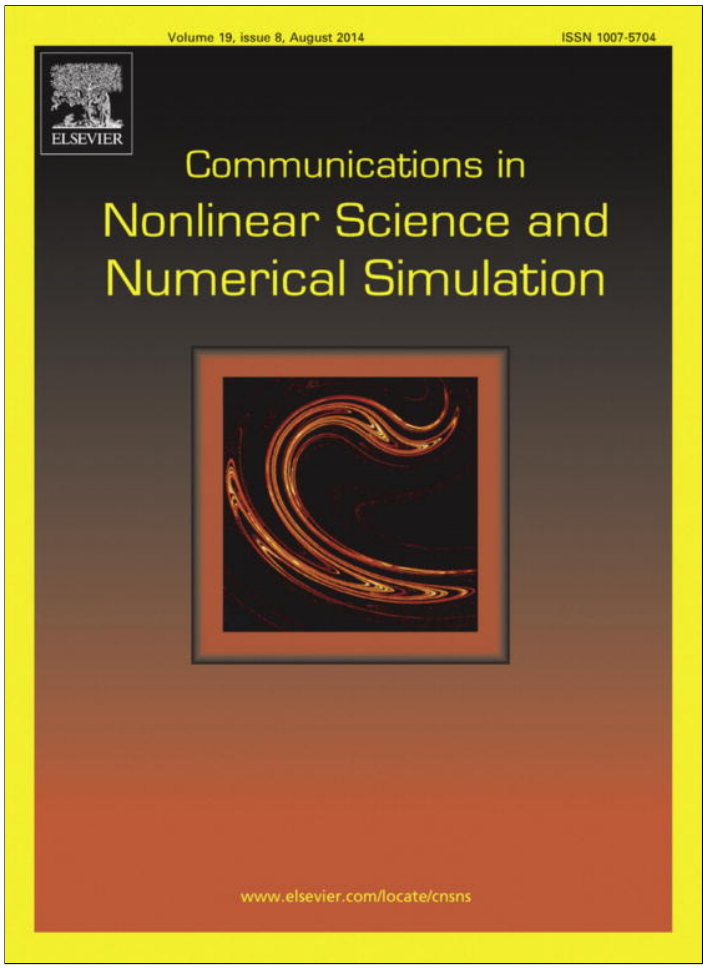

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Twelve limit cycles around a singular point in a planar cubic-degree polynomial system

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ABSTRACT

In this paper, we prove the existence of 12 small-amplitude limit cycles around a singular point in a planar cubic-degree polynomial system. Based on two previously developed cubic systems in the literature, which have been proved to exhibit 11 small-amplitude limit cycles, we applied a different method to show 11 limit cycles. Moreover, we show that one of the systems can actually have 12 small-amplitude limit cycles around a singular point. This is the best result so far obtained in cubic planar vector fields around a singular point.

1. Introduction

Studying bifurcation of limit cycles in planar polynomial systems is related to the second part of the well-known Hilbert's 16th problem [1]. The progress in the solution of the problem is very slow. It has not even solved the simplest quadratic systems after more than one century since the problem was posed by Hilbert at the Paris conference of the International Congress of Mathematicians in 1900. More precisely, the second part of Hilbert's 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that the following system,

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \tag{1}$$

can have, where $P_n(x,y)$, and $Q_n(x,y)$, represent nth-degree polynomials of x, and y. In early 1990's, Ilyashenko and Yakovenko [2], and Écalle [3] independently proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, the best result is $H(2) \geqslant 4$, obtained more than 30 years ago [4,5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$, is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geqslant 13$ [7,8]. Note that the 13 limit cycles obtained in [7,8] are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [9].

In order to help understand and attack Hilbert's 16th problem the so called weak Hilbert's 16th problem was posed by Arnold [10], which is closely related to the so-called near-Hamiltonian system [11]:

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$$\dot{x} = H_y(x,y) + \varepsilon p_n(x,y), \quad \dot{y} = -H_x(x,y) + \varepsilon q_n(x,y), \tag{2}$$

where $H(x,y)$, $p_n(x,y)$, and $q_n(x,y)$, are all polynomial functions of x, and y, and $0 < \varepsilon \ll 1$, represents a small perturbation. Investigating the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the (first-order) Melnikov function, given as an integral

$$M(h,\delta) = \oint_{H(x,y)=h} q_n(x,y)dx - p_n(x,y)dy, \tag{3}$$

along closed orbits $H(x,y) = h$ for $h \in (h_1, h_2)$, where δ denotes the parameters (or coefficients) involved in the polynomial functions q_n and p_n.

When we focus on the maximum number of small-amplitude limit cycles, $M(n)$, bifurcating from an elementary center or an elementary focus in system (1), one of the best-known results is $M(2) = 3$, which was solved by Bautin in 1952 [12]. For $n = 3$, a number of results have been obtained. Around an elemental focus, James and Lloyd [13] considered a particular class of cubic systems to obtain 8 limit cycles in 1991, and the systems were reinvestigated couple of years later by Ning et al. [14] to find another solution of 8 limit cycles. Yu and Corless [15] constructed a cubic system and combined symbolic and numerical computations to show 9 limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [16]. Another cubic system was also recently constructed by Lloyd and Pearson [17] to show 9 limit cycles with purely symbolic computation.

On the other hand, around a center, there are also a few results obtained in the past two decades. Żołądek studied classification of cubic centers and listed 17 cases for reversible centers and 35 cases for Darboux centers [18,19]. In 1995, Żołądek [20] first proposed a rational Darboux integral,

$$H_0 = \frac{f_1^5}{f_2^4} = \frac{(x^4 + 4x^2 + 4y)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^4}, \tag{4}$$

and used it to prove the existence of 11 small-amplitude limit cycles around a center. This result was extensively cited by many researchers in this area. After more than ten years, another two cubic systems are constructed to show 11 limit cycles [21,22]. Recently, the system defined by (4) was reinvestigated by Yu and Han with the method of focus value computation, who only obtained 9 limit cycles [23]. This obvious difference motivated a further investigation on this problem. Very recently, Tian and Yu [24] has proved that the 11 limit cycles obtained by Żołądek [20] are not correct, and the mistakes leading to the erroneous result have been identified.

In this paper, we will consider the two cubic systems proposed by Christopher [21], and Bondar and Sadovskii [22]. The first system discussed in [21] is determined by a Darboux first integral,

$$H_1 = \frac{(xy^2 + x + 1)^5}{x^3\left(xy^5 + \frac{5}{2}xy^3 + \frac{5}{2}y^3 + \frac{15}{8}xy + \frac{15}{4} + a\right)^2}, \tag{5}$$

where a is a parameter, from which we obtain the following dynamical system,

$$\begin{aligned} \dot{x} &= 10(32a^2 - 75)^2 x(-6 - 9x - 3x^2 + 8axy - 12y^2), \\ \dot{y} &= (32a^2 - 75)^2(24a - 16ax + 90y + 15xy - 16axy^2 + 60y^3), (32a^2 - 75 \neq 0). \end{aligned} \tag{6}$$

System (6) has an equilibrium point, given by

$$x_e = \frac{6(8a^2 + 25)}{32a^2 - 75}, \quad y_e = \frac{70a}{32a^2 - 75}. \tag{7}$$

Shifting the equilibrium point (x_e, y_e) to the origin and setting $a = 2$ yields the system:

$$\begin{aligned} \dot{x} &= -10(342 + 53x)(289x - 2112y + 159x^2 - 848xy + 636y^2), \\ \dot{y} &= -605788x + 988380y - 432745xy + 755568y^2 - 89888xy^2 + 168540y^3, \end{aligned} \tag{8}$$

which has been studied in [21] to show 11 small-amplitude limit cycles around the origin (i.e., around the equilibrium point (x_e, y_e) of system (6)).

The second system given in [22] is described by

$$\begin{aligned} \dot{x} &= y[1 - 2r(3r^2 + 5)x + (r^2 + 3)(3r^2 + 1)^2 x^2] \equiv \tilde{f}_1(x,y), \\ \dot{y} &= -x(1 - 8rx)[1 - 3r(r^2 + 3)x] + 2[2(3r^2 - 1) - r(r^2 + 3)(15r^2 - 7)x]xy - [r(r^2 + 11) - (r^2 + 3)(3r^4 + 22r^2 - 1)x]y^2 \\ &\quad + 2r(r^2 + 3)(r^2 - 1)y^3 \equiv \tilde{f}_2(x,y), \end{aligned} \tag{9}$$

where r is a parameter. It can be shown that the origin of system (9) is a center [22].

To find the small-amplitude limit cycles bifurcating from the origin of the systems (8) and (9), in general we may apply perturbations to the systems and then compute the Melnikov functions around the loops defined by the first integral

$H(x,y)=h$. For system (8), we may use H_1, while for system (9), we need to find the first integral, which is not an easy job. Even we have these H functions, it is difficulty to compute the Melnikov functions. Therefore, we turn to using focus value computation to analyze bifurcation of limit cycles around the origin. Suppose the focus values around the origin of the system (8) or (9) are given in the following form:

$$V=\sum_{i\geqslant 0}\varepsilon^i V_i,\quad \text{where}\quad V_i=\{v_{i0},v_{i1},v_{i2},\ldots\},\tag{10}$$

where ε is a small perturbation parameter. We call v_{ij} the jth ε^i-order focus value of the system, and note that $v_{0j}=0,\ j=0,1,2,\ldots$ since the origin is a center of these two systems. We use $M(n)$ to denote the number of limit cycles bifurcating from a singular point, where n is the order of the system.

The rest of the paper is organized as follows. In the next section, we use the method of focus value computation to show that there are 11 small-amplitude limit cycles around the origin of the systems (8) and (9). In Section 3, we use system (6) with the free coefficient a to prove that there exist 12 small-amplitude limit cycles around the origin. Conclusion is drawn in Section 4.

2. 11 Limit cycles in systems (8) and (9)

In this section, we will use the method of focus value computation to show that the systems (8) and (9) can have 11 small-amplitude limit cycles bifurcating from the origin, i.e., $M(3)\geqslant 11$. First, we consider system (8) and have the following result.

Theorem 1. *System (8) can have 11 small-amplitude limit cycles bifurcating from the origin by proper cubic perturbations.*

Proof. Adding perturbations to the non-perturbed system, we have two choices: either to the original system (6) (with $a=2$) or after the shifting the equilibrium (7) to the origin plus a linear transformation applied such that the Jacobian of the resulting system is in the Jordan canonical form. These two choices are equivalent, giving the same result on the number of limit cycles, but the latter is simpler. Therefore, we take the second choice. We first apply a linear transformation and a time scaling, given by

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{bmatrix}1 & 0\\ \frac{289}{2112} & \frac{12\sqrt{23602332855}}{7223040}\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix},\quad t\to\frac{t}{12\sqrt{23602332855}},$$

into system (8) to obtain

$$\begin{aligned}\dot{x}&=y-\frac{38483312\sqrt{23602332855}}{6969827487744}x^2+\frac{5022227}{10593792}xy-\frac{5312\sqrt{23602332855}}{1271255040}y^2-\frac{2039614912\sqrt{23602332855}}{2383681000808448}x^3\\&\quad+\frac{1047757}{21187584}x^2y-\frac{280912\sqrt{23602332855}}{434769223680}xy^2+\varepsilon p_3(x,y)\equiv f_1(x,y)+\varepsilon p_3(x,y),\\ \dot{y}&=-x-\frac{15317}{371712}x^2-\frac{15481312\sqrt{23602332855}}{757589944320}xy+\frac{5149003}{35312640}y^2+\frac{4490071331}{774425276416}x^3\\&\quad-\frac{44892595312\sqrt{23602332855}}{23836810008084480}x^2y+\frac{165731}{47083520}xy^2+\frac{280912\sqrt{23602332855}}{869538447360}y^3+\varepsilon q_3(x,y)\equiv f_2(x,y)+\varepsilon q_3(x,y),\end{aligned}\tag{11}$$

where the linear part of the unperturbed system is now in the Jordan canonical form, and the cubic polynomial perturbations have been added, given in the general form:

$$\begin{aligned}p_3(x,y)&=a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3,\\ q_3(x,y)&=b_{10}x+b_{01}y+b_{20}x^2+b_{11}xy+b_{02}y^2+b_{30}x^3+b_{21}x^2y+b_{12}xy^2+b_{03}y^3.\end{aligned}\tag{12}$$

To make the origin of the system be an elementary center, it requires that $a_{10}+b_{01}=0$, or $b_{01}=-a_{10}$. To further simplify the analysis, introducing another linear transformation and a time scaling, given by

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{bmatrix}\varepsilon a_{10} & 1+\varepsilon a_{01}\\ -1+\varepsilon b_{10} & -\varepsilon a_{10}\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix},\quad t\to\frac{t}{\omega_c},$$

where $\omega_c=\sqrt{1+\varepsilon(a_{01}-b_{10})-\varepsilon^2(a_{10}^2+a_{01}b_{10})}$, into system (11) yields

$$\begin{aligned}\dot{x}&=f_1(x,y)+\varepsilon\bar{p}_3(x,y),\\ \dot{y}&=f_2(x,y)+\varepsilon\bar{q}_3(x,y),\end{aligned}\tag{13}$$

where higher-order ε terms have been dropped since system (11) only has ε-order terms added, f_1 and f_2 are given in (11), and $\bar{p}_3$ and $\bar{q}_3$ are given below,

$$\begin{aligned}
\bar{p}_3(x,y) &= \left[a_{20} - \frac{5022227}{10593792}a_{10} + \frac{1937634155}{247808\sqrt{23602332855}}(a_{01}-b_{10})\right]x^2 + \left[a_{11} + \frac{53\sqrt{23602332855}}{52968960}a_{10} - \frac{5022227}{10593792}a_{01}\right]xy \\
&+ \left[a_{02} + \frac{4389205759}{743424\sqrt{23602332855}}(3a_{01}+b_{10})\right]y^2 + \left[a_{30} - \frac{1047757}{21187584}a_{10} + \frac{5404979485}{4460544\sqrt{23602332855}}(a_{01}-b_{10})\right]x^3 \\
&+ \left[a_{21} + \frac{2809\sqrt{23602332855}}{18115384320}a_{10} - \frac{1047757}{21187584}a_{01}\right]x^2y + \left[a_{12} + \frac{232627905227}{254251008\sqrt{23602332855}}(3a_{01}+b_{10})\right]xy^2 + a_{03}y^3, \\
\bar{q}_3(x,y) &= \left[b_{20} + \frac{5233613631}{123904\sqrt{23602332855}}a_{10} - \frac{15317}{371712}b_{10}\right]x^2 + \left[b_{11} + \frac{4832063}{26484480}a_{10}\frac{3585623893}{123904\sqrt{23602332855}}(a_{01}-b_{10})\right]xy \\
&+ \left[b_{02} - \frac{53\sqrt{23602332855}}{105937920}a_{10} - \frac{5149003}{35312640}a_{01}\right]y^2 + \left[b_{30} + \frac{4327705513}{1486848\sqrt{23602332855}}a_{10} + \frac{4490071331}{774425276416}b_{10}\right]x^3 \\
&+ \left[b_{21} + \frac{8985991}{211875840}a_{10} + \frac{448925953\sqrt{23602332855}}{3972801668014080}(a_{01}+b_{10})\right]x^2y \\
&+ \left[b_{12} - \frac{2809\sqrt{23602332855}}{14492307456}a_{10} - \frac{165731}{47083520}a_{01}\right]xy^2 + \left[b_{03} - \frac{2809\sqrt{23602332855}}{144923074560}(3a_{01}+b_{10})\right]y^3.
\end{aligned} \tag{14}$$

Now, based on system (13), we use focus value computation (e.g., using the Maple program developed in [25]) to obtain the ε-order focus values $v_{1j},\ j = 1,2,\ldots$, all of them are linear functions of a_{ij} and b_{ij}. For example, v_{11} is given by

$$\begin{aligned}
v_{11} &= \frac{8776116145175287}{1132248475384012800}a_{10} - \frac{215539887437\sqrt{23602332855}}{2264496950768025600}a_{01} - \frac{440628167\sqrt{23602332855}}{13242672226713600}b_{10} \\
&- \frac{4187}{60192}a_{20} + \frac{24857\sqrt{23602332855}}{171005581440}a_{11} - \frac{57611}{601920}a_{02} - \frac{2597\sqrt{23602332855}}{5500179520}b_{20} + \frac{159}{12160}b_{11} \\
&- \frac{371\sqrt{23602332855}}{2044631952}b_{02} + \frac{1}{8}(a_{12}+b_{21}+3a_{30}+3b_{03}).
\end{aligned}$$

Thus, we can use any one of the parameters, say b_{03}, to solve $v_{11} = 0$ to obtain b_{03} expressed in terms of other parameters. Similarly, we can solve the equation $v_{12} = 0$ for b_{12}, $v_{13} = 0$ for b_{21}, $v_{14} = 0$ for b_{30}, $v_{15} = 0$ for b_{02}, $v_{16} = 0$ for b_{11}, $v_{17} = 0$ for b_{20}, $v_{18} = 0$ for b_{10}, $v_{19} = 0$ for a_{30}, $v_{110} = 0$ for a_{21}, respectively. Finally, under theses solutions, the 11th- and 12th-order focus values are given by $v_{111} = c_{11}F_{10}$ and $v_{112} = c_{12}F_{10}$, where c_{11} and c_{12} are some constants, and the common factor F_{10} is a function of a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{12} and a_{03}, given by

$$\begin{aligned}
F_{10} &= 691234068956115\sqrt{23602332855}a_{10} + 7770354418542535 7945a_{01} - 2095858834724574 \\
&\times \sqrt{23602332855}a_{20} - 302178196047882421530a_{11} + 3804860775858270\sqrt{23602332855}a_{02} \\
&- 42463100922992640\sqrt{23602332855}a_{12} - 2181474002491403811 8400a_{03}.
\end{aligned}$$

This shows that the best result we can obtain is $v_{1j} = 0,\ j = 0,1,2,\ldots,10$ but $v_{111} \neq 0$, implying that system (11) can have at most 11 small-amplitude limit cycles bifurcating from the origin. The above function F_{10} also implies that besides the b_{ij} parameters, only two a_{ij} parameters are used to solve the focus values. In other words, we may leave one free parameter, say a_{10}, which can be used to scale the focus values, and set all other parameters zero, $a_{01} = a_{20} = a_{11} = a_{02} = a_{12} = a_{03} = 0$. Certainly, one can choose other possible combinations of the parameters a_{ij} and b_{ij} to show the same result. Thus, without loss of generality, we may assume $0 < v_{111} \ll 1$.

Finally, taking small perturbations in backward order on a_{21} for v_{110}, on a_{30} for v_{19}, on b_{10} for v_{18}, on b_{20} for v_{17}, on b_{11} for v_{16}, on b_{02} for v_{15}, on b_{30} for v_{14}, on b_{21} for v_{13}, on b_{12} for v_{12}, on b_{03} for v_{11}, and on b_{01} for v_{10} so that

$$v_{1j}v_{1(j+1)} < 0, \quad |v_{1j}| \ll |v_{1(j+1)}| \quad \text{for } j = 0,1,2,\ldots,10. \tag{15}$$

This shows that there exist 11 small-amplitude limit cycles around the origin of system (8). □

Next, we consider system (9) and have the following result.

Theorem 2. *System (9) can have 11 small-amplitude limit cycles bifurcating from the origin by proper cubic perturbations.*

Proof. According to [22], the perturbations added to system (9) are given by

$$\begin{aligned}
\tilde{p}_3(x,y) &= a_0x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2, \\
\tilde{q}_3(x,y) &= a_0y + a_7x^2 + a_8xy + a_9y^2 + a_{10}x^3 + a_{11}x^2y,
\end{aligned} \tag{16}$$

and so the perturbed system (9) becomes

$$\begin{aligned}\dot{x} &= \tilde{f}_1(x,y) + \varepsilon\tilde{p}_3(x,y),\\ \dot{y} &= \tilde{f}_2(x,y) + \varepsilon\tilde{q}_3(x,y),\end{aligned} \tag{17}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are given in (9). First note that the zero-order ε-order focus value $v_{10} = a_{10}$. Letting $a_{10} = 0$ and then executing the Maple program for computing the focus values [25] results in

$$v_{11} = \frac{1}{8}\left[4r(3r^2+11)a_1 + 8r(r^2+4)a_3 + 3a_4 + a6 + 4(3r^2-1)(a_7+a_9) + 2r(r^2+3)a_8 + a_{11}\right],$$

$$\begin{aligned}v_{12} = -\frac{1}{96}\Big[&4r(729r^8+7816r^6+26110r^4+27656r^2+89)a_1 - 24r^2(3r^2+1)(r^2+3)^2a_2\\ &+8r(245r^8+2773r^6+9669r^4+10643r^2-2)a_3 + (709r^6+5107r^4+7227r^2+45)a_4 - 4r(3r^4+58r^2-13)a_5\\ &+(215r^6+1809r^4+3177r^2+15)a_6 + 4(759r^8+4548r^6-2596r^2+5022r^4+11)a_7\\ &+2r(r^2+3)(263r^6+1601r^4+2585r^2-1)a_8 + 4(735r^8+4744r^6+5778r^4-2744r^2-1)a_9\\ &+20r(3r^4-22r^2+3)a_{10} + (263r^6+1553r^4+2649r^2-17)a_{11}\Big],\\ &\vdots\end{aligned}$$

Similarly, we can linearly solve the equations $v_{1i} = 0$ for $a_i,\ i = 1,2,\ldots,10$ one by one. Then, the 11th and 12th-order focus values are obtained as $v_{111} = 2340\frac{F_{20N}(r)}{F_{20D}(r)}F_{21}(r)$ and $v_{112} = -120\frac{F_{20N}(r)}{F_{20D}(r)}F_{22}(r)$, where F_{20N}, F_{20D}, F_{21} and F_{22} are all polynomials of r, given by

$$F_{20N} = a_{11}r^{12}(r^2-1)^{11}(r^2+3)^8(3r^2+1)^8(3r^6-31r^4-19r^2-1),$$

$$\begin{aligned}F_{20D} = {}&1323377757r^{34} + 26996002137r^{32} + 37965987792r^{30} - 3268655299296r^{28} - 16760339520156r^{26}\\ &+ 12963886884900r^{24} + 38642548810176r^{22} + 3147430151568r^{20} + 6604961423494r^{18}\\ &+ 38723884663134r^{16} + 6683800535760r^{14} - 3528405585600r^{12} - 58333647916r^{10} + 237931831540r^8\\ &+ 32335049504r^6 + 6091835792r^4 + 715493989r^2 + 26239665,\end{aligned}$$

$$F_{21} = 63r^{12} + 868r^{10} + 1407r^8 - 1232r^6 - 803r^4 + 588r^2 + 133,$$

$$\begin{aligned}F_{22} = {}&355761r^{18} + 9537262r^{16} + 77877345r^{14} + 179927041r^{12} + 41987927r^{10} - 169741431r^8 - 36416005r^6\\ &+ 63798403r^4 + 14431452r^2 + 383173.\end{aligned}$$

In order to obtain maximal number of limit cycles, we may solve $F_{21} = 0$ for r. However, unfortunately, $F_{21} = 0$ has no real solutions for r. Thus, the best result we can get is $v_{1i} = 0,\ i = 0,1,2,\ldots,10$, but $v_{111} \neq 0$, implying that system (17) can have at most 11 small-amplitude limit cycles bifurcating from the origin. Again, without loss of generality, we may use the parameter a_{11} to scale the focus values such that $0 < v_{111} \ll 1$. Further, by perturbing the parameters, in backward order, on a_i for $v_{1i},\ i = 10,9,\ldots,0$, such that the relations given in (15) hold, which implies that system (17) exhibits 11 small-amplitude limit cycles around the origin. □

3. 12 Limit cycles in system (6)

Now we return to system (6) and let the parameter a be free to vary. In this case, we have the following result.

Theorem 3. *System (6) can have 12 small-amplitude limit cycles bifurcating from the origin by using proper cubic perturbations with a properly chosen value of a, i.e.,* $M(3) \geqslant 12$.

In order to prove Theorem 3, we need a lemma. Consider the following generally perturbed system:

$$\begin{aligned}\dot{x} &= P(x,y,\delta_1) + \varepsilon p(x,y,\delta_2),\\ \dot{y} &= Q(x,y,\delta_1) + \varepsilon q(x,y,\delta_2),\end{aligned} \tag{18}$$

where $P,\ Q,\ p$ and q are polynomials of x and y. Suppose that the vector parameter δ_1 involved in P and Q is m dimension, while the vector parameter δ_2 involved in p and q is l dimension. It is further assumed that when $\varepsilon = 0$ system (18) is

integrable. In Hamiltonian case, $P = H_y,\ Q = -H_x$, where H is a Hamiltonian. If it is not a Hamiltonian system one needs to multiply the integrating factor in order to use Melnikov function. This increases complexity of computation. While with the method of focus value computation, it does not need to find the integrating factor and thus greatly simplifies the computation. For system (18), we have the following lemma.

Lemma 1. *By properly choosing the parameters δ_1 and δ_2 in system (18), k small-amplitude limit cycles exist around the origin of the system, satisfying $k \leqslant m + l$. The exact number of the limit cycles depends upon how many parameters can be chosen independently to solve the focus value equations (or to determine the zeros of Melnikov functions).*

The proof can follow the proof for Theorem 3 in [26].

We first consider ε-order focus values (equivalent to first-order Melnikov function), and then consider ε^2-order focus values. We will show that using ε^2-order focus values does not increase the number of limit cycles.

3.1. Based on ε-order focus values

Proof. Following the procedure in the proof for Theorem 1, for system (6), we first shift the equilibrium point defined in (7) to the origin, and then apply a liner transformation with a proper time scaling such that the Jacobian of the resulting system evaluated at the origin is in the form of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The time scaling is taken as $t \to \frac{t}{12\omega_c}$, where ω_c is given by

$$\omega_c = \sqrt{5(8a^2+25)(32a^2-75)(16384a^6-14400a^4+165000a^2+84375)}. \tag{19}$$

Then, adding the ε-order perturbation terms $p_3(x,y)$ and $q_3(x,y)$, given by

$$\begin{aligned} p_3(x,y) &= a_{101}x + a_{011}y + a_{201}x^2 + a_{111}xy + a_{021}y^2 + a_{301}x^3 + a_{211}x^2y + a_{121}xy^2 + a_{031}y^3, \\ q_3(x,y) &= b_{101}x + b_{011}y + b_{201}x^2 + b_{111}xy + b_{021}y^2 + b_{301}x^3 + b_{211}x^2y + b_{121}xy^2 + b_{031}y^3, \end{aligned} \tag{20}$$

into system (6), resulting a new system. Here, note that we add one more sub-index "1" to explicitly indicate the ε-order perturbations in order to distinct from the ε^2-order perturbations considered in the next subsection. Further, under the condition $b_{011} = -a_{101}$, we apply another linear transformation and a second time scaling, given below,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{bmatrix} 1 & 0 \\ \frac{-\varepsilon a_{101}}{1+\varepsilon a_{011}} & \frac{\omega_c}{1+\varepsilon a_{011}} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \to \frac{t}{\omega_c^*},$$

where $\omega_c^* = \sqrt{1 + \varepsilon(a_{011} - b_{101}) - \varepsilon^2(a_{101}^2 + a_{011}b_{101})}$, and obtain the final system, given by

$$\begin{aligned} \dot{x} = {} & -\frac{\sqrt{5}C_a}{768a^2(4a^2-5)^2\omega_c}x^2 + \frac{(32a^2-75)A_a}{384B_a}xy - \frac{\sqrt{5}(32a^2-75)\omega_c}{3840B_a}y^2 - \frac{\sqrt{5}(32a^2-75)C_a}{4608B_a\omega_c}x^3 \\ & + \frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}x^2y - \frac{\sqrt{5}(32a^2-75)^2\omega_c}{23040(8a^2+25)B_a}xy^2 \\ & + \varepsilon\Bigg\{\left[a_{201} - \frac{(32a^2-75)A_a}{384B_a}a_{101} + \frac{\sqrt{5}(8a^2+25)C_a}{1536B_a\omega_c}(a_{011}-b_{101})\right]x^2 \\ & + \left[a_{111} - \frac{(32a^2-75)A_a}{384B_a}a_{011} + \frac{\sqrt{5}(32a^2-75)\omega_c}{1920B_a}a_{101}\right]xy + \left[a_{021} + \frac{\sqrt{5}(32a^2-75)\omega_c}{7680B_a}(3a_{011}+b_{101})\right]y^2 \\ & + \left[a_{301} - \frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}a_{101} + \frac{\sqrt{5}(32a^2-75)C_a}{9216B_a\omega_c}(a_{011}-b_{101})\right]x^3 \\ & + \left[a_{211} - \frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}a_{011} + \frac{\sqrt{5}(32a^2-75)^2\omega_c}{11520(8a^2+25)B_a}a_{101}\right]x^2y \\ & + \left[a_{121} + \frac{\sqrt{5}(32a^2-75)^2\omega_c}{46080(8a^2+25)B_a}(3a_{011}+b_{101})\right]xy^2 + a_{031}y^3\Bigg\}, \end{aligned}$$

$$
\begin{aligned}
\dot{y} = & -\frac{(8a^2+25)(32a^2-75)(16a^2+225)}{768B_a}x^2 - \frac{\sqrt{5}(8a^2+25)^2(32a^2-75)^2(4288a^4-1200a^2+3375)}{1920B_a\omega_c}xy \\
& -\frac{(32a^2-75)(2048a^6-30400a^4+23000a^2-28125)}{3840B_a}y^2 + \frac{(8a^2+25)^2(16a^2+225)(32a^2-75)^3(3328a^4-8320a^2-3375)}{3072B_a\omega_c^2}x^3 \\
& +\frac{\sqrt{5}(32a^2-75)^2\omega_c}{46080(8a^2+25)B_a}y^3 - \frac{7\sqrt{5}C_a(16a^2+15)(16a^2+225)}{46080(32a^2+9)B_a\omega_c}x^2y - \frac{(32a^2-75)^2(512a^4-1040a^2-5625)}{46080B_a}xy^2 \\
& +\varepsilon\Bigg\{\left[b_{201} - \frac{5(8a^2+25)(16a^2+225)(32a^2-75)}{3840B_a}b_{101} + \frac{\sqrt{5}(8a^2+25)^2(32a^2-75)^2(1152a^4+2720a^2+3375)}{1280B_a\omega_c}a_{101}\right]x^2 \\
& +\left[b_{111} + \frac{(32a^2-75)(6144a^6-17120a^4-28125)/(960B_a)}{a}{}_{101} + \frac{\sqrt{5}(8a^2+25)^2(32a^2-75)^2(4288a^4-1200a^2+3375)}{3840B_a\omega_c)}(a_{011}-b_{101})\right]xy \\
& +\left[b_{021} + \frac{(32a^2-75)(2048a^6-30400a^4+23000a^2-28125)}{3840B_a}a_{011} - \frac{\sqrt{5}(32a^2-75)\omega_c}{3840B_a}a_{101}\right]y^2 \\
& +\left[b_{301} + \frac{5(8a^2+25)(16a^2+225)(3328a^4-8320a^2-3375)C_a}{15360(32a^2+9)B_a\omega_c^2}b_{101} - \frac{\sqrt{5}(2816a^4-16000a^2-10125)C_a}{15360(32a^2+9)B_a\omega_c}a_{101}\right]x^3 \\
& +\left[b_{211} + \frac{7(32a^2-75)^2(256a^4-400a^2-1125)}{23040B_a}a_{101} + \frac{7\sqrt{5}(16a^2+15)(16a^2+225)C_a}{92160(32a^2+9)B_a\omega_c}(a_{011}-b_{101})\right]x^2y \\
& +\left[b_{121} + \frac{(32a^2-75)^2(512a^4-1040a^2-5625)}{46080B_a}a_{011} - \frac{\sqrt{5}(32a^2-75)^2\omega_c}{9216(8a^2+25)B_a}a_{101}\right]xy^2 + \left[b_{031} - \frac{\sqrt{5}(32a^2-75)^2\omega_c}{92160(8a^2+25)B_a}(3a_{011}+b_{101})\right]y^3\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
A_a &= 2048a^6 - 768a^4 - 4600a^2 - 5625, \\
B_a &= a^2(4a^2-5)^2(8a^2+25), \\
C_a &= (8a^2+25)(32a^2+9)(32a^2-75)^3,
\end{aligned} \tag{21}
$$

and ω_c is given in (19). Now applying the method of focus value computation (e.g., executing the Maple program in [25]) yields $v_{1j},\ j = 1, 2, \ldots$. For example,

$$
\begin{aligned}
v_{11} = & \frac{1}{8}[a_{121}+b_{211}+3(a_{301}+b_{031})] + \Bigg[\frac{(32a^2-75)^3(805306368a^{12}+3501916160a^{10}-8707840000a^8-6744000000a^6)}{(460800a^2(4a^2-5)^2(8a^2+25)\omega_c^2} \\
& +\frac{(32a^2-75)^3(-55188750000a^4-33328125000a^2-83056640625)}{(460800a^2(4a^2-5)^2(8a^2+25)\omega_c^2}\Bigg]a_{101} \\
& -\frac{7\sqrt{5}(32a^2-75)^3(111616a^6+270400a^4-232500a^2+421875)}{921600B_a\omega_c}a_{011} \\
& -\frac{21\sqrt{5}(32a^2-75)^3(1024a^4+100a^2+5625)}{921600a^2(4a^2-5)^2\omega_c}b_{101} - \frac{(16a^2+15)(32a^2-75)}{96(4a^2-5)(8a^2+25)}a_{201} + \frac{7\sqrt{5}(32a^2-75)^2(128a^2+225)}{1920(4a^2-5)\omega_c}a_{111} \\
& -\frac{(32a^2-75)(128a^2+575)}{960(4a^2-5)(8a^2+25)}a_{021} - \frac{49\sqrt{5}(8a^2+25)(32a^2-75)^2}{320(4a^2-5)\omega_c}b_{201} - \frac{(32a^2-425)(32a^2-75)}{1920(4a^2-5)(8a^2+25)}b_{111} \\
& -\frac{7\sqrt{5}(2a^2+15)(32a^2-75)^2}{48(4a^2-5)\omega_c}b_{021}.
\end{aligned}
$$

Similarly, we linearly solve the polynomial equations one by one for $v_{11} = 0$ using b_{031}, for $v_{12} = 0$ using b_{121}, for $v_{13} = 0$ using b_{211}, for $v_{14} = 0$ using b_{301}, for $v_{15} = 0$ using b_{021}, for $v_{16} = 0$ using b_{111}, for $v_{17} = 0$ using b_{201}, for $v_{18} = 0$ using b_{101}, for $v_{19} = 0$ using a_{301}, for $v_{110} = 0$ using a_{211}, and then obtain

$$
v_{111} = 12103\frac{F_{30N}}{F_{30D}}F_{31}, \quad v_{112} = -931\frac{F_{30N}}{F_{30D}}F_{32}, \tag{22}
$$

where F_{30D} is a 62th-degree polynomial of a, and F_{30N} is given by

$$
\begin{aligned}
F_{30N} = & \frac{15(32a^2-75)^3 2(2048a^6-3200a^4-13500a^2+5625)}{a^{12}(4a^2-5)^{11}(8a^2+25)^1 1\omega_c^2}a_{101} \\
& +\frac{\sqrt{5}(16a^2+15)(32a^2-75)^3 3(2048a^6+7360a^4-55300a^2+16875)}{a^{12}(4a^2-5)^{11}(8a^2+25)^1 0\omega_c^3}a_{011} \\
& -\frac{6(8a^2+25)(32a^2-75)^3 1(17408a^6-40640a^4-67500a^2+28125)}{a^{12}(4a^2-5)^{11}(8a^2+25)^{11}\omega_c^2}a_{201}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{6\sqrt{5}(32a^2-75)^{32}(16384a^8+109056a^6-558080a^4-52500a^2+84375)}{a^{12}(4a^2-5)^{11}(8a^2+25)^9\omega_c^3}a_{111}\\
&+\left[\frac{30(32a^2-75)^{32}(67108864a^{14}-6291456a^{12}-2059927552a^{10}+5214044160a^8)}{a^{12}(4a^2-5)^{11}(8a^2+25)^9\omega_c^4}\right.\\
&\left.+\frac{30(32a^2-75)^{32}(-809408000a^6-7945500000a^4+1307812500a^2+474609375)}{a^{12}(4a^2-5)^{11}(8a^2+25)^9\omega_c^4}\right]a_{021}\\
&-\frac{23040(32a^2-75)^{31}(16a^2+15)(2048a^6+7360a^4-55300a^2+16875)}{a^{10}(4a^2-5)^9(8a^2+25)^8\omega_c^4}a_{121}\\
&-\left[\frac{7680\sqrt{5}(32a^2-75)^{32}(44171264a^{10}+194867200a^8-849510400*a^6)}{a^{10}(4a^2-5)^9(8a^2+25)^8\omega_c^5}\right.\\
&\left.+\frac{7680\sqrt{5}(32a^2-75)^{32}(-1291260000a^4+47362500a^2+170859375)}{a^{10}(4a^2-5)^9(8a^2+25)^8\omega_c^5}\right]a_{031}.
\end{aligned}
$$

The lengthy expressions for F_{31} and F_{32} are given in Appendix A. It can be shown by using the Groebner basis reduction procedure that $F_{32}|_{F_{31}=0}\neq 0$ and $F_{30D}|_{F_{31}=0}\neq 0$. The polynomial equation $F_{31}=0$ has three real solutions for a^2 (up to 1000 digit points, but only list 50 digit points here):

$$
\begin{aligned}
a^2 = {} & 4.0800973527117710361029748439520196435462690445802 1\ldots,\\
& 55.4186330411036726081995166213765432023414346476832 1\ldots,\\
& 244.1815793145813410974746372740248973294610865416749 8\ldots,
\end{aligned}
$$

and all of them satisfy $\omega_c>0$ (see the expression of ω_c given in (19)). So there are in a total six solutions. Taking the positive value of the second solution for a:

$$a = 7.4443692171401381002446239854626739506334765029627 2\ldots$$

and setting the non-used parameters $a_{011}=a_{201}=a_{111}=a_{021}=a_{121}=a_{031}=0$, and $a_{101}=1$, we obtain the following critical parameter values:

$$
\begin{aligned}
b_{031} &= -0.19166152145498089355202548357797946751132495935848\ldots,\\
b_{121} &= 0.11417084514014593144202950698087209236746686969414\ldots,\\
b_{211} &= -1.21440862253395164484193434261717547030904451575014\ldots,\\
b_{301} &= 0.13315882740016516186295687278609694132824695861367\ldots,\\
b_{021} &= 0.39631189749427819043808615679104642347912703580286\ldots,\\
b_{111} &= -5.32984926540348883870841754242680645572842701602586\ldots,\\
b_{201} &= -0.12480581817579714308657357120962081536587770395955\ldots,\\
b_{101} &= -19.01617439016444745664609221188921421020650255893027\ldots,\\
a_{301} &= -0.34039103269248693441579309816048517526296907212322\ldots,\\
a_{211} &= -0.15448521013159245023264811458682922516993711282715\ldots,
\end{aligned}
$$

under which the focus values become

$$
\begin{aligned}
&v_{11}=-0.6\times 10^{-999},\quad v_{12}=0.4\times 10^{-999},\quad v_{13}=-0.6\times 10^{-999},\\
&v_{14}=0.12\times 10^{-998},\quad v_{15}=-0.13\times 10^{-997},\quad v_{16}=0.8\times 10^{-997},\\
&v_{17}=0.4\times 10^{-996},\quad v_{18}=0.8\times 10^{-995},\quad v_{19}=-0.2\times 10^{-994},\\
&v_{110}=0.16\times 10^{-992},\quad v_{111}=-0.5\times 10^{-991},\\
&v_{112}=0.3444862811176206155109830801 64\ldots\times 10^{-18}.
\end{aligned}
$$

Therefore, we can take perturbations in backward order on a for v_{111}, on a_{211} for v_{110}, on a_{301} for v_{19}, on b_{101} for v_{18}, on b_{201} for v_{17}, on b_{111} for v_{16}, on b_{021} for v_{15}, on b_{301} for v_{14}, on b_{211} for v_{13}, on b_{121} for v_{12}, on b_{031} for v_{11}, on b_{011} for v_{10}, to obtain 12 small-amplitude limit cycles bifurcating from the origin. □

3.2. Based on ε^2-order focus value

Proof. In order to show that higher-order Melnikov functions will not generate more limit cycles, equivalently, here we use the ε^2-order focus values to prove this. To achieve this, we change the perturbations to include the ε^2-order perturbations, given in the form of

$$\begin{aligned} p_3(x,y) &= a_{101}x + a_{011}y + a_{201}x^2 + a_{111}xy + a_{021}y^2 + a_{301}x^3 + a_{211}x^2y + a_{121}xy^2 + a_{031}y^3 + \varepsilon[a_{102}x + a_{012}y + a_{202}x^2 \\ &\quad + a_{112}xy + a_{022}y^2 + a_{302}x^3 + a_{212}x^2y + a_{122}xy^2 + a_{032}y^3], \\ q_3(x,y) &= b_{101}x + b_{011}y + b_{201}x^2 + b_{111}xy + b_{021}y^2 + b_{301}x^3 + b_{211}x^2y + b_{121}xy^2 + b_{031}y^3 + \varepsilon[b_{102}x + b_{012}y + b_{202}x^2 \\ &\quad + b_{112}xy + b_{022}y^2 + b_{302}x^3 + b_{212}x^2y + b_{122}xy^2 + b_{032}y^3]. \end{aligned} \tag{23}$$

In order to make the origin an elementary center, we have $b_{011} = -a_{101}$ and $b_{012} = -a_{102}$. Also, in order to have all ε-order focus values to vanish, we may solve a_{121} from the equation $F_{30N} = 0$. Then, all the solutions for the ε-order perturbation parameters are obtained, as given in Appendix B. Now, we use these parameter expressions to simplify the ε^2-order focus values $v_{2j},\ j = 1, 2, \ldots$ and then linearly solve the polynomial equations one by one for $v_{21} = 0$ using b_{032}, for $v_{22} = 0$ using b_{122}, for $v_{23} = 0$ using b_{212}, for $v_{24} = 0$ using b_{302}, for $v_{25} = 0$ using b_{022}, for $v_{26} = 0$ using b_{112}, for $v_{27} = 0$ using b_{202}, for $v_{28} = 0$ using b_{102}, for $v_{29} = 0$ using a_{302}, and for $v_{210} = 0$ using a_{212}. Finally, we obtain

$$\begin{aligned} v_{211} &= -146381281540702208\frac{F_{40N}}{5F_{30D}}F_{31}, \\ v_{212} &= 180161577280864256\frac{F_{40N}}{5F_{30D}}F_{32} \end{aligned} \tag{24}$$

where the common factor F_{40N} is a function of the unused ε-order perturbation parameters, $a_{101},\ a_{011},\ a_{201},\ a_{111},\ a_{021},\ a_{031}$ and ε^2-order perturbation parameters, $a_{102},\ a_{012},\ a_{202},\ a_{112},\ a_{022},\ a_{122},\ a_{032}$, while the polynomial functions, F_{30D}, F_{31} and F_{32} are exactly the same as that obtained from using the ε-order focus values, see Eq. (22). Therefore, the best result we can have is $v_{2j} = 0,\ j = 0, 1, 2, \ldots, 10$, but $v_{211} \neq 0$, indicating that using ε^2-order perturbations still gives 12 small-amplitude limit cycles bifurcating from the origin. This suggests that using higher-order perturbations may not increase the number of limit cycles. □

4. Conclusion

In this paper, we have applied the method of focus value computation to confirm the results of 11 small-amplitude limit cycles around a singular point in two existing systems in the literature. Further, we used one of the two systems with a free parameter to obtain 12 small-amplitude limit cycles. This is the best result so far obtained in cubic planar vector fields around a singular point.

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This work was supported by the Natural Science and Engineering Research Council of Canada (NSERC).

Appendix A

In this appendix, we list the expressions for F_{31}, F_{32} in Eq. (22)..

$$\begin{aligned} F_{31} = {} & \frac{1000026716592044048191270590349312}{124556484375}a^{52} + \frac{819686924091392323696339765486944256}{124556484375}a^{50} \\ & + \frac{631565347915542732579532606804066304}{332150625}a^{48} - \frac{1444916542179021261406741104134413352 96}{199290375}a^{46} \\ & - \frac{1155070693768529876482855013072911728 64}{39858075}a^{44} \\ & - \frac{673280660493096641039914753995927715840}{531441}a^{42} \\ & - \frac{867565289921678101848586262362595239526400}{177147}a^{40} \\ & + \frac{1635541261270023148372887981613624852480000}{59049}a^{38} \\ & + \frac{7023522799360115593377651085025896038400000 0}{59049}a^{36} \\ & + \frac{3768999767982326590470655765207433216000000 00}{6561}a^{34} \\ & + \frac{1716840349613790147437200519764208000000000 00}{243}a^{32} \\ & - \frac{2596904471315354305194736912364454250000000 00}{2187}a^{30} \\ & + \frac{2298402431789470657144226366532002092285156 2500}{243}a^{28} \end{aligned}$$

$$
\begin{aligned}
&+\frac{908740714146641997798947934773427473907470703125}{324}a^{26}\\
&+\frac{4125233345223780686551133296904906654521942138671875}{165888}a^{24}\\
&+\frac{1075678198271930277811482872121156714215278625488281 25}{1179648}a^{22}\\
&+\frac{2849995705806023431084272992491276653122901916503906 25}{50331648}a^{20}\\
&-\frac{7041956332442795969869675026556977300061583518981933 59375}{536870912}a^{18}\\
&-\frac{4108532797322625013679635425957694539333924651145935058 59375}{68719476736}a^{16}\\
&-\frac{1625039201196595284837734125504377515189286321401596069335 9375}{1099511627776}a^{14}\\
&-\frac{8347931071955375676942522836771027591779036447405815124511 71875}{35184372088832}a^{12}\\
&-\frac{7301053088429557753325367797887383441864349879324436187744 140625}{281474976710656}a^{10}\\
&-\frac{1392720382042689388626305375788253537783790961839258670806 884765625}{72057594037927936}a^{8}\\
&-\frac{1090784349176422278436971579871882796762105135712772607803 3447265625}{1152921504606846976}a^{6}\\
&-\frac{5134794477451035667138574554177181763327644148375838994979 8583984375}{18446744073709551616}a^{4}\\
&-\frac{1209026166451524997276912796619257521505915065063163638114 92919921875}{29514790517935282585 6}a^{2}\\
&-\frac{6641108029952768765569085330070167137276939683943055570125579833984375}{3022314549036572936765 44}
\end{aligned}
$$

$$
\begin{aligned}
F_{32} =\ &\frac{58017949360473565698575840235528646893240 32}{84075626953125}a^{66}\\
&+\frac{4871211304076925410039730660578079195954937856}{84075626953125}a^{64}\\
&+\frac{9811561701770134762901213332237710167822984806 4}{5605041796875}a^{62}\\
&-\frac{1962468978666360643254373266776181402601021610 3936}{3363025078125}a^{60}\\
&-\frac{2125659190521304240539622185463297620029911492 1984}{134521003125}a^{58}\\
&-\frac{1952636774243223717088615524147792905554359599759 36}{1793613375}a^{56}\\
&-\frac{7011943597709914198500295736221723652294678493003776}{1076168025}a^{54}\\
&+\frac{5772105063198199391608064251261116917800389025202176}{43046721}a^{52}\\
&+\frac{7099203873714475705835654093966317126395109656494080 0}{4782969}a^{50}\\
&+\frac{1042963490507502149127717283982071424932992218300416 0000}{14348907}a^{48}\\
&+\frac{8385315027121568675222022631226494619530119505510400 0000}{4782969}a^{46}\\
&+\frac{8743796218046757949645763496050292431213076152320000 0000}{531441}a^{44}\\
&+\frac{5826786881587727341647331096975595433857419827200000 000000}{531441}a^{42}\\
&+\frac{2129248889639611408384650654516989568052818765500000 000000000}{531441}a^{40}\\
&+\frac{4724000936731734865616882474478522351715383043085937 5000000}{59049}a^{38}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1091949162042667875059900783614431515362949461654663085937 5}{1458}a^{36}\\
&+\frac{8914012837692734767094214440472186181129530564042282104492187 5}{2519424}a^{34}\\
&+\frac{6296912664284268845753057522531046117627256985972404479980468 75}{11943936}a^{32}\\
&-\frac{1301864027204367658036176263281189260151266012057805061340332031 25}{382205952*a^{3}0}\\
&-\frac{1191602731163460126952671814354391550383867397084074616432189941 40625}{48922361856}a^{28}\\
&-\frac{1658476439415273982904742793005386943419111250971034169197082519 53125}{21743271936}a^{26}\\
&-\frac{2440766420396022071583631338583161198395057803944002836942672729 4921875}{1855425871872}a^{24}\\
&-\frac{2682298283129296324144186713659500177601558668601047247648239135 7421875}{3298534883328}a^{22}\\
&+\frac{1237239683429339123586281174037416881980503446010383777320384979 248046875}{70368744177664}a^{20}\\
&+\frac{7253324110342472446370228429078979222726981398995849303901195526 123046875}{140737488355328}a^{18}\\
&+\frac{4154323473408135675207380714496250685227393743325475952588021755 218505859375}{72057594037927936}a^{16}\\
&+\frac{4232947777440561112067041403216392524222804188581176276784390211 1053466796875}{2305843009213693952}a^{14}\\
&-\frac{9872142202266738724544947588328934558180174750800733818323351442 813873291015625}{29514790517935282585 6}a^{12}\\
&-\frac{3075933919260594108729245979125965632504422969309030122531112283 4682464599609375}{59029581035870565171 2}a^{10}\\
&-\frac{1051741833916907265828111726023730566444697954171402329848206136 3756656646728515625}{3022314549036572936765 44}a^{8}\\
&-\frac{1461985206934809381156176030888797606798934586089288245602801907 8075885772705078125}{1208925819614629174706176}a^{6}\\
&-\frac{9278618885203336290965302082672859519981102111803039633741718716 9194221496582031 25}{4835703278458516698824704}a^{4}\\
&-\frac{3928798540956683603865726163827286688006574670861681397582287900 14982223510742187 5}{3868562622766813359059763 2}a^{2}\\
&+\frac{3414593142353645268625120533478459958765926507417987068038200959 5632553100585937 5}{6189700196426901374495621 12}.
\end{aligned}
$$

Appendix B

In this appendix, we list the solutions for the ε-order perturbation parameters under which all the ε-order focus values vanish.

$$
\begin{aligned}
b_{031} = &-\frac{(32a^2-75)(2048a^6-3200a^4-13500a^2+5625)\omega_c^2}{3072(8a^2+25)^2(16a^2+15)B_aD_a}a_{101}-\frac{\sqrt{5}(32a^2-75)^2\omega_c}{46080(8a^2+25)B_a}a_{011}\\
&+\frac{(17408a^6-40640a^4-67500a^2+28125)\omega_c^2}{7680(8a^2+25)(16a^2+15)B_aD_a}a_{201}\\
&+\frac{\sqrt{5}(32a^2-75)(16384a^8+109056a^6-558080a^4-52500a^2+84375)\omega_c}{7680(16a^2+15)B_aD_a}a_{111}\\
&-\frac{32a^2-75}{1536(16a^2+15)}\left[\frac{67108864a^{14}-6291456a^{12}-2059927552a^{10}+5214044160a^8}{B_aD_a}\right.\\
&\left.-\frac{809408000a^6+7945500000a^4-1307812500a^2-474609375}{B_aD_a}\right]a_{021}
\end{aligned}
$$

$$-\frac{\sqrt{5}}{15}\left[\frac{603979776a^14+304087040a^12-19492044800a^10+53643008000a^8}{(16a^2+15)D_a)\omega_c}\right.$$
$$\left.-\frac{7845600000a^6+155209500000a^4+3585937500a^2-21357421875}{(16a^2+15)D_a\omega_c}\right]a_{031},$$

$$b_{121}=\frac{\sqrt{5}\omega_c}{15360}\left[\frac{14680064a^{12}+195297280a^{10}-106700800a^8-1193728000a^6}{(32a^2-75)(8a^2+25)^2(16a^2+15)B_aD_a}\right.$$
$$\left.+\frac{2031300000a^4+1746562500a^2-791015625}{(32a^2-75)(8a^2+25)^2(16a^2+15)B_aD_a}\right]a_{101}+\frac{(32a^2-75)^2(512a^4-1040a^2-5625)}{46080B_a}a_{011}$$
$$-\frac{\sqrt{5}\omega_c}{38400}\left[\frac{22020096a^{12}+985661440a^{10}-81203200a^8-4414080000a^6}{(8a^2+25)(16a^2+15)B_aD_a}\right.$$
$$\left.+\frac{10975500000a^4+8732812500a^2-3955078125}{(8a^2+25)(16a^2+15)B_aD_a}\right]a_{201}$$
$$-\frac{(32a^2-75)(8a^2+25)}{7680(16a^2+15)}\left[\frac{16777216a^{12}+17301504a^{10}-299581440a^8+495603200a^6}{B_aD_a}\right.$$
$$\left.+\frac{1821720000a^4+761062500a^2-474609375}{B_aD_a}\right]a_{111}$$
$$-\frac{\sqrt{5}(8a^2+25)(32a^2+9)(32a^2-75)^2}{16a^2+15}\left[\frac{22020096a^{12}+985661440a^{10}-81203200a^8-4414080000a^6}{D_a}\right.$$
$$\left.+\frac{10975500000a^4+8732812500a^2-3955078125}{D_a}\right]a_{021}$$
$$+\frac{(32a^2-75)}{3(16a^2+15)}\left[\frac{19058917376a^{16}-78454456320a^{14}-480496844800a^{12}+4597039104000a^{10}}{D_a\omega_c^2}\right.$$
$$\left.+\frac{3557792000000a^8-45933048000000a^6-40921650000000a^4+5299804687500a^2+4805419921875}{D_a\omega_c^2}\right]a_{031}$$

$$b_{211}=-\frac{35(32a^2-75)}{46080(8a^2+25)}\left[\frac{1073741824a^{16}+10536091648a^{14}-14659092480a^{12}-130029977600a^{10}}{(16a^2+15)B_aD_a}\right.$$
$$\left.+\frac{206808064000a^8+250732800000a^6-557212500000a^4-176554687500a^2+106787109375}{(16a^2+15)B_aD_a}\right]a_{101}$$
$$+\frac{7\sqrt{5}(16a^2+15)(16a^2+225)(32a^2-75)^3}{(46080a^2(4a^2-5)^2\omega_c}a_{011}+\frac{7}{7680}\left[\frac{536870912a^{16}+17012097024a^{14}}{(16a^2+15)B_aD_a}\right.$$
$$+\frac{536870912a^{16}+17012097024a^{14}+13662945280a^{12}-259778150400a^{10}+308864000000a^8}{7680(16a^2+15)B_aD_a}$$
$$\left.+\frac{317592000000a^6-914512500000a^4-294257812500a^2+177978515625}{7680(16a^2+15)B_aD_a}\right]a_{201}$$
$$-\frac{\sqrt{5}(8a^2+25)^2(32a^2-75)^2}{7680(16a^2+15)}\left[\frac{4194304a^{12}-52297728a^{10}-169410560a^8-1519603200a^6}{B_aD_a\omega_c}\right.$$
$$\left.-\frac{884160000a^4+420187500a^2-284765625}{B_aD_a\omega_c}\right]a_{111}$$
$$+\frac{7(8a^2+25)^3(32a^2-75)^2}{1536(16a^2+15)}\left[\frac{32078036992a^{14}-48473571328a^{12}+40654602240a^{10}}{B_aD_a\omega_c^2}\right.$$
$$\left.+\frac{151542988800a^8-581091840000a^6-411075000000a^4-7973437500a^2+64072265625}{B_aD_a\omega_c^2}\right]a_{021}$$
$$-\frac{\sqrt{5}(8a^2+25)(32a^2-75)}{3(16a^2+15)}\left[\frac{3848290697216a^{20}+12685185908736a^{18}-98385621155840a^{16}}{D_a\omega_c^3}\right.$$
$$+\frac{46400222003200a^{14}+874754277376000a^{12}-4888711833600000a^{10}-2960110272000000a^8}{D_a\omega_c^3}$$
$$\left.+\frac{18303755400000000a^6+1074814312500000ja^4-2100858398437500a^2-1081219482421875}{D_a\omega_c^3}\right]a_{031}$$

$$
\begin{aligned}
b_{301} = {} & \frac{\sqrt{5}(32a^2-75)}{3072(16a^2+15)}\left[\frac{240518168576a^{18}+1523102777344a^{16}-12233610362880a^{14}-12036918476800a^{12}}{B_aD_a\omega_c}\right. \\
& + \frac{151498719232000a^{10}-200093644800000a^8-99326640000000a^6}{B_aD_a\omega_c} \\
& \left. + \frac{274416187500000ja^4+2705273437500a^2-24027099609375}{B_aD_a\omega_c}\right]a_{101} \\
& - \frac{(8a^2+25)(32a^2-75)^3(16a^2+225)(3328a^4-8320a^2-3375)}{3072a^2(4a^2-5)^2\omega_c^2}a_{011} \\
& - \frac{\sqrt{5}(8a^2+25)}{2560(16*a^2+15)}\left[\frac{223338299392a^{18}+4358586499072a^{16}-16686451261440a^{14}}{B_aD_a\omega_c}\right. \\
& - \frac{80077180108800a^{12}-390446694400000a^{10}+410136384000000a^8+223834800000000a^6}{B_aD_a\omega_c} \\
& \left. + \frac{574332187500000a^4-25866210937500a^2-40045166015625}{B_aD_a\omega_c}\right]a_{201} \\
& - \frac{(8a^2+25)^3(32a^2-75)^2}{512(16a^2+15)}\left[\frac{21646802944a^{12}-72642068480a^{10}+179301990400a^8}{B_aD_a\omega_c^2}\right. \\
& \left. - \frac{429881280000a^6+500337000000a^4+1898437500a^2-64072265625}{B_aD_a\omega_c^2}\right]a_{111} \\
& - \frac{\sqrt{5}(8a^2+25)^3(32a^2-75)^2A_a}{512(16a^2+15)}\left[\frac{21646802944a^{12}-72642068480a^{10}+179301990400a^8}{B_aD_a\omega_c^3}\right. \\
& \left. - \frac{429881280000a^6+500337000000a^4+1898437500a^2-64072265625}{B_aD_a\omega_c^3}\right]a_{021} \\
& + \frac{5(8a^2+25)^2(32a^2-75)}{16a^2+15}\left[\frac{123145302310912a^{22}+178533200560128a^{20}-3617892545331200a^{18}}{D_a\omega_c^4}\right. \\
& + \frac{2980885141913600a^{16}+21154374352896000a^{14}-149298441420800000a^{12}}{D_a\omega_c^4} \\
& + \frac{381401047552000000a^{10}+1608755736000000000a^8-7948350450000000000a^6}{D_a\omega_c^4} \\
& \left. - \frac{302276812500000000a^4-83774487304687500a^2-27030487060546875}{D_a\omega_c^4}\right]a_{031}
\end{aligned}
$$

$$
\begin{aligned}
b_{021} = {} & \frac{\sqrt{5}(135168a^6+227840a^4+432000a^2+646875)\omega_c}{80(8a^2+25)^2(16a^2+15)D_a}a_{101} \\
& - \frac{\sqrt{5}(3792896a^8-7065600a^6+18484000a^4+18675000a^2-16171875)\omega_c}{200(4a^2-5)(8a^2+25)(16a^2+15)(32a^2-75)D_a}a_{201} \\
& - \frac{524288a^{10}+9306112a^8-42483200a^6+189188000a^4+45975000a^2-73828125}{40(4a^2-5)(16a^2+15)D_a}a_{111} \\
& - \frac{\sqrt{5}(8a^2+25)(32a^2-75)(32a^2+9)(3792896a^8-7065600a^6+18484000a^4+18675000a^2-16171875)}{40(4a^2-5)(16a^2+15)D_a\omega_c}a_{021} \\
& - \frac{1536(10616832a^{10}+14177280a^8-174312000a^6-6425000a^4+555703125a^2+284765625)B_a}{(16a^2+15)D_a\omega_c^2}a_{031}
\end{aligned}
$$

$$
\begin{aligned}
b_{111} = {} & -\frac{24117248a^{10}+129826816a^8-414392320a^6-28512000a^4-342450000a^2-436640625}{24(8a^2+25)(16a^2+15)D_a}a_{101} \\
& + \left[\frac{140509184a^{12}+632684544a^{10}-3217612800a^8+4179008000a^6-5355000000a^4}{20(4a^2-5)(16a^2+15)(32a^2-75)D_a}\right. \\
& \left. - \frac{4865625000a^2-3955078125}{20(4a^2-5)(16a^2+15)(32a^2-75)D_a}\right]a_{201} + \frac{147\sqrt{5}(8a^2+25)^3(32a^2-75)(3328a^4+5400a^2-3375)}{20(4a^2-5)(16a^2+15)D_a\omega_c}a_{111} \\
& + \frac{147(8a^2+25)^3(32a^2-75)(3328a^4+5400a^2-3375)A_a}{4(4a^2-5)(16a^2+15)D_a\omega_c^2}a_{021}
\end{aligned}
$$

$$-\frac{1024\sqrt{5}(8a^2+25)B_a}{16*a^2+15}\left[\frac{1543503872a^{14}+6099828736a^{12}-36852367360a^{10}-19248000000a^8}{D_a\omega_c^3}\right.$$
$$\left.-\frac{104343000000a^6-135160875000a^4-352571484375a^2-192216796875}{D_a\omega_c^3}\right]a_{031}$$

$$b_{201}=-\frac{\sqrt{5}}{48}\left[\frac{7516192768a^{14}+108095602688a^{12}+242027069440a^{10}-491692032000a^8}{(8a^2+25)(16a^2+15)D_a\omega_c}\right.$$
$$\left.-\frac{764928000000a^6+1623960000000ja^4+3909515625000a^2+2456103515625}{(8a^2+25)(16a^2+15)D_a\omega_c}\right]a_{101}$$
$$+\frac{\sqrt{5}}{40}\left[\frac{17179869184a^{16}+389936054272a^{14}+381681664000a^{12}+1990696960000a^{10}+15888230400000a^8}{(4a^2-5)(16a^2+15)(32a^2-75)D_a\omega_c}\right.$$
$$\left.-\frac{69810300000000a^6+27306562500000a^4-49485937500000a^2-2669677734375}{(4a^2-5)(16a^2+15)(32a^2-75)D_a\omega_c}\right]a_{201}$$
$$+\frac{(32a^2-75)(8a^2+25)^2}{8(4a^2-5)(16a^2+15)}\left[\frac{67108864a^{12}+2878603264a^{10}+6557491200a^8+89104320000a^6}{D_a\omega_c^2}\right.$$
$$\left.+\frac{149073300000a^4+1468125000a^2-55529296875}{D_a\omega_c^2}\right]a_{111}$$
$$+\frac{\sqrt{5}(8a^2+25)^2(32a^2-75)}{8(4a^2-5)(16a^2+15)}\left[\frac{3580928983040a^{16}+9305952616448a^{14}+196061429760000a^{12}}{D_a\omega_c^3}\right.$$
$$+\frac{78571315200000a^{10}-47351270400000a^8-1235033100000000a^6}{D_a\omega_c^3}$$
$$\left.+\frac{-1516122562500000a^4+152634375000000a^2+408460693359375}{D_a\omega_c^3}\right]a_{021}$$
$$-\frac{2560a^2(8a^2+25)^2(4a^2-5)^2}{16a^2+15}\left[\frac{481036337152a^{18}+5655398187008a^{16}+16872028241920a^{14}}{D_a\omega_c^4}\right.$$
$$-\frac{51098320896000a^{12}+274010419200000a^{10}+661372152000000a^8+403593750000000a^6}{D_a\omega_c^4}$$
$$\left.+\frac{1894665937500000a^4+2928636474609375a^2+1081219482421875}{D_a\omega_c^4}\right]a_{031}$$

$$b_{101}=-a_{011}-\frac{2\sqrt{5}}{5}\left[\frac{34359738368a^{18}+36238786560a^{16}+443023360000a^{14}+2845179904000a^{12}}{(8a^2+25)(16a^2+15)(32a^2-75)D_a\omega_c}\right.$$
$$-\frac{12819968000000a^{10}+17001600000000a^8-25578000000000a^6}{(8a^2+25)(16a^2+15)(32a^2-75)D_a\omega_c}$$
$$\left.+\frac{46800000000000a^4+20012695312500a^2-13348388671875}{(8a^2+25)(16a^2+15)(32a^2-75)D_a\omega_c}\right]a_{101}$$
$$+\frac{96\sqrt{5}a^2(4a^2-5)}{5(16a^2+15)}\left[\frac{536870912a^{14}+96468992a^{12}+6935347200a^{10}+48501760000a^8}{(32a^2-75)^2D_a\omega_c}\right.$$
$$\left.-\frac{193125600000a^6+70267500000a^4-171843750000a^2+11865234375}{(32a^2-75)^2D_a\omega_c}\right]a_{201}$$
$$-\frac{672a^2(4a^2-5)(8a^2+25)^2(2326528a^8-41164800a^6-31260000a^4-8775000a^2+18984375)}{(16a^2+15)D_a\omega_c^2}a_{111}$$
$$-\frac{96\sqrt{5}a^2(4a^2-5)(8a^2+25)^2}{16*a^2+15}\left[\frac{17179869184a^{16}+74625056768a^{14}-982492643328a^{12}+1010963251200a^{10}}{D_a\omega_c^3}\right.$$
$$\left.-\frac{3191193600000a^8-1906850400000a^6-7689307500000a^4-635343750000a^2+1601806640625}{D_a\omega_c^3}\right]a_{021}$$
$$-\frac{491520B_a^2}{16*a^2+15}\left[\frac{51539607552a^{16}+428691423232a^{14}-470936453120a^{12}-4153282560000a^{10}}{(32a^2-75)D_a\omega_c^4}\right.$$
$$\left.-\frac{6479961600000a^8+10387608000000a^6-15258600000000a^4-25445390625000a^2-10144775390625}{(32a^2-75)D_a\omega_c^4}\right]a_{031}$$

$$
\begin{aligned}
a_{301} = &-\frac{(32a^2-75)^2}{4608(8a^2+25)}\left[\frac{67108864a^{14}-115343360a^{12}-2115502080a^{10}+7158784000a^8}{(16a^2+15)B_aD_a}\right.\\
&\left.-\frac{298880000a^6+16416300000a^4-2303437500a^2-1423828125}{(16a^2+15)B_aD_a}\right]a_{101}+\frac{\sqrt{5}(32a^2+9)(32a^2-75)^4}{(4608a^2(4a^2-5)^2\omega_c}a_{011}\\
&+\frac{(32a^2-75)}{768(16a^2+15)}\left[\frac{67108864a^{14}-6291456a^{12}-2059927552a^{10}}{768(16a^2+15)B_aD_a}\right.\\
&\left.+\frac{5214044160a^8-809408000a^6-7945500000a^4+1307812500a^2+474609375}{768(16a^2+15)B_aD_a}\right]a_{201}\\
&-\frac{\sqrt{5}(8a^2+25)^2(32a^2+9)(32a^2-75)^3(16384a^8+109056a^6-558080a^4-52500a^2+84375)}{768(16a^2+15)B_aD_a\omega_c}a_{111}\\
&-\frac{5(8a^2+25)^3(32a^2+9)^2(32a^2-75)^4(17408a^6-40640a^4-67500a^2+28125)}{768(16a^2+15)B_aD_a\omega_c^2}a_{02}\\
&-\frac{\sqrt{5}(8a^2+25)(32a^2-75)^3}{3(16a^2+15)}\left[\frac{2147483648a^{16}+28689039360a^{14}+83844136960a^{12}-49192140800a^{10}}{D_a\omega_c^3}\right.\\
&\left.-\frac{1668308480000a^8+2393026080000a^6+137097000000a^4-376776562500a^2-64072265625}{D_a\omega_c^3}\right]a_{031}
\end{aligned}
$$

$$
\begin{aligned}
a_{211} = &-\frac{\sqrt{5}(32a^2-75)^2(136a^2+75)(2048a^6-3200a^4-13500a^2+5625)\omega_c}{768(8a^2+25)^2(16a^2+15)B_aD_a}a_{101}\\
&-\frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}a_{011}\\
&+\frac{\sqrt{5}(32a^2-75)(136a^2+75)(17408a^6-40640a^4-67500a^2+28125)\omega_c}{1920(8a^2+25)(16a^2+15)B_aD_a}a_{201}\\
&+\frac{(8a^2+25)(32a^2-75)}{384(16a^2+15)}\left[\frac{4194304a^{12}+4325376a^{10}-110247936a^8+128967680a^6}{B_aD_a}\right.\\
&\left.+\frac{92424000a^4+7762500a^2-18984375}{B_aD_a}\right]a_{111}\\
&+\frac{\sqrt{5}(8a^2+25)(32a^2+9)(32a^2-75)^3(136a^2+75)(17408a^6-40640a^4-67500a^2+28125)}{384(16a^2+15)B_aD_a\omega_c}a_{021}\\
&+\frac{5(32a^2-75)^2}{3*(16*a^2+15)}\left[\frac{335544320*a^14+7241465856*a^12+17690132480*a^10-85375795200*a^8}{D_a\omega_c^2}\right.\\
&\left.-\frac{246324832000*a^6+96446400000*a^4-39479062500*a^2-12814453125}{D_a\omega_c^2}\right]a_{031}
\end{aligned}
$$

$$
\begin{aligned}
a_{121} = &\frac{(32a^2-75)(2048a^6-3200a^4-13500a^2+5625)\omega_c^2}{1536(8a^2+25)^2(16a^2+15)B_aD_a}a_{101}+\frac{\sqrt{5}(32a^2-75)^2\omega_c}{23040(8a^2+25)B_a}a_{011}\\
&-\frac{(17408a^6-40640a^4-67500a^2+28125)\omega_c^2}{3840(8a^2+25)(16a^2+15)B_aD_a}a_{201}\\
&-\frac{\sqrt{5}(32a^2-75)(16384a^8+109056a^6-558080a^4-52500a^2+84375)\omega_c}{3840(16a^2+15)B_aD_a}a_{111}\\
&+\frac{(32a^2-75)}{768(16a^2+15)}\left[\frac{67108864a^{14}-6291456a^{12}-2059927552a^{10}+5214044160a^8}{B_aD_a}\right.\\
&\left.-\frac{809408000a^6+7945500000a^4-1307812500a^2-474609375}{B_aD_a}\right]a_{021}\\
&-\frac{\sqrt{5}(32a^2-75)}{3(16a^2+15)}\left[\frac{44171264a^{10}+194867200a^8-849510400a^6-1291260000a^4+47362500a^2+170859375)}{D_a\omega_c}\right]a_{031}
\end{aligned}
$$

where ω_c is given in (19), A_a, B_a and C_a are given in (21), and $D_a = 2048a^67360a^4 - 55300a^2 + 16875$.

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