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On the Melnikov functions and limit cycles near a double homoclinic loop with a nilpotent saddle of order $\hat{m}$ ☆

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Abstract

For a centrally symmetric near-Hamiltonian system, we develop a method for computing all the coefficients in the expansions of three Melnikov functions near a double homoclinic loop. Moreover, we give a new estimation on the lower bound of $H(2\hat{n}, 5)$ for $11 \leq \hat{n} \leq 23$, where $H(2\hat{n}, 5)$ is the maximal number of limit cycles for a kind of Liénard system, $\dot{x} = y$, $\dot{y} = -g(x) + \varepsilon f(x)y$, with $\deg g(x) = 5$ and $\deg f(x) = 2\hat{n}$.

Keywords: Melnikov function; Limit cycle; Bifurcation; Liénard system

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1. Introduction

In 1900, Hilbert posed 23 mathematical problems in the Second International Congress of Mathematicians, among which the 16th problem has not been completely solved. The second part of the Hilbert's 16th problem is to find the maximal number of limit cycles, denoted by $H(n)$, and their distribution for a polynomial system,

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y),$$

where $P_n(x, y)$ and $Q_n(x, y)$ are polynomials in x, y with $\max\{\deg P_n, \deg Q_n\} = n$. Many results have been obtained on this problem for relevant works and introductions (see, for example, [1–4]). The Hilbert's 16th problem is extremely difficult and is even not completely solved for the simplest case $n = 2$. For this problem, Smale [5] pointed out that "Except for the Riemann hypothesis, it seems to be the most elusive of Hilbert's problems". Thus, the weakened Hilbert's 16th problem was proposed by Arnold [6] to reduce the difficulty of solving the problem, which is described below.

Consider the following near-Hamiltonian system,

$$\dot{x} = H_y + \varepsilon p(x, y, \delta), \quad \dot{y} = -H_x + \varepsilon q(x, y, \delta), \tag{1.1}$$

where $\varepsilon \in \mathbb{R}$ is a bounded small parameter, H, p and q are analytic functions in x and y, $\delta \in \mathbb{R}^n (n \in N^*)$ is a vector parameter. When $\varepsilon = 0$, system (1.1) becomes a Hamiltonian system,

$$\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.2}$$

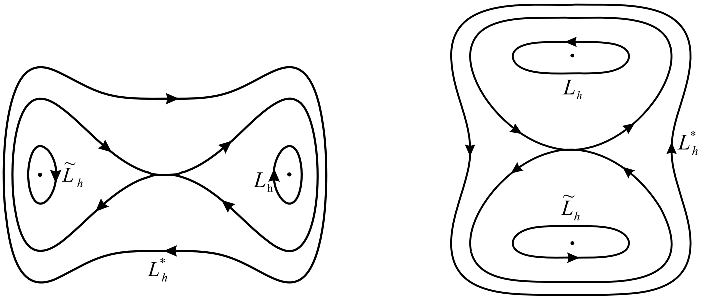
Suppose system (1.2) has at least a family of periodic orbits, defined by $H(x, y) = h, h \in J$ with J an open interval. The first order Melnikov function of system (1.1) is

$$M(h, \delta) = \oint_{H(x,y)=h} q\mathrm{d}x - p\mathrm{d}y.$$

The weakened Hilbert's 16th problem is to find the maximal number of isolated zeros of the above Melnikov function, which has a close relation with the number of limit cycles of system (1.1). One of the efficient methods of finding the number of isolated zeros of the Melnikov function is to study the expansion of $M(h, \delta)$ near a center or a closed orbit. This method has been generalized to consider piecewise smooth systems, for example, see [7–9]. General theory on the expansions and the computation of the first few coefficients can be found in the survey article [10]. Many works about the expansion of the Melnikov function have been done (see, for example, [11–17]). In the following, we briefly summarize some results on the expansion of the Melnikov function near a homoclinic loop and a double homoclinic loop, respectively.

Suppose system (1.2) has a homoclinic loop L_0 passing through a hyperbolic saddle at the origin, and $H(x, y) = h, 0 < -h \ll 1$ defines a family of periodic orbits. Roussarie [13] proved that $M(h, \delta)$ has the following expansion near the homoclinic loop,

$$M(h, \delta) = \sum_{j \geq 0} (c_{2j} + c_{2j+1} h \ln |h|) h^j, \quad 0 < -h \ll 1.$$



(a) Cuspidal type (b) Smooth type

Fig. 1. Two possibilities of a double homoclinic loop with a nilpotent saddle [17].

It is easy to see that $c_0 = \oint_{L_0} q\mathrm{d}x - p\mathrm{d}y$. The formulas for the coefficients c_1 and c_2 were obtained by Han and Ye in [14], and c_3 by Han et al. in [15]. Then no progress was reported on the coefficients c_j, $j \geq 4$, for almost one decade. In 2017, Tian and Han [18] developed an approach to compute the coefficients c_j for $j \geq 4$ under some restrictions. The method is also valid for studying double homoclinic bifurcation.

In this paper, we generalize this method to a degenerate case. For this purpose, suppose system (1.2) has a nilpotent saddle of order $\hat{m}$ ($\hat{m} \in N^*$) at the origin. Without loss of generality, further assume the Hamiltonian function has the following form,

$$H(x, y) = \frac{1}{2}y^2 + \sum_{i+j\geq 3} h_{ij}x^i y^j, \tag{1.3}$$

for (x, y) near the origin. There exists a unique analytic function $\varphi(x) = \sum_{j\geq 2} e_j x^j$ such that $H_y(x, \varphi(x)) = 0$ for $|x|$ small. By Definition 1 in Han et al. [19] we know that for $|x|$ small $H(x, \varphi(x))$ can be expressed as

$$H(x, \varphi(x)) = \sum_{j\geq k} h_j x^j, \quad h_k < 0, \ k = 2\hat{m} + 2. \tag{1.4}$$

Suppose the equation $H(x, y) = 0$ defines a double homoclinic loop L_0^* passing through the nilpotent saddle. By [17], we know that there are two possibilities for L_0^*, i.e., cuspidal type and smooth type. See Fig. 1. Consequently, system (1.1) has three Melnikov functions,

$$M(h, \delta) = \oint_{L_h} q\mathrm{d}x - p\mathrm{d}y, \ \widetilde{M}(h, \delta) = \oint_{\widetilde{L}_h} q\mathrm{d}x - p\mathrm{d}y, \ M^*(h, \delta) = \oint_{L_h^*} q\mathrm{d}x - p\mathrm{d}y. \tag{1.5}$$

In 2012, Han et al. [17] gave the formal expansions of $M, \widetilde{M}$ and M^* for $0 < |h| \ll 1$. In particular, for $\hat{m} = 1$, the authors obtained the formulas for the first seven coefficients in the expansions of $M, \widetilde{M}$, and the formulas for the first five coefficients in the expansion of M^*. Later, for any $\hat{m} \geq 1$, Yang et al. [20] obtained the formulas for the first few coefficients in the expansions of $M, \widetilde{M}$ and M^*. It is hard to get higher-order coefficients, and there are no any progress reported so far.

In this paper, we develop a method for computing all the coefficients in the expansions of the three Melnikov functions $M, \widetilde{M}$ and M^* for $0 < |h| \ll 1$ for the case that system (1.1) is centrally symmetric and has a double homoclinic loop L_0^*. The idea on deriving the formulas for the coefficients of h^j, $j \geq 2$ is motivated by the work of Tian and Han [18]. We further give general conditions for generating limit cycles bifurcating from the double homoclinic loop L_0^*.

Let $H(n, m)$ denote the maximal number of limit cycles for a generalized Liénard system [21],

$$\dot{x} = y, \quad \dot{y} = -g(x) - \varepsilon f(x) y, \tag{1.6}$$

where $\varepsilon > 0$ is sufficiently small, $f(x)$ and $g(x)$ are polynomials in x with $\deg g = m$ and $\deg f = n$.

There have been many studies on the lower bound of $H(n, m)$, see [22–26] and references therein. For $m = 5$, Kazemi et al. [27] proved that $H(4, 5) \geq 3$. Xu and Li [28,29] gave some lower bounds of $H(n, 5)$ for $n = 2, 4, 6, 8$ and 10. Xiong [30] studied the number of limit cycles near a compound loop with a cusp and a hyperbolic saddle and gave lower bounds of $H(n, 5)$ for $1 \leq n \leq 10$. Recently, Li and Yang [31] gave new estimations on the lower bounds of $H(n, 5)$ for $10 \leq n \leq 20$.

However, for $n \geq 21$, to the best of our knowledge, there are no any results for the lower bounds of $H(n, 5)$. As an application of our main results, we obtain the lower bounds of $H(n, 5)$ for $22 \leq n \leq 46$ with n even.

In the next section, we present some lemmas which are needed for proving our main results. The main results and their proofs are given in Section 3. In Section 4, an example is particularly considered to demonstrate the applicability of our theoretical results.

2. Preliminaries

For the asymptotic property of the three Melnikov functions in (1.5) at $h = 0$, Han et al. [17] obtained the following results.

Lemma 2.1. *([17]) Let (1.3) and (1.4) hold, and L_0^*, defined by $H(x, y) = 0$, be a double homoclinic loop of cuspidal type. Then the functions $M(h, \delta)$, $\widetilde{M}(h, \delta)$ and $M^*(h, \delta)$ in (1.5) have the following expansions.*

(i) *For $0 < -h \ll 1$,*

$$\begin{aligned} M(h, \delta) &= \varphi(h, \delta) - \frac{h \ln |h|}{2k} I^*_{1, \frac{k}{2}-1}(h) + |h|^{\frac{1}{k}+\frac{1}{2}} \sum_{\substack{r=0 \\ r \neq \frac{k}{2}-1}}^{k-2} \tilde{A}_r I^*_{1,r}(h) |h|^{\frac{r}{k}}, \\ \widetilde{M}(h, \delta) &= \widetilde{\varphi}(h, \delta) + (-1)^{k/2} \frac{h \ln |h|}{2k} I^*_{1, \frac{k}{2}-1}(h) + |h|^{\frac{1}{k}+\frac{1}{2}} \sum_{\substack{r=0 \\ r \neq \frac{k}{2}-1}}^{k-2} (-1)^r \tilde{A}_r I^*_{1,r}(h) |h|^{\frac{r}{k}}, \end{aligned} \tag{2.1}$$

where $\varphi(h,\delta)\in C^{\omega}$, $\widetilde{A}_r$ *are constants satisfying Lemma 2.5 in [17] and for* $0\le r\le k-1$ $I^*_{1,r}$ *satisfy*

$$I^*_{1,r}(h) \;=\; \sum_{m,j\ge 0}\widetilde{r}_{mk+r,j}\,\alpha^*_{mk+r,j}\,\beta^*_{mk+r}\,h^{j+m}, \tag{2.2}$$

with $\widetilde{r}_{mk+r,j}$ *given by* (18) *in [17] and*

$$\begin{aligned}
\alpha^*_{i,j}&=\begin{cases}\dfrac{\frac{3}{2}k\cdot\frac{5}{2}k\cdots\frac{2j+1}{2}k}{(\frac{3}{2}k+i+1)\cdots(\frac{2j+1}{2}k+i+1)}, & i\ge 0,\;\; j\ge 1,\\[2mm] 1, & i\ge 0,\;\; j=0,\end{cases}\\
\beta^*_{mk+r}&=\begin{cases}\dfrac{(-1)^m(r+1)(k+r+1)\cdots((m-1)k+r+1)}{(\frac{3}{2}k+r+1)(\frac{5}{2}k+r+1)\cdots(\frac{2m+1}{2}k+r+1)}, & m\ge 1,\;\; 0\le r\le k-1,\\[2mm] 1, & m=0,\;\; 0\le r\le k-1.\end{cases}
\end{aligned} \tag{2.3}$$

(ii) *For* $0<h\ll 1$,

$$M^*(h,\delta)=\begin{cases}\varphi^*(h,\delta)+2h^{\frac{1}{k}+\frac{1}{2}}\displaystyle\sum_{r=0}^{\frac{k}{2}-1}\bar{A}_r h^{\frac{2r}{k}}J^*_{1,r}(h), \text{ if } r_1 \text{ is not an integer,}\\[4mm] \varphi^*(h,\delta)-\dfrac{h\ln h}{k}J^*_{1,r_1}(h)+2h^{\frac{1}{k}+\frac{1}{2}}\displaystyle\sum_{\substack{r=0\\ r\ne r_1}}^{\frac{k}{2}-1}\bar{A}_r h^{\frac{2r}{k}}J^*_{1,r}(h), \text{ if } r_1 \text{ is an integer,}\end{cases} \tag{2.4}$$

where $2r_1=\frac{k}{2}-1$, $\varphi^*(h,\delta)\in C^{\omega}$, $\bar{A}_r$ *are constants satisfying Lemma 2.5 in [17], and for* $0\le r\le\frac{k}{2}-1$, $J^*_{1,r}$ *satisfy*

$$J^*_{1,r}(h) \;=\; \sum_{\substack{i=\frac{mk}{2}+r\\ m\ge 0,\, j\ge 1\, odd}} r^{(1)}_{i,j}\,\bar{\alpha}_{i,j}\,\bar{\beta}_i\,h^{m+[\frac{j}{2}]}, \tag{2.5}$$

with $r^{(1)}_{i,j}$ *given by* (20) *in [17] and*

$$\begin{aligned}
\bar{\alpha}_{i,j}&=\begin{cases}1, & i\ge 0,\;\; j=1,\\[2mm] \dfrac{\frac{3}{2}k\cdot\frac{5}{2}k\cdots\frac{j}{2}k}{(\frac{3}{2}k+2i+1)(\frac{5}{2}k+2i+1)\cdots(\frac{j}{2}k+2i+1)}, & i\ge 0,\;\; j\ge 3\; odd,\end{cases}\\
\bar{\beta}_i&=\begin{cases}1, & 0\le i\le\dfrac{k}{2}-1,\\[2mm] \dfrac{(-1)^m(2i+1-k)(2i+1-2k)\cdots(2i+1-mk)}{(2i+1+\frac{1}{2}k)(2i+1-\frac{1}{2}k)\cdots(2i+1-\frac{2m-3}{2}k)}, & i=\dfrac{mk}{2}+r,\;\; m\ge 1,\;\; 0\le r\le\dfrac{k}{2}-1.\end{cases}
\end{aligned} \tag{2.6}$$

Further, Yang et al. [20] gave the formulas of the first few coefficients in the expansions of M, $\widetilde{M}$ and M^*. The expansions of $M(h,\delta)$ and $\widetilde{M}(h,\delta)$ are given in the following lemma.

Lemma 2.2. *([20]) Let (1.3) and (1.4) hold with $h_k<0$ ($k=2\hat{m}+2, \hat{m}\geq 1$) and $L_0^*=L_0\cup\widetilde{L}_0$ be a double homoclinic loop of cuspidal type. Then for the expansions of $M(h,\delta)$ and $\widetilde{M}(h,\delta)$ in (2.1) we have*

$$\begin{aligned}
M(h,\delta)=&c_0+\sum_{r=0}^{\frac{k}{2}-2}c_{r+1}|h|^{\frac{r+1}{k}+\frac{1}{2}}+c_{\frac{k}{2}}h\ln|h|+c_{\frac{k}{2}+1}h+\sum_{r=\frac{k}{2}}^{k-2}c_{r+2}|h|^{\frac{r+1}{k}+\frac{1}{2}}\\
&+\sum_{r=0}^{\frac{k}{2}-2}c_{k+1+r}|h|^{\frac{r+1}{k}+\frac{3}{2}}+c_{\frac{3}{2}k}h^2\ln|h|+O(h^2),\ 0<-h\ll 1,\\
\widetilde{M}(h,\delta)=&\widetilde{c}_0+\sum_{r=0}^{\frac{k}{2}-2}(-1)^r c_{r+1}|h|^{\frac{r+1}{k}+\frac{1}{2}}+(-1)^{\frac{k}{2}+1}c_{\frac{k}{2}}h\ln|h|+\widetilde{c}_{\frac{k}{2}+1}h\\
&+\sum_{r=\frac{k}{2}}^{k-2}(-1)^r c_{r+2}|h|^{\frac{r+1}{k}+\frac{1}{2}}+\sum_{r=0}^{\frac{k}{2}-2}(-1)^r c_{k+1+r}|h|^{\frac{r+1}{k}+\frac{3}{2}}\\
&+(-1)^{\frac{k}{2}+1}c_{\frac{3}{2}k}h^2\ln|h|+O(h^2),\ 0<-h\ll 1,
\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}
&c_0=M(0,\delta)=\oint_{L_0}qdx-pdy,\ c_{r+1}=\widetilde{A}_r\widetilde{r}_{r,0},\ \ r=0,1,\cdots,\frac{k}{2}-2,\ \ c_{\frac{k}{2}}=-\frac{1}{2k}\widetilde{r}_{\frac{k}{2}-1,0},\\
&c_{\frac{k}{2}+1}=\oint_{L_0}\left(p_x+q_y-\bar{b}_{0,0}-\bar{b}_{1,0}x-\bar{b}_{2,0}x^2-\cdots-\bar{b}_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1}\right)\mathrm{d}t+\sum_{i=1}^{\frac{k}{2}}O_1(c_i),\\
&c_{r+2}=\widetilde{A}_r\widetilde{r}_{r,0},\ \ r=\frac{k}{2},\frac{k}{2}+1,\cdots,k-2,\\
&c_{k+1+r}=\frac{\widetilde{A}_r}{\frac{3}{2}k+r+1}\left((r+1)\widetilde{r}_{k+r,0}-\frac{3}{2}k\widetilde{r}_{r,1}\right),\ \ r=0,1,\cdots,\frac{k}{2}-2,\\
&c_{\frac{3}{2}k}=\frac{1}{8k}\left(\widetilde{r}_{\frac{3k}{2}-1,0}-3\widetilde{r}_{\frac{k}{2}-1,1}\right),\\
&\widetilde{c}_0=\widetilde{M}(0,\delta)=\oint_{\widetilde{L}_0}q\mathrm{d}x-p\mathrm{d}y,\\
&\widetilde{c}_{\frac{k}{2}+1}=\oint_{\widetilde{L}_0}\left(p_x+q_y-\bar{b}_{0,0}-\bar{b}_{1,0}x-\bar{b}_{2,0}x^2-\cdots-\bar{b}_{\frac{k}{2}-1,0}x^{\frac{k}{2}-1}\right)\mathrm{d}t+\sum_{i=1}^{\frac{k}{2}}O_1(c_i),
\end{aligned}\tag{2.8}$$

in which $O_1(c)$ denotes c multiplied by a constant, and based on the programs in [16], $\widetilde{r}_{i,0}, \widetilde{r}_{i,1}$ and $\bar{b}_{i,0}$ can be obtained by the method given in [17]. For example, some $\bar{b}_{i,0}$ are given as follows:

$$\begin{aligned} &\bar{b}_{0,0} = a_{10} + b_{01}, \ \bar{b}_{1,0} = 2a_{20} + b_{11}, \ \bar{b}_{2,0} = 3a_{30} + b_{21} - h_{21}(a_{11} + 2b_{02}), \\ &\bar{b}_{3,0} = 4a_{40} + b_{31} - 2h_{21}(a_{21} + b_{12}) + (2h_{12}h_{21} - h_{31})(a_{11} + 2b_{02}), \cdots . \end{aligned} \tag{2.9}$$

Suppose the conditions in Lemma 2.2 hold, and further assume system (1.2) has two elementary centers C_1 and C_2, surrounded by $\widetilde{L_0}$ and L_0, respectively. Let $h_{c_i} \equiv H(C_i), i = 1, 2$. By Han [11], one knows that for $0 < h - h_{c_2} \ll 1$, $M(h, \delta)$ has an expansion of the form,

$$M(h, \delta) = \sum_{l \geq 0} b_l (h - h_{c_2})^{l+1}. \tag{2.10}$$

The coefficients can be obtained by using the programs in [32], which show that

$$b_0 = k_0 (p_x + q_y)(C_2, \delta), \tag{2.11}$$

where k_0 is a constant.

For $\hat{m} = 1$, in the process of computing $c_i(\delta)$ for $i = 1, 2, 4, 5, 6$, the necessary step is to get $\widetilde{r}_{0,0}, \widetilde{r}_{1,0}, \widetilde{r}_{2,0}, \widetilde{r}_{4,0}, \widetilde{r}_{5,0}, \widetilde{r}_{0,1}$ and $\widetilde{r}_{1,1}$. To obtain more coefficients in the expansions of the three Melnikov functions, the values of $\widetilde{r}_{i,j}$ in (2.2) and $r^{(1)}_{i,j}$ in (2.5) need to be obtained first. The results given in Yang et al. [20] show the relationship between $\widetilde{r}_{i,j}$ and $r^{(1)}_{i,j}$.

Lemma 2.3. *([20]) Let (1.3) and (1.4) hold. Then for $\widetilde{r}_{i,j}$ in (2.2) and $r^{(1)}_{i,j}$ in (2.5), we have*

$$r^{(1)}_{i,2j+1} = \widetilde{r}_{2i,j}, \quad i \geq 0, \ j \geq 0. \tag{2.12}$$

In this paper, $\frac{\partial M}{\partial h}$ is needed to obtain more coefficients in the expansion of the Melnikov function. In 1994, Han [33] proved the following result.

Lemma 2.4. *For the Melnikov function $M(h, \delta)$, we have*

$$\frac{\partial M}{\partial h} = \oint_{L_h} (p_x + q_y) \mathrm{d}t.$$

In the next section, we will use the above lemmas to derive the formulas of the coefficients in the expansions of the three Melnikov functions M, $\widetilde{M}$ and M^*.

3. Main results

In the case that $H(x, y) = 0$ defines a double homoclinic loop L_0^* of cuspidal type, except the coefficients of the terms h^i for $i \geq 2$, we give the formulas for all the other coefficients in the expansions of $M(h, \delta)$, $\widetilde{M}(h, \delta)$ and $M^*(h, \delta)$. Especially, if system (1.1) is centrally symmetric, under some conditions we present the formulas of the coefficients of h^i for $i \geq 2$.

We first consider the coefficients in the expansions of $M(h, \delta)$ and $\widetilde{M}(h, \delta)$.

Lemma 3.1. *Let (1.3) and (1.4) hold and L_0^* be a double homoclinic loop of cuspidal type as shown in Fig. 1(a). The expansions of $M(h,\delta)$ and $\widetilde{M}(h,\delta)$ in (1.5) are given below:*

$$
\begin{aligned}
M(h,\delta)=&c_0+\sum_{r=0}^{\frac{k}{2}-2} c_{r+1}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}}+c_{\frac{k}{2}}h\ln|h| \\
&+\sum_{i\ge 0}\left(c_{\frac{k}{2}+1+ki}h^{i+1}+\sum_{r=\frac{k}{2}}^{k-2} c_{r+2+ki}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}+i}\right. \\
&\left.+\sum_{r=0}^{\frac{k}{2}-2} c_{k+1+r+ki}|h|^{\frac{r+1}{k}+\frac{3}{2}+i}+c_{\frac{3}{2}k+ki}h^{2+i}\ln|h|\right), \\
\widetilde{M}(h,\delta)=&\widetilde{c}_0+\sum_{r=0}^{\frac{k}{2}-2}(-1)^r c_{r+1}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}}+(-1)^{\frac{k}{2}+1}c_{\frac{k}{2}}h\ln|h| \\
&+\sum_{i\ge 0}\left(\widetilde{c}_{\frac{k}{2}+1+ki}h^{i+1}+\sum_{r=\frac{k}{2}}^{k-2}(-1)^r c_{r+2+ki}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}+i}\right. \\
&\left.+\sum_{r=0}^{\frac{k}{2}-2}(-1)^r c_{k+1+r+ki}|h|^{\frac{r+1}{k}+\frac{3}{2}+i}+(-1)^{\frac{k}{2}+1}c_{\frac{3}{2}k+ki}h^{2+i}\ln|h|\right),
\end{aligned}
\tag{3.1}
$$

for $0<-h\ll 1$, where c_j, $j=0,1,\cdots,\frac{3}{2}k$ and $\widetilde{c}_0, \widetilde{c}_{\frac{k}{2}+1}$ are given in (2.8), and

$$
\begin{aligned}
c_{r+2+ki}&=(-1)^i\widetilde{A}_r\sum_{\substack{m+j=i\\ m,j\ge 0}}\widetilde{r}_{mk+r,j}\alpha^*_{mk+r,j}\beta^*_{mk+r},\ r=\frac{k}{2},\frac{k}{2}+1,\cdots,k-2, \\
c_{k+1+r+ki}&=(-1)^{i+1}\widetilde{A}_r\sum_{\substack{m+j=i+1\\ m,j\ge 0}}\widetilde{r}_{mk+r,j}\alpha^*_{mk+r,j}\beta^*_{mk+r},\ r=0,1,\cdots,\frac{k}{2}-2, \\
c_{\frac{3}{2}k+ki}&=-\frac{1}{2k}\sum_{\substack{m+j=i+1\\ m,j\ge 0}}\widetilde{r}_{mk+\frac{k}{2}-1,j}\alpha^*_{mk+\frac{k}{2}-1,j}\beta^*_{mk+\frac{k}{2}-1}.
\end{aligned}
\tag{3.2}
$$

Proof. By (2.3) and (2.2) we know that

$$
\begin{aligned}
&\alpha^*_{mk+r,j}=1,\quad \beta^*_{mk+r}=1,\quad \text{for } m=0,\ j=0, \\
&I^*_{1,r}(h)=\widetilde{r}_{r,0}+\sum_{\substack{m+j\ge 1\\ m,j\ge 0}}\widetilde{r}_{mk+r,j}\alpha^*_{mk+r,j}\beta^*_{mk+r}h^{j+m},
\end{aligned}
$$

where $0 \le r \le k-1$. Then, we have

$$
\begin{aligned}
& |h|^{\frac{1}{k}+\frac{1}{2}} \sum_{\substack{r=0\\ r\neq\frac{k}{2}-1}}^{k-2} \tilde{A}_r I^*_{1,r}(h)|h|^{\frac{r}{k}} \\
= & \sum_{r=0}^{\frac{k}{2}-2} \tilde{A}_r \widetilde{r}_{r,0}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}} + \sum_{r=\frac{k}{2}}^{k-2} \tilde{A}_r \widetilde{r}_{r,0}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}} \\
& + \sum_{\substack{r=0\\ r\neq\frac{k}{2}-1}}^{k-2} \tilde{A}_r \sum_{i\ge 1} \sum_{\substack{m+j=i\\ m,j\ge 0}} \widetilde{r}_{mk+r,j}\alpha^*_{mk+r,j}\beta^*_{mk+r}(-1)^i|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}+i} \\
= & \sum_{r=0}^{\frac{k}{2}-2} \tilde{A}_r \widetilde{r}_{r,0}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}} + \sum_{i\ge 0}\left(\sum_{r=\frac{k}{2}}^{k-2} \sum_{\substack{m+j=i\\ m,j\ge 0}} \widetilde{r}_{mk+r,j}\tilde{A}_r\alpha^*_{mk+r,j}\beta^*_{mk+r}(-1)^i|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}+i} \right. \\
& \left. \sum_{r=0}^{\frac{k}{2}-2} \sum_{\substack{m+j=i+1\\ m,j\ge 0}} \widetilde{r}_{mk+r,j}\tilde{A}_r\alpha^*_{mk+r,j}\beta^*_{mk+r}(-1)^{i+1}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{3}{2}+i} \right)
\end{aligned}
$$

and

$$
\frac{h\ln|h|}{2k} I^*_{1,\frac{k}{2}-1} = \frac{h\ln|h|}{2k}\widetilde{r}_{\frac{k}{2}-1,0} + \frac{1}{2k}\sum_{i\ge 1}\sum_{\substack{m+j=i\\ m,j\ge 0}} \widetilde{r}_{mk+\frac{k}{2}-1,j}\alpha^*_{mk+\frac{k}{2}-1,j}\beta^*_{mk+\frac{k}{2}-1}h^{i+1}\ln|h|.
$$

Thus, by (2.1), we get

$$
\begin{aligned}
M(h,\delta) = \; & c_0 + \sum_{r=0}^{\frac{k}{2}-2} \tilde{A}_r \widetilde{r}_{r,0}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}} - \frac{h\ln|h|}{2k}\widetilde{r}_{\frac{k}{2}-1,0} \\
& + \sum_{i\ge 0}\left(c_{\frac{k}{2}+1+ki}h^{i+1} + \sum_{r=\frac{k}{2}}^{k-2} \sum_{\substack{m+j=i\\ m,j\ge 0}} \widetilde{r}_{mk+r,j}\tilde{A}_r\alpha^*_{mk+r,j}\beta^*_{mk+r}(-1)^i|h|^{\frac{r}{k}+\frac{1}{k}+\frac{1}{2}+i} \right. \\
& \sum_{r=0}^{\frac{k}{2}-2} \sum_{\substack{m+j=i+1\\ m,j\ge 0}} \widetilde{r}_{mk+r,j}\tilde{A}_r\alpha^*_{mk+r,j}\beta^*_{mk+r}(-1)^{i+1}|h|^{\frac{r}{k}+\frac{1}{k}+\frac{3}{2}+i} \\
& \left. -\frac{1}{2k}\sum_{\substack{m+j=i+1\\ m,j\ge 0}} \widetilde{r}_{mk+\frac{k}{2}-1,j}\alpha^*_{mk+\frac{k}{2}-1,j}\beta^*_{mk+\frac{k}{2}-1}h^{i+2}\ln|h| \right).
\end{aligned}
$$

It is easy to see that (3.2) hold. Similarly, a direct computation using the formula of $\widetilde{M}$ in (2.1) yields the expansion of $\widetilde{M}$ in (3.1). The proof is complete. □

Next we study the coefficients in the expansion of $M^*(h,\delta)$.

Lemma 3.2. *Let (1.3) and (1.4) hold and L_0^* be a double homoclinic loop of cuspidal type. For $0<h\ll 1$, if r_1 is not an integer (i.e., $\hat{m}$ is odd), the expansion of $M^*(h,\delta)$ in (1.5) has the form,*

$$\begin{aligned} M^*(h,\delta) \;=\; & c_0^* + \sum_{r=0}^{\frac{k}{4}-1} c_{r+1}^* h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}} + \sum_{i\ge 0}\Bigg(c^*_{\frac{k}{4}+1+(\frac{k}{2}+1)i} h^{i+1} \\ & + \sum_{r=\frac{k}{4}}^{\frac{k}{2}-1} c^*_{r+2+(\frac{k}{2}+1)i} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i} + \sum_{r=0}^{\frac{k}{4}-1} c^*_{\frac{k}{2}+r+2+(\frac{k}{2}+1)i} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i+1}\Bigg), \end{aligned} \tag{3.3}$$

where c_0^*, c_{r+1}^* and $c^*_{\frac{k}{4}+1}$ are given in [17] and [20] respectively, with

$$\begin{aligned} & c_0^* = c_0 + \widetilde{c}_0,\;\; c_{r+1}^* = 2\bar{A}_r \widetilde{r}_{2r,0} = \frac{2\bar{A}_r}{\widetilde{A}_{2r}} c_{2r+1},\;\; r=0,\cdots,\frac{k}{4}-1, \\ & c^*_{\frac{k}{4}+1} = \oint_{L_0^*} \Big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-2} \bar{b}_{i,0} x^i\Big)\mathrm{d}t + \sum_{i=1}^{\frac{k}{2}-1} O_1(c_i), \end{aligned}$$

and

$$\begin{aligned} & c^*_{r+2+(\frac{k}{2}+1)i} = (-1)^i \frac{2\bar{A}_r}{\widetilde{A}_{2r}} c_{2r+2+ki},\;\; r=\frac{k}{4},\cdots,\frac{k}{2}-1, \\ & c^*_{\frac{k}{2}+r+2+(\frac{k}{2}+1)i} = (-1)^{i+1} \frac{2\bar{A}_r}{\widetilde{A}_{2r}} c_{k+1+2r+ki},\;\; r=0,\cdots,\frac{k}{4}-1. \end{aligned} \tag{3.4}$$

If r_1 is an integer (i.e., $\hat{m}$ is even), for $0<h\ll 1$, $M^*(h,\delta)$ in (1.5) has the expansion of the form,

$$\begin{aligned} M^*(h,\delta) \;=\; & c_0^* + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} c_{r+1}^* h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}} + c^*_{\frac{k}{4}+\frac{1}{2}} h\ln h \\ & + \sum_{i\ge 0}\Bigg(c^*_{\frac{k}{4}+\frac{3}{2}+(\frac{k}{2}+1)i} h^{i+1} + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} c^*_{r+2+(\frac{k}{2}+1)i} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i} \\ & + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} c^*_{\frac{k}{2}+r+2+(\frac{k}{2}+1)i} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i+1} + c^*_{\frac{3}{4}k+\frac{3}{2}+(\frac{k}{2}+1)i} h^{i+2}\ln h \Bigg), \end{aligned} \tag{3.5}$$

where

$$c_0^* = c_0 + \widetilde{c}_0,\ c_{r+1}^* = 2\bar{A}_r \widetilde{r}_{2r,0} = \frac{2\bar{A}_r}{\widetilde{A}_{2r}} c_{2r+1},\ r = 0, \cdots, \frac{k}{4} - \frac{3}{2},\ c_{\frac{k}{4}+\frac{1}{2}}^* = 2c_{\frac{k}{2}},$$
$$c_{\frac{k}{4}+\frac{3}{2}}^* = \oint_{L_0^*} \Big(p_x + q_y - \sum_{i=0}^{\frac{k}{2}-1} \bar{b}_{i,0} x^i\Big) \mathrm{d}t + \sum_{i=1}^{\frac{k}{2}} O_1(c_i),$$

are given in [20], and

$$\begin{aligned}
c_{r+2+(\frac{k}{2}+1)i}^* &= \frac{2\bar{A}_r}{\widetilde{A}_{2r}} (-1)^i c_{2r+2+ki},\ r = \frac{k}{4} + \frac{1}{2}, \cdots, \frac{k}{2} - 1, \\
c_{\frac{k}{2}+r+2+(\frac{k}{2}+1)i}^* &= \frac{2\bar{A}_r}{\widetilde{A}_{2r}} (-1)^{i+1} c_{k+1+2r+ki},\ r = 0, \cdots, \frac{k}{4} - \frac{3}{2}, \\
c_{\frac{3}{4}k+\frac{3}{2}+(\frac{k}{2}+1)i}^* &= 2c_{\frac{3}{2}k+ki}.
\end{aligned} \tag{3.6}$$

Proof. For any $k = 2\hat{m} + 2$, $\hat{m} \in N^*$ and $0 \le r \le \frac{k}{2} - 1$, by (2.3) and (2.6) we obtain that

$$\bar{\alpha}_{\frac{mk}{2}+r,2j+1} = \frac{\frac{3}{2}k \cdot \frac{5}{2}k \cdots \frac{2j+1}{2}k}{(\frac{3}{2}k + mk + 2r + 1)(\frac{5}{2}k + mk + 2r + 1) \cdots (\frac{2j+1}{2}k + mk + 2r + 1)} = \alpha_{mk+2r,j}^*,$$
$$\text{for } m \ge 0,\ j \ge 1,$$
$$\bar{\alpha}_{\frac{mk}{2}+r,2j+1} = \alpha_{mk+2r,j}^* = 1,\ \text{for } m \ge 0,\ j = 0,$$

and

$$\begin{aligned}
\bar{\beta}_{\frac{mk}{2}+r} &= \beta_{mk+2r}^* = 1,\ \text{for } m = 0, \\
\bar{\beta}_{\frac{mk}{2}+r} &= \frac{(-1)^m (mk + 2r + 1 - k)(mk + 2r + 1 - 2k) \cdots (mk + 2r + 1 - (m-1)k)(mk + 2r + 1 - mk)}{(mk + 2r + 1 + \frac{1}{2}k)(mk + 2r + 1 - \frac{1}{2}k) \cdots (mk + 2r + 1 - \frac{2m-3}{2}k)} \\
&= \frac{(-1)^m ((m-1)k + 2r + 1)((m-2)k + 2r + 1) \cdots (k + 2r + 1)(2r + 1)}{(\frac{2m+1}{2}k + 2r + 1)(\frac{2m-1}{2}k + 2r + 1) \cdots (\frac{3}{2}k + 2r + 1)} \\
&= \beta_{mk+2r}^*,\ \text{for } m \ge 1.
\end{aligned}$$

Further, by (2.5) and (2.12) we get the relation between $J_{1,r}^*(h)$ and $I_{1,2r}^*(h)$ as follows:

$$\begin{aligned}
J_{1,r}^*(h) &= \sum_{\substack{m \ge 0, \\ j \ge 1\, odd}} r_{\frac{mk}{2}+r,j}^{(1)} \bar{\alpha}_{\frac{mk}{2}+r,j} \bar{\beta}_{\frac{mk}{2}+r} h^{m+[\frac{j}{2}]} \\
&= \sum_{\substack{m \ge 0, \\ j \ge 1\, odd}} \widetilde{r}_{mk+2r,\frac{j-1}{2}} \bar{\alpha}_{\frac{mk}{2}+r,j} \bar{\beta}_{\frac{mk}{2}+r} h^{m+[\frac{j}{2}]}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\geq 0,\, j\geq 0} \widetilde{r}_{mk+2r,j}\bar{\alpha}_{\frac{mk}{2}+r,2j+1}\bar{\beta}_{\frac{mk}{2}+r}h^{m+j} \\
&= \sum_{m\geq 0,\, j\geq 0} \widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r}h^{m+j} \\
&= I^*_{1,2r}(h).
\end{aligned}
\tag{3.7}
$$

Then, by (2.4) and (2.2) we can write $M^*(h,\delta)$ as

$$
\begin{aligned}
M^*(h,\delta) &= \varphi^*(h,\delta) + 2h^{\frac{1}{k}+\frac{1}{2}}\sum_{r=0}^{\frac{k}{2}-1}\bar{A}_r h^{\frac{2r}{k}} I^*_{1,2r}(h) \\
&= \varphi^*(h,\delta) + 2\sum_{r=0}^{\frac{k}{4}-1}\bar{A}_r\widetilde{r}_{2r,0}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}} + 2\sum_{r=\frac{k}{4}}^{\frac{k}{2}-1}\bar{A}_r\widetilde{r}_{2r,0}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}} \\
&\quad + 2\sum_{r=0}^{\frac{k}{4}-1}\bar{A}_r\sum_{i\geq 1}\sum_{\substack{m+j=i\\ m,j\geq 0}}\widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i} \\
&\quad + 2\sum_{r=\frac{k}{4}}^{\frac{k}{2}-1}\bar{A}_r\sum_{i\geq 1}\sum_{\substack{m+j=i\\ m,j\geq 0}}\widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i} \\
&= \varphi^*(h,\delta) + 2\sum_{r=0}^{\frac{k}{4}-1}\bar{A}_r\widetilde{r}_{2r,0}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}} \\
&\quad + 2\sum_{i\geq 0}\left(\sum_{r=\frac{k}{4}}^{\frac{k}{2}-1}\bar{A}_r\sum_{\substack{m+j=i\\ m,j\geq 0}}\widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i}\right. \\
&\quad \left. + \sum_{r=0}^{\frac{k}{4}-1}\bar{A}_r\sum_{\substack{m+j=i+1\\ m,j\geq 0}}\widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i+1}\right),
\end{aligned}
$$

if r_1 is not an integer. This shows that in this case $M^*(h,\delta)$ can be written in the form of (3.3), where

$$
\begin{aligned}
&c^*_{r+1} = 2\bar{A}_r\widetilde{r}_{2r,0},\ r = 0,\cdots,\frac{k}{4}-1, \\
&c^*_{r+2+(\frac{k}{2}+1)i} = 2\bar{A}_r\sum_{\substack{m+j=i\\ m,j\geq 0}}\widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r},\ r = \frac{k}{4},\cdots,\frac{k}{2}-1,
\end{aligned}
$$

$$c^*_{\frac{k}{2}+r+2+(\frac{k}{2}+1)i} = 2\bar{A}_r \sum_{\substack{m+j=i+1\\ m,j\geq 0}} \widetilde{r}_{mk+2r,j}\,\alpha^*_{mk+2r,j}\,\beta^*_{mk+2r}, \quad r=0,\cdots,\frac{k}{4}-1. \tag{3.8}$$

Comparing the above formulas with that in (3.2) leads to (3.4).

If r_1 is an integer, by (2.2), (2.4) and (3.7) we have

$$\begin{aligned}
M^*(h,\delta) &= \varphi^*(h,\delta) - \frac{h\ln h}{k} I^*_{1,2r_1}(h) + 2h^{\frac{1}{k}+\frac{1}{2}} \sum_{\substack{r=0\\ r\neq r_1}}^{\frac{k}{2}-1} \bar{A}_r h^{\frac{2r}{k}} I^*_{1,2r}(h)\\
&= \varphi^*(h,\delta) - \frac{h\ln h}{k} \sum_{i\geq 0} \sum_{\substack{m+j=i\\ m,j\geq 0}} \widetilde{r}_{mk+2r_1,j}\,\alpha^*_{mk+2r_1,j}\,\beta^*_{mk+2r_1} h^i\\
&\quad +2\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \bar{A}_r \widetilde{r}_{2r,0} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}} + 2\sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \bar{A}_r \widetilde{r}_{2r,0} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}}\\
&\quad +2\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \bar{A}_r \sum_{i\geq 1} \sum_{\substack{m+j=i\\ m,j\geq 0}} \widetilde{r}_{mk+2r,j}\,\alpha^*_{mk+2r,j}\,\beta^*_{mk+2r} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i}\\
&\quad +2\sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \bar{A}_r \sum_{i\geq 1} \sum_{\substack{m+j=i\\ m,j\geq 0}} \widetilde{r}_{mk+2r,j}\,\alpha^*_{mk+2r,j}\,\beta^*_{mk+2r} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i}\\
&= \varphi^*(0,\delta) + 2\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \bar{A}_r \widetilde{r}_{2r,0} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}} - \frac{h\ln h}{k}\widetilde{r}_{2r_1,0}\\
&\quad + \sum_{i\geq 0}\left(c^*_{\frac{k}{4}+\frac{3}{2}+(\frac{k}{2}+1)i} h^{i+1} + 2\sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} \sum_{\substack{m+j=i\\ m,j\geq 0}} \bar{A}_r \widetilde{r}_{mk+2r,j}\,\alpha^*_{mk+2r,j}\,\beta^*_{mk+2r} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i}\right.\\
&\quad +2\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \sum_{\substack{m+j=i+1\\ m,j\geq 0}} \bar{A}_r \widetilde{r}_{mk+2r,j}\,\alpha^*_{mk+2r,j}\,\beta^*_{mk+2r} h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i+1}\\
&\quad \left. -\frac{h\ln h}{k} \sum_{\substack{m+j=i+1\\ m,j\geq 0}} \widetilde{r}_{mk+2r_1,j}\,\alpha^*_{mk+2r_1,j}\,\beta^*_{mk+2r_1} h^{i+1}\right),
\end{aligned}$$

which indicates that in this case $M^*(h,\delta)$ can be written in the form of (3.5), where

$$c^*_{r+2+(\frac{k}{2}+1)i} = 2\sum_{\substack{m+j=i\\ m,j\geq 0}} \bar{A}_r \widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r}, \quad r=\frac{k}{4}+\frac{1}{2},\cdots,\frac{k}{2}-1,$$

$$c^*_{\frac{k}{2}+r+2+(\frac{k}{2}+1)i} = 2\sum_{\substack{m+j=i+1\\ m,j\geq 0}} \bar{A}_r \widetilde{r}_{mk+2r,j}\alpha^*_{mk+2r,j}\beta^*_{mk+2r}, \quad r=0,1,\cdots,\frac{k}{4}-\frac{3}{2},$$

$$c^*_{\frac{3}{4}k+\frac{3}{2}+(\frac{k}{2}+1)i} = -\frac{1}{k}\sum_{\substack{m+j=i+1\\ m,j\geq 0}} \widetilde{r}_{mk+2r_1,j}\alpha^*_{mk+2r_1,j}\beta^*_{mk+2r_1}.$$

Comparing the above formulas with that in (3.2), we obtain (3.6).

The proof is finished. □

The result for $k=4$ is then obtained directly, as given in the following corollary.

Corollary 3.1. *Let (1.4) hold with $k=4$ and L_0^* be a double homoclinic loop of cuspidal type. For the Melnikov functions $M(h,\delta)$, $\widetilde{M}(h,\delta)$ and $M^*(h,\delta)$ in (1.5), we have*

$$\begin{aligned}
M(h,\delta) &= c_0 + c_1|h|^{\frac{3}{4}} + c_2 h\ln|h| \\
&\quad + \sum_{i\geq 1}\Big(c_{4i-1}h^i + c_{4i}|h|^{\frac{4i+1}{4}} + c_{4i+1}|h|^{\frac{4i+3}{4}} + c_{4i+2}h^{i+1}\ln|h|\Big), \ 0<-h\ll 1,\\
\widetilde{M}(h,\delta) &= \widetilde{c}_0 + c_1|h|^{\frac{3}{4}} - c_2 h\ln|h| \\
&\quad + \sum_{i\geq 1}\Big(\widetilde{c}_{4i-1}h^i + c_{4i}|h|^{\frac{4i+1}{4}} + c_{4i+1}|h|^{\frac{4i+3}{4}} - c_{4i+2}h^{i+1}\ln|h|\Big), \ 0<-h\ll 1,\\
M^*(h,\delta) &= c_0^* + c_1^* h^{\frac{3}{4}} + \sum_{i\geq 1}\Big(c^*_{3i-1}h^i + c^*_{3i}h^{\frac{4i+1}{4}} + c^*_{3i+1}h^{\frac{4i+3}{4}}\Big), \ 0<h\ll 1,
\end{aligned} \tag{3.9}$$

where $c_0, \widetilde{c}_0, c_1, c_2, c_0^*, c_1^*$ are given in [17] (or see (2.8) and Lemma 3.2), and for $i\geq 1$

$$\begin{aligned}
c_{4i} &= (-1)^{i-1}\widetilde{A}_2 \sum_{\substack{m+j=i-1\\ m,j\geq 0}} \widetilde{r}_{4m+2,j}\alpha^*_{4m+2,j}\beta^*_{4m+2},\\
c_{4i+1} &= (-1)^{i}\widetilde{A}_0 \sum_{\substack{m+j=i\\ m,j\geq 0}} \widetilde{r}_{4m,j}\alpha^*_{4m,j}\beta^*_{4m},\\
c_{4i+2} &= -\frac{1}{8}\sum_{\substack{m+j=i\\ m,j\geq 0}} \widetilde{r}_{4m+1,j}\alpha^*_{4m+1,j}\beta^*_{4m+1},\\
c^*_{3i} &= (-1)^{i-1}\frac{2\bar{A}_1}{\widetilde{A}_2}c_{4i}, \quad c^*_{3i+1} = (-1)^{i}\frac{2\bar{A}_0}{\widetilde{A}_0}c_{4i+1}.
\end{aligned} \tag{3.10}$$

For $i \geq 1$, we give a Maple program (see Appendix A), by which the coefficients $c_{4i}, c_{4i+1}, c_{4i+2}$ in (3.10) can be obtained as follows:

$$
\begin{aligned}
c_8 &= \widetilde{A}_2\left(\frac{1}{3}\widetilde{r}_{6,0}-\frac{2}{3}\widetilde{r}_{2,1}\right),\ c_9=\widetilde{A}_0\left(\frac{5}{77}\widetilde{r}_{8,0}-\frac{6}{77}\widetilde{r}_{4,1}+\frac{60}{77}\widetilde{r}_{0,2}\right),\\
c_{10} &= -\frac{1}{64}\left(\widetilde{r}_{9,0}-\widetilde{r}_{5,1}+5\widetilde{r}_{1,2}\right),\ c_{12}=\widetilde{A}_2\left(\frac{7}{39}\widetilde{r}_{10,0}-\frac{2}{13}\widetilde{r}_{6,1}+\frac{20}{39}\widetilde{r}_{2,2}\right),\\
c_{13} &= -\widetilde{A}_0\left(-\frac{3}{77}\widetilde{r}_{12,0}+\frac{2}{77}\widetilde{r}_{8,1}-\frac{4}{77}\widetilde{r}_{4,2}+\frac{8}{11}\widetilde{r}_{0,3}\right),\\
c_{14} &= -\frac{1}{8}\left(-\frac{5}{64}\widetilde{r}_{13,0}+\frac{3}{64}\widetilde{r}_{9,1}-\frac{5}{64}\widetilde{r}_{5,2}+\frac{35}{64}\widetilde{r}_{1,3}\right),\\
c_{16} &= -\widetilde{A}_2\left(-\frac{77}{663}\widetilde{r}_{14,0}+\frac{14}{221}\widetilde{r}_{10,1}-\frac{20}{221}\widetilde{r}_{6,2}+\frac{280}{663}\widetilde{r}_{2,3}\right),\\
c_{17} &= \widetilde{A}_0\left(\frac{39}{1463}\widetilde{r}_{16,0}-\frac{18}{1463}\widetilde{r}_{12,1}+\frac{20}{1463}\widetilde{r}_{8,2}-\frac{8}{209}\widetilde{r}_{4,3}+\frac{144}{209}\widetilde{r}_{0,4}\right),\\
c_{18} &= -\frac{1}{8}\left(\frac{7}{128}\widetilde{r}_{17,0}-\frac{3}{128}\widetilde{r}_{13,1}+\frac{3}{128}\widetilde{r}_{9,2}-\frac{7}{128}\widetilde{r}_{5,3}+\frac{63}{128}\widetilde{r}_{1,4}\right),\\
c_{20} &= \widetilde{A}_2\left(\frac{55}{663}\widetilde{r}_{18,0}-\frac{22}{663}\widetilde{r}_{14,1}+\frac{20}{663}\widetilde{r}_{10,2}-\frac{40}{663}\widetilde{r}_{6,3}+\frac{80}{221}\widetilde{r}_{2,4}\right),\\
c_{21} &= -\widetilde{A}_0\left(-\frac{663}{33649}\widetilde{r}_{20,0}+\frac{234}{33649}\widetilde{r}_{16,1}-\frac{180}{33649}\widetilde{r}_{12,2}+\frac{40}{4807}\widetilde{r}_{8,3}h^5\right.\\
&\qquad\left.-\frac{144}{4807}\widetilde{r}_{4,4}+\frac{288}{437}\widetilde{r}_{0,5}\right),\\
c_{22} &= -\frac{1}{4096}\left(-21\widetilde{r}_{21,0}+7\widetilde{r}_{17,1}-5\widetilde{r}_{13,2}+7\widetilde{r}_{9,3}-21\widetilde{r}_{5,4}+231\widetilde{r}_{1,5}\right),\\
c_{24} &= \widetilde{A}_2\left(-\frac{352}{1105}\widetilde{r}_{2,5}+\frac{48}{1105}\widetilde{r}_{6,4}-\frac{56}{3315}\widetilde{r}_{10,3}+\frac{44}{3315}\widetilde{r}_{14,2}\right.\\
&\qquad\left.-\frac{22}{1105}\widetilde{r}_{18,1}+\frac{209}{3315}\widetilde{r}_{22,0}\right),\cdots\cdots.
\end{aligned}
\tag{3.11}
$$

Now we suppose that the equation $H(x,y)=0$ defines a double homoclinic loop of cuspidal type, denoted by L_0^*, and system (1.1) is centrally symmetric. We will give the coefficients of h^i for $i \geq 2$. In this case, we have $\widetilde{M}(h,\delta)=M(h,\delta)$. By (3.1) and Lemma 3.2, the expansions of $M(h,\delta)$ and $M^*(h,\delta)$ are given by

$$M(h,\delta)=c_0+\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} c_{2r+1}|h|^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}}+c_{\frac{k}{2}}h\ln|h|$$

$$+\sum_{i\geq 0}\left(c_{\frac{k}{2}+1+ki}h^{i+1}+\sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1}c_{2r+2+ki}|h|^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i}\right.$$
$$\left.+\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}}c_{k+1+2r+ki}|h|^{\frac{1}{k}+\frac{2r}{k}+\frac{3}{2}+i}+c_{\frac{3}{2}k+ki}h^{2+i}\ln|h|\right),$$

$$M^*(h,\delta)=c_0^*+\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}}\frac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{2r+1}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}}+2c_{\frac{k}{2}}h\ln h$$
$$+\sum_{i\geq 0}\left(c^*_{\frac{k}{4}+\frac{3}{2}+(\frac{k}{2}+1)i}h^{i+1}+\sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1}\frac{2\bar{A}_r}{\widetilde{A}_{2r}}(-1)^i c_{2r+2+ki}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i}\right.$$
$$\left.+\sum_{r=0}^{\frac{k}{4}-\frac{3}{2}}\frac{2\bar{A}_r}{\widetilde{A}_{2r}}(-1)^{i+1}c_{k+1+2r+ki}h^{\frac{1}{k}+\frac{2r}{k}+\frac{3}{2}+i}+2c_{\frac{3}{2}k+ki}h^{2+i}\ln h\right),\qquad(3.12)$$

if $\hat{m}$ is even, and

$$M(h,\delta)=c_0+\sum_{r=0}^{\frac{k}{4}-1}c_{2r+1}|h|^{\frac{2r}{k}+\frac{1}{k}+\frac{1}{2}}+\sum_{i\geq 0}\left(c_{\frac{k}{2}+1+ki}h^{i+1}\right.$$
$$\left.+\sum_{r=\frac{k}{4}}^{\frac{k}{2}-1}c_{2r+2+ki}|h|^{\frac{2r}{k}+\frac{1}{k}+\frac{1}{2}+i}+\sum_{r=0}^{\frac{k}{4}-1}c_{k+1+2r+ki}|h|^{\frac{2r}{k}+\frac{1}{k}+\frac{3}{2}+i}\right),$$
$$M^*(h,\delta)=c_0^*+\sum_{r=0}^{\frac{k}{4}-1}\frac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{2r+1}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}}+\sum_{i\geq 0}\left(c^*_{\frac{k}{4}+1+(\frac{k}{2}+1)i}h^{i+1}\right.$$
$$\left.+\sum_{r=\frac{k}{4}}^{\frac{k}{2}-1}(-1)^i\frac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{2r+2+ki}h^{\frac{1}{k}+\frac{1}{2}+\frac{2r}{k}+i}+\sum_{r=0}^{\frac{k}{4}-1}(-1)^{i+1}\frac{2\bar{A}_r}{\widetilde{A}_{2r}}c_{k+1+2r+ki}h^{\frac{1}{k}+\frac{3}{2}+\frac{2r}{k}+i}\right),\qquad(3.13)$$

if $\hat{m}$ is odd.

It should be noted that the coefficients of h^i, $i\geq 2$ are not given in Lemmas 3.1 and 3.2. They are given in the following theorem.

Theorem 3.1. *Consider centrally symmetric system* (1.1). *Let* (1.4) *hold and* L_0^*, *defined by* $H(x,y)=0$, *be a double homoclinic loop of cuspidal type. For convenience, let* $P_0(x,y,\delta)=p(x,y,\delta)$, $Q_0(x,y,\delta)=q(x,y,\delta)$. *Suppose there exist analytic functions* $P_s(x,y,\delta)$ *and* $Q_s(x,y,\delta)$ *for* $s=1,2,\cdots,n$ *and* $(x,y)\in U$ *with*

$$U = U_1 \cup U_2, \quad U_1 = \bigcup_{h_{c_2} \le h \le 0} (L_h \cup \widetilde{L}_h), \quad U_2 = \bigcup_{0 \le h \le h_0} L_h^*, \ h_0 > 0,$$

such that the following equality holds

$$(P_{s-1})_x + (Q_{s-1})_y = H_x P_s(x, y, \delta) + H_y(x, y) Q_s(x, y, \delta), \tag{3.14}$$

for

$$\begin{aligned} b_0 = \cdots = b_{s-1} = 0, \ c_{\frac{3}{2}k+ki} = 0, \\ c_{k+1+2r+k*i} = 0, \quad r = 0, \cdots, \frac{k}{4} - \frac{3}{2}, \\ c_{2r+2+k*(i+1)} = 0, \quad r = \frac{k}{4} + \frac{1}{2}, \cdots, \frac{k}{2} - 1, \end{aligned} \tag{3.15}$$

if $\hat{m}$ *is even, or*

$$\begin{aligned} b_0 = \cdots = b_{s-1} = 0, \\ c_{k+1+2r+k*i} = 0, \quad r = 0, \cdots, \frac{k}{4} - 1, \\ c_{2r+2+k*(i+1)} = 0, \quad r = \frac{k}{4}, \cdots, \frac{k}{2} - 1, \end{aligned}$$

if $\hat{m}$ *is odd, where* $i = -1, 0, 1, \cdots, s-2$.

Suppose $P_s(x, y, \delta)$ *and* $Q_s(x, y, \delta)$ *can be written as the form,*

$$P_s(x, y, \delta) = \sum_{\substack{i+j \ge 0 \\ i+j \, is \, odd}} a_{ij}^{[s]} x^i y^j, \ Q_s(x, y, \delta) = \sum_{\substack{i+j \ge 0 \\ i+j \, is \, odd}} b_{ij}^{[s]} x^i y^j, \tag{3.16}$$

for (x, y) *near the origin. Then,*

$$\begin{aligned} c_{\frac{k}{2}+1+kn} &= \frac{1}{(n+1)!} \bar{c}_{\frac{k}{2}+1}^{[n]} + \sum_{r=0}^{\frac{k}{4}-\frac{1}{2}} O_1(c_{kn+1+2r}), \\ c^*_{\frac{k}{4}+\frac{3}{2}+(\frac{k}{2}+1)n} &= \frac{1}{(n+1)!} c^{*[n]}_{\frac{k}{4}+\frac{3}{2}} + \sum_{r=0}^{\frac{k}{4}-\frac{1}{2}} O_1(c_{kn+1+2r}), \end{aligned} \tag{3.17}$$

if $\hat{m}$ *is even, or*

$$\begin{aligned} c_{\frac{k}{2}+1+kn} &= \frac{1}{(n+1)!} \bar{c}_{\frac{k}{2}+1}^{[n]} + \sum_{r=0}^{\frac{k}{4}-1} O_1(c_{kn+1+2r}), \\ c^*_{\frac{k}{4}+1+(\frac{k}{2}+1)n} &= \frac{1}{(n+1)!} \bar{c}^{*[n]}_{\frac{k}{4}+1} + \sum_{r=0}^{\frac{k}{4}-1} O_1(c_{kn+1+2r}), \end{aligned} \tag{3.18}$$

if $\hat{m}$ is odd, where

$$\bar{c}^{[n]}_{\frac{k}{2}+1} = \oint\limits_{L_0} [(P_n)_x + (Q_n)_y - \sum_{i=0}^{\frac{k}{2}-1} \bar{b}^{[n]}_{i,0} x^i] \mathrm{d}t,$$

$$\bar{c}^{*[n]}_{\frac{k}{4}+1} = 2\oint\limits_{L_0} \big[(P_n)_x + (Q_n)_y - \sum_{i=0}^{\frac{k}{2}-2} \bar{b}^{[n]}_{i,0} x^i\big] \mathrm{d}t,$$

$$\bar{c}^{*}_{\frac{k}{4}+\frac{3}{2}} = 2\oint\limits_{L_0} \big[(P_n)_x + (Q_n)_y - \sum_{i=0}^{\frac{k}{2}-1} \bar{b}^{[n]}_{i,0} x^i\big] \mathrm{d}t,$$

and $\bar{b}^{[n]}_{i,0}$ satisfies (2.9), with $a_{i,j}$ and $b_{i,j}$ replaced by $a^{[n]}_{i,j}$ and $b^{[n]}_{i,j}$, respectively.

Proof. If $\hat{m}$ is even and (3.15) holds, by induction we can get

$$\begin{aligned}\frac{\partial^n M}{\partial h^n} &= n! c_{\frac{k}{2}+1+k(n-1)} + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} \eta^{[1]}_n(k,r) c_{kn+1+2r} |h|^{\frac{2r}{k}+\frac{1}{k}+\frac{1}{2}} \\ &\quad + (n+1)! c_{\frac{k}{2}+kn} h \ln|h| + [\eta^{[2]}_n c_{\frac{k}{2}+kn} + (n+1)! c_{\frac{k}{2}+1+kn}] h + O(|h|^{1+\frac{2}{k}}),\end{aligned} \tag{3.19}$$

where $\eta^{[1]}_n(k,r)$ is determined by k and r, $\eta^{[2]}_n$ is a constant, and

$$\frac{\partial^n M}{\partial h^n} = \oint\limits_{L_h} Q_n \mathrm{d}x - P_n \mathrm{d}y \equiv M_n(h,\delta).$$

So, $M_n(h,\delta)$ can be written as

$$\begin{aligned} M_n(h,\delta) &= c^{[n]}_0 + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} c^{[n]}_{2r+1} |h|^{\frac{2r}{k}+\frac{1}{k}+\frac{1}{2}} + c^{[n]}_{\frac{k}{2}} h \ln|h| \\ &\quad + c^{[n]}_{\frac{k}{2}+1} h + \sum_{r=\frac{k}{4}+\frac{1}{2}}^{\frac{k}{2}-1} c^{[n]}_{2r+2} |h|^{\frac{2r}{k}+\frac{1}{k}+\frac{1}{2}} \\ &\quad + \sum_{r=0}^{\frac{k}{4}-\frac{3}{2}} c^{[n]}_{k+1+2r} |h|^{\frac{2r+1}{k}+\frac{3}{2}} + c^{[n]}_{\frac{3}{2}k} h^2 \ln|h| + O(h^2). \end{aligned} \tag{3.20}$$

Further, by (2.7) and (2.8), we know that

$$c^{[n]}_{\frac{k}{2}+1} = \bar{c}^{[n]}_{\frac{k}{2}+1} + \sum_{r=0}^{\frac{k}{4}-\frac{1}{2}} O_1(c_{kn+1+2r}).$$

Comparing (3.19) and (3.20), we obtain that

$$c^{[n]}_{\frac{k}{2}+1} = \eta^{[2]}_n c_{\frac{k}{2}+kn} + (n+1)! c_{\frac{k}{2}+1+kn},$$

which yields the formula of $c_{\frac{k}{2}+1+kn}$ in (3.17). The formula of $c^*_{\frac{k}{4}+\frac{3}{2}+(\frac{k}{2}+1)n}$ can be obtained similarly.

If $\hat{m}$ is odd, the proof is similar to the above. This ends the proof. □

Especially, for $k = 4$, by (3.13) we directly get the expansions of M and M^* as follows:

$$\begin{aligned} M(h,\delta) &= c_0 + c_1|h|^{\frac{3}{4}} + \sum_{i\geq 1}\left(c_{4i-1}h^i + c_{4i}|h|^{\frac{4i+1}{4}} + c_{4i+1}|h|^{\frac{4i+3}{4}}\right), \\ M^*(h,\delta) &= 2c_0 + \frac{2\bar{A}_0}{\widetilde{A}_0}c_1 h^{\frac{3}{4}} + \sum_{i\geq 1}\left(c^*_{3i-1}h^i + (-1)^{i-1}\frac{2\bar{A}_1}{\widetilde{A}_2}c_{4i}h^{\frac{4i+1}{4}} + (-1)^i\frac{2\bar{A}_0}{\widetilde{A}_0}c_{4i+1}h^{\frac{4i+3}{4}}\right), \end{aligned} \tag{3.21}$$

where $\bar{A}_0 > 0$, $\bar{A}_1 < 0$, $\widetilde{A}_0 < 0$, $\widetilde{A}_2 > 0$, c_{4i} and c_{4i+1} for $i \geq 1$ satisfy (3.10), and

$$\begin{aligned} c_3 &= \oint_{L_0}[(p_x + q_y) - (a_{1,0} + b_{0,1})]\mathrm{d}t + O_1(c_1), \\ c^*_3 &= 2\oint_{L_0}[(p_x + q_y) - (a_{1,0} + b_{0,1})]\mathrm{d}t + O_1(c_1). \end{aligned} \tag{3.22}$$

Suppose the conditions in Theorem 3.1 hold with $k = 4$. Then, when

$$b_0 = b_1 = \cdots = b_{n-1} = 0, c_{4i+1} = c_{4i+4} = 0, \quad i = 0, 1, \cdots, n-1,$$

we have

$$c_{4n+3} = \frac{1}{(n+1)!}\bar{c}^{[n]}_3 + O_1(c_{4n+1}), \quad c^*_{3n+2} = \frac{2}{(n+1)!}\bar{c}^{*[n]}_2 + O_1(c_{4n+1}), \tag{3.23}$$

where

$$\begin{aligned} \bar{c}^{[n]}_3 &= \oint_{L_0}[(P_n)_x + (Q_n)_y - (a^{[n]}_{10} + b^{[n]}_{01})]\mathrm{d}t, \\ \bar{c}^{*[n]}_2 &= 2\oint_{L_0}[(P_n)_x + (Q_n)_y - (a^{[n]}_{10} + b^{[n]}_{01})]\mathrm{d}t. \end{aligned}$$

Theorem 3.2. *If there exists* δ_0 *such that*

$$\begin{gathered}
c_0(\delta_0) = c_1(\delta_0) = 0, \\
c_{4i-1}(\delta_0) = c_{4i}(\delta_0) = c_{4i+1}(\delta_0) = 0,\ i = 1, 2, \cdots, \bar{k}-1,\ \bar{k} \ge 2, \\
c_{4\bar{k}-1}(\delta_0) = 0,\ c_{4\bar{k}}(\delta_0) = 0,\ c_{4\bar{k}+1}(\delta_0) \ne 0, \\
b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{j-1}(\delta_0) = 0,\ b_j(\delta_0) \ne 0,\ j \ge \bar{k}-1,
\end{gathered} \tag{3.24}$$

$$\operatorname{rank} \frac{\partial(c_0, c_1, c_3, c_4, c_5, c_7, \cdots, c_{4\bar{k}}, b_0, b_1, \cdots, b_{j-1})}{\partial\delta}\bigg|_{\delta=\delta_0} = 3\bar{k}+1+j, \tag{3.25}$$

and

$$b_j(\delta_0)c_{4\bar{k}+1}(\delta_0) < 0\ \ (or\ b_j(\delta_0)c_{4\bar{k}+1}(\delta_0) > 0),$$

then, system (1.1) *can have* $8\bar{k}+2j+4$ (*or* $8\bar{k}+2j+2$) *limit cycles for some* (ε, δ) *near* $(0, \delta_0)$.

Proof. Under the conditions (3.24), we have

$$\begin{aligned}
M(h, \delta_0) &= c_{4\bar{k}+1}(\delta_0)|h|^{\frac{4\bar{k}+3}{4}} + O(h^{\bar{k}+1}),\ 0 < -h \ll 1, \\
M(h, \delta_0) &= b_j(\delta_0)(h-h_{c_2})^{\bar{k}+1} + O((h-h_{c_2})^{\bar{k}+2}),\ 0 < h-h_{c_2} \ll 1.
\end{aligned}$$

Thus, if $b_j(\delta_0)c_{4\bar{k}+1}(\delta_0) < 0$, then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$M(-\varepsilon_1, \delta_0)M(h_{c_2}+\varepsilon_2, \delta_0) < 0,$$

which means that $M(h, \delta_0)$ has a simple zero h_0^* with $h_0^* \in (h_{c_2}, 0)$. If $b_j(\delta_0)c_{4\bar{k}+1}(\delta_0) > 0$, we can not find a simple zero of $M(h, \delta_0)$ in $(h_{c_2}, 0)$.

Without loss of generality, suppose $(-1)^{\bar{k}}c_{4\bar{k}+1}(\delta_0) > 0$. By (3.25), c_0, c_1, c_3, c_4, c_5, c_7, $\cdots$, $c_{4\bar{k}}$ and $b_0, b_1, \cdots, b_{j-1}$ can be taken as free parameters. Let them satisfy

$$\begin{aligned}
0 < -c_0 \ll c_1 \ \ &\ll |c_3| \ll |c_4| \ll |c_5| \\
&\ll |c_7| \ll |c_8| \ll |c_9| \\
&\ll \cdots \\
&\ll |c_{4\bar{k}-5}| \ll |c_{4\bar{k}-4}| \ll |c_{4\bar{k}-3}| \\
&\ll |c_{4\bar{k}-1}| \ll |c_{4\bar{k}}| \ll |c_{4\bar{k}+1}(\delta_0)|,
\end{aligned}$$

and

$$0 < |b_0| \ll |b_1| \ll \cdots \ll |b_{j-1}| \ll |b_j(\delta_0)|,$$

with

$$\begin{gathered}
c_{4i-1} > 0,\ (-1)^i c_{4i} < 0,\ \ i = 1, 2, \cdots, \bar{k}, \\
(-1)^i c_{4i+1} > 0,\ \ i = 1, 2, \cdots, \bar{k}-1, \\
b_i b_{i-1} < 0,\ i = 1, 2, 3, \cdots, j-1,\ b_{j-1}b_j(\delta_0) < 0,
\end{gathered}$$

which ensure that $M(h,\delta)$ has $3\bar{k}+1$ simple zeros near $h=0$ and j simple zeros near h_{c_2}, $M^*(h,\delta)$ has $2\bar{k}$ simple zeros near $h=0$. If $b_{\bar{k}-1}(\delta_0)c_{4\bar{k}+1}(\delta_0)<0$, $M(h,\delta)$ has another simple zero near h_0^*.

By taking ε sufficiently small, for system (1.1), we can obtain $3\bar{k}+1$ limit cycles respectively near L_0 and $\widetilde{L}_0$, j limit cycles respectively near C_1 and C_2, and $2\bar{k}$ limit cycles near L_0^* for some (ε,δ) near $(0,\delta_0)$. Further, if $b_j(\delta_0)c_{4\bar{k}+1}(\delta_0)<0$, then system (1.1) has another 2 limit cycles, one of which surrounds the j limit cycles near C_2 and is surrounded by the $3\bar{k}+1$ limit cycles near L_0, the other one surrounds the j limit cycles near C_1 and is surrounded by the $3\bar{k}+1$ limit cycles near $\widetilde{L}_0$.

Thus, if $b_{\bar{k}j}(\delta_0)c_{4\bar{k}+1}(\delta_0)<0$ (or $b_j(\delta_0)c_{4\bar{k}+1}(\delta_0)>0$), for some (ε,δ) near $(0,\delta_0)$ system (1.1) can have $8\bar{k}+2j+4$ (or $8\bar{k}+2j+2$) limit cycles. □

Theorem 3.3. *Suppose there exists δ_0 such that*

$$\begin{aligned} &c_0(\delta_0)=c_1(\delta_0)=0,\\ &c_{4i-1}(\delta_0)=c_{4i}(\delta_0)=c_{4i+1}(\delta_0)=0,\ i=1,2,\cdots,\bar{k}-1,\ \bar{k}\geq 2,\\ &b_0(\delta_0)=b_1(\delta_0)=\cdots=b_{j-1}(\delta_0)=0,\ b_j(\delta_0)\neq 0,\ j\geq \bar{k}-1. \end{aligned} \tag{3.26}$$

(1) *If*

$$\begin{aligned} &c_{4\bar{k}-1}(\delta_0)=0,\ c_{4\bar{k}}(\delta_0)\neq 0,\\ &\operatorname{rank}\frac{\partial(c_0,c_1,c_3,c_4,c_5,c_7,\cdots,c_{4\bar{k}-1},b_0,b_1,\cdots,b_{j-1})}{\partial\delta}\bigg|_{\delta=\delta_0}=3\bar{k}+j, \end{aligned} \tag{3.27}$$

and

$$b_j(\delta_0)c_{4\bar{k}}(\delta_0)<0\ (\,or\ b_j(\delta_0)c_{4\bar{k}}(\delta_0)>0),$$

then, system (1.1) *can have* $8\bar{k}+2j+2$ *(or* $8\bar{k}+2j$*) limit cycles for some* (ε,δ) *near* $(0,\delta_0)$.

(2) *If*

$$\begin{aligned} &c_{4\bar{k}-1}(\delta_0)\neq 0,\\ &\operatorname{rank}\frac{\partial(c_0,c_1,c_3,c_4,c_5,c_7,\cdots,c_{4\bar{k}-3},b_0,b_1,\cdots,b_{j-1})}{\partial\delta}\bigg|_{\delta=\delta_0}=3\bar{k}+j-1, \end{aligned} \tag{3.28}$$

and

$$(-1)^{\bar{k}}b_j(\delta_0)c_{4\bar{k}-1}(\delta_0)<0\ (\,or\ (-1)^{\bar{k}}b_j(\delta_0)c_{4\bar{k}-1}(\delta_0)>0),$$

then, system (1.1) *can have* $8\bar{k}+2j-1$ *(or* $8\bar{k}+2j-3$*) limit cycles for some* (ε,δ) *near* $(0,\delta_0)$.

4. An application

Consider the system

$$\dot{x} = y, \quad \dot{y} = x^3(1-x^2) + \varepsilon \sum_{i=0}^{\hat{n}} a_i x^{2i} y, \ 11 \le \hat{n} \le 23, \tag{4.1}$$

which has the Hamiltonian function,

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{6}x^6.$$

The phase portrait of system $(4.1)|_{\varepsilon=0}$ is shown in Fig. 1 (a).

We apply the formulas derived in the previous section to obtain the coefficients in $M(h,\delta)$ and $M^*(h,\delta)$. By (2.8), we have

$$\begin{aligned} c_0 &= \sum_{i=0}^{\hat{n}} a_i \oint_{L_0} x^{2i} y \mathrm{d}x = \sum_{i=0}^{\hat{n}} 2a_i \int_0^{\frac{\sqrt{6}}{2}} x^{2i+2} \sqrt{\frac{1}{2} - \frac{1}{3}x^2} \mathrm{d}x, \\ c_3 &= \oint_{L_0} (\sum_{i=0}^{\hat{n}} a_i x^{2i} - a_0) \mathrm{d}t = \sum_{i=1}^{\hat{n}} 2a_i \int_0^{\frac{\sqrt{6}}{2}} \frac{x^{2i-2}}{\sqrt{\frac{1}{2} - \frac{1}{3}x^2}} \mathrm{d}x. \end{aligned} \tag{4.2}$$

Further, by (2.8), (3.11) and the programs in [16], we can obtain the coefficients c_1 and c_{4i}, c_{4i+1} for $i = 1, 2 \cdots, 5$. Since their expressions are long, we put them in Appendix A.

With a transformation of variables, $u = \sqrt{2}(x-1)$, $v = y$, and a time scaling $t \to \sqrt{2}t$, system (4.1) becomes

$$\dot{u} = v, \quad \dot{v} = \frac{\sqrt{2}}{2}\left(\frac{u}{\sqrt{2}} + 1\right)^3 \left(1 - (\frac{u}{\sqrt{2}} + 1)\right)^3 + \frac{\sqrt{2}}{2}\varepsilon \sum_{i=0}^{\hat{n}} a_i (\frac{u}{\sqrt{2}} + 1)^{2i} v.$$

When $\varepsilon = 0$, the above system has a Hamiltonian function,

$$H_1(u, v) = \frac{1}{2}(v^2 + u^2) + \frac{1}{48}u^6 + \frac{1}{8}\sqrt{2}u^5 + \frac{9}{16}u^4 + \frac{7}{12}\sqrt{2}u^3.$$

We consider the case $\hat{n} = 23$. Using the programs in [32], we can get the coefficients $b_l(\delta)$ in (2.10) (we put them in Appendix A since they are long).

From the equations $b_0 = c_1 = c_4 = 0$ we get

$$a_0 = 0, \ a_1 = 0, \ a_2 = -\sum_{i=3}^{23} a_i. \tag{4.3}$$

Then, by the formulas in [18], we obtain $P_1(x, y, \delta)$ and $Q_1(x, y, \delta)$ as follows:

$$P_1(x,y,\delta)|_{b_0=c_1=c_4=0} = \frac{\sum\limits_{i\geq 0}^{23} a_i x^{2i}}{-x^3(1-x^2)}\Big|_{b_0=c_1=c_4=0} = \sum_{j=3}^{23}\sum_{i=j}^{23} a_i x^{2j-5},\ \ Q_1(x,y,\delta)=0,$$

such that (3.14) holds for $s=1$. Thus, by (3.23) and (4.3), we obtain

$$\begin{aligned}
c_7|_{b_0=c_1=c_4=0} &= \oint_{L_0} [(P_1)_x - (P_1)_x|_{x=0}]\mathrm{d}t \\
&= \sqrt{3}\pi\Big(3\,a_4 + \frac{27}{4}a_5 + \frac{405}{32}a_6 + \frac{2835}{128}a_7 + \frac{76545}{2048}a_8 \\
&\quad + \frac{505197}{8192}a_9 + \frac{6567561}{65536}a_{10} \\
&\quad + \frac{42220035}{262144}a_{11} + \frac{2153221785}{8388608}a_{12} + \frac{13637071305}{33554432}a_{13} + \frac{171827098443}{268435456}a_{14} \\
&\quad + \frac{1077824526597}{1073741824}a_{15} + \frac{26945613164925}{17179869184}a_{16} + \frac{167891897412225}{68719476736}a_{17} \\
&\quad + \frac{2086656439266225}{549755813888}a_{18} + \frac{12937269923450595}{2199023255552}a_{19} \\
&\quad + \frac{12807897224216089 05}{140737488355328}a_{20} \\
&\quad + \frac{7910760050251113825}{562949953421312}a_{21} + \frac{97566040619763737175}{4503599627370496}a_{22} \\
&\quad + \frac{600801408026966171025}{18014398509481984}a_{23}\Big).
\end{aligned}$$

Based on (4.3) we solve the equations $b_1 = c_5 = c_8 = 0$ for a_3, a_4 and a_5, and get

$$\begin{aligned}
a_3 &= 0, \\
a_4 &= \frac{7}{5}a_6 + \frac{16}{5}a_7 + \frac{27}{5}a_8 + 8\,a_9 + 11\,a_{10} + \frac{72}{5}a_{11} + \frac{91}{5}a_{12} + \frac{112}{5}a_{13} \\
&\quad + 27\,a_{14} + 32\,a_{15} + \frac{187}{5}a_{16} + \frac{216}{5}a_{17} + \frac{247}{5}a_{18} + 56\,a_{19} \\
&\quad + 63\,a_{20} + \frac{352}{5}a_{21} + \frac{391}{5}a_{22} + \frac{432}{5}a_{23}, \\
a_5 &= -\frac{252}{5}a_{18} - \frac{12}{5}a_6 - \frac{21}{5}a_7 - \frac{32}{5}a_8 - 9\,a_9 - 12\,a_{10} - \frac{77}{5}a_{11} - \frac{96}{5}a_{12} \\
&\quad - \frac{117}{5}a_{13} - 28\,a_{14} - 33\,a_{15} - \frac{192}{5}a_{16} - \frac{221}{5}a_{17} - \frac{396}{5}a_{22} \\
&\quad - 57\,a_{19} - 64\,a_{20} - \frac{357}{5}a_{21} - \frac{437}{5}a_{23}.
\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3), we get $a_2 = 0$.

Similarly, for $b_0 = b_1 = c_1 = c_4 = c_5 = c_8 = 0$, we can find

$$P_2(x, y, \delta) = \frac{(P_1)_x}{x^3(x^2-1)}, \quad Q_2(x, y) = 0,$$

such that (3.14) holds with $s = 2$. Then, under the conditions (4.4) and (4.3), i.e., $b_0 = b_1 = c_1 = c_4 = c_5 = c_8 = 0$, we have

$$\begin{aligned}
c_{11}|_{b_0=b_1=c_1=c_4=c_5=c_8=0} &= \oint_{L_0} [(P_2)_x - (P_2)_x|_{x=0}]\mathrm{d}t \\
= \sqrt{3}\pi \Big(& 27\,a_7 + \frac{405}{4}\,a_8 + \frac{8505}{32}\,a_9 + \frac{76545}{128}\,a_{10} + \frac{2525985}{2048}\,a_{11} \\
& + \frac{19702683}{8192}\,a_{12} + \frac{295540245}{65536}\,a_{13} \\
& + \frac{2153221785}{262144}\,a_{14} + \frac{122733641745}{8388608}\,a_{15} + \frac{859135492215}{33554432}\,a_{16} + \frac{11856069792567}{268435456}\,a_{17} \\
& + \frac{80836839494775}{1073741824}\,a_{18} + \frac{2182594666358925}{17179869184}\,a_{19} \\
& + \frac{14606595074863575}{68719476736}\,a_{20} + \frac{194059048851758925}{549755813888}\,a_{21} \\
& + \frac{1280789722421608905}{2199023255552}\,a_{22} + \frac{134482920854268935025}{140737488355328}\,a_{23} \Big).
\end{aligned}$$

Similarly, for

$$b_0 = \cdots = b_{s-1} = 0, \;\; c_{4i+5} = c_{4i+8} = 0, i = -1, 0, \cdots, s-2,$$

we can find $P_s(x, y, \delta)$ and $Q_s(x, y, \delta)$ as follows:

$$P_s(x, y, \delta) = \frac{(P_{s-1})_x}{x^3(x^2-1)}, \quad Q_s(x, y, \delta) = 0, \;\; s = 3, 4, 5$$

such that (3.14) holds. Similarly, we obtain

$$\begin{aligned}
c_{15}|_{b_0=b_1=b_2=c_1=c_4=c_5=c_8=c_9=c_{12}=0} &= \oint_{L_0} [(P_3)_x - (P_3)_x|_{x=0}]\mathrm{d}t \\
= \sqrt{3}\pi \Big(& \frac{8505}{4}\,a_{11} + 405\,a_{10} + \frac{229635}{32}\,a_{12} + \frac{2525985}{128}\,a_{13} + \frac{98513415}{2048}\,a_{14} \\
& + \frac{886620735}{8192}\,a_{15} + \frac{15072552495}{65536}\,a_{16} + \frac{122733641745}{262144}\,a_{17} + \frac{7732219429935}{8388608}\,a_{18} \\
& + \frac{59280348962835}{33554432}\,a_{19} + \frac{889205234442525}{268435456}\,a_{20} + \frac{6547783999076775}{1073741824}\,a_{21}
\end{aligned}$$

$$
\left.+\frac{189885735973226475}{17179869184}\,a_{22}+\frac{1358413341962312475}{68719476736}\,a_{23}\right),
$$

$$
\begin{aligned}
&c_{19}|_{b_0=b_1=b_2=b_3=c_1=c_4=c_5=c_8=c_9=c_{12}=c_{13}=c_{16}=0}\\
&=\oint_{L_0}[(P_4)_x-(P_4)_x|_{x=0}]\mathrm{d}t\\
&=\sqrt{3}\pi\left(\frac{229635}{4}\,a_{14}+8505\,a_{13}+\frac{7577955}{32}\,a_{15}+\frac{98513415}{128}\,a_{16}+\frac{4433103675}{2048}\,a_{17}\right.\\
&\quad+\frac{45217657485}{8192}\,a_{18}+\frac{859135492215}{65536}\,a_{19}+\frac{7732219429935}{262144}\,a_{20}+\frac{533523140665515}{8388608}\,a_{21}\\
&\quad\left.+\frac{4446026172212625}{33554432}\,a_{22}+\frac{72025623989844525}{268435456}\,a_{23}\right),\\
&c_{23}|_{b_0=b_1=b_2=b_3=b_4=c_1=c_4=c_5=c_8=c_9=c_{12}=c_{13}=c_{16}=c_{17}=c_{20}=0}\\
&=\oint_{L_0}[(P_5)_x-(P_5)_x|_{x=0}]\mathrm{d}t\\
&=\sqrt{3}\pi\left(\frac{7577955}{4}\,a_{17}+229635\,a_{16}+\frac{295540245}{32}\,a_{18}+\frac{4433103675}{128}\,a_{19}\right.\\
&\quad+\frac{226088287425}{2048}\,a_{20}+\frac{2577406476645}{8192}\,a_{21}\\
&\quad\left.+\frac{54125536009545}{65536}\,a_{22}+\frac{533523140665515}{262144}\,a_{23}\right).
\end{aligned}
$$

For $\delta_0=(a_0,a_1,\cdots,a_{23})$ satisfying

$$
\begin{aligned}
&a_0=a_1=\cdots=a_{10}=0,\\
&a_{11}=\frac{112970457}{219152384}\,a_{23},\ a_{12}=\frac{41796853983}{2848980992}\,a_{23},\ a_{13}=-\frac{217348128711}{1424490496}\,a_{23},\\
&a_{14}=\frac{3044761832889}{7122452480}\,a_{23},\ a_{15}=-\frac{5582471316303}{14244904960}\,a_{23},\ a_{16}=-\frac{420059945607}{1294991360}\,a_{23},\\
&a_{17}=\frac{693245743852039}{596149272576}\,a_{23},\ a_{18}=-\frac{1472075464589699}{1117779886080}\,a_{23},\ a_{19}=\frac{3639524693975}{4299153408}\,a_{23},\\
&a_{20}=-\frac{16534088582225}{48466237248}\,a_{23},\ a_{21}=\frac{3174058863809}{36349677936}\,a_{23},\ a_{22}=-\frac{5620913}{416988}\,a_{23},
\end{aligned}
$$

we have

$$
\begin{gathered}
b_0(\delta_0)=b_1(\delta_0)=\cdots=b_4(\delta_0)=0,\\
c_0(\delta_0)=c_1(\delta_0)=c_2(\delta_0)=c_{4i-1}(\delta_0)=c_{4i}(\delta_0)=c_{4i+1}(\delta_0),\,i=1,2,\cdots,5,\ c_{23}(\delta_0)=0,
\end{gathered}
$$

and

$$
b_5(\delta_0)=\frac{3254293}{150994944}\sqrt{2}a_{23}\pi,\ c_{24}(\delta_0)=\frac{2215107}{8320}\,\widetilde{A}_2a_{23}.
$$

Note that $b_5(\delta_0)c_{24}(\delta_0) > 0$ if $a_{23} \neq 0$. By Theorem 3.3 (1) with $\bar{k} = 6$, $j = 5$, for some (ε, δ) near $(0, \delta_0)$, system (4.1) has 58 limit cycles. This proves that $H(46, 5) \geq 58$.

Similarly, for $\hat{n} = 22, 18, 14$, we can find $\delta_0^{[\hat{n}]}$ such that (3.26) and (3.28) hold with $(\bar{k}, j) = (6, 5)$, $(5, 4)$ and $(4, 3)$, respectively, and $(-1)^{\bar{k}} c_{4\bar{k}-1}(\delta_0) b_j(\delta_0) > 0$ if $a_{\hat{n}} \neq 0$. Thus, by Theorem 3.3 (2), for some (ε, δ) near $(0, \delta_0^{[\hat{n}]})$, system (4.1) has $8\bar{k} + 2j - 3$ limit cycles which means that

$$H(44, 5) \geq 55, \ H(36, 5) \geq 45, \ H(28, 5) \geq 35.$$

For $\hat{n} = 21, 17, 13$ (or $\hat{n} = 20, 16, 12$), we can find $\delta_0^{[\hat{n}]}$ such that (3.24) and (3.25) hold with $(\bar{k}, j) = (5, 5), (4, 4)$ and $(3, 3)$ (or $(\bar{k}, j) = (5, 4), (4, 3)$ and $(3, 2)$), respectively, and $c_{4\bar{k}+1}(\delta_0) b_j(\delta_0) > 0$ if $a_{\hat{n}} \neq 0$. By Theorem 3.2, for some (ε, δ) near $(0, \delta_0^{[\hat{n}]})$, system (4.1) has $8\bar{k} + 2j + 2$ limit cycles which means that

$$H(42, 5) \geq 52, \ H(34, 5) \geq 42, \ H(26, 5) \geq 32,$$
$$H(40, 5) \geq 50, \ H(32, 5) \geq 40, \ H(24, 5) \geq 30.$$

For $\hat{n} = 19, 15, 11$, we can find $\delta_0^{[\hat{n}]}$ such that (3.26) and (3.27) hold with $(\bar{k}, j) = (5, 4), (4, 3)$ and $(3, 2)$, respectively, and $(-1)^{\bar{k}} c_{4\bar{k}}(\delta_0) b_j(\delta_0) > 0$ if $a_{\hat{n}} \neq 0$. Thus, by Theorem 3.3 (1), for some (ε, δ) near $(0, \delta_0^{[\hat{n}]})$, system (4.1) has $8\bar{k} + 2j$ limit cycles which means that

$$H(38, 5) \geq 48, \ H(30, 5) \geq 38, \ H(22, 5) \geq 28.$$

Summarizing the above results gives the following theorem.

Theorem 4.1. *For the lower bounds of $H(n, 5)$ with $n = 2\hat{n}$ $(11 \leq \hat{n} \leq 23)$, we have*

$$H(2\hat{n}, 5) \geq \frac{5}{2}\hat{n}, \ \hat{n} = 12, 14, 16, 18, 20, 22,$$

$$H(2\hat{n}, 5) \geq \frac{5\hat{n} - 1}{2}, \ \hat{n} = 13, 17, 21, \ H(2\hat{n}, 5) \geq \frac{5\hat{n} + 1}{2}, \ \hat{n} = 11, 15, 19.$$

For system (1.6), by Theorem 4.1 we directly have

$$H(2\hat{n} + 1, 5) \geq H(2\hat{n}, 5), \ 11 \leq \hat{n} \leq 23,$$

which shows the lower bound of $H(n, 5)$ for $22 \leq n \leq 46$ odd.

Appendix A

The following Maple program can be used to calculate the coefficients $c_{4i}, c_{4i+1}, c_{4i+2}$ in (3.10) for $i \geq 1$.

```
> restart;with(LinearAlgebra):i:=6:
> for r from 0 to 3 do
>  beta[4*0+r]:=1:
```

```
>    for m from 1 to i do
>      beta[4*m+r]:=beta[4*(m-1)+r]*(-1)*((m-1)*4+r+1)/((2*m+1)*2+r+1):
>    od:
> od:
> for r from 0 to 3 do
>   for m from 0 to i do
>     alpha[m*4+r,0]:=1:
>       for j from 1 to i do
>         alpha[4*m+r,j]:=alpha[4*m+r,j-1]*((2*j+1)*4/2)/((2*j+1)*4/2+(4*m+r)+1):
>       od:
>   od:
> od:
> for n from 1 to i do
  c[4*n+2]:=0:c[4*n+1]:=0:c[4*(n+1)]:=0:
  c[4]:=tilA[2]*tilr[2,0]*alpha[2,0]*beta[2]:
>  for m from 0 to n do
>   c[4*n+2]:=c[4*n+2]-1/8*tilr[4*m+1,n-m]*alpha[4*m+1,n-m]*beta[4*m+1]:
>   c[4*n+1]:=c[4*n+1]+(-1)^n*tilA[0]*tilr[4*m,n-m]*alpha[4*m,n-m]*beta[4*m]:
>   c[4*(n+1)]:=c[4*(n+1)]+(-1)^(n)*tilA[2]*tilr[4*m+2,n-m]*alpha[4*m+2,n-m]
                *beta[4*m+2]:
>  od:
> od:
```

For system (4.1), the following are the formulas of some coefficients in the expansion of $M(h,\delta)$ for $0 < -h \ll 1$.

$$c_1 = 4\,\widetilde{A}_0 a_0,\ c_4 = \widetilde{A}_2\,(4\,a_0 + 8\,a_1)\,,\ c_5 = \widetilde{A}_0\left(\frac{10}{7}\,a_0 + \frac{40}{21}\,a_1 + \frac{16}{7}\,a_2\right),$$

$$c_8 = \widetilde{A}_2\left(\frac{770}{81}\,a_0 + \frac{308}{27}\,a_1 + \frac{112}{9}\,a_2 + \frac{32}{3}\,a_3\right),$$

$$c_9 = \widetilde{A}_0\left(\frac{1105}{198}\,a_0 + \frac{4420}{693}\,a_1 + \frac{520}{77}\,a_2 + \frac{480}{77}\,a_3 + \frac{320}{77}\,a_4\right),$$

$$c_{12} = \widetilde{A}_2\left(\frac{33649}{702}\,a_0 + \frac{168245}{3159}\,a_1 + \frac{58520}{1053}\,a_2 + \frac{6160}{117}\,a_3 + \frac{4928}{117}\,a_4 + \frac{896}{39}\,a_5\right),$$

$$c_{13} = \widetilde{A}_0\left(\frac{32045}{972}\,a_0 + \frac{32045}{891}\,a_1 + \frac{11050}{297}\,a_2 + \frac{3536}{99}\,a_3 + \frac{7072}{231}\,a_4 + \frac{1664}{77}\,a_5 + \frac{768}{77}\,a_6\right),$$

$$c_{16} = \widetilde{A}_2\left(\frac{5215595}{16524}\,a_0 + \frac{36509165}{107406}\,a_1 + \frac{2086238}{5967}\,a_2 + \frac{672980}{1989}\,a_3 + \frac{5383840}{17901}\,a_4 \right.$$
$$\left. + \frac{468160}{1989}\,a_5 + \frac{98560}{663}\,a_6 + \frac{39424}{663}\,a_7\right),$$

$$c_{17} = \widetilde{A}_0\left(\frac{243061325}{1034208}\,a_0 + \frac{48612265}{193914}\,a_1 + \frac{1185665}{4617}\,a_2 + \frac{128180}{513}\,a_3 + \frac{1281800}{5643}\,a_4 \right.$$
$$\left. + \frac{353600}{1881}\,a_5 + \frac{28288}{209}\,a_6 + \frac{113152}{1463}\,a_7 + \frac{39936}{1463}\,a_8\right),$$

$$c_{20} = \widetilde{A}_2\left(\frac{1505816785}{629856}\,a_0 + \frac{1505816785}{594864}\,a_1 + \frac{32038655}{12393}\,a_2 + \frac{10431190}{4131}\,a_3 + \frac{41724760}{17901}\,a_4\right.$$

$$+\frac{11921360}{5967}a_5+\frac{3076480}{1989}a_6+\frac{6152960}{5967}a_7+\frac{1070080}{1989}a_8+\frac{112640}{663}a_9\Big),$$

$$c_{21}=\widetilde{A}_0\Big(\frac{18035150315}{9657792}a_0+\frac{90175751575}{45874512}a_1+\frac{1701429275}{849528}a_2+\frac{486122650}{247779}a_3$$
$$+\frac{194449060}{106191}a_4+\frac{18970640}{11799}a_5+\frac{5127200}{3933}a_6+\frac{41017600}{43263}a_7$$
$$+\frac{2828800}{4807}a_8+\frac{1357824}{4807}a_9+\frac{2715648}{33649}a_{10}\Big).$$

For system (4.1), the following are the formulas of some coefficients in the expansion of $M(h,\delta)$ for $0<h-h_{c_2}\ll 1$.

$$b_0=\sqrt{2}\pi\sum_{i=0}^{23}a_i,$$

$$b_1=\frac{\pi}{24}\sqrt{2}\,(41\,a_0+5\,a_1-7\,a_2+5\,a_3+41\,a_4+101\,a_5+185\,a_6+293\,a_7$$
$$+425\,a_8+581\,a_9+761\,a_{10}+965\,a_{11}+1193\,a_{12}+1445\,a_{13}+1721\,a_{14}$$
$$+2021\,a_{15}+2345\,a_{16}+2693\,a_{17}+3065\,a_{18}+3461\,a_{19}+3881\,a_{20}$$
$$+4325\,a_{21}+4793\,a_{22}+5285\,a_{23}),$$

$$b_2=\frac{\pi}{1728}\sqrt{2}\,(14785\,a_0+1465\,a_1-959\,a_2+25\,a_3+385\,a_4-455\,a_5+385\,a_6$$
$$+9241\,a_7+35905\,a_8+93625\,a_9+199105\,a_{10}+372505\,a_{11}+637441\,a_{12}$$
$$+1020985\,a_{13}+1553665\,a_{14}+2269465\,a_{15}+3205825\,a_{16}+4403641\,a_{17}$$
$$+5907265\,a_{18}+7764505\,a_{19}+10026625\,a_{20}+12748345\,a_{21}$$
$$+15987841\,a_{22}+19806745\,a_{23}).$$

$$b_3=\frac{5\pi}{248832}\sqrt{2}\,(2867893\,a_0+246841\,a_1-112763\,a_2-6311\,a_3+25333\,a_4$$
$$-9191\,a_5-7931\,a_6+17017\,a_7-19019\,a_8+17017\,a_9+1015045\,a_{10}$$
$$+5409817\,a_{11}+18217717\,a_{12}+48322393\,a_{13}+110009221\,a_{14}$$
$$+224748601\,a_{15}+423228085\,a_{16}+747633337\,a_{17}+1254177925\,a_{18}$$
$$+2015881945\,a_{19}+3125599477\,a_{20}+4699294873\,a_{21}+6879567877\,a_{22}$$
$$+9839427577\,a_{23}),$$

$$\cdots\cdots.$$

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