Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 266 (2019) 455–492

Journal of Differential Equations

www.elsevier.com/locate/jde

Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle of order m

Junmin Yang [a,b,*], Pei Yu [b], Maoan Han [c,d]

[a] College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei, 050024, China

[b] Department of Applied Mathematics, Western University, London, Ontario, N6A 5B7, Canada

[c] Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, China

[d] Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China

Received 25 May 2018
Available online 24 July 2018

Abstract

In this paper, we study the explicit expansion of the first order Melnikov function near a double homoclinic loop passing through a nilpotent saddle of order m in a near-Hamiltonian system. For any positive integer $m (m \geq 1)$, we derive the formulas of the coefficients in the expansion, which can be used to study the limit cycle bifurcations for near-Hamiltonian systems. In particular, for $m = 2$, we use the coefficients to consider the limit cycle bifurcations of general near-Hamiltonian systems and give the existence conditions for 10, 11, 13, 15 and 16 (11, 13 and 16, respectively) limit cycles in the case that the homoclinic loop is of cuspidal type (smooth type, respectively) and their distributions. As an application, we consider a near-Hamiltonian system with a nilpotent saddle of order 2 and obtain the lower bounds of the maximal number of limit cycles.

Keywords: Limit cycle; Bifurcation; Nilpotent saddle

* Corresponding author at: College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei, 050024, China.

E-mail address: jmyang@hebtu.edu.cn (J. Yang).

https://doi.org/10.1016/j.jde.2018.07.042
0022-0396/

1. Introduction

As we know, the study of the number of limit cycles for a near Hamiltonian system is closely related to the known Hilbert's 16th problem presented by D. Hilbert more than one hundred years ago [1]. There have been many studies on the number of limit cycles for the following general near-Hamiltonian system

$$\dot{x} = H_y + \varepsilon p(x, y, \varepsilon, \delta), \quad \dot{y} = -H_x + \varepsilon q(x, y, \varepsilon, \delta), \tag{1.1}$$

where p, q and H are C^∞ functions, $\varepsilon \geq 0$ is a small perturbation parameter and δ is a parameter vector, $\delta \in D \subset \mathbf{R}^{\varpi}$ with $\varpi \in \mathbf{Z}^+$ and D compact. When $\varepsilon = 0$, system (1.1) becomes the following Hamiltonian system,

$$\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.2}$$

Suppose that system (1.2) has a nilpotent singular point, which, without loss of generality, is assumed at the origin. That is to say, the function $H(x, y)$ satisfies $H_x(0, 0) = 0$, $H_y(0, 0) = 0$, and

$$\frac{\partial(H_y, -H_x)}{\partial(x, y)} \neq \mathbf{0}, \quad \det \frac{\partial(H_y, -H_x)}{\partial(x, y)} = 0.$$

Without loss of generality, we further suppose that

$$H_{yy}(0, 0) = 1, \quad H_{xy}(0, 0) = H_{xx}(0, 0) = 0,$$

which means that the expansion of H at the origin can be written as

$$H(x, y) = \frac{1}{2} y^2 + \sum_{i+j \geq 3} h_{i,j} x^i y^j. \tag{1.3}$$

By the implicit function theorem we can show that there exists a unique C^∞ function $\varphi(x) = \sum_{j \geq 2} e_j x^j$ such that

$$H_y(x, \varphi(x)) = 0 \tag{1.4}$$

for $|x|$ small. Then, $H(x, \varphi(x))$ can be expanded in the form of

$$H(x, \varphi(x)) = \sum_{j \geq 3} \bar{h}_j x^j.$$

Let $k \geq 3$ be an integer such that

$$\bar{h}_k \neq 0, \quad \bar{h}_j = 0, \quad \text{for} \quad j < k. \tag{1.5}$$

For cubic Hamiltonian systems, Han et al. [2] gave the following definition by using a well known result introduced in [3].

Definition 1. ([2]) Consider system (1.1) with (1.5) being satisfied. Then the origin of system (1.2) is called a cusp of order m if $k = 2m + 1$. It is called a nilpotent center of order m (a nilpotent saddle of order m, respectively) if $k = 2m + 2$ and $h_k > 0$ (if $k = 2m + 2$ and $h_k < 0$, respectively).

Suppose the level curves $H(x, y) = h, h \in J$ of the Hamiltonian system (1.2) contain at least a family of closed orbits with clockwise orientation denoted by Γ_h, where J denotes an open interval having an endpoint $h = 0$. Introduce ([4])

$$M(h, \delta) = \oint_{\Gamma_h} q dx - p dy|_{\varepsilon=0},$$

which is called the Abelian integral or the first order Melnikov function of system (1.1). The so-called weak Hilbert's 16th problem is to find the maximal number of isolated zeros of $M(h, \delta)$, which is closely related to the maximal number of limit cycles of system (1.1) (see [5], [6]). Many studies have been done on the expansion of $M(h, \delta)$ and limit cycle bifurcations of system (1.1). In the following, we briefly summarize some of these works.

Let (1.3) and (1.5) hold. If the origin of system (1.2) is a nilpotent center of order m, i.e. $\bar{h}_k > 0 (k = 2m + 2)$, Han et al. [7] derived the asymptotic expansions of $M(h, \delta)$ at $h = 0$ for system (1.1), given by

$$M(h, \delta) = h^{\frac{2+m}{2(m+1)}} \sum_{l \geq 0} b_l(\delta) h^{\frac{l}{m+1}}, \quad \text{for } 0 < h \ll 1. \tag{1.6}$$

The method of computing b_l given in [7] is hard to be extended to derive explicit formulas of b_l for larger l. Then in [8], the authors developed a new approach to obtain (1.6) and established an efficient algorithm for computing the coefficients b_l for any $m \geq 1$.

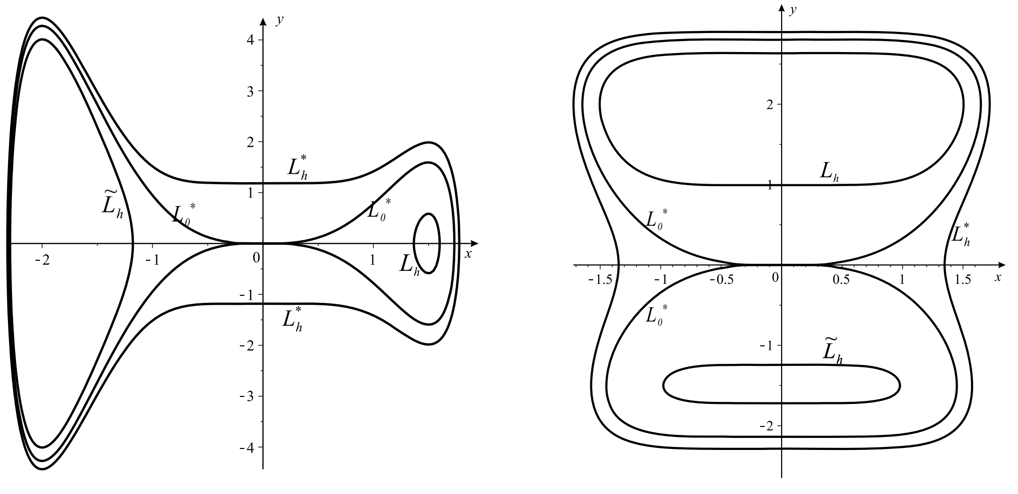
If the origin of (1.2) is a nilpotent cusp of order m, i.e. $\bar{h}_k \neq 0$ $(k = 2m + 1)$, there have also been many studies. For the special cubic Hamiltonian system,

$$\dot{x} = y, \quad \dot{y} = -x^2(x - 1),$$

which has a homoclinic loop passing through a nilpotent cusp of order 1 at the origin, Dumortier and Li [9], and Zhao and Zhang [10] studied the property of $M(h, \delta)$ near a cuspidal loop under different perturbations. Later, for general near-Hamiltonian system (1.1), Han et al. [11] derived the expansion of $M(h, \delta)$ and developed symbolic programs to calculate the coefficients of $M(h, \delta)$. Later, by using the method given in [11], Atabaigi et al. [12] and Xiong [13] obtained the first seven and eleven coefficients in the expansion of $M(h, \delta)$ for the cases $m = 2$ and $m = 3$, respectively.

If the origin of (1.2) is a nilpotent saddle, then it is a cuspidal type or a smooth type as shown in [14]. Yang and Zhao [15] obtained the upper bound of the number of zeros of Abelian integral for a quartic Hamiltonian with figure-of-eight loop through a nilpotent saddle. Zhao [16] studied the upper bound of the number of zeros of Abelian integral for a class of hyper-elliptic Hamilton systems with a double homoclinic loop through a nilpotent saddle. Suppose

$$H(x, y) = -\frac{1}{4}x^4 + \sum_{i \geq 5} h_{i,0} x^i + y^2 \sum_{i+j \geq 0} h_{i,j} x^i y^j, \tag{1.7}$$



(a) Cuspidal type (b) Smooth type

Fig. 1. A double homoclinic loop with a nilpotent saddle of order 2.

Zang et al. [17] gave the expansion of $M(h,\delta)$ at $h=0$, and obtained the formulas of the first seven coefficients of the expansion in the case of cuspidal type. The following system

$$\dot{x}=y,\quad \dot{y}=-g(x)-\varepsilon f(x)y,$$

is called a Liénard system of type (m,n), where $f(x), g(x)$ are polynomials in x with $\deg f = n, \deg g = m$. Using Chebyshev property and the asymptotic expansions of $M(h,\delta)$ given by Zang et al. [17], Wang and Xiao [18] studied small perturbations on Hamiltonian vector field with a hyper-elliptic Hamiltonian of degree five, which is a Liénard system of type $(4,3)$. They proved that this system can have at most three limit cycles in the plane for sufficiently small positive ε. Then, using the expansion of $M(h,\delta)$ given by Zang et al. [17], Kazemi and Zangemeh [19] gave the expansion of $M(h,\delta)$ for a planar near-Hamiltonian system near a heteroclinic loop connecting two nilpotent saddles of order 1 and studied the bifurcation of limit cycles in such systems. Asheghi and Bakhshalizadeh [20] studied the sharp upper bound of the maximal number of isolated zeros of $M(h,\delta)$ for a Liénard system of type $(6,5)$ by using Chebyshev property and the expansion of $M(h,\delta)$ given in Zang et al. [17].

However, it has been noted in [11] that, if $H(x,y)$ does not satisfy (1.7), the result given in [17] is not applicable.

For general system (1.1), suppose it has a double homoclinic loop L_0^* passing through a nilpotent saddle of order m (see Fig. 1). It can be seen that there are three families of periodic orbits denoted by L_h, $\widetilde{L}_h$ and L_h^*, respectively. Correspondingly, there are three Melnikov functions as follows

$$M(h,\delta)=\oint_{L_h} q\,dx-p\,dy|_{\varepsilon=0},\quad \widetilde{M}(h,\delta)=\oint_{\widetilde{L}_h} q\,dx-p\,dy|_{\varepsilon=0},\quad M^*(h,\delta)=\oint_{L_h^*} q\,dx-p\,dy|_{\varepsilon=0}. \tag{1.8}$$

Han et al. [14] gave the expansions of $M(h,\delta)$, $\widetilde{M}(h,\delta)$ and $M^*(h,\delta)$ for the cases of cuspidal type and smooth type. As we know, to study the limit cycle bifurcations we need to know the first several coefficients in the expansions. But, only for $m=1$, Han et al. [14] gave the formulas of the first seven coefficients in the expansions of $M(h,\delta)$ and $\widetilde{M}(h,\delta)$ and the first five coefficients in the expansion of $M^*(h,\delta)$. Then, based on the results given in [14], Sun [21] studied the expansion of the first order Melnikov function near a heteroclinic loop connecting two nilpotent saddles of order 1, and Tian and Han [22] studied the expansion of the first order Melnikov function near a compound loop passing through a nilpotent saddle of order 1 and a nilpotent cusp of order 1.

The coefficients for $m\geq 2$ in the expansions of $M(h,\delta)$, $\widetilde{M}(h,\delta)$ and $M^*(h,\delta)$ were not given in [14] and no any relative results exist so far. For general $m\in\mathbf{Z}^+$, not a concrete value of m, in this paper, we derive the explicit expansions of the above three Melnikov functions in both cases of cuspidal type and smooth type, and thus, the limit cycle bifurcations for near Hamiltonian systems with one or more nilpotent saddles of any order can be studied further. The main results of this paper are stated in Theorems 3.1 and 3.2.

Especially, for $m=2$, we present some sufficient conditions to determine the number and distribution of limit cycles. For example, in the case of cuspidal type, we give the existence conditions for 16, 15, 13, 11 and 10 limit cycles and their distributions. In the case of smooth type, we obtain the existence conditions for 16, 13 and 11 limit cycles and their distributions.

This paper is organized as follows. We first present some preliminary lemmas in Section 2. Then, we give the explicit expansions of $M(h,\delta)$, $\widetilde{M}(h,\delta)$, and $M^*(h,\delta)$, and their first several coefficients in Section 3. In Section 4, we further study the limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle of order 2, give the conditions that limit cycles exist and obtain the distribution of these limit cycles. As an application, in Section 5, we consider limit cycle bifurcations of a type of system with a nilpotent saddle of cuspidal type.

2. Preliminary lemmas

If the origin of system (1.2) is a nilpotent saddle of order m, the expansions of $M(h,\delta)$, $\widetilde{M}(h,\delta)$ and $M^*(h,\delta)$ in both cases of cuspidal type and smooth type are obtained by Han et al. in [14] as shown in the following two lemmas.

Lemma 2.1. *([14]) Let (1.5) hold with $h_k<0$ and $k\geq 4$ even, and L_0^* be a double homoclinic loop defined by $H(x,y)=0$ of cuspidal type. Then the functions $M(h,\delta)$, $\widetilde{M}(h,\delta)$ and $M^*(h,\delta)$ in (1.8) have the following expansions:*

(i) *for $0<-h\ll 1$,*

$$\begin{aligned} M(h,\delta)&=\varphi(h,\delta)-\frac{h\ln|h|}{2k}I^*_{1,\frac{k}{2}-1}(h)+|h|^{\frac{1}{k}+\frac{1}{2}}\sum_{\substack{r=0\\ r\neq\frac{k}{2}-1}}^{k-2}\tilde{A}_rI^*_{1r}(h)|h|^{\frac{r}{k}},\\ \widetilde{M}(h,\delta)&=\widetilde{\varphi}(h,\delta)+(-1)^{k/2}\frac{h\ln|h|}{2k}I^*_{1,\frac{k}{2}-1}(h)+|h|^{\frac{1}{k}+\frac{1}{2}}\sum_{\substack{r=0\\ r\neq\frac{k}{2}-1}}^{k-2}(-1)^r\tilde{A}_rI^*_{1r}(h)|h|^{\frac{r}{k}}, \end{aligned}\tag{2.1}$$

where $\varphi(h,\delta)\in C^\omega$ for $0\leq -h\ll 1$, $\widetilde{A}_r$ $(0\leq r\leq k-2, r\neq\frac{k}{2}-1)$ are constants satisfying

$$\tilde{A}_r = \begin{cases} -\frac{k}{k+2(1+r)} \int_0^1 \frac{v^{\frac{k}{2}-r-2}}{\sqrt{1-v^k}} dv, & \text{for } r < \frac{k}{2}-1, \\ -\frac{k}{k+2(1+r)} \left[\frac{2}{k-2(r+1)} + \int_0^1 \frac{v^{\frac{3}{2}k-r-2}}{\sqrt{1-v^k}(1+\sqrt{1-v^k})} dv \right], & \text{for } \frac{k}{2}-1 < r < k-1, \\ 0, & \text{for } r = k-1, \end{cases} \tag{2.2}$$

and I_{1r}^* $(0 \le r \le k-2)$ *satisfy*

$$I_{1r}^*(h) = \sum_{\bar{m}, j \ge 0} \widetilde{r}_{\bar{m}k+r,j} \alpha_{\bar{m}k+r,j}^* \beta_{\bar{m}k+r}^* h^{j+\bar{m}}, \tag{2.3}$$

with $\widetilde{r}_{\bar{m}k+r,j}$ *given by* (18) *in [14] and*

$$\alpha_{ij}^* = \begin{cases} \frac{\frac{3}{2}k \cdot \frac{5}{2}k \cdots \frac{2j+1}{2}k}{(\frac{3}{2}k+i+1)\cdots(\frac{2j+1}{2}k+i+1)}, & i \ge 0,\ j \ge 1, \\ 1, & i \ge 0,\ j = 0, \end{cases}$$
$$\beta_{\bar{m}k+r}^* = \begin{cases} \frac{(-1)^{\bar{m}}(r+1)(k+r+1)\cdots((\bar{m}-1)k+r+1)}{(\frac{3}{2}k+r+1)(\frac{5}{2}k+r+1)\cdots(\frac{2\bar{m}+1}{2}k+r+1)}, & \bar{m} \ge 1,\ 0 \le r \le k-1, \\ 1, & \bar{m} = 0,\ 0 \le r \le k-1; \end{cases} \tag{2.4}$$

(ii) for $0 \le h \ll 1$,

$$M^*(h,\delta) = \begin{cases} \varphi^*(h,\delta) + 2h^{\frac{1}{k}+\frac{1}{2}} \sum_{r=0}^{\frac{k}{2}-1} \bar{A}_r h^{\frac{2r}{k}} J_{1r}^*(h), \text{ if } r_1 \text{ is not an integer}, \\ \varphi^*(h,\delta) - \frac{h\ln h}{k} J_{1r_1}^*(h) + 2h^{\frac{1}{k}+\frac{1}{2}} \sum_{\substack{r=0 \\ r \ne r_1}}^{\frac{k}{2}-1} \bar{A}_r h^{\frac{2r}{k}} J_{1r}^*(h), \text{ if } r_1 \text{ is an integer}, \end{cases} \tag{2.5}$$

where $2r_1 = \frac{k}{2} - 1$, $\varphi^*(h,\delta) \in C^\omega$ *for* $0 \le h \ll 1$, $\bar{A}_r (0 \le r \le \frac{k}{2}-1, r \ne r_1)$ *are constants satisfying*

$$\bar{A}_r = \begin{cases} \frac{k}{k+2(1+2r)} \int_0^\infty \frac{v^{2r}dv}{\sqrt{1+v^k}}, & \text{for } \quad 2r < \frac{k}{2}-1, \\ -\frac{k}{k+2(1+2r)} \int_0^\infty \frac{v^{2r}dv}{\sqrt{v^k(1+v^k)}[\sqrt{v^k}+\sqrt{1+v^k}]}, & \text{for } \quad \frac{k}{2}-1 < 2r, \end{cases} \tag{2.6}$$

and J_{1r}^* $(0 \le r \le \frac{k}{2}-1)$ *satisfy*

$$J_{1r}^*(h) = \sum_{\substack{i=\frac{\bar{m}k}{2}+r \\ \bar{m}\geq 0,\, j\geq 1\, odd}} r_{ij}^{(1)} \bar{\alpha}_{ij} \bar{\beta}_i h^{\bar{m}+[\frac{j}{2}]}, \tag{2.7}$$

with $r_{ij}^{(1)}$ *given by* (20) *in [14] and*

$$\bar{\alpha}_{ij} = \begin{cases} 1, & i \geq 0,\ j = 1, \\ \dfrac{\frac{3}{2}k \cdot \frac{5}{2}k \cdots \frac{j}{2}k}{(\frac{3}{2}k+2i+1)(\frac{5}{2}k+2i+1)\cdots(\frac{j}{2}k+2i+1)}, & i \geq 0,\ j \geq 3\ odd, \end{cases}$$

$$\bar{\beta}_i = \begin{cases} 1, & 0 \leq i \leq \frac{k}{2}-1, \\ \dfrac{(-1)^{\bar{m}}(2i+1-k)(2i+1-2k)\cdots(2i+1-\bar{m}k)}{(2i+1+\frac{1}{2}k)(2i+1-\frac{1}{2}k)\cdots(2i+1-\frac{2\bar{m}-3}{2}k)}, & i = \frac{\bar{m}k}{2}+r,\ \bar{m}\geq 1, \\ & 0 \leq r \leq \frac{k}{2}-1. \end{cases}$$

Lemma 2.2. *([14]) Let* (1.5) *hold with* $h_k < 0$ *and* $k \geq 4$ *even, and* L_0^* *be a double homoclinic loop of smooth type. The functions* $M(h,\delta)$, $\widetilde{M}(h,\delta)$ *and* $M^*(h,\delta)$ *in* (1.8) *have the following expansions:*

(i) for $0 < h \ll 1$,

$$M(h,\delta) = \begin{cases} \varphi(h,\delta) + h^{\frac{1}{k}+\frac{1}{2}} \displaystyle\sum_{r=0}^{\frac{k}{2}-1} \bar{A}_r h^{\frac{2r}{k}} J_{1r}^*(h), & \text{if } r_1 \text{ is not an integer}, \\ \varphi(h,\delta) - \dfrac{h\ln h}{2k} J_{1r_1}^*(h) + h^{\frac{1}{k}+\frac{1}{2}} \displaystyle\sum_{\substack{r=0 \\ r\neq r_1}}^{\frac{k}{2}-1} \bar{A}_r h^{\frac{2r}{k}} J_{1r}^*(h), & \text{if } r_1 \text{ is an integer}, \end{cases} \tag{2.8}$$

$$\widetilde{M}(h,\delta) = \begin{cases} \widetilde{\varphi}(h,\delta) + h^{\frac{1}{k}+\frac{1}{2}} \displaystyle\sum_{r=0}^{\frac{k}{2}-1} \bar{A}_r h^{\frac{2r}{k}} J_{1r}^*(h), & \text{if } r_1 \text{ is not an integer}, \\ \widetilde{\varphi}(h,\delta) - \dfrac{h\ln h}{2k} J_{1r_1}^*(h) + h^{\frac{1}{k}+\frac{1}{2}} \displaystyle\sum_{\substack{r=0 \\ r\neq r_1}}^{\frac{k}{2}-1} \bar{A}_r h^{\frac{2r}{k}} J_{1r}^*(h), & \text{if } r_1 \text{ is an integer}, \end{cases} \tag{2.9}$$

where $\varphi, \widetilde{\varphi} \in C^\infty$ *for* $0 \leq h \ll 1$, J_{1r}^* $(0 \leq r \leq \frac{k}{2}-1)$ *satisfy* (2.7), $\bar{A}_r$ $(0 \leq r \leq \frac{k}{2}-1, r \neq r_1)$ *are constants satisfying* (2.6), *and* $2r_1 = \frac{k}{2}-1$;

(ii) for $0 < -h \ll 1$,

$$M^*(h,\delta) = \begin{cases} \varphi^*(h,\delta) + 2|h|^{\frac{1}{k}+\frac{1}{2}} \sum_{r=0}^{\left[\frac{k-1}{2}\right]} \widetilde{A}_{2r}|h|^{\frac{2r}{k}} I^*_{1,2r}(h), \text{ if } r_1 \text{ is not an integer}, \\ \varphi^*(h,\delta) - \dfrac{h\ln|h|}{k} I^*_{1,2r_1}(h) + 2|h|^{\frac{1}{k}+\frac{1}{2}} \sum_{\substack{r=0 \\ r\neq r_1}}^{\left[\frac{k-1}{2}\right]} \widetilde{A}_{2r}|h|^{\frac{2r}{k}} I^*_{1,2r}(h), \text{ if } r_1 \text{ is an integer}, \end{cases} \tag{2.10}$$

where $\varphi^* \in C^\omega$ *for* $0 < -h \ll 1$, $I^*_{1,2r}$ *satisfy* (2.3) *and* $\widetilde{A}_{2r}$ *are constants satisfying* (2.2).

By (2.1), (2.3), (2.7), (2.8)–(2.10), it is easy to see that in order to obtain the coefficients in the expansions of M, $\widetilde{M}$ and M^* in Lemma 2.1 and Lemma 2.2, we need to compute $\widetilde{r}_{ij}$ in (2.3) and $r^{(1)}_{ij}$ in (2.7). For general $k \geq 3$, we first present the algorithm for computing $\widetilde{r}_{ij}$ and $r^{(1)}_{ij}$ given in [14] to give the relationship between $\widetilde{r}_{ij}$ and $r^{(1)}_{ij}$.

Consider system (1.1), where $H(x, y)$ satisfies (1.3) and

$$p(x,y,\varepsilon,\delta) = \sum_{i+j=0}^{n} a_{i,j}x^i y^j, \quad q(x,y,\varepsilon,\delta) = \sum_{i+j=0}^{n} b_{i,j}x^i y^j.$$

Suppose system (1.1) has a nilpotent singular point of order m at the origin.

Introduce a new variable

$$v = y - \varphi(x). \tag{2.11}$$

Then system (1.1) becomes

$$\dot{x} = H^*_v(x,v) + \varepsilon p^*(x,v,\varepsilon,\delta), \quad \dot{v} = -H^*_x(x,v) + \varepsilon q^*(x,v,\varepsilon,\delta),$$

where

$$H^*(x,v) = H(x, v+\varphi(x)), \quad p^*(x,v,\varepsilon,\delta) = p(x, v+\varphi(x), \varepsilon, \delta),$$
$$q^*(x,v,\varepsilon,\delta) = q(x, v+\varphi(x), \varepsilon, \delta) - \varphi'(x)p^*(x,v,\varepsilon,\delta).$$

By (1.3) and (1.4) we know that $H^*(x, v)$ can be written as

$$H^*(x,v) = H^*_0(x) + \sum_{j\geq 1} H^*_j(x)v^{j+1},$$

where

$$H^*_0(x) = H(x,\varphi(x)), \quad H^*_j(x) = \frac{1}{j+1}\frac{\partial^{j+1}H}{\partial y^{j+1}}(x,\varphi(x)), \quad j \geq 1.$$

Let $\bar{q}(x, v, \delta) = q^*(x, v, 0, \delta) - q^*(x, 0, 0, \delta) + \int_0^v p_x^*(x, u, 0, \delta)du$. Then it can be shown that

$$\bar{q}_v = (p_x^* + q_v^*)|_{\varepsilon=0} \tag{2.12}$$

and $\bar{q}(x, 0, \delta) = 0$. So we can write

$$\bar{q}(x, v, \delta) = v \sum_{i+j\geq 0} \bar{b}_{ij} x^i v^j = \sum_{j\geq 1} q_j(x) v^j, \tag{2.13}$$

where

$$q_{j+1}(x) \quad = \quad \frac{1}{(j+1)!}\frac{\partial^j}{\partial y^j}(p_x + q_y)(x, \varphi(x), 0, \delta) = \sum_{i\geq 0} \bar{b}_{ij} x^i, \quad j \geq 0. \tag{2.14}$$

Note that the periodic orbits of system (1.1) is in clockwise orientation. So, if the origin is a nilpotent center of order m, the equation $H(x, y) = h, h > 0$ (or $H^*(x, v) = h, h > 0$) defines a family of periodic orbits surrounding the origin. If the origin is a nilpotent cusp of order m, the equation $H(x, y) = h, h < 0$ or $H(x, y) = h, h > 0$ defines a family of periodic orbits inside or outside the cuspidal loop. And if the origin is a nilpotent saddle of order m as shown in Fig. 1, the equation $H(x, y) = h, h > 0$ (or $h < 0$) defines a family of periodic orbits L_h^* (or two families of periodic orbits L_h and $\widetilde{L}_h$). By [8], [11] and [14] we find that for any $k \geq 3$, the equation $H^*(x, v) = h$ has exactly two C^∞ solutions $v_1(x, w)$ and $v_2(x, w)$ in v satisfying $v_2(x, w) = v_1(x, -w)$, where $w = \sqrt{h - H_0^*(x)}$. We then write

$$\bar{q}(x, v_1, \delta) - \bar{q}(x, v_2, \delta) = \sum_{j\geq 0} \bar{q}_j(x) w^{2j+1}. \tag{2.15}$$

Let $\psi(x) = \operatorname{sgn}(x)\left[H_0^*(x)\right]^{\frac{1}{k}}$ if the origin is a nilpotent center, $\psi(x) = \left[H_0^*(x)\right]^{\frac{1}{k}}$ if the origin is a nilpotent cusp or $\psi(x) = \operatorname{sgn}(x)\left[-H_0^*(x)\right]^{\frac{1}{k}}$ if the origin is a nilpotent saddle, and

$$\widetilde{q}_j(u) = \left.\frac{\bar{q}_j(x)}{\psi'(x)}\right|_{x=\psi^{-1}(u)} = \sum_{i\geq 0} \widetilde{r}_{i,j} u^i. \tag{2.16}$$

We further let

$$\widetilde{q}_j(u) + \widetilde{q}_j(-u) = \sum_{i\geq 0} r_{i,j} u^{2i} \tag{2.17}$$

and

$$\begin{gathered} \bar{q}(x, v_l(x, w), \delta) = \sum_{j\geq 1} \bar{q}_{lj}(x) w^j, \\ \widetilde{q}_{lj}(u) = \left.\frac{\bar{q}_{lj}(x)}{\psi'(x)}\right|_{x=\psi^{-1}(u)}, \quad \widetilde{q}_{lj}(u) + \widetilde{q}_{lj}(-u) = \sum_{i\geq 0} r_{i,j}^{(l)} u^{2i}, \quad l = 1, 2, \end{gathered} \tag{2.18}$$

where $\bar{q}_{2j}(x) = (-1)^j \bar{q}_{1j}(x), \widetilde{q}_{2j}(u) = (-1)^j \widetilde{q}_{1j}(u)$ which gives $r_{i,j}^{(2)} = (-1)^j r_{i,j}^{(1)}$ by [14].

From the process of computing $\widetilde{r}_{ij}$ and $\widetilde{r}_{ij}^{(1)}$, it can be seen that the computation is very complicated. Now, at first, we give a relationship between $\widetilde{r}_{ij}$ and $r_{ij}^{(1)}$. Further, note that the programs in [11] with appropriate modifying can be used to compute $\widetilde{r}_{ij}$ in this paper. Then, based on the programs in [11] and the following Lemma, for $k = 2m + 2(m \geq 1, m \in \mathbf{Z}^+), \bar{h}_k < 0$ we can compute some coefficients of the expansions of M, $\widetilde{M}$ and M^* in (2.1), (2.5), (2.8)–(2.10), respectively.

Lemma 2.3. *Let $k \geq 3$ be an integer and (1.5) hold. Then for $r_{ij}, \widetilde{r}_{i,j}$ and $r_{i,j}^{(1)}$ in (2.16), (2.17) and (2.18), we have*

$$r_{ij} = 2\widetilde{r}_{2i,j}, \quad r_{i,2j+1}^{(1)} = \widetilde{r}_{2i,j}, \quad i \geq 0, \quad j \geq 0.$$

Proof. By (2.16) and (2.17), we have

$$\widetilde{q}_j(u) + \widetilde{q}_j(-u) = \sum_{i\geq 0} \widetilde{r}_{i,j} u^i + \sum_{i\geq 0} \widetilde{r}_{i,j} (-u)^i = \sum_{i\geq 0} 2\widetilde{r}_{2i,j} u^{2i}, \quad j \geq 0. \tag{2.19}$$

Note that

$$\widetilde{q}_j(u) + \widetilde{q}_j(-u) = \sum_{i\geq 0} r_{ij} u^{2i}, \quad j \geq 0.$$

We easily get $r_{ij} = 2\widetilde{r}_{2i,j}$.

By (2.18), we have

$$\begin{aligned} \bar{q}(x, v_1, \delta) - \bar{q}(x, v_2, \delta) &= \sum_{j\geq 0} \bar{q}_{1j}(x) w^j - \sum_{j\geq 0} \bar{q}_{2j}(x) w^j \\ &= \sum_{j\geq 0} \bar{q}_{1j}(x) w^j - \sum_{j\geq 0} (-1)^j \bar{q}_{1j}(x) w^j \\ &= \sum_{j\geq 0} 2\bar{q}_{1,2j+1}(x) w^{2j+1}, \end{aligned}$$

which, together with (2.15), gives

$$\bar{q}_j(x) = 2\bar{q}_{1,2j+1}(x), \quad j \geq 0.$$

By (2.15), (2.16) and (2.18), we further obtain

$$\begin{aligned} \widetilde{q}_j(u) + \widetilde{q}_j(-u) &= \left.\frac{\bar{q}_j(x)}{\psi'(x)}\right|_{x=\psi^{-1}(u)} + \left.\frac{\bar{q}_j(x)}{\psi'(x)}\right|_{x=\psi^{-1}(-u)} \\ &= \left.\frac{2\bar{q}_{1,2j+1}(x)}{\psi'(x)}\right|_{x=\psi^{-1}(u)} + \left.\frac{2\bar{q}_{1,2j+1}(x)}{\psi'(x)}\right|_{x=\psi^{-1}(-u)} \\ &= 2\widetilde{q}_{1,2j+1}(u) + 2\widetilde{q}_{1,2j+1}(-u) \\ &= 2\sum_{i\geq 0} r_{i,2j+1}^{(1)} u^{2i}. \end{aligned}$$

It follows from (2.19) that $r_{i,2j+1}^{(1)} = \widetilde{r}_{2i,j}$. This ends the proof. □

3. The coefficients in the expansions of M, $\widetilde{M}$ and M^*

Suppose the origin of (1.2) is a nilpotent saddle of order m ($m \geq 1$). In the following, we will give the formulas of some coefficients in the expansions of $M(h,\delta)$, $\widetilde{M}(h,\delta)$ and $M^*(h,\delta)$. At first, in the cuspidal case we have the following theorem.

Theorem 3.1. *Let (1.5) hold with $\bar{h}_k < 0$ ($k = 2m+2, m \geq 1$) and $L_0^* = L_0 \cup \widetilde{L}_0$ be a double homoclinic loop of cuspidal type. Then for the expansions of $M(h,\delta)$ and $\widetilde{M}(h,\delta)$ in (2.1) we have*

$$\begin{aligned}
M(h,\delta) &= c_0 + \sum_{r=0}^{\frac{k}{2}-2} c_{r+1}|h|^{\frac{r+1}{k}+\frac{1}{2}} + c_{\frac{k}{2}} h \ln|h| + c_{\frac{k}{2}+1} h + \sum_{r=\frac{k}{2}}^{k-2} c_{r+2}|h|^{\frac{r+1}{k}+\frac{1}{2}} \\
&\quad + \sum_{r=0}^{\frac{k}{2}-2} c_{k+1+r}|h|^{\frac{r+1}{k}+\frac{3}{2}} + c_{\frac{3}{2}k} h^2 \ln|h| + O(h^2),\ 0 < -h \ll 1, \\
\widetilde{M}(h,\delta) &= \widetilde{c}_0 + \sum_{r=0}^{\frac{k}{2}-2} (-1)^r c_{r+1}|h|^{\frac{r+1}{k}+\frac{1}{2}} + (-1)^{\frac{k}{2}+1} c_{\frac{k}{2}} h \ln|h| + \widetilde{c}_{\frac{k}{2}+1} h + \sum_{r=\frac{k}{2}}^{k-2} (-1)^r c_{r+2}|h|^{\frac{r+1}{k}+\frac{1}{2}} \\
&\quad + \sum_{r=0}^{\frac{k}{2}-2} (-1)^r c_{k+1+r}|h|^{\frac{r+1}{k}+\frac{3}{2}} + (-1)^{\frac{k}{2}+1} c_{\frac{3}{2}k} h^2 \ln|h| + O(h^2),\ 0 < -h \ll 1,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&c_0 = M(0,\delta) = \oint_{L_0} q\,dx - p\,dy,\ c_{r+1} = \widetilde{A}_r \widetilde{r}_{r,0},\quad r = 0, 1, \cdots, \frac{k}{2} - 2,\quad c_{\frac{k}{2}} = -\frac{1}{2k}\widetilde{r}_{\frac{k}{2}-1,0}, \\
&c_{\frac{k}{2}+1} = \oint_{L_0} \left(p_x + q_y - \bar{b}_{0,0} - \bar{b}_{1,0}x - \bar{b}_{2,0}x^2 - \cdots - \bar{b}_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1}\right) dt + \sum_{i=1}^{\frac{k}{2}} O_1(c_i), \\
&c_{r+2} = \widetilde{A}_r \widetilde{r}_{r,0},\quad r = \tfrac{k}{2}, \tfrac{k}{2}+1, \cdots, k-2, \\
&c_{k+1+r} = \frac{\widetilde{A}_r}{\frac{3}{2}k+r+1}\left((r+1)\widetilde{r}_{k+r,0} - \tfrac{3}{2}k\,\widetilde{r}_{r,1}\right),\quad r = 0, 1, \cdots, \tfrac{k}{2} - 2, \\
&c_{\frac{3}{2}k} = \tfrac{1}{8k}\left(\widetilde{r}_{\frac{3k}{2}-1,0} - 3\widetilde{r}_{\frac{k}{2}-1,1}\right), \\
&\widetilde{c}_{\frac{k}{2}+1} = \oint_{\widetilde{L}_0} \left(p_x + q_y - \bar{b}_{0,0} - \bar{b}_{1,0}x - \bar{b}_{2,0}x^2 - \cdots - \bar{b}_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1}\right) dt + \sum_{i=1}^{\frac{k}{2}} O_1(c_i),
\end{aligned} \tag{3.2}$$

where $\widetilde{r}_{i,0}, \widetilde{r}_{i,1}$ and $\bar{b}_{i,0}$ can be obtained by [14], based on the programs in [11]. For example, some $\bar{b}_{i,0}$ are shown as follows:

$$\begin{aligned}
&\bar{b}_{0,0} = a_{10} + b_{01},\ \bar{b}_{1,0} = 2a_{20} + b_{11},\ \bar{b}_{2,0} = 3a_{30} + b_{21} - h_{21}(a_{11} + 2b_{02}), \\
&\bar{b}_{3,0} = 4a_{40} + b_{31} - 2h_{21}(a_{21} + b_{12}) + (3h_{12}h_{21} - h_{31})(a_{11} + 2b_{02}), \cdots
\end{aligned}$$

$O_1(c)$ denotes c multiplied by a constant and $\widetilde{A}_r$ satisfies (2.2).

For the expansion of $M^*(h,\delta)$ *in* (2.5) *we have*

$$M^*(h,\delta) = c_0^* + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \hat{A}_r c_{2r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + 2c_{\frac{k}{2}} h\ln h + c^*_{\frac{k}{4}+\frac{3}{2}} h + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \hat{A}_r c_{2r+2} h^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$- \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \hat{A}_r c_{k+1+2r} h^{\frac{2r+1}{k}+\frac{3}{2}} + 2c_{\frac{3}{2}k} h^2 \ln h + O(h^2),\ 0<h\ll 1,$$

if m is even, or

$$M^*(h,\delta) = c_0^* + \sum_{r=0}^{\frac{k}{4}-1} \hat{A}_r c_{2r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + \bar{c}^*_{\frac{k}{4}+1} h + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} \hat{A}_r c_{2r+2} h^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$- \sum_{r=0}^{\frac{k}{4}-1} \hat{A}_r c_{k+1+2r} h^{\frac{2r+1}{k}+\frac{3}{2}} + O(h^2),$$

if m is odd, where

$$\begin{aligned} &c_0^* = c_0 + \widetilde{c}_0,\quad \hat{A}_r = \frac{2\bar{A}_r}{\widetilde{A}_{2r}},\\ &c^*_{\frac{k}{4}+\frac{3}{2}} = \oint_{L_0^*} \left(p_x + q_y - \bar{b}_{0,0} - \bar{b}_{1,0}x - \bar{b}_{2,0}x^2 - \cdots - \bar{b}_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1}\right)dt + \sum_{i=1}^{\frac{k}{2}} O_1(c_i),\\ &\bar{c}^*_{\frac{k}{4}+1} = \oint_{L_0^*} \left(p_x + q_y - \bar{b}_{0,0} - \bar{b}_{1,0}x - \bar{b}_{2,0}x^2 - \cdots - \bar{b}_{\frac{k}{2}-2,0}x^{\frac{k}{2}-2}\right)dt + \sum_{i=1}^{\frac{k}{2}-1} O_1(c_i), \end{aligned} \tag{3.3}$$

and each $\bar{A}_r$ *satisfies* (2.6).

Proof. By (2.3), $I^*_{1r}(h)$ can be written as

$$\begin{aligned} I^*_{1r}(h) &= \widetilde{r}_{r,0}\alpha^*_{r,0}\beta^*_r + (\widetilde{r}_{k+r,0}\alpha^*_{k+r,0}\beta^*_{k+r} + \widetilde{r}_{r,1}\alpha^*_{r,1}\beta^*_r)h + O(h^2),\\ &= \widetilde{r}_{r,0} + (\widetilde{r}_{k+r,0}\beta^*_{k+r} + \widetilde{r}_{r,1}\alpha^*_{r,1})h + O(h^2),\quad 0\le r\le k-2, \end{aligned} \tag{3.4}$$

by (2.4), where

$$\alpha^*_{r,0}=1,\ \beta^*_r=1,\ \alpha^*_{k+r,0}=1,\ \beta^*_{k+r} = -\frac{r+1}{\frac{3}{2}k+r+1},\ \alpha^*_{r,1} = \frac{\frac{3}{2}k}{\frac{3}{2}k+r+1}.$$

Substituting (3.4) into the formula of $M(h,\delta)$ in (2.1) we get

$$
\begin{aligned}
M(h,\delta) &= \varphi(0,\delta)+\varphi_h(0,\delta)h-\frac{h\ln|h|}{2k}\left[\widetilde{r}_{\frac{k}{2}-1,0}+(\widetilde{r}_{\frac{3k}{2}-1,0}\beta^*_{\frac{3k}{2}-1}+\widetilde{r}_{\frac{k}{2}-1,1}\alpha^*_{\frac{k}{2}-1,1})h\right]\\
&\quad+\sum_{r=0}^{\frac{k}{2}-2}\left[\widetilde{A}_r\widetilde{r}_{r,0}|h|^{\frac{r+1}{k}+\frac{1}{2}}-\widetilde{A}_r\left(\widetilde{r}_{k+r,0}\beta^*_{k+r}+\widetilde{r}_{r,1}\alpha^*_{r,1}\right)|h|^{\frac{r+1}{k}+\frac{3}{2}}\right]\\
&\quad+\sum_{r=\frac{k}{2}}^{k-2}\widetilde{A}_r\widetilde{r}_{r,0}|h|^{\frac{r+1}{k}+\frac{1}{2}}+O(h^2)\\
&=\varphi(0,\delta)+\sum_{r=0}^{\frac{k}{2}-2}\widetilde{A}_r\widetilde{r}_{r,0}|h|^{\frac{r+1}{k}+\frac{1}{2}}-\frac{1}{2k}\widetilde{r}_{\frac{k}{2}-1,0}h\ln|h|+\varphi_h(0,\delta)h\\
&\quad+\sum_{r=\frac{k}{2}}^{k-2}\widetilde{A}_r\widetilde{r}_{r,0}|h|^{\frac{r+1}{k}+\frac{1}{2}}-\sum_{r=0}^{\frac{k}{2}-2}\widetilde{A}_r\left(\widetilde{r}_{k+r,0}\beta^*_{k+r}+\widetilde{r}_{r,1}\alpha^*_{r,1}\right)|h|^{\frac{r+1}{k}+\frac{3}{2}}\\
&\quad-\frac{1}{2k}\left(\widetilde{r}_{\frac{3k}{2}-1,0}\beta^*_{\frac{3k}{2}-1}+\widetilde{r}_{\frac{k}{2}-1,1}\alpha^*_{\frac{k}{2}-1,1}\right)h^2\ln|h|+O(h^2).
\end{aligned}
\tag{3.5}
$$

Comparing (3.5) with the expansion of $M(h,\delta)$ in (3.1) yields the formulas of c_i, $0\le i\le \frac{3}{2}k$, $i\ne \frac{k}{2}+1$ as shown in (3.2).

Similarly, substituting (3.4) into the formula of $\widetilde{M}(h,\delta)$ in (2.1) we have

$$
\begin{aligned}
\widetilde{M}(h,\delta) &= \widetilde{\varphi}(0,\delta)+\sum_{r=0}^{\frac{k}{2}-2}(-1)^r\widetilde{A}_r\widetilde{r}_{r,0}|h|^{\frac{r+1}{k}+\frac{1}{2}}+(-1)^{\frac{k}{2}}\frac{1}{2k}\widetilde{r}_{\frac{k}{2}-1,0}h\ln|h|+\widetilde{\varphi}_h(0,\delta)h\\
&\quad+\sum_{r=\frac{k}{2}}^{k-2}(-1)^r\widetilde{A}_r\widetilde{r}_{r,0}|h|^{\frac{r+1}{k}+\frac{1}{2}}-\sum_{r=0}^{\frac{k}{2}-2}(-1)^r\widetilde{A}_r\left(\widetilde{r}_{k+r,0}\beta^*_{k+r}+\widetilde{r}_{r,1}\alpha^*_{r,1}\right)|h|^{\frac{r+1}{k}+\frac{3}{2}}\\
&\quad+(-1)^{\frac{k}{2}}\frac{1}{2k}(\widetilde{r}_{\frac{3k}{2}-1,0}\beta^*_{\frac{3k}{2}-1}+\widetilde{r}_{\frac{k}{2}-1,1}\alpha^*_{\frac{k}{2}-1,1})h^2\ln|h|+O(h^2)\\
&=\widetilde{c}_0+\sum_{r=0}^{\frac{k}{2}-2}\widetilde{c}_{r+1}|h|^{\frac{r+1}{k}+\frac{1}{2}}+\widetilde{c}_{\frac{k}{2}}h\ln|h|+\widetilde{c}_{\frac{k}{2}+1}h+\sum_{r=\frac{k}{2}}^{k-2}\widetilde{c}_{r+2}|h|^{\frac{r+1}{k}+\frac{1}{2}}\\
&\quad+\sum_{r=0}^{\frac{k}{2}-2}\widetilde{c}_{k+1+r}|h|^{\frac{r+1}{k}+\frac{3}{2}}+\widetilde{c}_{\frac{3}{2}k}h^2\ln|h|+O(h^2),
\end{aligned}
$$

where

$$
\begin{aligned}
&\widetilde{c}_0=\widetilde{M}(0,\delta)=\oint_{\widetilde{L}_0}q\,dx-p\,dy,\\
&\widetilde{c}_{r+1}=(-1)^r c_{r+1},\quad r=0,1,\cdots,\tfrac{k}{2}-2,\quad \widetilde{c}_{\frac{k}{2}}=(-1)^{\frac{k}{2}+1}c_{\frac{k}{2}},\\
&\widetilde{c}_{r+2}=(-1)^r c_{r+2},\quad r=\tfrac{k}{2},\tfrac{k}{2}+1,\cdots,k-2,\\
&\widetilde{c}_{k+1+r}=(-1)^r c_{k+1+r},\quad r=0,1,\cdots,\tfrac{k}{2}-2,\ \widetilde{c}_{\frac{3}{2}k}=(-1)^{\frac{k}{2}+1}c_{\frac{3}{2}k}.
\end{aligned}
$$

Next, we derive the formulas of $c_{\frac{k}{2}+1}$ and $\widetilde{c}_{\frac{k}{2}+1}$. It follows from (2.12), (2.13) and (2.14) that

$$\begin{aligned} p_x^* + q_v^* &= \bar{q}_v = \sum_{j\geq 1} j q_j(x) v^{j-1} \\ &= \bar{b}_{0,0} + \bar{b}_{1,0}x + \bar{b}_{2,0}x^2 + \cdots + \bar{b}_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1} + x^{\frac{k}{2}}\phi_1(x) + v\phi_2(x,v), \end{aligned}$$

where $\phi_1(x)$ is a polynomial in x and $\phi_2(x, v)$ is a polynomial in (x, v).

Note that $p_x + q_y = p_x^* + q_v^*$ and by [23],

$$M_h'(h,\delta) = \oint_{L_h} (p_x + q_y)dt. \tag{3.6}$$

Then, $M(h, \delta)$ can be rewritten as

$$\begin{aligned} M(h,\delta) &= M(0,\delta) + \int_0^h M_h'(h,\delta)dh \\ &= M(0,\delta) + \int_0^h \left[\oint_{L_h} (p_x + q_y)dt\right] dh \\ &= M(0,\delta) + \int_0^h \left[\oint_{L_h^*} (p_x^* + q_v^*)dt\right] dh \\ &= M(0,\delta) + \sum_{i=0}^{\frac{k}{2}-1} \bar{b}_{i,0} m_i(h) + m_{\frac{k}{2}}(h) + m_{\frac{k}{2}+1}(h), \end{aligned} \tag{3.7}$$

where

$$m_0(h) = \int_0^h T(h)dh, \quad m_i(h) = \int_0^h \left(\oint_{L_h^*} x^i dt\right) dh, \quad i = 1, 2, \cdots, \frac{k}{2} - 1,$$

$$m_{\frac{k}{2}}(h) = \int_0^h \left(\oint_{L_h^*} x^{\frac{k}{2}}\phi_1(x)dt\right) dh, \quad m_{\frac{k}{2}+1}(h) = \int_0^h \left(\oint_{L_h^*} v\phi_2(x,v)dt\right) dh,$$

and $T(h)$ denotes the period of L_h^*. Note that

$$M(h,\delta) = \oint_{H(x,y)=h} q dx - p dy = \oint_{H^*(x,v)=h} q^* dx - p^* dv.$$

Taking $p^*=0$ and $q^*=v$, $x^i v$ $(i=1,2,\cdots,\frac{k}{2}-1)$, $x^{\frac{k}{2}}\phi_1(x)v$, $\int_0^v s\phi_2(x,s)ds$ respectively in (3.7), we obtain

$$\begin{aligned}
m_0(h) &= \oint\limits_{L_h^*} v dx - \oint\limits_{L_0^*} v dx,\\
m_i(h) &= \oint\limits_{L_h^*} x^i v dx - \oint\limits_{L_0^*} x^i v dx,\ i=1,2,\cdots,\frac{k}{2}-1,\\
m_{\frac{k}{2}}(h) &= \oint\limits_{L_h^*} x^{\frac{k}{2}}\phi_1(x) dx - \oint\limits_{L_0^*} x^{\frac{k}{2}}\phi_1(x) dx,\\
m_{\frac{k}{2}+1}(h) &= \oint\limits_{L_h^*} \widetilde{\phi}_2(x,v) dx - \oint\limits_{L_0^*} \widetilde{\phi}_2(x,v) dx,
\end{aligned}$$

where $\widetilde{\phi}_2(x,v)=\int_0^v s\phi_2(x,s)ds$. By the expansion of $M(h,\delta)$ in (3.1), we may assume

$$m_j(h)=\sum_{r=0}^{\frac{k}{2}-2}\lambda_{j,r+1}|h|^{\frac{r+1}{k}+\frac{1}{2}}+\lambda_{j,\frac{k}{2}}h\ln|h|+\lambda_{j,\frac{k}{2}+1}h+O(|h|^{1+\frac{1}{k}}),\ j=0,1,\cdots,\frac{k}{2}+1, \tag{3.8}$$

where $\lambda_{j,r+1}(r=0,1,\cdots,\frac{k}{2})$ are some constants if $j=0,1,\cdots,\frac{k}{2}-1$, are polynomials in $\bar{b}_{i,0}$ $(i\geq\frac{k}{2})$ if $j=\frac{k}{2}$, and are polynomials in $\bar{b}_{is}$ $(i\geq 0, s\geq 1)$ if $j=\frac{k}{2}+1$.

Substituting (3.8) into (3.7), we get

$$\begin{aligned}
&M(h,\delta)\\
&=M(0,\delta)+\left(\sum_{i=0}^{\frac{k}{2}-1}\bar{b}_{i,0}\lambda_{i,1}+\lambda_{\frac{k}{2},1}+\lambda_{\frac{k}{2}+1,1}\right)|h|^{\frac{1}{k}+\frac{1}{2}}+\left(\sum_{i=0}^{\frac{k}{2}-1}\bar{b}_{i,0}\lambda_{i,2}+\lambda_{\frac{k}{2},2}+\lambda_{\frac{k}{2}+1,2}\right)|h|^{\frac{2}{k}+\frac{1}{2}}\\
&\quad+\cdots\\
&\quad+\left(\sum_{i=0}^{\frac{k}{2}-1}\bar{b}_{i,0}\lambda_{i,\frac{k}{2}-1}+\lambda_{\frac{k}{2},\frac{k}{2}-1}+\lambda_{\frac{k}{2}+1,\frac{k}{2}-1}\right)|h|^{1-\frac{1}{k}}+\left(\sum_{i=0}^{\frac{k}{2}-1}\bar{b}_{i,0}\lambda_{i,\frac{k}{2}}+\lambda_{\frac{k}{2},\frac{k}{2}}+\lambda_{\frac{k}{2}+1,\frac{k}{2}}\right)h\ln|h|\\
&\quad+\left(\sum_{i=0}^{\frac{k}{2}-1}\bar{b}_{i,0}\lambda_{i,\frac{k}{2}+1}+\lambda_{\frac{k}{2},\frac{k}{2}+1}+\lambda_{\frac{k}{2}+1,\frac{k}{2}+1}\right)h+O(|h|^{1+\frac{1}{k}}).
\end{aligned}\tag{3.9}$$

By (2.18), (2.19) and (2.20) in [11], we know that

$$\bar{q}_0(x)=2q_1(x)a_1(x)=\sum_{i\geq 0}\alpha_{i,0}x^i,$$

where $a_1(x) = \sum_{i\geq 0} \bar{a}_{i,1} x^i$, $q_1(x) = \sum_{i\geq 0} \bar{b}_{i0} x^i$, where $\bar{a}_{i,1}$ is a constant for each i. It follows that $\alpha_{i,0} = \sum_{s=0}^{i} \Psi_s \bar{b}_{s0}$, where Ψ_s, $s = 0, 1, \cdots, i$ are some constants. Further, by (2.16), we know that $\widetilde{r}_{r,0}$ can be written as

$$\widetilde{r}_{r,0} = \sum_{l=0}^{r} \bar{\Psi}_l \alpha_{l,0} = \sum_{l=0}^{r} \widetilde{\Psi}_l \bar{b}_{l,0},$$

where $\bar{\Psi}_l$, $\widetilde{\Psi}_l$, $l = 0, 1, 2, \cdots, r$ are some constants.

By the formulas of c_{r+1}, $r = 0, 1, \cdots, \frac{k}{2} - 1$ in (3.2), it is seen that c_{r+1} has the form of

$$c_{r+1} = \sum_{l=0}^{r} \hat{\Psi}_l \bar{b}_{l,0}, \quad r = 0, 1, \cdots, \frac{k}{2} - 1, \tag{3.10}$$

where $\hat{\Psi}_l$, $l = 0, 1, 2, \cdots, r$ are some constants. Then comparing the expansion of $M(h, \delta)$ in (3.1) and (3.9), we obtain

$$\lambda_{ij} = 0, \quad j = 1, 2, 3, \cdots, \frac{k}{2}, \quad \text{and } i = j, j+1, \cdots, \frac{k}{2} + 1. \tag{3.11}$$

Noticing $\oint_{L_h} x^i dt = \oint_{L_h^*} x^i dt$ by (2.11), we further have

$$\begin{aligned}
&\oint\limits_{L_h} \left(p_x + q_y - \bar{b}_{0,0} - \bar{b}_{1,0} x - \bar{b}_{2,0} x^2 - \cdots - \bar{b}_{\frac{k}{2}-1,0} x^{\frac{k}{2}-1}\right) dt \\
&= \oint\limits_{L_h^*} \left(p_x^* + q_v^* - \bar{b}_{0,0} - \bar{b}_{1,0} x - \bar{b}_{2,0} x^2 - \cdots - \bar{b}_{\frac{k}{2}-1,0} x^{\frac{k}{2}-1}\right) dt \\
&= \oint\limits_{L_h^*} \left(x^{\frac{k}{2}} \phi_1(x) + v \phi_2(x, v)\right) dt \\
&= \left(m_{\frac{k}{2}}(h) + m_{\frac{k}{2}+1}(h)\right)'.
\end{aligned}$$

Thus, by (3.6), (3.8) and (3.11) we have

$$\begin{aligned}
&\oint_{L_0}\left(p_x+q_y-\bar b_{0,0}-\bar b_{1,0}x-\bar b_{2,0}x^2-\cdots-\bar b_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1}\right)dt\\
&=\lim_{h\to 0}\left(m_{\frac{k}{2}}(h)+m_{\frac{k}{2}+1}(h)\right)'\\
&=\lim_{h\to 0}\left(\lambda_{\frac{k}{2},\frac{k}{2}+1}h+\lambda_{\frac{k}{2}+1,\frac{k}{2}+1}h+O(|h|^{1+\frac{1}{k}})\right)'\\
&=\lim_{h\to 0}\left(\lambda_{\frac{k}{2},\frac{k}{2}+1}+\lambda_{\frac{k}{2}+1,\frac{k}{2}+1}+O(|h|^{\frac{1}{k}})\right)\\
&=\lambda_{\frac{k}{2},\frac{k}{2}+1}+\lambda_{\frac{k}{2}+1,\frac{k}{2}+1}.
\end{aligned}\tag{3.12}$$

On the other hand, by (3.1), (3.9) and (3.10) we obtain

$$\begin{aligned}
c_{\frac{k}{2}+1}&=\bar b_{0,0}\lambda_{0,\frac{k}{2}+1}+\bar b_{1,0}\lambda_{1,\frac{k}{2}+1}+\cdots+\bar b_{\frac{k}{2}-1,0}\lambda_{\frac{k}{2}-1,\frac{k}{2}+1}+\lambda_{\frac{k}{2},\frac{k}{2}+1}+\lambda_{\frac{k}{2}+1,\frac{k}{2}+1}\\
&=\lambda_{\frac{k}{2},\frac{k}{2}+1}+\lambda_{\frac{k}{2}+1,\frac{k}{2}+1}+\sum_{i=1}^{\frac{k}{2}}\bar\mu_i c_i,
\end{aligned}$$

where $\bar\mu_i$ are some constants. This, together with (3.12), implies that

$$c_{\frac{k}{2}+1}=\oint_{L_0}\left(p_x+q_y-\bar b_{0,0}-\bar b_{1,0}x-\bar b_{2,0}x^2-\cdots-\bar b_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1}\right)dt+\sum_{i=1}^{\frac{k}{2}}O_1(c_i).$$

The formula of $\widetilde c_{\frac{k}{2}+1}$ can be obtained in the same way.

In the following, we derive the formulas of the coefficients in the expansion of $M^*(h,\delta)$. By Lemma 2.1 (ii) we know that $r_1=\frac{1}{2}\left(\frac{k}{2}-1\right)=\frac{m}{2}$.

If m is even, then r_1 is an integer. For $0\le r\le\frac{k}{2}-1$, by (2.5) we have

$$M^*(h,\delta)=\varphi^*(h,\delta)-\frac{h\ln h}{k}J^*_{1,\frac{k}{4}-\frac{1}{2}}(h)+2h^{\frac{1}{k}+\frac{1}{2}}\sum_{\substack{r=0\\ r\ne\frac{k}{4}-\frac{1}{2}}}^{\frac{k}{2}-1}\bar A_r h^{\frac{2r}{k}}J^*_{1r}(h),\quad 0\le h\ll 1,\tag{3.13}$$

where J^*_{1r} can be written as

$$\begin{aligned}
J^*_{1r}&=r^{(1)}_{r,1}\bar\alpha_{r,1}\bar\beta_r+\left(r^{(1)}_{\frac{k}{2}+r,1}\bar\alpha_{\frac{k}{2}+r,1}\bar\beta_{\frac{k}{2}+r}+r^{(1)}_{r,3}\bar\alpha_{r,3}\bar\beta_r\right)h+O(h^2),\\
&=r^{(1)}_{r,1}+\left(r^{(1)}_{\frac{k}{2}+r,1}\bar\beta_{\frac{k}{2}+r}+r^{(1)}_{r,3}\bar\alpha_{r,3}\right)h+O(h^2),
\end{aligned}\tag{3.14}$$

with

$$\bar{\alpha}_{r,1}=1,\ \ \bar{\beta}_r=1,\ \ \bar{\alpha}_{\frac{k}{2}+r,1}=1,\ \ \bar{\alpha}_{r,3}=\frac{\frac{3}{2}k}{\frac{3}{2}k+2r+1},\ \ \bar{\beta}_{\frac{k}{2}+r}=-\frac{2r+1}{2r+1+\frac{3}{2}k}.$$

Substituting (3.14) into (3.13) yields

$$\begin{aligned}
M^*(h,\delta) &= \varphi^*(0,\delta)+\varphi_h^*(0,\delta)h-\tfrac{h\ln h}{k}\left(r^{(1)}_{\frac{k}{4}-\frac{1}{2},1}+\left(r^{(1)}_{\frac{3}{4}k-\frac{1}{2},1}\bar{\beta}_{\frac{3}{4}k-\frac{1}{2}}+r^{(1)}_{\frac{k}{4}-\frac{1}{2},3}\bar{\alpha}_{\frac{k}{4}-\frac{1}{2},3}\right)h\right)\\
&\quad+\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}}\left[2\bar{A}_r r^{(1)}_{r,1}h^{\frac{2r+1}{k}+\frac{1}{2}}+2\bar{A}_r\left(r^{(1)}_{\frac{k}{2}+r,1}\bar{\beta}_{\frac{k}{2}+r}+r^{(1)}_{r,3}\bar{\alpha}_{r,3}\right)h^{\frac{2r+1}{k}+\frac{3}{2}}\right]\\
&\quad+\sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1}\left(2\bar{A}_r r^{(1)}_{r,1}h^{\frac{2r+1}{k}+\frac{1}{2}}\right)+O(h^2)\\
&= c_0^*+\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}}c^*_{r+1}h^{\frac{2r+1}{k}+\frac{1}{2}}+c^*_{\frac{k}{4}+\frac{1}{2}}h\ln h+c^*_{\frac{k}{4}+\frac{3}{2}}h+\sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1}c^*_{r+2}h^{\frac{2r+1}{k}+\frac{1}{2}}\\
&\quad+\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}}c^*_{\frac{k}{2}+2+r}h^{\frac{2r+1}{k}+\frac{3}{2}}+c^*_{\frac{3}{4}k+\frac{3}{2}}h^2\ln h+O(h^2),
\end{aligned}$$

where

$$\begin{aligned}
c_0^* &= \varphi^*(0,\delta)=M^*(0,\delta)=c_0+\widetilde{c}_0,\\
c^*_{r+1} &= 2\bar{A}_r r^{(1)}_{r,1},\ \ r=0,1,\cdots,\tfrac{k}{4}-\tfrac{3}{2},\\
c^*_{\frac{k}{4}+\frac{1}{2}} &= -\tfrac{1}{k}r^{(1)}_{\frac{k}{4}-\frac{1}{2},1},\ \ c^*_{r+2}=2\bar{A}_r r^{(1)}_{r,1},\ \ r=\tfrac{k}{4}+\tfrac{1}{2},\tfrac{k}{4}+\tfrac{3}{2},\cdots,\tfrac{k}{2}-1,\\
c^*_{\frac{k}{2}+2+r} &= 2\bar{A}_r\left(r^{(1)}_{\frac{k}{2}+r,1}\bar{\beta}_{\frac{k}{2}+r}+r^{(1)}_{r,3}\bar{\alpha}_{r,3}\right)\\
&= \tfrac{2\bar{A}_r}{\frac{3}{2}k+2r+1}\left(\tfrac{3}{2}k\,r^{(1)}_{r,3}-(2r+1)r^{(1)}_{\frac{k}{2}+r,1}\right),\ \ r=0,1,\cdots,\tfrac{k}{4}-\tfrac{3}{2},\\
c^*_{\frac{3}{4}k+\frac{3}{2}} &= -\tfrac{1}{k}\left(r^{(1)}_{\frac{3}{4}k-\frac{1}{2},1}\bar{\beta}_{\frac{3}{4}k-\frac{1}{2}}+r^{(1)}_{\frac{k}{4}-\frac{1}{2},3}\bar{\alpha}_{\frac{k}{4}-\frac{1}{2},3}\right)=\tfrac{1}{4k}\left(r^{(1)}_{\frac{3}{4}k-\frac{1}{2},1}-3r^{(1)}_{\frac{k}{4}-\frac{1}{2},3}\right).
\end{aligned}\tag{3.15}$$

By Lemma 2.3, we know that

$$r^{(1)}_{i,1}=\widetilde{r}_{2i,0},\ \ r^{(1)}_{i,3}=\widetilde{r}_{2i,1},\ \ i\geq 0.$$

Further, we have

$$
\begin{aligned}
&c^*_{r+1}=2\bar{A}_r r^{(1)}_{r,1}=2\bar{A}_r\widetilde{r}_{2r,0}=\tfrac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{2r+1},\quad r=0,1,\cdots,\tfrac{k}{4}-\tfrac{3}{2},\\
&c^*_{\frac{k}{4}+\frac{1}{2}}=-\tfrac{1}{k}\widetilde{r}_{\frac{k}{2}-1,0}=2\,c_{\frac{k}{2}},\\
&c^*_{r+2}=2\bar{A}_r r^{(1)}_{r,1}=2\bar{A}_r\widetilde{r}_{2r,0}=\tfrac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{2r+2},\quad r=\tfrac{k}{4}+\tfrac{1}{2},\tfrac{k}{4}+\tfrac{3}{2},\cdots,\tfrac{k}{2}-1,\\
&c^*_{\frac{k}{2}+2+r}=\tfrac{2\bar{A}_r}{\frac{3}{2}k+2r+1}\left(\tfrac{3}{2}k\,\widetilde{r}_{2r,1}-(2r+1)\widetilde{r}_{k+2r,0}\right)=-\tfrac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{k+1+2r},\quad r=0,1,\cdots,\tfrac{k}{4}-\tfrac{3}{2},\\
&c^*_{\frac{3}{4}k+\frac{3}{2}}=\tfrac{1}{4k}\left(\widetilde{r}_{\frac{3}{2}k-1,0}-3\widetilde{r}_{\frac{k}{2}-1,1}\right)=2c_{\frac{3}{2}k}.
\end{aligned}
\tag{3.16}
$$

By using the similar procedure in obtaining $c_{\frac{k}{2}+1}$, we obtain the formula of $c^*_{\frac{k}{4}+\frac{3}{2}}$ as shown in (3.3).

If m is odd, then r_1 is not an integer. In this case, by (2.5) and (3.14) we have

$$
\begin{aligned}
&M^*(h,\delta)\\
&=\varphi^*(0,\delta)+\varphi^*_h(0,\delta)h+\sum_{r=0}^{\frac{k}{4}-1}\left[2\bar{A}_r r^{(1)}_{r,1}h^{\frac{2r+1}{k}+\frac{1}{2}}+2\bar{A}_r\left(r^{(1)}_{\frac{k}{2}+r,1}\bar{\beta}_{\frac{k}{2}+r}+r^{(1)}_{r,3}\bar{\alpha}_{r,3}\right)h^{\frac{2r+1}{k}+\frac{3}{2}}\right]\\
&\quad+\sum_{r=\frac{k}{4}}^{\frac{k}{2}-1}2\bar{A}_r r^{(1)}_{r,1}h^{\frac{2r+1}{k}+\frac{1}{2}}+O(h^2)\\
&=\bar{c}^*_0+\sum_{r=0}^{\frac{k}{4}-1}\bar{c}^*_{r+1}h^{\frac{2r+1}{k}+\frac{1}{2}}+\bar{c}^*_{\frac{k}{4}+1}h+\sum_{r=\frac{k}{4}}^{\frac{k}{2}-1}\bar{c}^*_{r+2}h^{\frac{2r+1}{k}+\frac{1}{2}}+\sum_{r=0}^{\frac{k}{4}-1}\bar{c}^*_{\frac{k}{2}+2+r}h^{\frac{2r+1}{k}+\frac{3}{2}}+O(h^2),
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{c}^*_0&=\varphi^*(0,\delta)=M^*(0,\delta)=c_0+\widetilde{c}_0,\\
\bar{c}^*_{r+1}&=2\bar{A}_r r^{(1)}_{r,1},\quad r=0,1,\cdots,\tfrac{k}{4}-1,\\
\bar{c}^*_{r+2}&=2\bar{A}_r r^{(1)}_{r,1},\quad r=\tfrac{k}{4},\tfrac{k}{4}+1,\cdots,\tfrac{k}{2}-1,\\
\bar{c}^*_{\frac{k}{2}+2+r}&=2\bar{A}_r\left(r^{(1)}_{\frac{k}{2}+r,1}\bar{\beta}_{\frac{k}{2}+r}+r^{(1)}_{r,3}\bar{\alpha}_{r,3}\right)\\
&=\tfrac{2\bar{A}_r}{\frac{3}{2}k+2r+1}\left(\tfrac{3}{2}k\,\widetilde{r}_{2r,1}-(2r+1)\widetilde{r}_{k+2r,0}\right),\quad r=0,1,\cdots,\tfrac{k}{4}-1.
\end{aligned}
\tag{3.17}
$$

Similar to (3.16), we further have

$$
\begin{aligned}
\bar{c}^*_{r+1}&=\tfrac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{2r+1},\quad r=0,1,\cdots,\tfrac{k}{4}-1,\\
\bar{c}^*_{r+2}&=\tfrac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{2r+2},\quad r=\tfrac{k}{4},\tfrac{k}{4}+1,\cdots,\tfrac{k}{2}-1,\\
\bar{c}^*_{\frac{k}{2}+2+r}&=-\tfrac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{k+1+2r},\quad r=0,1,\cdots,\tfrac{k}{4}-1.
\end{aligned}
\tag{3.18}
$$

The formula of $\bar{c}^*_{\frac{k}{4}+1}$ can be obtained by using a similar procedure as used in obtaining $c_{\frac{k}{2}+1}$. The proof is complete. □

When L_0^* is a double homoclinic loop of smooth type, we have the following result.

Theorem 3.2. *Let (1.5) hold with $\bar{h}_k < 0$ $(k = 2m+2, m \geq 1)$ and $L_0^* = L_0 \cup \widetilde{L}_0$ be a double homoclinic loop of smooth type which is defined by $H(x, y) = 0$. Then for the functions $M(h, \delta)$, $\widetilde{M}(h, \delta)$ and $M^*(h, \delta)$ in (1.8) we have*

$$\begin{aligned} M(h,\delta) &= c_0 + \sum_{r=0}^{\frac{k}{4}-1} \tfrac{1}{2}\hat{A}_r c_{2r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + \hat{c}_{\frac{k}{4}+1} h + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} \tfrac{1}{2}\hat{A}_r c_{2r+2} h^{\frac{2r+1}{k}+\frac{1}{2}} \\ &\quad - \sum_{r=0}^{\frac{k}{4}-1} \tfrac{1}{2}\hat{A}_r c_{k+1+2r} h^{\frac{2r+1}{k}+\frac{3}{2}} + O(h^2), \quad 0 < h \ll 1, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \widetilde{M}(h,\delta) &= \widetilde{c}_0 + \sum_{r=0}^{\frac{k}{4}-1} \tfrac{1}{2}\hat{A}_r c_{2r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + \widetilde{\hat{c}}_{\frac{k}{4}+1} h + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} \tfrac{1}{2}\hat{A}_r c_{2r+2} h^{\frac{2r+1}{k}+\frac{1}{2}} \\ &\quad - \sum_{r=0}^{\frac{k}{4}-1} \tfrac{1}{2}\hat{A}_r c_{k+1+2r} h^{\frac{2r+1}{k}+\frac{3}{2}} + O(h^2), \quad 0 < h \ll 1, \end{aligned} \tag{3.20}$$

$$\begin{aligned} M^*(h,\delta) &= c_0^* + \sum_{r=0}^{\frac{k}{4}-1} 2\, c_{2r+1} |h|^{\frac{2r+1}{k}+\frac{1}{2}} + \hat{c}^*_{\frac{k}{4}+1} h + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} 2\, c_{2r+2} |h|^{\frac{2r+1}{k}+\frac{1}{2}} \\ &\quad + \sum_{r=0}^{\frac{k}{4}-1} 2\, c_{k+1+2r} |h|^{\frac{2r+1}{k}+\frac{3}{2}} + O(h^2), \quad 0 < -h \ll 1, \end{aligned} \tag{3.21}$$

if m is odd, where $\hat{A}_r = \frac{2\bar{A}_r}{\widetilde{A}_{2r}}$ and

$$\hat{c}_{\frac{k}{4}+1} = \oint_{L_0} \big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-2} \bar{b}_{i,0} x^i\big) dt + \sum_{i=1}^{\frac{k}{2}-1} O_1(c_i),$$

$$\widetilde{\hat{c}}_{\frac{k}{4}+1} = \oint_{\widetilde{L}_0} \big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-2} \bar{b}_{i,0} x^i\big) dt + \sum_{i=1}^{\frac{k}{2}-1} O_1(c_i),$$

$$\hat{c}^*_{\frac{k}{4}+1} = \oint_{L_0^*} \big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-2} \bar{b}_{i,0} x^i\big) dt + \sum_{i=1}^{\frac{k}{2}-1} O_1(c_i),$$

and

$$\begin{aligned} M(h,\delta) &= c_0 + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \tfrac{1}{2}\hat{A}_r c_{2r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + c_{\frac{k}{2}} h \ln h + \bar{\bar{c}}_{\frac{k}{4}+\frac{3}{2}} h + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \tfrac{1}{2}\hat{A}_r c_{2r+2} h^{\frac{2r+1}{k}+\frac{1}{2}} \\ &\quad - \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \tfrac{1}{2}\hat{A}_r c_{k+1+2r} h^{\frac{2r+1}{k}+\frac{3}{2}} + c_{\frac{3}{2}k} h^2 \ln h + O(h^2), \quad 0 < h \ll 1, \end{aligned} \tag{3.22}$$

$$\widetilde{M}(h,\delta) = \widetilde{c}_0 + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \frac{1}{2}\hat{A}_r c_{2r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + c_{\frac{k}{2}} h\ln h + \widetilde{\bar{c}}_{\frac{k}{4}+\frac{3}{2}} h + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \frac{1}{2}\hat{A}_r c_{2r+2} h^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$- \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \frac{1}{2}\hat{A}_r c_{k+1+2r} h^{\frac{2r+1}{k}+\frac{3}{2}} + c_{\frac{3}{2}k} h^2 \ln h + O(h^2), \quad 0 < h \ll 1, \tag{3.23}$$

$$M^*(h,\delta) = c_0^* + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} 2c_{2r+1}|h|^{\frac{2r+1}{k}+\frac{1}{2}} + 2c_{\frac{k}{2}} h\ln|h| + \widetilde{\bar{c}}^*_{\frac{k}{4}+\frac{3}{2}} h + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} 2c_{2r+2}|h|^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$+ \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} 2c_{k+1+2r}|h|^{\frac{2r+1}{k}+\frac{3}{2}} + 2c_{\frac{3}{2}k} h^2 \ln|h| + O(h^2), \quad 0 < -h \ll 1, \tag{3.24}$$

if m is even, where

$$\bar{\bar{c}}_{\frac{k}{4}+\frac{3}{2}} = \oint_{L_0} \big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-1} \bar{b}_{i,0} x^i\big) dt + \sum_{i=1}^{\frac{k}{2}} O_1(c_i),$$
$$\widetilde{\bar{c}}_{\frac{k}{4}+\frac{3}{2}} = \oint_{\widetilde{L}_0} \big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-1} \bar{b}_{i,0} x^i\big) dt + \sum_{i=1}^{\frac{k}{2}} O_1(c_i),$$
$$\widetilde{\bar{c}}^*_{\frac{k}{4}+\frac{3}{2}} = \oint_{L_0^*} \big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-1} \bar{b}_{i,0} x^i\big) dt + \sum_{i=1}^{\frac{k}{2}} O_1(c_i).$$

Proof. Note that $r_1 = \frac{m}{2}$ by Lemma 2.2. If m is odd, then by (2.8) and (3.14) we have

$$M(h,\delta) = c_0 + \sum_{r=0}^{\frac{k}{4}-1} \bar{A}_r r_{r,1}^{(1)} h^{\frac{2r+1}{k}+\frac{1}{2}} + \hat{c}_{\frac{k}{4}+1} h + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} \bar{A}_r r_{r,1}^{(1)} h^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$+ \sum_{r=0}^{\frac{k}{4}-1} \bar{A}_r \left(r^{(1)}_{\frac{k}{2}+r,1} \bar{\beta}_{\frac{k}{2}+r} + r^{(1)}_{r,3} \bar{\alpha}_{r,3} \right) h^{\frac{2r+1}{k}+\frac{3}{2}} + O(h^2), \; 0 < h \ll 1,$$

where

$$c_0 = M(0,\delta) = \oint_{L_0} q\,dx - p\,dy.$$

Then by (3.17), we obtain

$$M(h,\delta) = c_0 + \sum_{r=0}^{\frac{k}{4}-1} \frac{1}{2}\bar{c}^*_{r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + \hat{c}_{\frac{k}{4}+1} h + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} \frac{1}{2}\bar{c}^*_{r+2} h^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$+ \sum_{r=0}^{\frac{k}{4}-1} \frac{1}{2}\bar{c}^*_{\frac{k}{2}+2+r} h^{\frac{2r+1}{k}+\frac{3}{2}} + O(h^2).$$

This, together with (3.18), gives (3.19). Similarly, we can get (3.20).

By (2.10) and (3.4) we have

$$M^*(h,\delta) = c_0^* + \sum_{r=0}^{\frac{k}{4}-1} 2\widetilde{A}_{2r}\widetilde{r}_{2r,0}|h|^{\frac{2r+1}{k}+\frac{1}{2}} + \varphi_h^*(0,\delta)h + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} 2\widetilde{A}_{2r}\widetilde{r}_{2r,0}|h|^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$- \sum_{r=0}^{\frac{k}{4}-1} 2\widetilde{A}_{2r}\left(\widetilde{r}_{k+2r,0}\beta^*_{k+2r} + \widetilde{r}_{2r,1}\alpha^*_{2r,1}\right)|h|^{\frac{2r+1}{k}+\frac{3}{2}} + O(h^2), \quad 0 < -h \ll 1,$$

where

$$c_0^* = M^*(0,\delta) = \oint_{L_0^*} q\,dx - p\,dy.$$

Then, (3.21) holds by the formulas of $c_{r+1}, c_{r+2}, c_{k+1+r}$ in (3.2).

If m is even, by (2.8) and (3.14) we have

$$M(h,\delta) = c_0 + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \bar{A}_r r^{(1)}_{r,1} h^{\frac{2r+1}{k}+\frac{1}{2}} - \frac{h\ln h}{2k} r^{(1)}_{\frac{k}{4}-\frac{1}{2},1} + \bar{\bar{c}}_{\frac{k}{4}+\frac{3}{2}} h + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \bar{A}_r r^{(1)}_{r,1} h^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$+ \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \bar{A}_r \left(r^{(1)}_{\frac{k}{2}+r,1}\bar{\beta}_{\frac{k}{2}+r} + r^{(1)}_{r,3}\bar{\alpha}_{r,3}\right) h^{\frac{2r+1}{k}+\frac{3}{2}}$$
$$- \frac{1}{2k} h^2 \ln h \left(r^{(1)}_{\frac{3}{4}k-\frac{1}{2},1}\bar{\beta}_{\frac{3}{4}k-\frac{1}{2}} + r^{(1)}_{\frac{k}{4}-\frac{1}{2},3}\bar{\alpha}_{\frac{k}{4}-\frac{1}{2},3}\right) + O(h^2), \ 0 < h \ll 1.$$

Then, by (3.15) we get

$$M(h,\delta) = c_0 + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \frac{1}{2}c^*_{r+1} h^{\frac{2r+1}{k}+\frac{1}{2}} + \frac{1}{2}c^*_{\frac{k}{4}+\frac{1}{2}} h\ln h + \bar{\bar{c}}_{\frac{k}{4}+\frac{3}{2}} h + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \frac{1}{2}c^*_{r+2} h^{\frac{2r+1}{k}+\frac{1}{2}}$$
$$+ \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \frac{1}{2}c^*_{\frac{k}{2}+2+r} h^{\frac{2r+1}{k}+\frac{3}{2}} + \frac{1}{2}c^*_{\frac{3}{4}k+\frac{3}{2}} h^2 \ln h + O(h^2).$$

Further by (3.16) we can get (3.22). Similarly, we can prove (3.23).

By (2.10) and (3.4), we have

$$\begin{aligned} M^*(h,\delta) &= c_0^* + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} 2\widetilde{A}_{2r}\widetilde{r}_{2r,0}|h|^{\frac{2r+1}{k}+\frac{1}{2}} - \frac{1}{k}\widetilde{r}_{\frac{k}{2}-1,0}h\ln|h| + \widetilde{\bar{c}}^*_{\frac{k}{4}+\frac{3}{2}}h \\ &\quad + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} 2\widetilde{A}_{2r}\widetilde{r}_{2r,0}|h|^{\frac{2r+1}{k}+\frac{1}{2}} - \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} 2\widetilde{A}_{2r}\left(\widetilde{r}_{k+2r,0}\beta^*_{k+2r} + \widetilde{r}_{2r,1}\alpha^*_{2r,1}\right)|h|^{\frac{2r+1}{k}+\frac{3}{2}} \\ &\quad - \tfrac{1}{k}h^2\ln|h|\left(\widetilde{r}_{\frac{3k}{2}-1,0}\beta^*_{\frac{3k}{2}-1} + \widetilde{r}_{\frac{k}{2}-1,1}\alpha^*_{\frac{k}{2}-1,1}\right) + O(h^2), \quad 0<-h\ll 1. \end{aligned}$$

Then (3.24) holds with the formulas of $c_{r+1}, c_{\frac{k}{2}}, c_{r+2}, c_{k+1+r}$ and $c_{\frac{3k}{2}}$ in (3.2).

Using a similar way in obtaining $c_{\frac{k}{2}+1}$ we can obtain the formulas of $\hat{c}_{\frac{k}{4}+1}, \widetilde{\hat{c}}_{\frac{k}{4}+1}, \hat{c}^*_{\frac{k}{4}+1}, \bar{\bar{c}}_{\frac{k}{4}+\frac{3}{2}}$, $\widetilde{\bar{c}}_{\frac{k}{4}+\frac{3}{2}}, \widetilde{\bar{c}}^*_{\frac{k}{4}+\frac{3}{2}}$. This completes the proof. □

Let L_0 be a homoclinic loop defined by $H(x,y)=0$. Then, for the expansion of $M(h,\delta)$, we have the following corollary.

Corollary 3.1. *Let* (1.5) *hold with* $\bar{h}_k < 0$ $(k=2m+2, m\geq 1)$ *and* L_0 *be a homoclinic loop defined by* $H(x,y)=0$. *Then,*

(i) if L_0 *is of cuspidal type, the expansion of* $M(h,\delta)$ *in* (3.1) *holds;*

(ii) if L_0 *is of smooth type, the expansion of* $M(h,\delta)$ *in* (3.19) *holds.*

4. Limit cycle bifurcation

In previous section, we have derived the coefficients in the expansions of $M(h,\delta)$, $\widetilde{M}(h,\delta)$ and $M^*(h,\delta)$. In this section, we study limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle of order 2 based on Theorems 3.1 and 3.2.

For $m=2, k=2m+2$, we first execute the symbolic programs in [14] with appropriate modifying and obtain

$$\begin{aligned} \widetilde{r}_{0,0} &= 2\sqrt{2}|\bar{h}_6|^{-\frac{1}{6}}(a_{1,0}+b_{0,1}) = 2\sqrt{2}|\bar{h}_6|^{-\frac{1}{6}}\bar{b}_{0,0}, \\ \widetilde{r}_{1,0} &= -\tfrac{2}{3}\sqrt{2}|\bar{h}_6|^{-\frac{4}{3}}\left[-3|\bar{h}_6|(2a_{2,0}+b_{1,1}) + (a_{1,0}+b_{0,1})(\bar{h}_7+3\bar{h}_6h_{1,2})\right] \\ &= 2\sqrt{2}|\bar{h}_6|^{-\frac{1}{3}}\bar{b}_{1,0} + O_1(\bar{b}_{0,0}), \\ \widetilde{r}_{2,0} &= \tfrac{1}{4}\sqrt{2}|\bar{h}_6|^{-\frac{5}{2}}\left\{8\bar{h}_6^2\left[(3a_{3,0}+b_{2,1}) - h_{2,1}\left(a_{1,1}+2b_{0,2}\right)\right)\right]\} \\ &\quad - (2a_{2,0}+b_{1,1})\left(4\bar{h}_6\bar{h}_7 + 8\bar{h}_6^2h_{1,2}\right) \\ &\quad + (a_{1,0}+b_{0,1})\left(3\bar{h}_7^2 + 4\bar{h}_6\bar{h}_7h_{1,2} + 24\bar{h}_6^2h_{0,3}h_{2,1} + 12\bar{h}_6^2h_{1,2}^2 - 8\bar{h}_6^2h_{2,2} - 4\bar{h}_6\bar{h}_8\right) \\ &= 2\sqrt{2}|\bar{h}_6|^{-\frac{1}{2}}\bar{b}_{2,0} + O_1(\bar{b}_{0,0}) + O_1(\bar{b}_{1,0}). \end{aligned}$$

Then by Theorem 3.1, we have the following result.

Theorem 4.1. *Let* (1.5) *hold with* $\bar{h}_k < 0$ *and* $k = 6$, *and* $L_0^* = L_0 \cup \widetilde{L}_0$ *be a double homoclinic loop of cuspidal type. Then for the expansions of* $M(h,\delta)$, $\widetilde{M}(h,\delta)$ *and* $M^*(h,\delta)$ *in* (2.1) *we have*

$$\begin{aligned} M(h,\delta) &= c_0 + c_1|h|^{\frac{2}{3}} + c_2|h|^{\frac{5}{6}} + c_3 h\ln|h| + c_4 h \\ &\quad + c_5|h|^{\frac{7}{6}} + c_6|h|^{\frac{4}{3}} + c_7|h|^{\frac{5}{3}} + c_8|h|^{\frac{11}{6}} + c_9 h^2\ln|h| + O(h^2), \;\; 0 < -h \ll 1, \\ \widetilde{M}(h,\delta) &= \widetilde{c}_0 + c_1|h|^{\frac{2}{3}} - c_2|h|^{\frac{5}{6}} + c_3 h\ln|h| + \widetilde{c}_4 h \\ &\quad - c_5|h|^{\frac{7}{6}} + c_6|h|^{\frac{4}{3}} + c_7|h|^{\frac{5}{3}} - c_8|h|^{\frac{11}{6}} + c_9 h^2\ln|h| + O(h^2), \;\; 0 < -h \ll 1, \\ M^*(h,\delta) &= c_0^* + c_1^* h^{\frac{2}{3}} + c_2^* h\ln h + c_3^* h + c_4^* h^{\frac{4}{3}} + c_5^* h^{\frac{5}{3}} + c_6^* h^2\ln h + O(h^2), \; 0 < h \ll 1, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &c_0 = M(0,\delta) = \oint_{L_0} q\,dx - p\,dy|_{\varepsilon=0}, \; c_1 = \widetilde{A}_0\widetilde{r}_{0,0} = 2\sqrt{2}|\bar{h}_6|^{-\frac{1}{6}}\widetilde{A}_0\bar{b}_{0,0}, \\ &c_2 = \widetilde{A}_1\widetilde{r}_{1,0} = 2\sqrt{2}|\bar{h}_6|^{-\frac{1}{3}}\widetilde{A}_1\bar{b}_{1,0} + O_1(c_1), \\ &c_3 = -\frac{1}{12}\widetilde{r}_{2,0} = -\frac{\sqrt{2}}{6}|\bar{h}_6|^{-\frac{1}{2}}\bar{b}_{2,0} + O_1(c_1) + O_1(c_2), \\ &c_4 = \oint_{L_0}[(p_x+q_y)|_{\varepsilon=0} - \bar{b}_{0,0} - \bar{b}_{1,0}x - \bar{b}_{2,0}x^2]dt + O_1(c_1) + O_1(c_2) + O_1(c_3), \\ &c_5 = \widetilde{A}_3\widetilde{r}_{3,0}, \; c_6 = \widetilde{A}_4\widetilde{r}_{4,0}, \; c_7 = -\frac{1}{10}\widetilde{A}_0\left(9\widetilde{r}_{0,1} - \widetilde{r}_{6,0}\right), \\ &c_8 = \frac{1}{11}\widetilde{A}_1\left(2\widetilde{r}_{7,0} - 9\widetilde{r}_{1,1}\right), \; c_9 = \frac{1}{48}\left(\widetilde{r}_{8,0} - 3\widetilde{r}_{2,1}\right), \\ &\widetilde{c}_0 = \widetilde{M}(0,\delta) = \oint_{\widetilde{L}_0} q\,dx - p\,dy|_{\varepsilon=0}, \\ &\widetilde{c}_4 = \oint_{\widetilde{L}_0}[(p_x+q_y)|_{\varepsilon=0} - \bar{b}_{0,0} - \bar{b}_{1,0}x - \bar{b}_{2,0}x^2]dt + O_1(c_1) + O_1(c_2) + O_1(c_3), \\ &c_0^* = c_0 + \widetilde{c}_0, \;\; c_1^* = 2\tfrac{\bar{A}_0}{\widetilde{A}_0}c_1, \;\; c_2^* = 2c_3, \\ &c_3^* = \oint_{L_0^*}[(p_x+q_y)|_{\varepsilon=0} - \bar{b}_{0,0} - \bar{b}_{1,0}x - \bar{b}_{2,0}x^2]dt + O_1(c_1) + O_1(c_2) + O_1(c_3), \\ &c_4^* = 2\tfrac{\bar{A}_2}{\widetilde{A}_4}c_6, \;\; c_5^* = -2\tfrac{\bar{A}_0}{\widetilde{A}_0}c_7, \;\; c_6^* = 2c_9, \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} &\widetilde{A}_0 = -0.5258182899\cdots, \; \widetilde{A}_1 = -0.7285951946\cdots, \; \widetilde{A}_3 = 0.3200718001\cdots, \\ &\widetilde{A}_4 = 0.0808471737\cdots, \; \bar{A}_0 = 1.051636580\cdots, \; \bar{A}_2 = -0.1616943474\cdots. \end{aligned} \tag{4.3}$$

Introduce the following notations

$$
\begin{aligned}
&\bar{c}_{01}=c_0,\ \ \bar{c}_{02}=\widetilde{c}_0,\ \ \bar{c}_1=\bar{b}_{0,0},\ \ \bar{c}_2=\bar{b}_{1,0},\ \ \bar{c}_3=\bar{b}_{2,0},\\
&\bar{c}_{41}=\oint_{L_0}[(p_x+q_y)|_{\varepsilon=0}-\bar{b}_{0,0}-\bar{b}_{1,0}x-\bar{b}_{2,0}x^2]dt,\\
&\bar{c}_{42}=\oint_{\widetilde{L}_0}[(p_x+q_y)|_{\varepsilon=0}-\bar{b}_{0,0}-\bar{b}_{1,0}x-\bar{b}_{2,0}x^2]dt,\\
&\bar{c}_3^*=\oint_{L_0^*}[(p_x+q_y)|_{\varepsilon=0}-\bar{b}_{0,0}-\bar{b}_{1,0}x-\bar{b}_{2,0}x^2]dt,\\
&\bar{c}_5=\widetilde{r}_{3,0},\ \ \bar{c}_6=\widetilde{r}_{4,0},\ \ \bar{c}_7=9\widetilde{r}_{0,1}-\widetilde{r}_{6,0},\ \ \bar{c}_8=9\widetilde{r}_{1,1}-2\widetilde{r}_{7,0},\ \ \bar{c}_9=\widetilde{r}_{8,0}-3\widetilde{r}_{2,1}.
\end{aligned}
\tag{4.4}
$$

Next, for system (1.1) we discuss the bifurcation of limit cycles near L_0^*. If system (1.1) has $i+j+k$ limit cycles near L_0^*, of which i limit cycles are near $\widetilde{L_0}$ inside, j limit cycles near L_0 inside and k limit cycles near L_0^* outside, then we say that system (1.1) has a distribution $(i,j)+k$ of limit cycles. We can prove the following theorem.

Theorem 4.2. *Let* (1.5) *hold with* $k=6$, $\bar{h}_k<0$ *and* L_0^* *be a double homoclinic loop of cuspidal type defined by* $H(x,y)=0$. *Suppose there exists a parameter* $\delta_0\in\mathbf{R}^{\varpi}$ $(\varpi>7)$ *such that*

$$
\begin{aligned}
&\bar{c}_{01}(\delta_0)=\bar{c}_{02}(\delta_0)=\bar{c}_1(\delta_0)=\bar{c}_2(\delta_0)=\bar{c}_3(\delta_0)=\bar{c}_{41}(\delta_0)=\bar{c}_{42}(\delta_0)=0,\\
&\bar{c}_5(\delta_0)\neq 0,\ \ \bar{c}_6(\delta_0)\neq 0,\\
&\det\frac{\partial(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_2,\bar{c}_3,\bar{c}_{41},\bar{c}_{42})}{\partial(\delta_1,\delta_2,\cdots,\delta_7)}\neq 0.
\end{aligned}
\tag{4.5}
$$

Then for some (ε,δ) *near* $(0,\delta_0)$*, system* (1.1) *has* 10 *limit cycles near* L_0^* *with five different distributions:* $(3,4)+3$, $(2,4)+4$, $(3,3)+4$, $(4,3)+3$ *and* $(4,2)+4$.

Proof. By (4.1), (4.2) and (4.4), $\widetilde{M}(h,\delta)$, $M(h,\delta)$, $M^*(h,\delta)$ can be expressed as

$$
\begin{aligned}
\widetilde{M}(h,\delta)\ =\ &\bar{c}_{02}-k_1\bar{c}_1|h|^{\frac{2}{3}}+(k_2\bar{c}_2+O_1(\bar{c}_1))|h|^{\frac{5}{6}}+[-k_3\bar{c}_3+O_1(\bar{c}_1)+O_1(\bar{c}_2)]h\ln|h|\\
&+(\bar{c}_{42}+O_1(\bar{c}_1)+O_1(\bar{c}_2)+O_1(\bar{c}_3))h-\bar{c}_5|\widetilde{A}_3||h|^{\frac{7}{6}}+\bar{c}_6|\widetilde{A}_4||h|^{\frac{4}{3}}\\
&+\tfrac{1}{10}\bar{c}_7|\widetilde{A}_0||h|^{\frac{5}{3}}-\tfrac{1}{11}\bar{c}_8|\widetilde{A}_1||h|^{\frac{11}{6}}+\tfrac{1}{48}\bar{c}_9h^2\ln|h|+O(h^2),\quad h<0,\\
M(h,\delta)\ =\ &\bar{c}_{01}-k_1\bar{c}_1|h|^{\frac{2}{3}}+[-k_2\bar{c}_2+O_1(\bar{c}_1)]|h|^{\frac{5}{6}}+[-k_3\bar{c}_3+O_1(\bar{c}_1)+O_1(\bar{c}_2)]h\ln|h|\\
&+[\bar{c}_{41}+O_1(\bar{c}_1)+O_1(\bar{c}_2)+O_1(\bar{c}_3)]h+\bar{c}_5|\widetilde{A}_3||h|^{\frac{7}{6}}+\bar{c}_6|\widetilde{A}_4||h|^{\frac{4}{3}}\\
&+\tfrac{1}{10}\bar{c}_7|\widetilde{A}_0||h|^{\frac{5}{3}}+\tfrac{1}{11}\bar{c}_8|\widetilde{A}_1|h|^{\frac{11}{6}}+\tfrac{1}{48}\bar{c}_9h^2\ln|h|+O(h^2),\quad h<0,\\
M^*(h,\delta)\ =\ &(\bar{c}_{01}+\bar{c}_{02})+k_1^*\bar{c}_1h^{\frac{2}{3}}+[-k_2^*\bar{c}_3+O_1(\bar{c}_1)+O_1(\bar{c}_2)]h\ln h\\
&+[\bar{c}_{41}+\bar{c}_{42}+O_1(\bar{c}_1)+O_1(\bar{c}_2)+O_1(\bar{c}_3)]h-2\bar{c}_6|\bar{A}_2|h^{\frac{4}{3}}\\
&+\tfrac{1}{5}\bar{c}_7|\bar{A}_0|h^{\frac{5}{3}}+\tfrac{1}{24}\bar{c}_9h^2\ln h+O(h^2),\quad h>0,
\end{aligned}
\tag{4.6}
$$

where

$$k_1 = 2\sqrt{2}|h_6|^{-\frac{1}{6}}|\widetilde{A}_0| > 0, \quad k_2 = 2\sqrt{2}|h_6|^{-\frac{1}{3}}|\widetilde{A}_1| > 0, \quad k_3 = \frac{\sqrt{2}}{6}|h_6|^{-\frac{1}{2}} > 0,$$
$$k_1^* = 4\sqrt{2}|h_6|^{-\frac{1}{6}}|\bar{A}_0| > 0, \quad k_2^* = \frac{\sqrt{2}}{3}|h_6|^{-\frac{1}{2}} > 0.$$

We first introduce the following five cases which will be used later to obtain the number of limit cycles that system (1.1) has.

Case 1. Suppose $\bar{c}_1 \neq 0$ and vary $(\bar{c}_{01}, \bar{c}_{02})$ near $(0, 0)$.

Let $(\bar{c}_{01}, \bar{c}_{02}) = (0, 0)$. By (4.6) we easily get

$$\bar{c}_1\widetilde{M} < 0, \quad \bar{c}_1 M < 0, \quad \text{for } 0 < -h \ll 1,$$
$$\bar{c}_1 M^* > 0, \quad \text{for } 0 < h \ll 1.$$

Then take $\bar{c}_{01}, \bar{c}_{02}$ as free parameters such that $0 < |\bar{c}_{0j}| \ll |\bar{c}_1|$, $j = 1, 2$. Then, one of the following four conclusions hold.

(1.i) If $\bar{c}_{01}\bar{c}_1 > 0, \bar{c}_{02}\bar{c}_1 > 0$, then

$$\bar{c}_1\widetilde{M} > 0, \quad \bar{c}_1 M > 0, \quad \text{for } 0 < -h \ll 1,$$
$$\bar{c}_1 M^* > 0, \quad \text{for } 0 < h \ll 1,$$

which indicates a distribution $(1, 1) + 0$ of 2 limit cycles.

(1.ii) If $\bar{c}_{01}\bar{c}_1 > 0, \bar{c}_{02}\bar{c}_1 < 0$ and $(\bar{c}_{01} + \bar{c}_{02})\bar{c}_1 < 0$, then

$$\bar{c}_1\widetilde{M} < 0, \quad \bar{c}_1 M > 0, \quad \text{for } 0 < -h \ll 1,$$
$$\bar{c}_1 M^* < 0, \quad \text{for } 0 < h \ll 1,$$

which means a distribution $(0, 1) + 1$ of 2 limit cycles.

(1.iii) Similarly, if $\bar{c}_{01}\bar{c}_1 < 0, \bar{c}_{02}\bar{c}_1 > 0$ and $(\bar{c}_{01} + \bar{c}_{02})\bar{c}_1 < 0$, then system (1.1) has a distribution $(1, 0) + 1$ of two limit cycles.

(1.iv) If $\bar{c}_{01}\bar{c}_1 < 0, \bar{c}_{02}\bar{c}_1 < 0$, then system (1.1) has a distribution $(0, 0) + 1$ of one limit cycle.

Case 2. Suppose $\bar{c}_{01} = \bar{c}_{02} = 0, \bar{c}_2 \neq 0, \bar{c}_3 \neq 0$ and vary $\bar{c}_1$ near zero.

First, let $\bar{c}_1 = 0$. By (4.6) we have

$$\bar{c}_2\widetilde{M} > 0, \quad \bar{c}_2 M < 0, \quad \text{for } 0 < -h \ll 1,$$
$$\bar{c}_3 M^* > 0, \quad \text{for } 0 < h \ll 1.$$

Then take $\bar{c}_1$ as a free parameter such that $0 < |\bar{c}_1| \ll \min\{|\bar{c}_2|, |\bar{c}_3|\}$. Varying $\bar{c}_1$ near zero gives the following results.

(2.i) If $\bar{c}_1\bar{c}_2 > 0, \bar{c}_1\bar{c}_3 < 0$, then

$$\bar{c}_2\widetilde{M} < 0, \quad \bar{c}_2 M < 0, \quad \text{for } 0 < -h \ll 1,$$
$$\bar{c}_3 M^* < 0, \quad \text{for } 0 < h \ll 1,$$

which implies a distribution $(1, 0) + 1$ of 2 limit cycles;

(2.ii) If $\bar{c}_1\bar{c}_2 < 0, \bar{c}_1\bar{c}_3 < 0$, then system (1.1) has a distribution $(0, 1) + 1$ of 2 limit cycles;

(2.iii) If $\bar{c}_1\bar{c}_2 > 0, \bar{c}_1\bar{c}_3 > 0$, then system (1.1) has a distribution $(1, 0) + 0$ of one limit cycle;

(2.iv) If $\bar{c}_1\bar{c}_2 < 0, \bar{c}_1\bar{c}_3 > 0$, then system (1.1) has a distribution $(0, 1) + 0$ of one limit cycle.

Case 3. Suppose $\bar{c}_{01} = \bar{c}_{02} = \bar{c}_1 = 0$, $\bar{c}_3 \neq 0$ and vary $\bar{c}_2$ near zero.

For $\bar{c}_2 = 0$, by (4.6) we have

$$\bar{c}_3 \widetilde{M} < 0, \quad \bar{c}_3 M < 0, \quad \text{for } \ 0 < -h \ll 1,$$
$$\bar{c}_3 M^* > 0, \quad \text{for } \ 0 < h \ll 1.$$

Then take c_2 as a free parameter such that $0 < |\bar{c}_2| \ll |\bar{c}_3|$. By varying $\bar{c}_2$ near zero we find that

(3.i) if $\bar{c}_2\bar{c}_3 > 0$, then system (1.1) has a distribution $(1, 0) + 0$ of one limit cycle;

(3.ii) if $\bar{c}_2\bar{c}_3 < 0$, then system (1.1) has a distribution $(0, 1) + 0$ of one limit cycle.

Case 4. Let $\bar{c}_{01} = \bar{c}_{02} = \bar{c}_1 = \bar{c}_2 = 0$, $\bar{c}_{41}\bar{c}_{42} \neq 0$ and $\bar{c}_3^* = \bar{c}_{41} + \bar{c}_{42} \neq 0$, and vary $\bar{c}_3$ near zero.

For $\bar{c}_3 = 0$, we have

$$\bar{c}_{42} \widetilde{M} < 0, \quad \bar{c}_{41} M < 0, \quad \text{for } \ 0 < -h \ll 1,$$
$$(\bar{c}_{41} + \bar{c}_{42}) M^* > 0, \quad \text{for } \ 0 < h \ll 1.$$

Now take $\bar{c}_3$ as a free parameter such that $0 < |\bar{c}_3| \ll |\bar{c}_{4j}|$ for $j = 1, 2$ and vary $\bar{c}_3$ near zero. We find that

(4.i) if $\bar{c}_{41}\bar{c}_{42} > 0, \bar{c}_{41}\bar{c}_3 < 0$, then system (1.1) has a distribution $(1, 1) + 1$ of 3 limit cycles;

(4.ii) if $\bar{c}_{41}\bar{c}_{42} < 0, \bar{c}_{41}\bar{c}_3 > 0$ and $(\bar{c}_{41} + \bar{c}_{42})\bar{c}_3 < 0$, then system (1.1) has a distribution $(1, 0) + 1$ of 2 limit cycles;

(4.iii) if $\bar{c}_{41}\bar{c}_{42} < 0, \bar{c}_{41}\bar{c}_3 < 0$ and $(\bar{c}_{41} + \bar{c}_{42})\bar{c}_3 < 0$, then system (1.1) has a distribution $(0, 1) + 1$ of 2 limit cycles;

(4.iv) if $\bar{c}_{41}\bar{c}_{42} < 0, \bar{c}_{41}\bar{c}_3 > 0$ and $(\bar{c}_{41} + \bar{c}_{42})\bar{c}_3 > 0$, then system (1.1) has a distribution $(1, 0) + 0$ of one limit cycle;

(4.v) if $\bar{c}_{41}\bar{c}_{42} < 0, \bar{c}_{41}\bar{c}_3 < 0$ and $(\bar{c}_{41} + \bar{c}_{42})\bar{c}_3 > 0$, then system (1.1) has a distribution $(0, 1) + 0$ of one limit cycle;

Case 5. Suppose $\bar{c}_{01} = \bar{c}_{02} = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = 0$, $\bar{c}_5 \neq 0$, $\bar{c}_6 \neq 0$ and vary $(\bar{c}_{41}, \bar{c}_{42})$ near $(0, 0)$.

First, for $\bar{c}_{41} = \bar{c}_{42} = 0$, by (4.6) we have

$$\bar{c}_5 \widetilde{M} < 0, \quad \bar{c}_5 M > 0, \quad \text{for } \ 0 < -h \ll 1,$$
$$\bar{c}_6 M^* < 0, \quad \text{for } \ 0 < h \ll 1.$$

Then take $\bar{c}_{41}$ and $\bar{c}_{42}$ as free parameters such that $0 < |\bar{c}_{4j}| \ll \min\{|\bar{c}_5|, |\bar{c}_6|\}$ for $j = 1, 2$ and vary $\bar{c}_{41}, \bar{c}_{42}$ near zero. We obtain that

(5.i) if $\bar{c}_{41}\bar{c}_{42} > 0, \bar{c}_5\bar{c}_{41} > 0, \bar{c}_{41}\bar{c}_6 > 0$, then system (1.1) has a distribution $(0, 1) + 1$ of 2 limit cycles;

(5.ii) if $\bar{c}_{41}\bar{c}_{42} > 0, \bar{c}_5\bar{c}_{41} < 0, \bar{c}_{41}\bar{c}_6 > 0$, then system (1.1) has a distribution $(1, 0) + 1$ of 2 limit cycles;

(5.iii) if $\bar{c}_{41}\bar{c}_{42} < 0, \bar{c}_5\bar{c}_{41} > 0, (\bar{c}_{41} + \bar{c}_{42})\bar{c}_6 > 0$, then system (1.1) has a distribution $(1, 1) + 1$ of 3 limit cycles;

(5.iv) if $\bar{c}_{41}\bar{c}_{42} < 0, \bar{c}_5\bar{c}_{41} < 0, (\bar{c}_{41} + \bar{c}_{42})\bar{c}_6 > 0$, then system (1.1) has a distribution $(0, 0) + 1$ of one limit cycle.

(5.v) if $\bar{c}_{41}\bar{c}_{42} > 0, \bar{c}_5\bar{c}_{41} > 0, \bar{c}_{41}\bar{c}_6 < 0$, then system (1.1) has a distribution $(0, 1) + 0$ of one limit cycle;

(5.vi) if $\bar{c}_{41}\bar{c}_{42} > 0, \bar{c}_5\bar{c}_{41} < 0, \bar{c}_{41}\bar{c}_6 < 0$, then system (1.1) has a distribution $(1, 0) + 0$ of one limit cycle;

(5.vii) if $\bar{c}_{41}\bar{c}_{42} < 0, \bar{c}_5\bar{c}_{41} > 0, (\bar{c}_{41} + \bar{c}_{42})\bar{c}_6 < 0$, then system (1.1) has a distribution $(1, 1) + 0$ of 2 limit cycles;

Next, we describe the detailed steps to obtain 10 limit cycles and their distributions.

By (4.5) we can take $\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}$ as free parameters such that

$$0 < |\bar{c}_{0j}| \ll |\bar{c}_1| \ll |\bar{c}_2| \ll |\bar{c}_3| \ll |\bar{c}_{4j}| \ll \min\{|\bar{c}_5|, |\bar{c}_6|\}. \tag{4.7}$$

Firstly, as in case 5, let $\bar{c}_{01} = \bar{c}_{02} = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = 0$. We vary $\bar{c}_{41}, \bar{c}_{42}$ near zero. As one of seven cases, we first suppose the condition $(5.i)$ holds, which implies a $(0, 1) + 1$-distribution of 2 limit cycles. Note that any of the conditions $(4.ii)$, $(4.iii)$, $(4.iv)$ and $(4.v)$ is not satisfied if $(5.i)$ holds since $\bar{c}_{41}\bar{c}_{42} > 0$. Secondly, we vary $\bar{c}_3$ near zero as in case 4 such that $(4.i)$ holds. Together with $(5.i)$, this gives a $(1, 2) + 2$-distribution of 5 limit cycles.

Thirdly, as in case 3, we vary $\bar{c}_2$ near zero based on the conditions $(5.i)$ and $(4.i)$. If $(3.i)$ holds, then system (1.1) has 6 limit cycles with a $(2, 2) + 2$-distribution. If $(3.ii)$ holds, then system (1.1) has 6 limit cycles with a $(1, 3) + 2$-distribution.

Fourthly, as in case 2, we vary $\bar{c}_1$ near zero. At first, if $(3.i)$ holds, we know that $\bar{c}_2\bar{c}_3 > 0$ which means that the conditions $(2.i)$ and $(2.iv)$ are not satisfied. So, we vary $\bar{c}_2$ near zero such that $(2.ii)$ or $(2.iii)$ holds. Then the conditions $(5.i)$, $(4.i)$, $(3.i)$ and $(2.ii)$ lead to a $(2, 3) + 3$-distribution of 8 limit cycles. However, the conditions $(5.i)$, $(4.i)$, $(3.i)$ and $(2.iii)$ only lead to a $(3, 2) + 2$-distribution of 7 limit cycles. Similarly, we vary $\bar{c}_1$ near zero such that $(2.i)$ or $(2.iv)$ holds. Then the conditions $(5.i)$, $(4.i)$, $(3.ii)$, $(2.i)$ lead to a $(2, 3) + 3$-distribution of 8 limit cycles, and the conditions $(5.i)$, $(4.i)$, $(3.ii)$, $(2.iv)$ lead to a $(1, 4) + 2$-distribution of 7 limit cycles, respectively.

Finally, we vary $\bar{c}_{01}, \bar{c}_{02}$ near zero as in case 1. If $(5.i)$, $(4.i)$, $(3.ii)$, $(2.iv)$ or $(5.i)$, $(4.i)$, $(3.i)$, $(2.iii)$ hold, we only get at most 9 limit cycles by varying $\bar{c}_{01}$ and $\bar{c}_{02}$. Based on the conditions $(5.i)$, $(4.i)$, $(3.ii)$ and $(2.i)$ we vary $\bar{c}_{01}, \bar{c}_{02}$ near zero such that $(1.i)$, $(1.ii)$ or $(1.iii)$ holds which implies a $(3, 4) + 3$, $(2, 4) + 4$ or $(3, 3) + 4$-distribution of 10 limit cycles, respectively. Similarly, if $(5.i)$, $(4.i)$, $(3.i)$ and $(2.ii)$ are satisfied, we vary $\bar{c}_{01}, \bar{c}_{02}$ near zero such that $(1.i)$, $(1.ii)$ or $(1.iii)$ holds. Then we can get a $(3, 4) + 3$, $(2, 4) + 4$ or $(3, 3) + 4$-distribution of 10 limit cycles, respectively.

The above steps are shown more clearly in the following flow chart.

$$\xrightarrow{(5.i)(4.i)} (1,2)+2 \begin{cases} \xrightarrow[(1,0)+0]{(3.i)} (2,2)+2 \xrightarrow[(0,1)+1]{(2.ii)} \searrow \\ \xrightarrow[(0,1)+0]{(3.ii)} (1,3)+2 \xrightarrow[(1,0)+1]{(2.i)} \nearrow \end{cases} (2,3)+3 \begin{cases} \xrightarrow[(1,1)+0]{(1.i)} (3,4)+3 \\ \xrightarrow[(0,1)+1]{(1.ii)} (2,4)+4 \\ \xrightarrow[(1,0)+1]{(1.iii)} (3,3)+4 \end{cases} \tag{4.8}$$

If $(5.ii)$ or $(5.iii)$ holds, the steps to obtain 10 limit cycles are shown in the following flow chart.

$$\xrightarrow{(5.iii)(4.ii)} (2,1)+2 \begin{cases} \xrightarrow[(1,0)+0]{(3.i)} (3,1)+2 \xrightarrow[(0,1)+1]{(2.ii)} \searrow \\ \xrightarrow[(0,1)+0]{(3.ii)} (2,2)+2 \xrightarrow[(1,0)+1]{(2.i)} \nearrow \end{cases} (3,2)+3 \begin{cases} \xrightarrow[(1,1)+0]{(1.i)} (4,3)+3 \\ \xrightarrow[(0,1)+1]{(1.ii)} (3,3)+4 \\ \xrightarrow[(1,0)+1]{(1.iii)} (4,2)+4 \end{cases} \tag{4.9}$$

$$\xrightarrow{(5.iii)(4.iii)} (1,2)+2 \begin{cases} \xrightarrow[(1,0)+0]{(3.i)} (2,2)+2 \xrightarrow[(0,1)+1]{(2.ii)} \searrow \\ \xrightarrow[(0,1)+0]{(3.ii)} (1,3)+2 \xrightarrow[(1,0)+1]{(2.i)} \nearrow \end{cases} (2,3)+3 \begin{cases} \xrightarrow[(1,1)+0]{(1.i)} (3,4)+3 \\ \xrightarrow[(0,1)+1]{(1.ii)} (2,4)+4 \\ \xrightarrow[(1,0)+1]{(1.iii)} (3,3)+4 \end{cases} \tag{4.10}$$

$$\xrightarrow{(5.ii)(4.i)} (2,1)+2 \begin{cases} \xrightarrow[(1,0)+0]{(3.i)} (3,1)+2 \xrightarrow[(0,1)+1]{(2.ii)} \searrow \\ \xrightarrow[(0,1)+0]{(3.ii)} (2,2)+2 \xrightarrow[(1,0)+1]{(2.i)} \nearrow \end{cases} (3,2)+3 \begin{cases} \xrightarrow[(1,1)+0]{(1.i)} (4,3)+3 \\ \xrightarrow[(0,1)+1]{(1.ii)} (3,3)+4 \\ \xrightarrow[(1,0)+1]{(1.iii)} (4,2)+4, \end{cases} \tag{4.11}$$

If (5.iv) holds, we can get at most 8 limit cycles using the similar steps as before. If (5.v), (5.vi) or (5.vii) holds, we can get at most 9 limit cycles.

If $\bar{c}_5\bar{c}_6 > 0$, then we can take $\bar{c}_{41}, \bar{c}_{42}$ such that one of (5.i) and (5.iii) holds. Then, for system (1.1), by (4.8), (4.9) and (4.10) we get 10 limit cycles near L_0^* with five different distributions: $(3,4)+3$, $(2,4)+4$, $(3,3)+4$, $(4,3)+3$ and $(4,2)+4$.

If $\bar{c}_5\bar{c}_6 < 0$, then we can take $\bar{c}_{41}, \bar{c}_{42}$ such that one of (5.ii) and (5.iii) holds. Similarly, we also can get 10 limit cycles near L_0^* with the same distributions as discussed above.

The proof is complete. □

Similarly, we have the following theorem.

Theorem 4.3. *Let* (1.5) *hold with* $k = 6$, $\bar{h}_k < 0$ *and* L_0^* *be a double homoclinic loop of cuspidal type defined by* $H(x, y) = 0$. *Suppose there exists a parameter* $\delta_0 \in \mathbf{R}^m$ $(m \geq 8)$ *such that*

$$\bar{c}_{01}(\delta_0) = \bar{c}_{02}(\delta_0) = \bar{c}_1(\delta_0) = \bar{c}_2(\delta_0) = \bar{c}_3(\delta_0) = \bar{c}_{41}(\delta_0) = \bar{c}_{42}(\delta_0) = 0.$$

Then the following conclusions hold.

(1) *If*

$$\begin{aligned} &\bar{c}_5(\delta_0) = \cdots = \bar{c}_{k_1-1}(\delta_0) = 0,\ \bar{c}_{k_1}(\delta_0) \neq 0, \quad k_1 = 6 \text{ or } 7, \\ &\operatorname{rank} \frac{\partial(\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}, \bar{c}_5, \cdots, \bar{c}_{k_1-1})}{\partial \delta} = k_1 + 2, \end{aligned} \tag{4.12}$$

then for some (ε, δ) *near* $(0, \delta_0)$*, system* (1.1) *has* 11 *(or* 13*) limit cycles near* L_0^* *if* $k_1 = 6$ *(or* $k_1 = 7$*) with distributions:* $(4,4)+3, (3,4)+4$ *and* $(4,3)+4$ *(or* $(5,5)+3, (4,5)+4$ *and* $(5,4)+4$*).*

Especially, if

$$\bar{c}_5(\delta_0) = 0,\ \bar{c}_6(\delta_0) \neq 0,$$
$$\mathrm{rank}\frac{\partial(\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}, \bar{c}_5)}{\partial\delta} = \mathrm{rank}\frac{\partial(\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42})}{\partial\delta} = 7,$$

then for some (ε, δ) *near* $(0, \delta_0)$*, system* (1.1) *has* 11 *limit cycles near* L_0^* *with three distributions:* $(4,4)+3, (3,4)+4$ *and* $(4,3)+4$*.*

(2) *If*

$$\bar{c}_5(\delta_0) = \bar{c}_6(\delta_0) = \bar{c}_7(\delta_0) = 0,\ \bar{c}_8(\delta_0) \neq 0,\quad \bar{c}_9(\delta_0) \neq 0,$$
$$\det\frac{\partial(\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}, \bar{c}_5, \bar{c}_6, \bar{c}_7)}{\partial(\delta_1, \delta_2, \cdots, \delta_{10})} \neq 0,$$

then for some (ε, δ) *near* $(0, \delta_0)$*, system* (1.1) *has* 15 *limit cycles near* L_0^* *with three distributions. Further,*

(i) if $\bar{c}_8\bar{c}_9 < 0$*, then the three distributions of* 15 *limit cycles are* $(5,6)+4, (4,6)+5, (5,5)+5$*;*

(ii) if $\bar{c}_8\bar{c}_9 > 0$*, then the seven distributions of* 15 *limit cycles are* $(6,5)+4, (5,5)+5, (6,4)+5$.

(3) *If one of the following conditions holds*

(i)

$$\bar{c}_5(\delta_0) = \bar{c}_6(\delta_0) = \bar{c}_7(\delta_0) = 0,\quad \bar{c}_8(\delta_0) = 0,\quad \bar{c}_9(\delta_0) \neq 0,$$
$$\mathrm{rank}\frac{\partial(\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}, \bar{c}_5, \cdots, \bar{c}_8)}{\partial\delta} = 10,$$
$$\mathrm{rank}\frac{\partial(\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}, \bar{c}_5, \bar{c}_6, \bar{c}_7)}{\partial\delta} = 10,$$

(ii)

$$\bar{c}_5(\delta_0) = \bar{c}_6(\delta_0) = \bar{c}_7(\delta_0) = \bar{c}_8(\delta_0) = 0,\quad \bar{c}_9(\delta_0) \neq 0,$$
$$\mathrm{rank}\frac{\partial(\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}, \bar{c}_5, \cdots, \bar{c}_8)}{\partial\delta} = 11,$$

then for some (ε, δ) *near* $(0, \delta_0)$*, system* (1.1) *has* 16 *limit cycles near* L_0^* *with three distributions:* $(6,6)+4, (5,6)+5$ *and* $(6,5)+5$.

Proof. (1) Suppose $k_1 = 6$ and $\bar{c}_{01} = \bar{c}_{02} = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_{41} = \bar{c}_{42} = \bar{c}_5 = 0$. Then

$$\bar{c}_6\widetilde{M} > 0,\ \bar{c}_6 M > 0,\ \bar{c}_6 M^* < 0.$$

By (4.12) we can take $\bar{c}_{01}, \bar{c}_{02}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{41}, \bar{c}_{42}, \bar{c}_5$ as free parameters such that

$$0 < |\bar{c}_{01}|, |\bar{c}_{02}| \ll \bar{c}_1| \ll \bar{c}_2| \ll |\bar{c}_3| \ll |\bar{c}_{41}|, |\bar{c}_{42}| \ll |\bar{c}_5| \ll |\bar{c}_6|.$$

As the first step, one of the following cases can occur.

(6.i) If $\bar{c}_5\bar{c}_6 > 0$, then we have

$$\bar{c}_6\widetilde{M} < 0,\ \bar{c}_6 M > 0,\ \bar{c}_6 M^* < 0,$$

which implies that system (1.1) has 1 limit cycle with the distribution $(1, 0) + 0$.

(6.ii) If $\bar{c}_5\bar{c}_6 < 0$, then we have

$$\bar{c}_6\widetilde{M} > 0,\ \bar{c}_6 M < 0,\ \bar{c}_6 M^* < 0,$$

which implies that system (1.1) has 1 limit cycle with the distribution $(0, 1) + 0$.

Note that (6.i) and (5.ii) can not hold simultaneously, and (6.i), (5.iii), (4.ii) also can not hold simultaneously. So, by (6.i) and (4.8), or (6.i) and (4.10) we can get 11 limit cycles with 3 distributions $(4, 4) + 3$, $(3, 4) + 4$ and $(4, 3) + 4$.

Similarly, by (6.ii) and (4.9), or (6.ii) and (4.11) we also can obtain 11 limit cycles with 3 distributions $(4, 4) + 3$, $(3, 4) + 4$ and $(4, 3) + 4$.

Suppose $k_1 = 7$. Then system (1.1) has 13 limit cycles near L_0^* with 3 distributions. The steps to obtain 13 limit cycles are shown in the following flow chart:

$$\begin{array}{l} \xrightarrow{(7.ii),\,(6.i),\,(4.8)} \searrow \\ \xrightarrow{(7.ii),\,(6.i),\,(4.10)} \searrow \\ \xrightarrow{(7.ii),\,(6.ii),\,(4.9)} \nearrow \\ \xrightarrow{(7.ii),\,(6.ii),\,(4.11)} \nearrow \end{array} \begin{cases} (5,5)+3 \\ (4,5)+5 \\ (5,4)+4 \end{cases} \tag{4.13}$$

Other cases can be proved similarly, and the proof is completed. □

Next, if the origin is a nilpotent saddle of smooth type, we can similarly prove the following theorems.

At first, it directly follows from Theorem 3.2 and (4.4) that the expansions of the functions $M(h, \delta)$, $\widetilde{M}(h, \delta)$, $M^*(h, \delta)$ in (1.8) are given below.

Theorem 4.4. *Let (1.5) hold with $k = 6$, $\bar{h}_k < 0$ and L_0^* be a double homoclinic loop of smooth type defined by $H(x, y) = 0$. We have*

$$\begin{aligned} M(h,\delta) &= \bar{c}_{01} + \tfrac{1}{2}\hat{A}_0 c_1 h^{\frac{2}{3}} + c_3 h\ln h + (\bar{c}_{41} + O_1(c_1) + O_1(c_2) + O_1(c_3))h + \tfrac{1}{2}\hat{A}_2 c_6 h^{\frac{4}{3}} \\ &\quad - \tfrac{1}{2}\hat{A}_0 c_7 h^{\frac{5}{3}} + c_9 h^2 \ln h + O(h^2), \\ \widetilde{M}(h,\delta) &= \bar{c}_{02} + \tfrac{1}{2}\hat{A}_0 c_1 h^{\frac{2}{3}} + c_3 h\ln h + (\bar{c}_{42} + O_1(c_1) + O_1(c_2) + O_1(c_3))h + \tfrac{1}{2}\hat{A}_2 c_6 h^{\frac{4}{3}} \\ &\quad - \tfrac{1}{2}\hat{A}_0 c_7 h^{\frac{5}{3}} + c_9 h^2 \ln h + O(h^2), \end{aligned} \tag{4.14}$$

for $0 < h \ll 1$, where $\hat{A}_0 < 0$, $\hat{A}_2 < 0$, and

$$M^*(h,\delta)=(\bar{c}_{01}+\bar{c}_{02})+2c_1|h|^{\frac{2}{3}}+2c_3 h\ln|h|+c_3^* h+2c_6|h|^{\frac{4}{3}}+2c_7|h|^{\frac{5}{3}}+2c_9 h^2\ln|h|+O(h^2),$$

for $0<-h\ll 1$.

Theorem 4.5. *Let* (1.5) *hold with* $k=6$, $\bar{h}_k<0$ *and* L_0^* *be a double homoclinic loop of smooth type defined by* $H(x,y)=0$. *Suppose there exists a parameter* $\delta_0\in\mathbf{R}^m$ $(m\geq 7)$ *such that*

$$\bar{c}_{01}(\delta_0)=\bar{c}_{02}(\delta_0)=\bar{c}_1(\delta_0)=\bar{c}_3(\delta_0)=\bar{c}_{41}(\delta_0)=\bar{c}_{42}(\delta_0)=0. \tag{4.15}$$

Then we have

(1) *If*

$$\bar{c}_6(\delta_0)\neq 0,\quad \det\frac{\partial(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_3,\bar{c}_{41},\bar{c}_{42})}{\partial(\delta_1,\delta_2,\cdots,\delta_6)}\neq 0,$$

then for some (ε,δ) *near* $(0,\delta_0)$*, system* (1.1) *has* 11 *limit cycles near* L_0^* *with three different distributions:* $(4,4)+3$, $(4,3)+4$ *and* $(3,4)+4$*;*

(2) *If*

$$\bar{c}_6(\delta_0)=0,\quad \bar{c}_7(\delta_0)\neq 0,\quad \det\frac{\partial(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_3,\bar{c}_{41},\bar{c}_{42},\bar{c}_6)}{\partial(\delta_1,\delta_2,\cdots,\delta_7)}\neq 0,$$

then for some (ε,δ) *near* $(0,\delta_0)$*, system* (1.1) *has* 13 *limit cycles near* L_0^* *with three different distributions:* $(5,5)+3$, $(5,4)+4$ *and* $(4,5)+4$*;*

(3) *If*

$$\bar{c}_6(\delta_0)=\bar{c}_7(\delta_0)=0,\quad \bar{c}_9(\delta_0)\neq 0,\quad \det\frac{\partial(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_3,\bar{c}_{41},\bar{c}_{42},\bar{c}_6,\bar{c}_7)}{\partial(\delta_1,\delta_2,\cdots,\delta_8)}\neq 0,$$

then for some (ε,δ) *near* $(0,\delta_0)$*, system* (1.1) *has* 16 *limit cycles near* L_0^* *with three different distributions:* $(6,6)+4$, $(6,5)+5$ *and* $(5,6)+5$.

5. Applications

Consider system

$$\dot{x}=y,\quad \dot{y}=x^5(1-x^2)-\varepsilon f(x,\delta)y,\quad n=8,9,10,11,12, \tag{5.1}$$

where $f(x,\delta)=\sum_{i=0}^{n}a_i x^i$ with $\delta=(a_0,a_1,\cdots,a_n)$. We have the following Theorem.

Theorem 5.1. *Let* $C_1^*(n,7)$ *denote the maximal number of limit cycles of system* (5.1)*. We have*

$$C_1^*(8,7)\geq 11,\quad C_1^*(9,7)\geq 11,\quad C_1^*(10,7)\geq 14,\quad C_1^*(11,7)\geq 15,\quad C_1^*(12,7)\geq 17.$$

We can prove the following theorem by Theorems 4.2 and 4.3.

Proof. For system (5.1)$|_{\varepsilon=0}$, it has a nilpotent saddle of order 2 at the origin and two elementary centers $(1,0)$ and $(-1,0)$. The Hamiltonian function of system (5.1)$|_{\varepsilon=0}$ is

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{6}x^6+\frac{1}{8}x^8,$$

and it is obvious that

$$H(0,0)=0,\ \ H(1,0)=H(-1,0)=-\frac{1}{24}.$$

Note that $\bar{b}_{0,0}=-a_0,\ \ \bar{b}_{1,0}=-a_1,\ \ \bar{b}_{2,0}=-a_2$. By (4.4) and (4.2), we get

$$\begin{aligned}
&\bar{c}_{02}=\bar{c}_{01}=-\oint\limits_{L_0} f(x)y\,dx=\sum_{i=0}^{n}a_iI_i,\\
&\bar{c}_{42}=\bar{c}_{41}=\oint\limits_{L_0}\left(-f(x)+a_0+a_1x+a_2x^2\right)dt=-\oint\limits_{L_0}\sum_{i=3}^{n}a_ix^i\,dt=\sum_{i=3}^{n}a_i\bar{I}_i,\\
&\bar{c}_1=-a_0,\ \ \bar{c}_2=-a_1,\ \ \bar{c}_3=-a_2,\ \ \bar{c}_5=-\sqrt{2}\,6^{\frac{2}{3}}\left(2a_3+a_1\right),\\
&\bar{c}_6=-\tfrac{1}{32}\sqrt[3]{2}\,3^{\frac{5}{6}}\left(80a_2+55a_0+128a_4\right),\\
&\bar{c}_7=\tfrac{1}{256}2^{\frac{2}{3}}\sqrt[6]{3}\left(2184a_2+1729a_0+3072a_6+2688a_4\right),\\
&\bar{c}_8=\tfrac{1}{2}\sqrt{2}\sqrt[3]{6}\left(48a_7+48a_5+42a_3+35a_1\right),\\
&\bar{c}_9=-\tfrac{3}{4096}\sqrt{3}\left(30240a_2+25515a_0+36864a_6+32768a_8+34560a_4\right),
\end{aligned}\tag{5.2}$$

where

$$I_i=-\oint\limits_{L_0}x^iy\,dx=-\frac{1}{3}\int_0^{x_2}x^{i+3}\sqrt{12-9x^2}\,dx,$$

$$\bar{I}_i=-\oint\limits_{L_0}x^i\,dt=-\oint\limits_{L_0}\frac{x^i}{y}dx=-12\int_0^{x_2}\frac{x^{i-3}}{\sqrt{12-9x^2}}dx.$$

For $n=12$, there exists a parameter $\delta_{12,0}=(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_8,a_9,a_{10},a_{11},a_{12})$ satisfying

$$a_0=a_1=a_2=a_3=a_4=a_6=a_9=a_{11}=0,\ \ a_5=-a_7,\ \ a_8=\tfrac{640}{459}a_{12},\ \ a_{10}=-\tfrac{368}{153}a_{12},$$

such that

$$(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_2,\bar{c}_3,\bar{c}_{41},\bar{c}_{42},\bar{c}_5,\bar{c}_6,\bar{c}_7,\bar{c}_8)(\delta_{12,0})=(0,0,0,0,0,0,0,0,0,0,0)\tag{5.3}$$

and

$$\bar{c}_9(\delta_{12,0}) = -\tfrac{5120}{153}\sqrt{3}a_{12}, \quad \det\tfrac{\partial(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_2,\bar{c}_3,\bar{c}_{41},\bar{c}_{42},\bar{c}_5,\bar{c}_6,\bar{c}_7,\bar{c}_8)}{\partial(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_8,a_9,a_{10},a_{11})} = \tfrac{1125899906842624}{3701802772395}\pi^2. \tag{5.4}$$

Then by Theorem 4.3 (3.ii), system (1.1) has 16 limit cycles near L_0^* for some (ε, δ) near $(0, \delta_{12,0})$ if $a_{12} \neq 0$, which have 3 distributions.

In the following we consider the sign of $M^*(h, \delta)$ for $h > 0$ to find more limit cycles.

Let $G(x) = -\frac{1}{6}x^6 + \frac{1}{8}x^8$ and $G(x_i(h)) = h,\ i = 1, 2$ with $x_2(h) < -\frac{2}{3}\sqrt{3} < \frac{2}{3}\sqrt{3} < x_1(h)$. Then using the similar way in Proposition 3.1 in [14], we have

$$\begin{aligned} M^*(h, \delta_{12,0}) &= -\oint\limits_{L_h} f(x, \delta)|_{\delta=\delta_{12,0}} y dx \\ &= -2 \int_{x_2(h)}^{x_1(h)} f(x, \delta_{12,0})\sqrt{2(h - G(x))}dx \\ &= -2 \int_{x_2(h)}^{x_1(h)} \frac{F(x)}{\sqrt{2(h - G(x))}} dG(x) \\ &= -2 \int_{0}^{x_1(h)} \frac{F(x) - F(-x)}{\sqrt{2(h - G(x))}} dG(x), \end{aligned} \tag{5.5}$$

where

$$f(x, \delta_{12,0}) = a_7(-x^5 + x^7) + a_{12}\left(\frac{640}{459}x^8 - \frac{368}{153}x^{10} + x^{12}\right),$$

$$F(x) = \int_0^x f(x, \delta_{12,0})dx, \quad F(x) - F(-x) = \left(\frac{1280}{4131}x^9 - \frac{736}{1683}x^{11} + \frac{2}{13}x^{13}\right)a_{12}.$$

There exists $x_0 > 0$ such that $\frac{1280}{4131}x^9 - \frac{736}{1683}x^{11} + \frac{2}{13}x^{13} > 1$ for $x_0 > 0$ since

$$\lim_{x\to+\infty}\left(\frac{1280}{4131}x^9 - \frac{736}{1683}x^{11} + \frac{2}{13}x^{13}\right) = +\infty.$$

Note that

$$\int_0^{x_0} \frac{\frac{1280}{4131}x^9 - \frac{736}{1683}x^{11} + \frac{2}{13}x^{13}}{\sqrt{2(h - G(x))}} dG(x) \to 0, \quad \text{as } h \to +\infty, \tag{5.6}$$

and

$$\begin{aligned}
&\int_{x_0}^{x_1(h)} \frac{\frac{1280}{4131}x^9 - \frac{736}{1683}x^{11} + \frac{2}{13}x^{13}}{\sqrt{2(h-G(x))}} dG(x) \\
&> \int_{x_0}^{x_1(h)} \frac{1}{\sqrt{2(h-G(x))}} dG(x) \\
&= \sqrt{2(h-G(x_0))} \to +\infty, \quad \text{as } h \to +\infty.
\end{aligned} \tag{5.7}$$

Then by (5.5), (5.6) and (5.7), we have

$$a_{12}M^*(h, \delta_{12,0}) \to -\infty, \quad \text{as } h \to +\infty.$$

By (4.1), (4.3), (4.4) and (5.3) we further have

$$a_{12}M^*(h, \delta_{12,0}) = -\frac{640}{459}\sqrt{3}\,a_{12}^2 h^2 \ln h + O(h^2) > 0, \quad \text{for } 0 < h \ll 1.$$

Thus, there is a zero of $M^*(h, \delta_{12,0})$ for $h > 0$ large denoted by h_0^* which leads to a limit cycle near $L_{h_0^*}$.

For $0 < h + \frac{1}{24} \ll 1$, by [24] and [25], we have

$$\widetilde{M}(h, \delta) = \sum_{i \geq 0} \widetilde{b}_{l,i}(h + \frac{1}{24})^{i+1}, \quad M(h, \delta) = \sum_{i \geq 0} b_{r,i}(h + \frac{1}{24})^{i+1},$$

where

$$b_{r,0} = -\sqrt{2}\pi \sum_{i=0}^{12} a_i, \quad \widetilde{b}_{l,0} = \sqrt{2}\pi \sum_{i=0}^{12} (-1)^{i+1} a_i.$$

We further have

$$b_{r,0}(\delta_{12,0}) = \widetilde{b}_{l,0}(\delta_{12,0}) = \frac{5}{459}\sqrt{2}\pi\, a_{12}.$$

Then

$$\begin{aligned}
a_{12}M(h, \delta_{12,0}) &= \tfrac{5}{459}\sqrt{2}\pi\, a_{12}^2(h + \tfrac{1}{24}) + O((h + \tfrac{1}{24})^2) > 0, \quad \text{for } 0 < h + \tfrac{1}{24} \ll 1, \\
a_{12}M(h, \delta_{12,0}) &= -\tfrac{320}{459}\sqrt{3}\,a_{12}^2 h^2 \ln|h| + O(h^2) > 0, \quad \text{for } 0 < -h \ll 1.
\end{aligned}$$

Here we can not find a zero of $M(h, \delta)$ for $0 < h + \frac{1}{24} \ll 1$. Similarly, we can not find a zero of $\widetilde{M}(h, \delta)$ for $0 < h + \frac{1}{24} \ll 1$.

From the above, system (1.1) has 17 limit cycles for some (ε, δ) near $(0, \delta_{12,0})$ if $a_{12} \neq 0$, of which 16 limit cycles near L_0^* and a large limit cycle near $L_{h_0^*}$ which means that $C^*(12, 7) \geq 17$.

For $n=11$, we can find $\delta_{11,0}=(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_8,a_9,a_{10})$ with

$$a_0=a_1=a_2=a_3=a_4=0,\quad a_5=\tfrac{55}{36}a_{11}-a_7,$$
$$a_6=-\tfrac{16}{9}a_{11},\quad a_8=\tfrac{8}{3}a_{11},\quad a_9=-\tfrac{61}{24}a_{11},\quad a_{10}=-\tfrac{7}{8}a_{11},$$

such that

$$\begin{aligned}&(\bar c_{01},\bar c_{02},\bar c_1,\bar c_2,\bar c_3,\bar c_4,\bar c_5,\bar c_6,b_{r,0})(\delta_{11,0})=(0,0,0,0,0,0,0,0,0),\\&\bar c_7(\delta_{11,0})=-\tfrac{64}{3}\sqrt{2}\sqrt[6]{6}a_{11},\quad b_{r,1}(\delta_{11,0})=-\tfrac{23}{576}\pi\sqrt{2}a_{11},\\&\det\frac{\partial(\bar c_{01},\bar c_{02},\bar c_1,\bar c_2,\bar c_3,\bar c_{41},\bar c_{42},\bar c_5,\bar c_6,b_{r0})}{\partial(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_8,a_9)}\\=&-\tfrac{4194304}{71744535}\pi^4+\tfrac{549755813888}{122159491489035}\pi^3\sqrt{3}+\tfrac{1152921504606846976}{2116413190047531375}\pi^2\\=&-0.07651730197\cdots.\end{aligned}\tag{5.8}$$

So, system (1.1) has 14 limit cycles with 3 distributions for some (ε,δ) near $(0,\delta_{11,0})$ if $a_{11}\neq 0$, of which 13 limit cycles near L_0^* by Theorem 4.3 (1), and 1 limit cycle near the right center.

Note that

$$[F(x)-F(-x)]_{\delta=\delta_{11,0}}=-\frac{7}{44}a_{11}x^{11}+\frac{16}{27}a_{11}x^9-\frac{32}{63}a_{11}x^7.$$

Similarly as the case $n=12$, we have

$$\begin{aligned}&a_{11}M^*(h,\delta_{11,0})=-\tfrac{64}{15}\,|A_0|\sqrt{2}\sqrt[6]{6}h^{\frac{5}{3}}a_{11}^2+O(h^2\ln h)<0,\quad \text{for } 0<h\ll 1,\\&a_{11}M^*(h,\delta_{11,0})\to+\infty,\quad \text{as } h\to+\infty.\end{aligned}$$

Thus, there is a zero of $M^*(h,\delta_{11,0})$ for $h>0$ large denoted by h_1^* which leads to a limit cycle near $L_{h_1^*}$ surrounding L_0^*.

We can not find a zero of $M(h,\delta)$ and $\widetilde{M}(h,\delta)$ for $-\frac{1}{24}<h<0$.

Thus, we have proved $C^*(11,7)\geq 15$.

For $n=10$, we can find $\delta_{10,0}=(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_8,a_9)$ with

$$a_0=a_1=a_2=a_3=a_4=a_9=0,$$
$$a_5=-a_7,\quad a_6=\tfrac{128}{63}a_{10},\quad a_8=-\tfrac{64}{21}a_{10},$$

such that

$$\begin{aligned}&(\bar c_{01},\bar c_{02},\bar c_1,\bar c_2,\bar c_3,\bar c_{41},\bar c_{42},\bar c_5,\bar c_6)(\delta_{10,0})=(0,0,0,0,0,0,0,0,0),\quad \bar c_7(\delta_{10,0})=\tfrac{512}{21}\sqrt{2}\sqrt[6]{6}a_{10},\\&\det\frac{\partial(\bar c_{01},\bar c_{02},\bar c_1,\bar c_2,\bar c_3,\bar c_{41},\bar c_{42},\bar c_5,\bar c_6)}{\partial(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_8,a_9)}=-\frac{2097152\sqrt{6}\pi^2}{745784441325}\Big(24255\sqrt{3}\pi-131072\Big)\neq 0.\end{aligned}\tag{5.9}$$

So, by Theorem 4.3 (1) system (1.1) has 13 limit cycles near L_0^* with 3 distributions for some (ε,δ) near $(0,\delta_{10,0})$ if $a_{10}\neq 0$.

Note that

$$[F(x)-F(-x)]_{\delta=\delta_{10,0}}=\frac{256}{441}a_{10}x^7-\frac{128}{189}a_{10}x^9+\frac{2}{11}a_{10}x^{11}.$$

Then, similarly as before we have

$$a_{10}M^*(h,\delta_{10,0})=\tfrac{512}{105}\left|\bar{A}_0\right|2^{\frac{2}{3}}3^{\frac{1}{6}}a_{10}^2h^{\frac{5}{3}}>0,$$
$$a_{10}M^*(h,\delta_{12,0})\to-\infty,\quad \text{as}\ \ h\to+\infty.$$

Thus, there exist 14 limit cycles for system (1.1) with 3 distributions which gives $C^*(10,7)\geq 14$.

For $n=9$, we can find $\delta_{9,0}$ with

$$(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9)=(0,0,0,0,-\frac{256}{585}a_8,-a_7,-\frac{112}{195}a_8,a_7,a_8,0)$$

such that

$$(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_2,\bar{c}_3,\bar{c}_{41},\bar{c}_{42},\bar{c}_5)(\delta_{9,0})=(0,0,0,0,0,0,0,0),\ \ \bar{c}_6(\delta_{9,0})=\tfrac{1024}{585}2^{\frac{1}{3}}3^{\frac{5}{6}}a_8, \tag{5.10}$$

and

$$\det\frac{\partial(\bar{c}_{01},\bar{c}_{02},\bar{c}_1,\bar{c}_2,\bar{c}_3,\bar{c}_{41},\bar{c}_{42},\bar{c}_5)}{\partial(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_8)}=-\frac{134217728}{49718962755}6^{\frac{2}{3}}\sqrt{2}\pi^2.$$

So, by Theorem 4.3 (1) system (1.1) has 11 limit cycles near L_0^* with 3 distributions for some (ε,δ) near $(0,\delta_{9,0})$ if $a_8\neq 0$. We can not find a zero of $M^*(h,\delta)$ for $h>0$. Thus we get $C^*(9,7)\geq 11$.

In the proof of $n=9$, take $a_9=0$. We easily prove that for $n=8$ system (1.1) has 11 limit cycles near L_0^* with 3 distributions for some (ε,δ) near $(0,\delta_{8,0})$ if $a_8\neq 0$.

This ends the proof. □

Acknowledgments

The first author is supported by National Natural Science Foundation of China (11571090) and Science Foundation of Hebei Normal University (L2017J01). The second author is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC No. R2686A02). The third author is supported by National Natural Science Foundation of China (11431008, 11771296) and Shanghai Rising-Star Program (No. 18QA1403300). J. Yang thanks Western University where she visited and finished this manuscript.

References

[1] D. Hilbert, Mathematical problems, Bull. Amer. Math. Soc. 8 (10) (1902) 437–479.
[2] M. Han, C. Shu, J. Yang, Polynomial Hamiltonian systems with a nilpotent critical point, Adv. Space Res. 46 (2010) 521–525.
[3] L. Perko, Differential Equations and Dynamical Systems, Texts in Applied Mathematics, vol. 7, Springer-Verlag, New York, 1991.

[4] V.K. Melnikov, On the stability of a center for time-periodic perturbations, Trudy Moskov. Mat. Obšč. 12 (1963) 3–52.
[5] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (1) (2003) 47–106.
[6] C. Christopher, C. Li, Limit Cycles of Differential Equations, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2007.
[7] M. Han, J. Jiang, H. Zhu, Limit cycle bifurcations in near-Hamiltonian systems by perturbing a nilpotent center, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 18 (10) (2008) 3013–3027.
[8] J. Yang, M. Han, Computation of expansion coefficients of Melnikov functions near a nilpotent center, Comput. Math. Appl. 64 (6) (2012) 1957–1974.
[9] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four. II. Cuspidal loop, J. Differential Equations 175 (2) (2001) 209–243.
[10] Y. Zhao, Z. Zhang, Abelian integrals for cubic vector fields, Ann. Mat. Pura Appl. (4) 176 (1999) 251–272.
[11] M. Han, H. Zang, J. Yang, Limit cycle bifurcations by perturbing a cuspidal loop in a Hamiltonian system, J. Differential Equations 246 (2009) 129–163.
[12] A. Atabaigi, H.R.Z. Zangeneh, R. Kazemi, Limit cycle bifurcation by perturbing a cuspidal loop of order 2 in a Hamiltonian system, Nonlinear Anal. 75 (4) (2012) 1945–1958.
[13] Y. Xiong, Limit cycle bifurcations by perturbing a Hamiltonian system with a cuspidal loop of order m, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 25 (6) (2015) 1550083, 13.
[14] M. Han, J. Yang, D. Xiao, Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 22 (8) (2012) 1250189, 33.
[15] J. Yang, L. Zhao, Zeros of Abelian integrals for a quartic Hamiltonian with figure-of-eight loop through a nilpotent saddle, Nonlinear Anal. Real World Appl. 27 (2016) 350–365.
[16] L. Zhao, The perturbations of a class of hyper-elliptic Hamilton systems with a double homoclinic loop through a nilpotent saddle, Nonlinear Anal. 95 (2014) 374–387.
[17] H. Zang, M. Han, D. Xiao, On Melnikov functions of a homoclinic loop through a nilpotent saddle for planar near-Hamiltonian systems, J. Differential Equations 245 (4) (2008) 1086–1111.
[18] J. Wang, D. Xiao, On the number of limit cycles in small perturbations of a class of hyper-elliptic Hamiltonian systems with one nilpotent saddle, J. Differential Equations 250 (4) (2011) 2227–2243.
[19] R. Kazemi, H.R.Z. Zangeneh, Bifurcation of limit cycles in small perturbations of a hyper-elliptic Hamiltonian system with two nilpotent saddles, J. Appl. Anal. Comput. 2 (4) (2012) 395–413.
[20] R. Asheghi, A. Bakhshalizadeh, On the distribution of limit cycles in a Liénard system with a nilpotent center and a nilpotent saddle, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 26 (2) (2016) 1650025, 15.
[21] X. Sun, Bifurcation of limit cycles from a Liénard system with a heteroclinic loop connecting two nilpotent saddles, Nonlinear Dynam. 73 (1–2) (2013) 869–880.
[22] Y. Tian, Limit cycle bifurcations by perturbing a compound loop with a cusp and a nilpotent saddle, Nonlinear Dynam. 1–2 (2014) 819798, 14.
[23] M. Han, Bifurcations of invariant tori and subharmonic solutions for periodic perturbed systems, Sci. China Ser. A 37 (11) (1994) 1325–1336.
[24] Y. Hou, M. Han, Melnikov functions for planar near-Hamiltonian systems and Hopf bifurcations, J. Shanghai Norm. Univ. Nat. Sci. 213 (2006) 1–10.
[25] M. Han, J. Yang, P. Yu, Hopf bifurcations for near-Hamiltonian systems, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 19 (12) (2009) 4117–4130.