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ANALYSIS ON TOPOLOGICAL PROPERTIES OF THE LORENZ AND THE CHEN ATTRACTORS USING GCM

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This letter reports a study on some topological properties of chaos using a generalized competitive mode (GCM). The Lorenz system and the Chen system are used as examples for comparison. It is shown that for typical parameter values used in the two systems, the Lorenz attractor has one pair of GCMs in competition, while the Chen attractor has two pairs of GCMs in competition. This explains why the two attractors are topologically different, and furthermore indicates that the Chen attractor is more complex than the Lorenz attractor from the dynamics point of view.

Keywords: Lorenz system; Chen system; chaotic attractor; GCM.

1. Introduction

A new concept called *Generalized Competitive Mode* (GCM) was recently proposed to study dynamical systems [Yao *et al.*, 2002, 2004, 2006]. In particular, it has been shown [Yao *et al.*, 2006] that a chaotic system has at least two GCMs. One application of GCM is to estimate a parameter regime in which a nonlinear system may exhibit chaotic motions. It has been demonstrated that chaos may be found by analyzing the GCMs of a system without numerical integrations. Another application of GCM is to create new chaotic systems by predesigning some GCMs. More examples showing various applications of GCM can be found in [Yao *et al.*, 2006].

In this note, we consider a new application of GCM, i.e. using GCM to study the topological properties of strange attractors obtained from different systems. In particular, we study the Lorenz and the Chen systems and compare their topological characteristics. The Lorenz system is described by

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[Lorenz, 1963, 1993]

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \alpha x - y - xz, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{1}$$

where σ, α and β are parameters, taking positive real values. The Chen system [Chen & Ueta, 1999; Ueta & Chen, 2000] is given by

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x + cy - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{2}$$

where a, b and c are positive real parameters. The Chen system (2) can be obtained from the Lorenz system (1) by letting $\sigma = a$, $\alpha = c$ and $\beta = b$, and moreover adding a linear feedback control given by

$$u = -ax + (c + 1)y \tag{3}$$

to the second equation of system (1).

Both the Lorenz and the Chen systems can exhibit chaotic motions for certain parameter values. However, it has been shown that the Chen attractor is quite different from the Lorenz attractor. Two typical attractors are shown in Figs. 1(a) and 1(b), with the corresponding parameter values:

$$\sigma = 10, \quad \alpha = 25, \quad \beta = \frac{8}{3}, \tag{4}$$

for the Lorenz system, and

$$a = 35, \quad c = 28, \quad b = 3, \tag{5}$$

for the Chen system.

A fundamental question arises: Are the Lorenz attractor and the Chen attractor topologically equivalent? In this note, we use GCM to investigate the topological difference between the two attractors.

2. Generalized Competitive Mode

First, we briefly introduce the concept of GCM. In physics, mode often means a single frequency. A linear oscillator is completely described by a mode, including the frequency, phase and amplitude. However, mode may not be necessary defined physically, it can be something else considered in competition. To generalize the idea of linear mode to nonlinear systems, consider the general nonlinear autonomous system

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n, \tag{6}$$

where $f_i \in C_1(\mathbf{R})$ and, for any integer $1 \leq j \leq n$, $|\partial f_i / \partial x_j|$ is bounded. Differentiating Eq. (6) with respect to time yields

$$\begin{aligned} \ddot{x}_i = \sum_{j=1}^{n} f_j \frac{\partial f_i}{\partial x_j} &= -x_i g_i(x_1, x_2, \ldots, x_i, \ldots, x_n) \\ &\quad + h_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \end{aligned} \tag{7}$$

where g_i and h_i are some nonlinear functions, and g_i is bounded. Comparing Eq. (7) to system (6) shows that h_i should not contain variable x_i.

It can be seen from Eq. (7) that the dynamical behavior of the component x_i is determined by the functions g_i and h_i, as well as the system's initial conditions. In this sense, and comparing to system (6), we may call g_i the *generalized competitive mode* (GCM) associated with the component x_i. Comparing the term $-x_i g_i$ with $\omega^2 x$ in the linear oscillator $\ddot{x} + \omega^2 x = 0$ suggests by analogy that the GCM should only exist when $g_i > 0$.

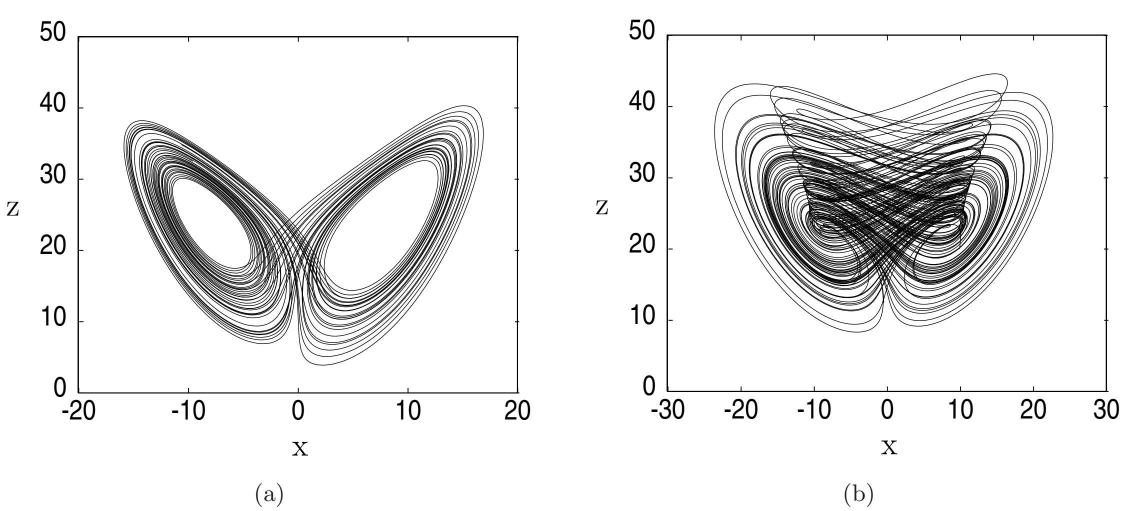


Fig. 1. (a) The Lorenz attractor for $\sigma = 10,\ \alpha = 25,\ \beta = 8/3$; and (b) the Chen attractor for $a = 35,\ c = 28,\ b = 3$.

When more than one GCM exists in a dynamical system, these GCMs may be viewed as being in competition with each other. If there exist at least two GCMs, then the competition between the GCMs *may* lead to complex motions such as chaos. However, if the system has at most one GCM, then complex phenomena like chaos cannot occur because of lacking competition. Based on the concept of mode competition and the definition of the GCM, necessary conditions for a system to have chaos have been proposed [Yao *et al.*, 2002, 2004, 2006]:

(1) There exist at least two GCMs.
(2) At least two such GCMs are competitive.
(3) At least one GCM is a function of evolutionary variables such as t.

The first condition is obvious since one mode can only generate one type of simple motion. The second one requires GCMs to compete with each other. The last condition excludes a system from being chaotic if all the GCMs are constants, such as in the linear case.

3. Main Results

Applying formula (7) to the Lorenz system (1) yields the following GCMs:

$$\begin{aligned} g_x &= -\sigma^2 - \sigma(\alpha - z), \\ g_y &= -1 - \sigma(\alpha - z) + x^2, \\ g_z &= x^2 - \beta^2. \end{aligned} \tag{8}$$

Obviously, $g_y > g_x$ if $\sigma > 1$. Thus, there is no GCM competition between the modes g_x and g_y for $\sigma > 1$. This indicates that in the commonly refereed parameter regime, there are at most two pairs of GCMs, (g_x, g_z) and (g_y, g_z), in competition. However, for the pair (g_x, g_z) to be in competition, the equation $g_z - g_x = 0$ must have at least one solution. But $g_z - g_x = x^2 + \sigma^2 + \sigma\alpha - \beta^2 - \sigma z = 0$ results in

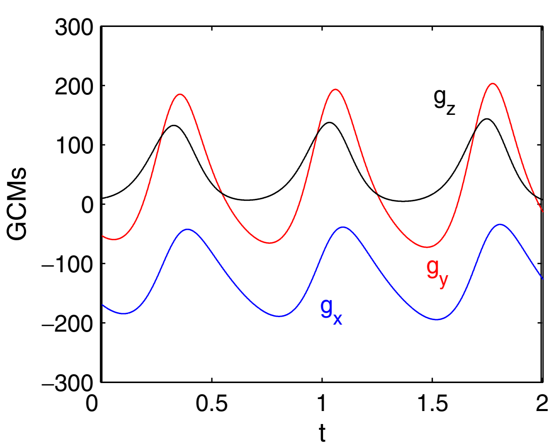
$$z = \frac{x^2}{\sigma} + \sigma + \alpha - \frac{\beta^2}{\sigma}, \tag{9}$$

which, together with $\alpha = 10$, $\alpha = 25$ and $\beta = 8/3$ (which are used for Fig. 1(a)), gives

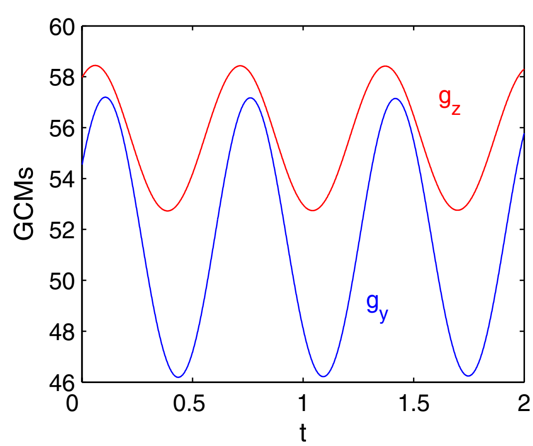
$$z = \frac{x^2}{10} + 35 - \frac{32}{45} > \frac{x^2}{10} + 34. \tag{10}$$

It is easy to see from the trajectories shown in Fig. 1(a) that z cannot reach the values specified by Eq. (10). Therefore, the Lorenz system has at most one pair of GCMs (i.e. the pair (g_y, g_z)) in competition for the parameter values $\alpha = 10$, $\alpha = 25$ and $\beta = 8/3$.

When this pair of GCMs is in competition, the Lorenz system may be chaotic; otherwise, no chaos exists. The GCMs of the Lorenz system for two sets of parameter values are shown in Fig. 2. Although only a very small change is given to parameter α (while σ and β are kept the same values for the two cases), qualitatively different behaviors have been noted. Figure 2(a) indicates one pair of GCMs in competition, implying the existence of chaotic motions (see the computer simulated phase portrait given in Fig. 1(a)). Figure 2(b), on the other hand, shows no GCM completion and hence this case cannot exhibit chaotic motions. Indeed, computer simulation yields periodic solutions (limit cycles), as shown in Fig. 3.



(a)



(b)

Fig. 2. The GCMs of the Lorenz system for $\sigma = 10, \beta = 8/3$, when (a) $\alpha = 25$; and (b) $\alpha = 24.5$.

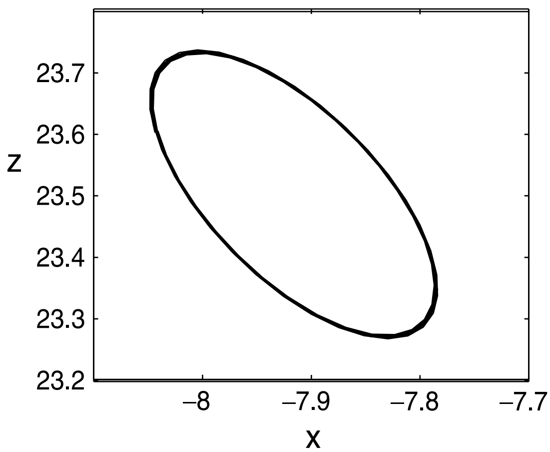


Fig. 3. The limit cycle for Fig. 2(b).

Next, we consider the Chen system (2) and obtain the corresponding GCMs:

$$\begin{aligned} g_x &= a(z-c), \\ g_y &= a(z-c) + x^2 + a^2 - c^2, \\ g_z &= x^2 - b^2. \end{aligned} \tag{11}$$

It is clear that $g_y > g_x$ when $a^2 > c^2$. Therefore, at most two pairs of GCMs, (g_x, g_z) and (g_y, g_z), are competitive. By choosing the parameter values given in Eq. (5), we use a numerical method to find the GCMs for the Chen system, as shown in Fig. 4. The following findings are observed from Fig. 4.

(i) There are two pairs of GCMs, (g_x, g_z) and (g_y, g_z), in competition for most time periods.
(ii) There exists a relatively long time between the competition periods, in which no GCM competition is found. Assume that the GCMs are in competition for, say, $t \in [t_0, t_1]$, then there is no GCMs in competition for $t \in [t_1, t_2]$, and then there is a period with competition, and then no competition, and so on. This process repeats, with a period ratio $(t_1 - t_0)/(t_2 - t_1) \approx 4$. The ratio is slightly varied with time. The time history for $z(t)$ is shown in Fig. 5. It is seen from Fig. 4 that when $t \in [0, 3] \cup [6.5, 10]$, there are two pairs of GCMs in competition; while when $t \in [3, 6.5]$, there is no GCM in competition.

Therefore, in period $t \in (t_0, t_1)$, the system's behavior is more complicated, as shown in Fig. 1(b).

We have used GCM to confirm the topological difference between the Lorenz and the Chen attractors: The Lorenz attractor has one pair of GCMs in competition, while the Chen attractor has two pairs of GCMs in competition. This explains why the Chen attractor looks more complex than the Lorenz attractor, and indicate that they are topologically different.

Next, we consider a system different from the Lorenz and the Chen systems, but showing similar behaviors. The system is given by

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= cy - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{12}$$

where a, b and c are positive real parameters. This system has been shown to exhibit attractors similar to the Lorenz and the Chen attractors [Lü & Chen, 2002]. In particular, under the fixed values $a = 36,\ b = 3$, this system is similar to the Lorenz attractor when $12.7 < c < 17.0$, has a transition

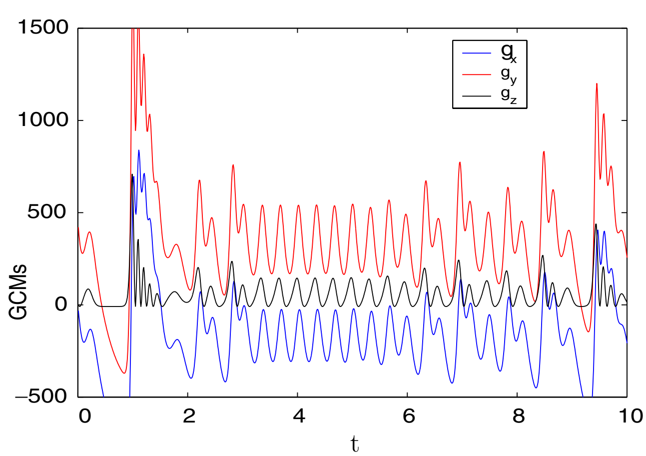


Fig. 4. The GCMs of the Chen system when $a = 35$, $c = 28$, $b = 3$.

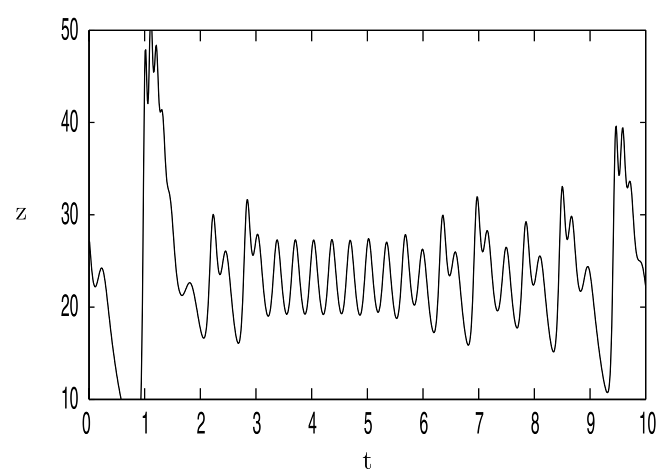
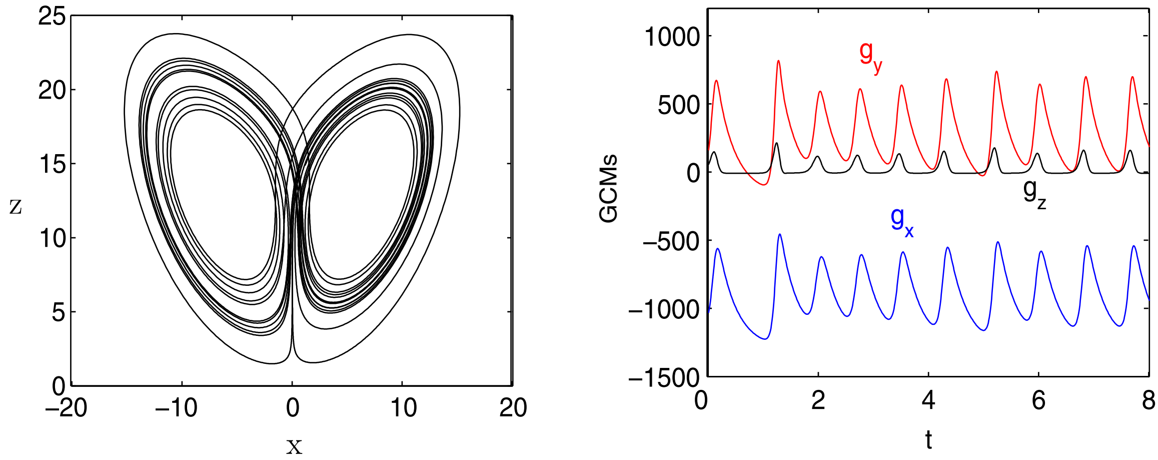
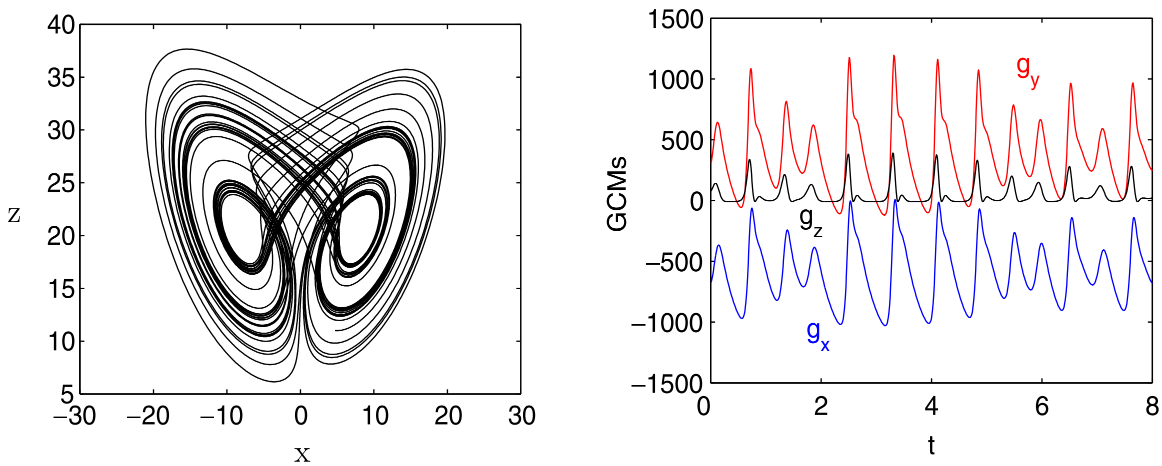


Fig. 5. The time history $y(t)$ of the Chen system when $a = 35$, $c = 28$, $b = 3$.



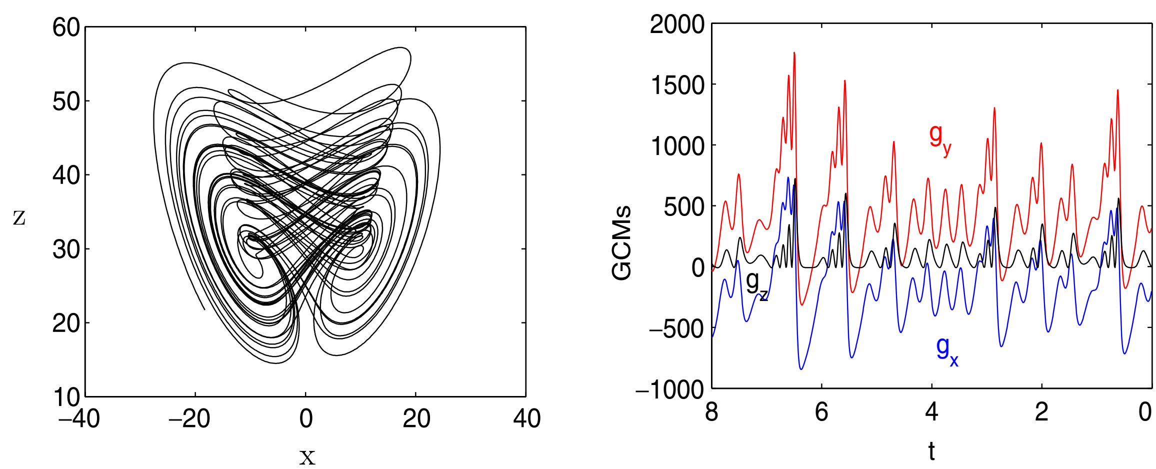
(a) (b)

Fig. 6. Computer simulation results for system (12) when $a = 36, b = 3, c = 13$: (a) the chaotic attractor; and (b) the GCMs.



(a) (b)

Fig. 7. Computer simulation results for system (12) when $a = 36, b = 3, c = 20$: (a) the chaotic attractor; and (b) the GCMs.



(a) (b)

Fig. 8. Computer simulation results for system (12) when $a = 36, b = 3, c = 13$: (a) the chaotic attractor; and (b) the GCMs.

shape when $18.0 < c < 22.0$, and then becomes similar to the Chen attractor when $23.0 < c < 28.5$. In the following, we choose $c = 13,\ 20$ and 28 for the three cases and apply GCM analysis to investigate the difference between the three cases.

Case 1. $a = 36,\ b = 3,\ c = 13$. The strange attractor is shown in Fig. 6(a), which is similar to the Lorenz attractor. The corresponding GCMs are depicted in Fig. 6(b), showing that there is only one pair of GCMs, (g_y, g_z), in competition, which agrees with what we have found in the Lorenz system.

Case 2. $a = 36,\ b = 3,\ c = 20$. For this case, the system exhibits a chaotic attractor that is slightly more complex than that of Case 1 [see Fig. 7(a)]. It is observed from the GCMs, shown in Fig. 7(b), that the pair of GCMs (g_y, g_z) keeps in competition as in Case 1, while the pair of GCMs (g_x, g_z) is about to become competitive.

Case 3. $a = 36,\ b = 3,\ c = 28$. The attractor for this case, as shown in Fig. 8(a), is similar to the Chen attractor [see Fig. 1(b)]. In fact, for this case, the two pairs of GCMs, (g_x, g_z) and (g_y, g_z), are both in stronger competition, as depicted in Fig. 8(b).

4. Conclusion

The GCM technique is employed to study chaotic attractors, showing that the Lorenz attractor and the Chen attractor are topologically (qualitatively) different because they have different numbers of GCMs. A related quadratic system has also been investigated, which further verifies and demonstrates the usefulness of GCMs in studying different types of chaotic systems.

Acknowledgments

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