 179301990400a^8 \\
&\quad - 4298812800000a^6 + 500337000000a^4 + 18984375000a^2 - 64072265625 \bigg] a_{111} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)^2(32a^2 - 75)^2}{123145302310912a^{12} + 178533200560128a^{10} - 361789254533120a^{18} \\
&\quad + 298085141913600a^{16} + 2115437435289600a^{14} - 14929844142080000a^{12} \\
&\quad + 3814010475520000000a^{10} + 1608755736000000000a^8 - 7948350450000000000a^6 \\
&\quad + 3022768125000000000a^4 - 83774487304687500a^2 - 27030487060546875 \bigg] a_{201} \\
&\quad - \frac{5(8a^2 + 25)^2(32a^2 - 75)}{16a^4 + 15} \left[ 123145302310912a^{12} + 178533200560128a^{10} - 361789254533120a^{18} \\
&\quad + 298085141913600a^{16} + 2115437435289600a^{14} - 14929844142080000a^{12} \\
&\quad + 3814010475520000000a^{10} + 1608755736000000000a^8 - 7948350450000000000a^6 \\
&\quad + 3022768125000000000a^4 - 83774487304687500a^2 - 27030487060546875 \bigg] a_{301} \\
\end{align*}
\]

\[
b_{301} \approx 5\left(1351686a^{18} + 2278400a^4 + 4320000a^2 + 646875 \right) \alpha_{101} \\
\approx \frac{\sqrt{5}(3792896a^8 - 70656000a^4 + 184840000a^2 + 18675000a^2 - 16171875) \alpha_{101}}{200(4a^2 - 5)(8a^2 + 25)(16a^2 + 15)|32a^2 - 75|D_a} \\
\approx \frac{524288a^{10} + 9306112a^8 - 42483200a^6 + 189188000a^4 + 4597500a^2 - 73828125 \alpha_{111}}{40(4a^2 - 5)(16a^2 + 15)D_a} \\
\approx \frac{\sqrt{5}(8a^2 + 25)(32a^2 - 75)(32a^2 + 9)(3792896a^8 - 7065600a^4 + 184840000a^2 + 18675000a^2 - 16171875)}{40(4a^2 - 5)(16a^2 + 15)D_a} \\
\approx \frac{1536(10616832a^{10} + 141772080a^8 - 174312000a^6 - 64250000a^4 + 555703125a^2 + 284765625)B_a}{(16a^2 + 15)D_a} \\
\approx \frac{21417248a^{10} + 129826816a^8 - 414392320a^6 - 28512000a^4 - 342450000a^2 - 436640625 \alpha_{101}}{24(8a^2 + 25)(16a^2 + 15)D_a} \\
\approx \frac{140509184a^{12} + 632684544a^{10} - 3217612800a^8 + 4179008000a^6 - 5355000000a^4}{20(4a^2 - 5)(16a^2 + 15)|32a^2 - 75|D_a} \\
\approx \frac{4865625000a^6 - 3955078125}{20(4a^2 - 5)(16a^2 + 15)|32a^2 - 75|D_a} \alpha_{201} + \frac{147\sqrt{5}(8a^2 + 25)(32a^2 - 75)(3328a^4 + 5400a^2 - 3375)}{20(4a^2 - 5)(16a^2 + 15)D_a} \alpha_{111} \\
\approx \frac{147(8a^2 + 25)(32a^2 - 75)(3328a^4 + 5400a^2 - 3375)A_a}{4(4a^2 - 5)(16a^2 + 15)D_a} \alpha_{201} \\
\end{align*}
\]
\[
\begin{align*}
\frac{1024\sqrt{5}(8a^2 + 25)B_e}{16 + a^2 + 15} & \left[ \frac{1543503872a^{14} + 6099828736a^{12} - 36852367360a^{10} - 19248000000a^8}{D_e \omega_t^2} \right] a_{011} \\
\frac{104343000000a^6 - 135160875000a^4 - 352571484375a^2 - 19221769875}{D_e \omega_t^2} & a_{011} \\
\end{align*}
\]

\[
b_{201} = \frac{\sqrt{5}}{48} \left[ \frac{7516192768a^{14} + 108095602688a^{12} + 242027069440a^{10} - 491692032000a^8}{(8a^2 + 25)(16a^2 + 15)D_e \omega_t} \right] a_{011} \\
+ \frac{\sqrt{5}}{40} \left[ \frac{17179869184a^{16} + 389936054272a^{14} + 381681664000a^{12} + 1990696960000a^{10} + 15888230400000a^8}{(4a^2 - 5)(16a^2 + 15)(32a^2 - 75)D_e \omega_t} \right] a_{011} \\
+ \frac{\sqrt{5}}{6981030000000a^6 + 273065625000000a^4 - 49485937500000a^2 - 266967773475}{(4a^2 - 5)(16a^2 + 15)(32a^2 - 75)D_e \omega_t} a_{201} \\
+ \frac{(32a^2 - 75)(8a^2 + 25)^2}{8(4a^2 - 5)(16a^2 + 15)} \left[ \frac{67108864a^{14} + 2878603264a^{12} + 6557491200a^8 + 89104320000a^6}{D_e \omega_t^2} \right] a_{111} \\
+ \frac{(32a^2 - 75)(32a^2 - 75)^2}{16a^2 + 15} \left[ \frac{3580928983040a^{16} + 9305952616448a^{14} + 196061429760000a^{12}}{D_e \omega_t^2} \right] a_{011} \\
+ \frac{(8a^2 + 25)(4a^2 - 5)^2}{481036337152a^{18} + 5655398187008a^{16} + 168720828241920a^{14}} \left[ \frac{785713520000a^{10} + 47351270400000a^8 - 123503310000000a^6}{D_e \omega_t^2} \right] a_{021} \\
+ \frac{2560a^2(8a^2 + 25)(4a^2 - 5)^3}{16a^2 + 15} \left[ \frac{481036337152a^{18} + 5655398187008a^{16} + 168720828241920a^{14}}{D_e \omega_t^2} \right] a_{011} \\
+ \frac{5109832089600a^{12} + 274010419200000a^{10} + 661377215200000a^8 + 40359375000000a^6}{D_e \omega_t^2} a_{111} \\
+ \frac{1894665937500000a^4 + 2928636474609375a^2 + 108121948241875}{D_e \omega_t^2} a_{031} \\
\end{align*}
\]

\[
b_{101} = -a_{011} - \frac{2\sqrt{5}}{5} \left[ \frac{34359738368a^{16} + 36238786560a^{14} + 443023360000a^{12} + 2845179904000a^{10}}{(8a^2 + 25)(16a^2 + 15)(32a^2 - 75)D_e \omega_t} \right] a_{011} \\
+ \frac{12819968000000a^{10} + 17001600000000a^8 - 25578000000000a^6}{(8a^2 + 25)(16a^2 + 15)(32a^2 - 75)D_e \omega_t} a_{101} \\
+ \frac{468000000000000a^4 + 200126953125000a^2 - 13348388671875}{(8a^2 + 25)(16a^2 + 15)(32a^2 - 75)D_e \omega_t} a_{101} \\
+ \frac{96\sqrt{5}a^4(4a^2 - 5)^5}{5(16a^2 + 15)} \left[ \frac{536870912a^{14} + 96468992a^{12} + 6935347200a^{10} + 4850176000a^8}{D_e \omega_t^2} \right] a_{111} \\
+ \frac{1931256000000a^6 + 70267500000a^4 - 171843750000a^2 + 11865234375}{(32a^2 - 75)^2D_e \omega_t} a_{201} \\
+ \frac{672a^2(4a^2 - 5)(8a^2 + 25)^2}{16a^2 + 15} \left[ \frac{2326528a^6 - 41164800a^4 - 31260000a^2 - 8775000a^2 + 18984375}{16a^2 + 15} D_e \omega_t^2 \right] a_{111} \\
+ \frac{96\sqrt{5}a^4(4a^2 - 5)(8a^2 + 25)^5}{16a^2 + 15} \left[ \frac{17179869184a^{16} + 74625056768a^{14} - 982492643328a^{12} + 1010963251200a^{10}}{(32a^2 - 75)D_e \omega_t^2} \right] a_{021} \\
+ \frac{3191193600000a^4 - 1906850400000a^2 - 768930750000a^2 - 635343750000a^2 + 1601806640625}{(32a^2 - 75)D_e \omega_t^2} a_{021} \\
+ \frac{491520a^2}{16a^2 + 15} \left[ \frac{51539607552a^{16} + 428691423232a^{14} + 470936453120a^{12} - 4153282560000a^{10}}{(32a^2 - 75)D_e \omega_t^2} \right] a_{031} \\
+ \frac{6479961600000a^8 + 1038760800000a^6 - 1525860000000a^4 - 2544539062500a^2 - 10144775390625}{(32a^2 - 75)D_e \omega_t^2} a_{031} \\
\end{align*}
\]
\[
\begin{align*}
a_{301} &= \frac{(32a^2 - 75)^2}{4608(8a^2 + 25)} \left[ 67108864a^4 - 115343360a^{12} - 2115502080a^{10} + 715878400a^8 \right. \\
&\quad \left. - \frac{298880000a^6 + 16416300000a^4 - 2303437500a^2 - 1423828125}{(16a^2 + 15)B_6 D_a} \right] a_{011} + \frac{\sqrt{5}(32a^2 + 9)(32a^2 - 75)^4}{(4608a^2(4a^2 - 5))^5} a_{011} \\
&\quad + \frac{(32a^2 - 75)}{768(16a^2 + 15)} \left[ \frac{1}{(16a^2 + 15)B_6 D_a} \right] (2704 P. Yu, Y. Tian / Commun Nonlinear Sci Numer Simulat 19 (2014) 2690–2705) \\
&\quad - \frac{52140444160a^8 - 8094080000a^6 - 7945500000a^4 + 1307812500a^2 + 474609375}{768(16a^2 + 15)B_6 D_a} \right] a_{201} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)^2(32a^2 + 9)(32a^2 - 75)^5}{768(16a^2 + 15)B_6 D_a} \right] a_{111} \\
&\quad - \frac{5(8a^2 + 25)^3(32a^2 + 9)^2(32a^2 - 75)(17408a^6 - 40640a^4 - 67500a^2 + 28125)}{768(16a^2 + 15)B_6 D_a} \right] a_{002} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)(32a^2 - 75)^3}{3(16a^2 + 15)} \left[ \frac{21474836464a^4 + 28689039360a^4 + 83844136960a^4 - 49192140800a^4}{D_a \epsilon \Omega^2} \right] \right] a_{031} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)(32a^2 + 9)(32a^2 - 75)(136a^2 + 75)(17408a^6 - 40640a^4 - 67500a^2 + 28125)}{384(16a^2 + 15)B_6 D_a} \right] a_{012} \\
&\quad + \frac{5(32a^2 - 75)^2}{3(16a^2 + 15)} \left[ \frac{335544320 * a^4 + 7241465856 * a^2 + 17690132480 * a^0 - 85375795200 * a^8}{D_a \epsilon \Omega^2} \right] \right] a_{031} \\
&\quad - \frac{246324832000 * a^4 + 96446400000 * a^4 + 39479062500 * a^2 - 12814453125}{D_a \epsilon \Omega^2} \right] a_{031} \\
&\quad = \frac{(32a^2 - 75)(2048a^6 - 3200a^4 - 13500a^2 + 5625)}{1536(8a^2 + 25)} \left[ \frac{1}{(16a^2 + 15)B_6 D_a} \right] a_{011} + \frac{\sqrt{5}(32a^2 - 75)^2}{23040(8a^2 + 25)} a_{011} \\
&\quad + \frac{(17408a^6 - 40640a^4 - 67500a^2 + 28125)}{3840(8a^2 + 25)} \left[ \frac{1}{(16a^2 + 15)B_6 D_a} \right] a_{021} \\
&\quad - \frac{\sqrt{5}(32a^2 - 75)(16384a^4 + 109056a^4 - 558080a^4 - 52500a^2 + 84375)}{3840(16a^2 + 15)B_6 D_a} \right] a_{111} \\
&\quad - \frac{3(16a^2 + 15)}{768(16a^2 + 15)} \left[ \frac{67108864a^{14} - 6291456a^{12} - 2059927552a^{10} + 52140444160a^8}{B_6 D_a} \right] a_{002} \\
&\quad - \frac{52140444160a^8 - 8094080000a^6 - 7945500000a^4 + 1307812500a^2 + 474609375}{768(16a^2 + 15)B_6 D_a} \right] a_{201} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)^2(32a^2 + 9)(32a^2 - 75)^5}{768(16a^2 + 15)B_6 D_a} \right] a_{111} \\
&\quad - \frac{5(8a^2 + 25)^3(32a^2 + 9)^2(32a^2 - 75)(17408a^6 - 40640a^4 - 67500a^2 + 28125)}{768(16a^2 + 15)B_6 D_a} \right] a_{002} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)(32a^2 - 75)^3}{3(16a^2 + 15)} \left[ \frac{21474836464a^4 + 28689039360a^4 + 83844136960a^4 - 49192140800a^4}{D_a \epsilon \Omega^2} \right] \right] a_{031} \\
&\quad - \frac{\sqrt{5}(8a^2 + 25)(32a^2 + 9)(32a^2 - 75)(136a^2 + 75)(17408a^6 - 40640a^4 - 67500a^2 + 28125)}{384(16a^2 + 15)B_6 D_a} \right] a_{012} \\
&\quad + \frac{5(32a^2 - 75)^2}{3(16a^2 + 15)} \left[ \frac{335544320 * a^4 + 7241465856 * a^2 + 17690132480 * a^0 - 85375795200 * a^8}{D_a \epsilon \Omega^2} \right] \right] a_{031} \\
&\quad - \frac{246324832000 * a^4 + 96446400000 * a^4 + 39479062500 * a^2 - 12814453125}{D_a \epsilon \Omega^2} \right] a_{031} \\
&\quad = \frac{(32a^2 - 75)(2048a^6 - 3200a^4 - 13500a^2 + 5625)}{1536(8a^2 + 25)} \left[ \frac{1}{(16a^2 + 15)B_6 D_a} \right] a_{011} + \frac{\sqrt{5}(32a^2 - 75)^2}{23040(8a^2 + 25)} a_{011} \end{align*}
\]

where \( \epsilon \) is given in (19), \( A_6 \), \( B_6 \) and \( C_6 \) are given in (21), and \( D_6 = 2048a^67360a^4 - 55300a^2 + 16875 \).

References
[12] Bautin N. On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. Math Sbornik (N.S.) 1952;30:181–96.