Tension of Conductor Under Concentrated Loads

A common difficulty in the analysis and design of transmission and distribution lines is to determine a conductor's tension and its static profile under concentrated loads. For relatively small concentrated loads (such as detuning pendulums on transmission lines), approximation methods may give good predictions. For large concentrated loads (such as fallen trees on distribution lines), however, exact solutions must be found. This paper presents methodologies to compute conductor tension and static profile in three-dimensional space using both approximate and exact solution procedures under concentrated loads with different boundary conditions. Practical engineering examples from galloping control of transmission lines and mechanical coordination of distribution lines are given to demonstrate the applicability of the theory.

1 Introduction

A necessary step in designing an electrical transmission line or distribution line is to compute the line component loads and the static profile. One important component load is the conductor tension because most of the other component loads and the static profile are related to it. These computations depend heavily upon environmental conditions which produce external forces on the line such as wind load and ice load. External loads are usually approximated as a distributed force on a line component (such as the conductor) so that closed-form solutions for tension and static profile can be obtained to greatly simplify the computation. However, in certain situations, the line has concentrated loads. One example is the detuning pendulum, an anti-galloping control device (Havard et al., 1985), which is attached to a transmission line or distribution line. The new conductor tension and the new static profile due to the detuning pendulums need to be calculated for new line design and line maintenance, because field measurements are impractical and may cause service interruptions. Another example of a concentrated load is a tree falling onto an overhead distribution line during a storm. In situations where the tree load exceeds the structural strength of the line, the calculation method developed for small concentrated loads (such as detuning pendulums) is not applicable. A large tree plus wind load can be up to 30 times heavier than the weight of a conductor span, whereas the total weight of a span's detuning pendulums is only up to 20 percent of that of a conductor span. Thus, an approach to calculate the conductor tension and static profile for large concentrated loads is desirable to aid in determining which components of a power line are most likely to fail. Ideally the structures should be designed with a component failure hierarchy (or "mechanical coordination") such that the most likely components to fail are also the easiest to repair.

Both an approximation formula and an exact solution procedure were considered by Irvine (1981) for a conductor structure in a two-dimensional (longitudinal and vertical) space. For the approximation formula, a closed form of the tension increase is found from a third-degree polynomial of one variable (tension increase), which makes it easy to use in practice. However, this formula is only valid for a single vertical concentrated load with the assumptions that the static profile without the concentrated load is parabolic and both ends of the cable are fixed. For the exact solution procedure, on the other hand, Irvine gave the solution formulae for multiple vertical concentrated masses with fixed boundary conditions, but they require information on the unstrained cable length which is usually unavailable in practice.

In this paper, both approximation and exact solution formulae are generalized for a three-dimensional (longitudinal, vertical, and transversal) space, which is closer to the reality where the transversal (horizontal) displacement (due to wind load, for example) cannot be neglected. The approximation formulae are generalized to compute the conductor tension and static profile for multiple inclined concentrated loads with different boundary conditions. For the exact solution, using an approach developed recently to calculate the unstrained length of a conductor (Wong and Yu, 1993), a comprehensive procedure is developed to compute exact solutions for conductor tension and static profile in three-dimensional space with various boundary conditions. However, when very large concentrated loads are encountered, an approximation to the exact solution by neglecting the concentrated loads may be acceptable, and the formulae become simpler. This approximation solution is also presented in this paper. Two practical examples, one from galloping control of transmission lines and the other from mechanical coordination of distribution lines, are presented to illustrate the applicability of the theory.

2 Formulation

A span of a typical conductor line is shown in Fig. 1 where x, y, and z indicate the coordinates along longitudinal, vertical, and horizontal directions, respectively. $s'$ is the Lagrangian coordinate under both distributed and concentrated loads and indicates a reference distance from the left end of the span. The concentrated loads are denoted by $F_i, i = 1, 2, \ldots, n$, each of which deviates from the vertical coordinate by an angle $\theta_i$.

2.1 Approximate Solution for Small Concentrated Loads. In order to compute the tension change due to concentrated loads, the first step is to find the solution of the static profile of the conductor without concentrated load.

Static Profile Without Concentrated Loads. The static profile without concentrated loads may be described by the following differential equations:

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Technical Editor, Prof. Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

\[ D = \frac{L_s}{2} + \left( \frac{H}{Q} \right) \times \sinh^{-1} \left[ \left( \frac{Q}{2H} \right) \frac{Q_s(y_{r0} - y_{i0}) + Q_z(z_{r0} - z_{i0})}{\sqrt{Q^2 + B^2 \sinh(QL_s/2H)}} \right]. \]  

The solution of \( s \) is found by integrating \( ds/dx = [1 + (dy_i/dx)^2 + (dz_i/dx)^2]^{1/2} \) as

\[ s(x) = 2H\sqrt{Q^2 + B^2} \sinh((Q/2H)x) \times \cosh((Q/2H)(2D - x)). \]

Therefore, given \( H, Q, Q_s \) and the boundary conditions, the static profile can be determined from (6) - (8). The span length, \( L_s \), equals \( s(L_s) \). Setting \( Q_s = y_{r0} = z_{r0} = y_{l0} = z_{l0} = 0 \) in (6) - (8) leads to the solutions given by Irvine (1981).

**Tension Increase and New Static Profile.** Next, suppose that the tension increase due to concentrated loads is expressed as \( \tau \), and that \( u, y, z \) are additional displacements produced by the inclined concentrated loads along longitudinal, vertical, and horizontal directions, respectively. The new static profile can be described by

\[ \frac{d}{ds} \left[ (T + \tau) \left( \frac{dx}{ds} \right)^2 \right] = 0 \]

\[ \frac{d}{ds} \left[ (T + \tau) \left( \frac{dy_i}{ds} \right)^2 \right] = -Q_y + \sum_{i=1}^{n} F_i \cos \theta_i \delta(s - s_i) \]

\[ \frac{d}{ds} \left[ (T + \tau) \left( \frac{dz_i}{ds} \right)^2 \right] = -Q_z - \sum_{i=1}^{n} F_i \sin \theta_i \delta(s - s_i) \]

\[ \tau = AE \left( \frac{dx'}{ds} - 1 \right) \]

where \( \delta(s - s_i) = 1 \) (or 0) if \( s = s_i \) (\( s \neq s_i \)), \( AE \) (A is the cross-sectional area of the conductor and \( E \) is the Young's modulus) is the axial rigidity of the conductor without concentrated loads, and \( s' \) indicates the new Lagrangian coordinate under both distributed and concentrated loads.

It is not possible to obtain a general analytic solution from (9) to (12). However, if the additional conductor displacement along the longitudinal direction due to the small concentrated loads is assumed to be small compared to the original displacement due to the distributed loads \( Q_y \) and \( Q_z \) (i.e., \( \frac{du}{ds} \approx 0 \) compared with \( \frac{dx}{ds} \)), then (9) becomes \((T + \tau)(dx/ds) = H + h \) which, by using (5), results in

\[ \tau = h(dx/ds) \]

where \( h \) is a constant. It should be noted here that the horizontal component, \( H \), of the conductor tension under the distributed load only is not the same as that given in (5) because the boundary conditions are now different from the case without concentrated load. However, it can be shown that if the difference of the end movements along all the three directions (e.g., \( u(L_s) - u(0) \)) are sufficiently small, the change in \( H \) is negligible.

Now, with the aid of (2), (3), (5), and (13), (10) and (11) can be rewritten with boundary conditions as

\[ \frac{d}{dx} \left[ \left( H + h \right) \frac{dy_i}{dx} + h \frac{dy_i}{dx} \right] = 0 \]

\[ (x_i < x < x_{i+1}, \ i = 0, 1, 2, \ldots n) \]
\[ y_i(x_i) = y_{i-1}(x_i) \quad \text{and} \quad \frac{dy_i(x_i)}{dx} - \frac{dy_{i-1}(x_i)}{dx} = \frac{1}{H + h} F_i \cos \theta_i \quad (i = 0, 1, 2, \ldots n) \] (14)

and

\[ \frac{d}{dx} \left[ (H + h) \frac{dz_2(x_i)}{dx} + h \frac{dz_1(x_i)}{dx} \right] = 0 \]

\( (x_i < x < x_{i+1}, i = 0, 1, 2, \ldots n) \)

\[ z_{2i}(x_i) = z_{2i-1}(x_i) \quad \text{and} \quad \frac{dz_{2i}(x_i)}{dx} - \frac{dz_{2i-1}(x_i)}{dx} = -\frac{1}{H + h} F_i \sin \theta_i \quad (i = 0, 1, 2, \ldots n), \]

respectively, where, as in the case without concentrated loads, boundary conditions \( y_{20}(0), y_{2n}(L_x), z_{20}(0), \) and \( z_{2n}(L_x) \) are not necessarily zero. The solutions of (14) and (15) are given by

\[ y_i(x) = \frac{-[h y_i - (c_i x + e_i)]}{(H + h)} \]

\[ z_i(x) = \frac{-[h z_i - (d_i x + f_i)]}{(H + h)} \] (16)

where

\[ c_i = c_{i1} h + c_{i0}, \quad d_i = d_{i1} h + d_{i0} \]

\[ c_{i1} = \frac{1}{L_x} [y_{i2n}(L_x) - y_{i20}(0)] + \sum_{k=1}^{n} \left( \frac{x_k}{L_x} - 1 \right) F_k \cos \theta_k + \sum_{k=0}^{i} F_k \cos \theta_k \]

\[ d_{i1} = [z_{2n}(L_x) - z_{20}(0) + z_{1L_x} - z_{10}]/L_x \]

\[ d_{i0} = \frac{1}{L_x} [z_{2n}(L_x) - z_{20}(0)] - \sum_{k=1}^{n} \left( \frac{x_k}{L_x} - 1 \right) F_k \sin \theta_k - \sum_{k=0}^{i} F_k \sin \theta_k \]

\[ e_i = H y_{20}(0) + h [y_{i20}(0) + y_{10}] - \sum_{k=0}^{i} x_k F_k \cos \theta_k \]

\[ f_i = H z_{20}(0) + h [z_{i20}(0) + z_{10}] + \sum_{k=0}^{i} x_k F_k \sin \theta_k \] (17)

for \( i = 1, 2, \ldots n. \) Here, \( F_0 = 0, \) introduced for convenience. The new static profile can be described as \( y(x) = y_i + y_{i-1} = [H y_i + (c_i x + e_i)]/(H + h) \) and \( z(x) = z_i + z_{i-1} = [H z_i + (d_i x + f_i)]/(H + h) \) if the tension change, \( h, \) is known.

Then, combining (13) and (18) and integrating the resulting equation from 0 to \( L_x \) yields a third-degree polynomial of \( h \)

\[ (h/H)^3 + a(h/H)^2 + b(h/H) - c = 0 \] (19)

in which

\[ a = 2 + \frac{1}{24} \lambda^2 + \left( \frac{AE}{HL_x} \right) \left[ u(0) - u(L_x) \right] \]

\[ + \frac{1}{2} \left( \frac{AE}{H^2 L_x} \right) \sum_{i=1}^{n} (c_i^2 + d_i^2) (x_{i+1} - x_i) \]

\[ b = 1 + \frac{1}{12} \lambda^2 + 2 \left( \frac{AE}{HL_x} \right) \left[ u(0) - u(L_x) \right] \]

\[ + \left( \frac{AE}{H^2 L_x} \right) \sum_{i=1}^{n} (c_i^2 + d_i^2) (x_{i+1} - x_i) \]

\[ + H \sum_{i=0}^{n} c_i [y_i(x_{i+1}) - y_i(x_i)] + H \sum_{i=0}^{n} d_i [z_i(x_{i+1}) - z_i(x_i)] \]

\[ - \left( \frac{AE}{HL_x} \right) \left[ u(0) - u(L_x) \right] \] (20)

where \( y_i(x_i) \) and \( z_i(x_i) \) are given in (6), and

\[ L_x = \int_{0}^{s_i} \left( \frac{dx}{ds} \right)^3 dx \]

\[ = \frac{2H Q^2 + B^2}{Q} \sinh \left( \frac{Q D}{H} \right) \left[ 1 + \frac{1}{3} \sin^2 \left( \frac{Q D}{H} \right) \right] \]

\[ \lambda^2 = 6 \left( \frac{AE}{HL_x} \right) \left[ \left( 1 + \frac{B^2}{Q^2} \right) \left( \frac{H}{2Q} \right) \sinh \left( \frac{2Q H}{2Q} \right) \right] \]

\[ + \sinh \left( \frac{2Q H}{2Q} \right) \left[ 1 - \frac{B^2}{Q^2} \left( L_x - D \right) \right] \] (21)

Given \( H, L_x, Q_x, \) and \( Q_z, \) the tension increase \( h \) can be found in a closed form from the third-degree polynomial (19) if \( u(0) \) and \( u(L_x) \) are known. How to find \( u(0) \) and \( u(L_x) \) is discussed in Section 2.3. If the boundary conditions are zero along all the directions, then all the three coefficients \( a, b, \) and \( c \) are positive because \( 0 < \theta_i < 90 \) deg, therefore, the polynomial (19) has only one positive root.

It is noted that the formulae given by Irvine (1981) for one vertical concentrated load without horizontal displacement can be deduced directly from (17), (19)-(21) by simply setting \( n = 1, Q_i = \theta_i = 0, \) boundary conditions to zero, and approximating \( L_x, \lambda^2 \) and \( y_i(x_i) \) up to second-order terms.

2.2 Exact Solution for Large Concentrated Loads. Suppose that the Lagrange’s coordinate of the unstretched conductor is denoted by \( s \) whereas the stretched conductor is given by \( s', \) as shown in Fig. 2. It should be noted here that the definition of \( s \) is different from that given in Section 2.1. Then, the static profile of the stretched conductor may be described by

\[ \tau = \frac{ds'}{ds} \left[ \frac{ds'}{ds} \right]^2 - \frac{dy_0}{ds} \frac{dy_2}{ds} \frac{dz_0}{ds} \frac{dz_2}{ds} \]

\[ + \frac{1}{2} \left[ \left( \frac{dy_0}{ds} \right)^2 + \left( \frac{dy_2}{ds} \right)^2 + \left( \frac{dz_0}{ds} \right)^2 + \left( \frac{dz_2}{ds} \right)^2 \right] \] (18)

\[ \frac{d}{ds} \left( \frac{ds'}{ds} \right) = 0 \]
and solving this equation together with (26) by eliminating \( Q \).

**New Tension and Static Profile.** Integrating (22) results in

\[
T' \frac{dx_i}{ds'} = H' \tag{28}
\]

\[
T' \frac{dy_i}{ds'} = -V - Q_i s + \sum_{k=0}^{i} F_k \cos \theta_k
\]

\[
T' \frac{dz_i}{ds'} = W - Q_i s - \sum_{k=0}^{i} F_k \sin \theta_k
\]

for \( s_i < s < s_{i+1}, i = 0, 1, 2, \ldots, n \), with boundary conditions

\[
\begin{align*}
x_0(0) &= u(0) , \quad x_i(L_o) = L_o + u(L_o) , \quad x_i(s) = x_{i-1}(s) \\
y_i(s) &= y_{i-1}(s) \quad \text{and} \quad z_i(s) = z_{i-1}(s)
\end{align*}
\]

where \( i = 1, 2, \ldots, n \). Boundary conditions \( y_0(0), y_n(L_o), z_0(0), \) and \( z_n(L_o) \) are not necessarily zero, \( H' \) is the unknown horizontal tension component of the stretched conductor; \( V \) and \( W \) represent the vertical and horizontal reactions, respectively, at the left support end; \( u(0) \) and \( u(L_o) \) are additional end displacements of the stretched conductor along the longitudinal direction under the distributed and concentrated loads (see Fig. 2). The following equation can be directly obtained from (28) with the aid of \((dx_i/ds')^2 + (dy_i/ds')^2 + (dz_i/ds')^2 = 1\):

\[
T' = Q \left( s + a_i \right)^2 + b_i^2 \tag{30}
\]

where

\[
\begin{align*}
a_i &= \frac{\left[ Q_y(V - E_i) - Q_z(W - G_i) \right]}{Q^2} \\
b_i &= \frac{\left[ Q^2H'^2 + \left[ Q_y(V - E_i) + Q_z(W - G_i) \right]^2 \right]}{Q^2}
\end{align*}
\]

\[
E_i = \sum_{k=0}^{i} F_k \cos \theta_k \quad \text{and} \quad G_i = \sum_{k=0}^{i} F_k \sin \theta_k
\]

for \( i = 0, 1, 2, \ldots, n \). Employing (23), (28), (29) and the relation

\[
\frac{dx_i}{ds} = \frac{dx_i}{ds'} + \frac{H'}{AE} + \frac{H'}{Q} \left( s + a_i \right)^2 + b_i^2
\]

one can find

\[
x_i(s) = \left( \frac{H'}{AE} \right) s + \left( \frac{H'}{Q} \right) \left[ \sinh^{-1} \left( \frac{\left[ Q_y(V - E_i) - Q_z(W - G_i) \right]}{Q^2} \right) \right]
\]

\[
(s_i < s < s_{i+1}, i = 0, 1, \ldots, n)
\]

where

\[
x_0(s) = u(0) - \left( \frac{H'}{Q} \right) \sinh^{-1} \left( \frac{\left[ Q_y(V - E_i) - Q_z(W - G_i) \right]}{Q^2} \right)
\]

\[
x_i = x_{i-1} + \left( \frac{H'}{Q} \right) \left[ \sinh^{-1} \left( \frac{\left[ Q_y(V - E_i) - Q_z(W - G_i) \right]}{Q^2} \right) \right]
\]

\[
(s_i < s < s_{i+1}, i = 1, 2, \ldots, n)
\]

in which the boundary condition \( x_0(0) = u(0) \) has been used. Similarly, \( y_i(s) \) is given by

\[
y_i(s) = -\left[ Q_y(s/2) + (V - E_i) s \right] \left( \frac{H'}{Q} \right) s/AE
\]

\[
- \left[ (V - E_i - Q_y a_i) \sinh^{-1} \left( \frac{\left[ Q_y(V - E_i) - Q_z(W - G_i) \right]}{Q^2} \right) \right]
\]

\[
+ Q_y \sqrt{\left( s + a_i \right)^2 + b_i^2} / Q + y_{i-1}
\]

\[
(s_i < s < s_{i+1}, i = 0, 1, \ldots, n)
\]

\[
y_0 = \left[ (V - E_0 - Q_y a_0) \sinh^{-1} \left( \frac{\left[ Q_y(V - E_i) - Q_z(W - G_i) \right]}{Q^2} \right) \right]
\]

\[
+ y_0(0)
\]

**Journal of Applied Mechanics**
\[
Y_i = Y_{i-1} - (E_i - E_{i-1})s_i/AE
+ \left\{ (V - E_i - Q\alpha_i) \sinh^{-1}\left( \frac{s_i + a_i}{b_{i-1}} \right) \right\}/Q
- (V - E_{i-1} - Q\alpha_{i-1}) \sinh^{-1}\left( \frac{s_i + a_i}{b_{i-1}} /b_{i-1} \right)/Q
+ \left\{ \sqrt{(s_i + a_i)^2 + b^2_i} - \sqrt{(s_i + a_{i-1})^2 + b^2_{i-1}} \right\}/Q
(\text{i} = 1, 2, \ldots n). \tag{35}
\]

\(Z_i(s)\) and \(Z_i\) can be obtained from (35) by substituting \(Q_i, V, E_i,\) and \(y_0(0)\) with \(Q_i, -W, -G_i\) and \(z_0(0)\), respectively. The expressions \(x_i(s), y_i(s),\) and \(z_i(s)\) describe the static profile under both distributed and concentrated loads if \(H', V\) and \(W\) are known.

Next, applying the boundary conditions \(x_i(L_0) = L_x + u(L_x),\) \(y_i(Lo)\) and \(z_i(Lo)\) to the static solutions produces the following equations:

\[
0 = \left[ u(0) - u(L_0) - L_x + \left( \frac{H'}{AE} \right) L_0 \right] + \sum_{i=1}^{n+1} \sinh^{-1}\left( \frac{s_i + a_{i-1}}{b_{i-1}} \right) - \sum_{i=0}^{n} \sinh^{-1}\left( \frac{s_i + a_i}{b_i} \right) \tag{36}
\]

\[
0 = \frac{1}{AE} \left\{ QV + QW \right\} L_0
- \sum_{i=1}^{n+1} \left[ L_0 - s_i \right] F_i \left( Q \cos \theta_i - Q \sin \theta_i \right)
+ \frac{1}{Q} \sum_{i=1}^{n} \left\{ Q \left( V - E_{i-1} \right) + Q \left( W - G_{i-1} \right) \right\}
\times \sinh^{-1}\left( \frac{s_i + a_{i-1}}{b_{i-1}} \right) - \sum_{i=0}^{n} \left[ Q \left( V - E_i \right) \right]
+ Q_i \left[ (W - G_i) \right] \sinh^{-1}\left( \frac{s_i + a_i}{b_i} \right)
+ Q_i \left[ y_0(L_0) - y_0(0) \right] - Q_i \left[ z_0(L_0) - z_0(0) \right] \tag{37}
\]

and

\[
0 = \frac{1}{AE} \left[ \frac{1}{2} Q^2 L_0 + Q,V - Q,W \right] L_0
- \sum_{i=1}^{n} \left( L_0 - s_i \right) F_i \left( Q \cos \theta_i - Q \sin \theta_i \right)
+ Q \sum_{i=1}^{n} \sqrt{(a_i)^2 + b_i^2} - \sum_{i=0}^{n} \sqrt{(a_i + a_{i+1})^2 + b_{i+1}^2}
+ Q_i \left[ y_0(L_0) - y_0(0) \right] + Q_i \left[ z_0(L_0) - z_0(0) \right] \tag{38}
\]

where the identity \(Q_i (V - E_i) - Q_i (W - G_i) - Q_i^2 a_i = 0 (i = 0, 1, 2, \ldots n)\) has been used. Now, the unknown variables \(H', V,\) and \(W\) can be solved numerically (e.g., by using Newton-Raphson method) from (36)–(38) if \(u(0)\) and \(u(L_0)\) are known.

### 2.3 Approximate Solution for Very Large Concentrated Loads

When the concentrated loads are very large compared to the distributed loads (e.g., their ratio is over 200 percent), then an approximation solution may be found from the results presented in the previous section by neglecting the distributed loads (i.e., letting \(Q_i = Q_i = 0\)). Thus, the procedure becomes simpler and the formulae of (36)–(38) are reduced to

\[
0 = [u(0) - u(L_0) - L_x] + \sum_{i=0}^{n} \left( s_{i+1} - s_i \right) \Psi_i \tag{39}
\]

\(\Psi_i = 1/AE + [H'^2 + (V - E_i)^2 + (W - G_i)^2]^{-1/2}\). Although (39) is much simpler than (36)–(38), a numerical iteration scheme is still needed to solve the nonlinear coupled equations. Having found \(V, W,\) and \(H',\) then

\[
T_i(s) = \sqrt{H'^2 + (V - E_i)^2 + (W - G_i)^2}
\]

\[
y_i(s) = \Psi_i (V - E_i) s + \gamma_i
\]

\[
z_i(s) = \Psi_i (W - G_i) s + \zeta_i
\]

\(s_i < s < s_{i+1}, i = 0, 1, \ldots n\). It is noted that the tension, \(T_i(s)\), is now approximated by piecewise constants whereas \(y_i(s)\) and \(z_i(s)\) are approximated by piecewise linear functions of \(s\).

### 2.4 Boundary Conditions

To solve for \(h\) in Section 2.1 or \(H'\) in Section 2.2 (or Section 2.3), either the displacements \(u(0)\) and \(u(L_0)\), or their relations to other variables are needed. As shown in Fig. 3, the adjacent spans’ horizontal components of tension and displacements are denoted by \(H_i\) and \(u_i\), respectively. Thus, \(u_1 = u(0)\) and \(u_2 = -u(L_0)\). Furthermore, the relationship between the force \(P_i\) and displacement \(u_i\) can be found as follows. For a transmission line having supporting towers, the end support components are usually suspension insulator springs and the Hooke’s law can be employed to establish a linear approximation

\[
P_i = \beta u_i \quad i = 1, 2 \quad \tag{41}
\]

where \(\beta\) is the spring constant. For a distribution line, the force
\( P_i \) is directly produced by the bending of the supporting poles due to the unbalanced tensions of the adjacent conductor spans. The relation between the force and deflection (bending displacement) can be described as \( P_i = -ka_i \) if the pole is treated as an elastic material. Here \( k \) is called the modulus and the minus sign indicates that the force acts at the opposite direction of the deflection (Salvadori and Schwarz, 1954). Constant \( k \) is usually obtained from bending strength tests and can be found from many industrial product manuals (e.g., CSA, 1990). Therefore, Eq. (41) can be used as a general formula. Note that the special case where \( u_i = 0 \), \( i = 1,2 \) (i.e., \( u(0) = u(L_x) = 0 \)) implies that the equations given in Section 2.1 (Section 2.2 or Section 2.3) are sufficient for solving \( h(H', V, W) \). There are two general cases: tangent line (including dead end) and angle line.

**Case 1: Tangent Line.** Suppose that the adjacent spans’ length are given by \( L_1^{(i)} \) and \( L_2^{(i)} \), respectively, and the unstrained conductor lengths are obtained from Eq. (26) as \( L_0^{(i)} \) and \( L_2^{(i)} \), respectively. Then equation (25) yields

\[
L_i^{(0)} + u_i = (H_i/\alpha E) L_i^{(0)} + (2H_i/Q) \sinh^{-1} (QL_i^{(1)}/2H_i), \quad i = 1,2.
\]

From Fig. 3(a) it is seen that

\[
H' = H_1 + P_i, \quad i = 1,2.
\]

Combining (41)-(43) gives

\[
L_i^{(0)} + u_i = (H' - \beta u_i) \times \frac{(QL_i^{(1)}/\alpha E + 2(Q/\alpha E) \sinh^{-1}(QL_i^{(1)}/2(H' - \beta u_i))]}{i = 1,2}.
\]

For the exact solution procedure given in Section 2.2 (or approximation in Section 2.3), the above two equations (or one equation if one of \( u_i \) equals 0) can now be directly added to (36)-(38) (or (39)), and \( H', V, W, \) and \( u_i \) can be solved simultaneously by numerical iteration. For the approximate solution procedure given in Section 2.1, note that \( H' = H + h \) and that an approximate solution of \( u_i \) from (44) is required. This approximation is given by

\[
u_i = \left[ L_i^{(0)} h + (AE + H) L_i^{(0)} - AE L_i^{(1)} \right]/\left[ \alpha E + \beta L_i^{(0)} \right]
\]

and hence

\[
u(0) - u(L_x) = u_1 + u_2 = (d h + H f) L_x/\alpha E
\]

\[
\begin{align*}
d &= \sum_{i=1}^{3} \frac{L_i^{(0)}/\alpha E}{1 + \beta L_i^{(0)}/\alpha E} \\
f &= \sum_{i=1}^{3} \frac{(AE/H)(L_i^{(0)} - L_i^{(1)})/L_x + L_i^{(0)}/L_x}{1 + \beta L_i^{(0)}/\alpha E}.
\end{align*}
\]

Then, the coefficients, \( a, b, \) and \( c \), of the third-degree polynomial of \( h \) can be obtained from (20) if \( \lambda \) and the coefficient of the first term of \( c \) are divided by \((1 + d)\), and the term, \((AE/HL_x) \{u(0) - u(L_x)\}\), is replaced by \( f(1 + d)\).

**Case 2: Angle Line.** It can be seen from Fig. 3(b) that at the supporting end of the angle line, the balance of the resultant forces along the direction perpendicular to guy wires results in

\[
H' \cos(\alpha/2) = (H_1 + P_2 - Q_x) \cos(\alpha/2),
\]

\[i.e., \quad H' = H_1^2 + P_2 - Q_x,
\]

where \( Q_x \) is the wind load contributed from the angle line due to the wind perpendicular to the central span. \( Q_x \) is along the longitudinal direction of the angle line, given by

\[
Q_x = Q_x L_x \sin \alpha.
\]

The formulae given for Case 1, however, can be employed here again provided that (44) is modified so that \((H' - \beta u_1)\) and \(Q_x \) (included in \( Q \)) are replaced by \((H' - \beta u_1 + Q_x)\) and \(Q_x \cos \alpha\), respectively.

### Table 1

<table>
<thead>
<tr>
<th>Store</th>
<th>Transmission line</th>
<th>Distribution line</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rx</td>
<td>125.000</td>
<td>100.000</td>
</tr>
<tr>
<td>H (kN)</td>
<td>15.000</td>
<td>2.190</td>
</tr>
<tr>
<td>A (mm²)</td>
<td>402.900</td>
<td>33.690</td>
</tr>
<tr>
<td>E (kN/mm²)</td>
<td>63.358</td>
<td>80.000</td>
</tr>
<tr>
<td>M (kg/m)</td>
<td>1.683</td>
<td>0.186</td>
</tr>
<tr>
<td>Qy (N/m)</td>
<td>-16.714</td>
<td>-1.334</td>
</tr>
<tr>
<td>Qz (N/m)</td>
<td>1.755</td>
<td>5.680</td>
</tr>
</tbody>
</table>

### Table 1 Properties of lines

<table>
<thead>
<tr>
<th>Property</th>
<th>Value 1</th>
<th>Value 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P (kg)</td>
<td>14.000</td>
<td>Tree Branch of tree</td>
</tr>
<tr>
<td>X/Lx</td>
<td>1/4, 5/12, 1/3</td>
<td>Fb (kg)</td>
</tr>
<tr>
<td>θ (deg)</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>X/Lx</td>
<td>1/2</td>
<td>1/4</td>
</tr>
</tbody>
</table>

### 3 Practical Applications

The formulae derived in previous sections are applied to practical examples. One example is chosen from the galloping control of transmission lines by using detuning pendulums (Havard et al., 1985) and the other one is chosen from the mechanical coordination of distribution lines hit by fallen trees (Wong and Yu, 1993). The main focus is on the conductor’s tension, length, and static profile.

#### 3.1 Detuning Pendulum

Galloping is a low-frequency, high-amplitude oscillation which can occur on an iced electrical transmission line in a steady side wind. The inability to prevent galloping can lead to severe disruptions in the electrical supply and even a cascading collapse of a line’s supporting towers. This problem has been observed since at least the 1930’s and research on galloping control has been conducted extensively during the past two decades in North America, Europe, and Japan (Havard et al., 1985). While a broad range of control devices were considered, the major one tested in North America is the detuning pendulum (Havard, 1981). It consists of a mass suspended below and rigidly clamped to the conductor. The detuning pendulum is an economical control device and can provide trouble-free service once installed because it has no moving parts. Hence, many North American utilities have been using and still prefer to use it in galloping control of transmission lines. However, the installation of detuning pendulums alters the load distribution of the conductor and therefore, changes the conductor’s tension as well as the static profile.

The relative parameters of the transmission line and detuning pendulum are listed in Table 1. The line is assumed to have an iced shape C11 (CEA, 1992) plus a side wind of 10 m/s. The effect of ice load and wind load has been included in \( A, Q_y, \) and \( Q_z \). Three identical detuning pendulums are assumed to be vertically installed on one span of the line. Moreover, it is assumed that the adjacent spans are treated in the same way, i.e., \( u(0) = u(L_x) = 0 \). Both the approximation and exact solution of the static profile are shown in Fig. 4 with main parameters given in Table 2 where \( T_L, Y_L, \) and \( Z_L \) represent maximum conductor tension, maximum vertical, and horizontal displacements, respectively. Here, the notations \((S)\) and \((L)\) represent the approximation methods for Small and Large concentrated loads, respectively. It clearly indicates that the approximation excellently agrees with the exact solution. This is because the concentrated load is small compared to the conductor’s weight (the ratio of the total weight of three detuning pendulums to the total iced conductor’s weight of one span is about 20 percent.

#### 3.2 Fallen Tree

Trees inevitably fall onto overhead distribution lines during storm conditions in heavily treed areas. Because it is not environmentally acceptable to remove all poles
potential danger trees adjacent to distribution lines, the line should be designed so that the component most likely to fail mechanically is also the one that can be repaired with minimum cost and time. This leads to mechanical coordination studies for new line designs as well as the improvement of existing lines. Such a study first requires the computation of component loads due to fallen trees. A tree falling onto the conductor will increase the conductor tension and the tension increase will be transferred to other components of the distribution line including the pole and associated hardware. The new loads for these components can be calculated using basic principles of mechanics on the static profile. The properties of the distribution line are given in Table 1. Also given in this table are data for the case of a whole small tree and the case of a tree branch. The load from a fallen tree consists of two parts, namely the weight of the tree, \( F_V \) (vertical) and the wind load on the tree, \( F_H \) (horizontal). (The details for estimating these two loads can be found from the report (Wong and Yu, 1993).) The static profiles computed by using the approximation method and the exact solution procedure are given in Fig. 5 for a small tree and in Fig. 6 for a tree branch. The main parameters of the static profile are listed in Table 2.

Figure 5 presents the exact solution with the approximation \((L)\) only because the approximation \((S)\) is inapplicable for this case due to the ratio of concentrated load to distributed load being very large. It is shown that this approximation \((L)\) method produces results which agree well with the exact solution (with error less than two percent) and they are indeed piecewise linear functions (i.e., straight lines). For the case of a typical \#2 ACSR conductor branch line, the ultimate tension strength (UTS) of the line is 12.40 kN (Wong and Yu, 1993) and this is much less than the maximum conductor tension caused by the fallen tree case (73.10 kN, see Table 2). Both the approximation and exact results indicate that the line will fail (e.g., the conductor, the pole or other structural component will break). For the tree branch case, on the other hand, both the two approximations as well as the exact solution are demonstrated in Fig. 6 for a complete comparison. It is seen that the discrepancy between the approximation \((S)\) results and the exact solution is very large. The vertical displacement given by approximation \((L)\) seems a reasonable approximation (the largest error is about 15 percent appearing at 65 m from the left end support) whereas the horizontal approximation cannot be accepted (the largest error > 58 percent). Moreover, the exact solution yields an increase in tension of about 58 percent and 21 percent compared to approximations \((S)\) and \((L)\), respectively. The exact solution.

![Fig. 4](image1)

**Fig. 4** Static profile of line with detuning pendulums; (a) vertical direction, (b) horizontal direction

![Fig. 5](image2)

**Fig. 5** Static profile of line with fallen tree at center; (a) vertical direction, (b) horizontal direction

<table>
<thead>
<tr>
<th>Case</th>
<th>Method</th>
<th>( L ) (m)</th>
<th>( H' ) (kN)</th>
<th>( F_V ) (kN)</th>
<th>( Y_M ) (m)</th>
<th>( Z_M ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Detuning tree</td>
<td>(I)</td>
<td>125.12</td>
<td>17.73</td>
<td>17.80</td>
<td>2.33</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>(II)</td>
<td>125.10</td>
<td>17.79</td>
<td>17.84</td>
<td>2.32</td>
<td>0.19</td>
</tr>
<tr>
<td>Fallen tree</td>
<td>(I)</td>
<td>100.02</td>
<td>70.84</td>
<td>70.88</td>
<td>1.16</td>
<td>1.16</td>
</tr>
<tr>
<td></td>
<td>(II)</td>
<td>100.03</td>
<td>73.06</td>
<td>73.10</td>
<td>1.15</td>
<td>1.18</td>
</tr>
<tr>
<td>Branch tree</td>
<td>(I)</td>
<td>100.09</td>
<td>8.91</td>
<td>8.94</td>
<td>1.26</td>
<td>1.47</td>
</tr>
<tr>
<td></td>
<td>(II)</td>
<td>100.01</td>
<td>10.43</td>
<td>10.44</td>
<td>0.85</td>
<td>0.85</td>
</tr>
<tr>
<td></td>
<td>(III)</td>
<td>100.02</td>
<td>12.62</td>
<td>12.63</td>
<td>0.80</td>
<td>0.93</td>
</tr>
</tbody>
</table>

![Fig. 6](image3)

**Fig. 6** Static profile of line with tree branch at quarter point; (a) vertical direction, (b) horizontal direction

808 / Vol. 62, SEPTEMBER 1995

Transactions of the ASME
indicates that the conductor will break while both approximation results suggest that the line will remain in service. The approximation (S) method failed in this case because the load of the tree branch is relatively large (the ratio of the tree branch’s weight to the distributed load is about ½) and located at one point, compared to the case of detuning pendulum where 20 percent concentrated load is distributed to three positions. The approximation (L) method failed, however, due to the neglect of the distributed loads, \( Q_y \) and \( Q_z \), which are large, compared to the concentrated loads, especially in the horizontal direction. It is interesting to note from Fig. 6 that the static profile is flat with a ratio of sag to span length of less than \( \frac{1}{8} \). For such a low conductor sag, the approximation (S) method should produce very good results for distributed loads only (Irvine, 1981), but this method cannot achieve a reasonable approximation for relatively large concentrated loads.

4 Conclusions

Two approximation and one exact solution methods have been developed to compute conductor tension and static profile under distributed and multiple concentrated loads. Formulae are given explicitly in three-dimensional space with different boundary conditions. The applications to two practical engineering examples show that the approximation (S) approach gives very accurate results for relatively small concentrated loads and the approximation (L) method produces excellent responses for very large concentrated loads, but the exact solution procedure must be used if concentrated loads are comparable to the distributed loads. It is further shown that the approximation (S) method, which is usually employed for a flat line (a sag to span length ratio of less than 1:8) with only distributed loads, does not provide reasonable results if large concentrated loads are superimposed, and the exact solution procedure must be used. However, the approximation (L) method may be used applied to the exact solution if the concentrated loads are very large.

References


