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On the independent perturbation parameters and the number of limit cycles of a type of Liénard system [☆]

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we study a type of polynomial Liénard system of degree m $(m \geq 2)$ with polynomial perturbations of degree n. We prove that the first order Melnikov function of such system has at most $n+1-\left[\frac{n+1}{m+1}\right]$ independent perturbation parameters which can be used to simplify this kind of systems. As an application, we study a type of Lienard systems for $m = 4, n = 19, 28$ and obtain the new lower bounds of the maximal number of limit cycles.

1. Introduction and main results

The phenomenon of limit cycles is observed in almost all fields of science and engineering, and studying such phenomenon is not only significant in theoretical development but also important in applications. Limit cycle theory is closely related to the well-known Hilbert's 16th problem [14], which is one of the 23 mathematical problems presented by D. Hilbert in the Second International Congress of Mathematics in 1900. More precisely, the second part of the 16th problem is to find the maximal number and relative positions of limit cycles for the following planar polynomial differential system,

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \tag{1.1}$$

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where P_n and Q_n are nth-degree polynomials in x and y. This problem is extremely difficult as Smale [21] pointed out that "except for the Riemann hypothesis it seems to be the most elusive of Hilbert's problems". After more than 100 years, the problem is even not completely solved for the case $n = 2$, though many works have been published, see survey articles [4], [8], [15], [16], [18], [20] and references therein.

To overcome the difficulty in solving Hilbert's 16th problem, later Arnold [1] proposed the so-called weak Hilbert's 16th problem, described as follows. Consider the following near-Hamiltonian system,

$$\dot{x} = H_y + \varepsilon p(x, y, \varepsilon, \delta), \quad \dot{y} = -H_x + \varepsilon q(x, y, \varepsilon, \delta), \tag{1.2}$$

where H, p and q are C^∞ functions, ε is a small positive perturbation parameter, $0 < \varepsilon \ll 1$, and $\delta \in D \subset \mathbf{R}^m$ is a vector parameter with D compact. We assume that the level curves $H(x, y) = h, h_1 \le h \le h_2$ of the Hamiltonian system (1.2)$|_{\varepsilon=0}$ contain at least one family of closed orbits denoted by Γ_h. The weak Hilbert's 16th problem is to find the maximal number of isolated zeros of the Abelian integral or the first order Melnikov function [19],

$$M(h, \delta) = \oint_{\Gamma_h} q dx - p dy.$$

The number of zeros of the Melnikov function is closely related to the maximal number of limit cycles of system (1.2) (e.g. see [4], [16]).

In this paper we consider the polynomial Liénard system of the form,

$$\dot{x} = y, \quad \dot{y} = -g(x) - \varepsilon f(x) y, \tag{1.3}$$

where $0 < \varepsilon \ll 1$, $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x)$ is a polynomial in x with $\deg g(x) = m$. This system is a simplified version of Hilbert's 16th problem. It has been studied by many researchers to find the maximal number of limit cycles with respect to the degrees of $f(x)$ and $g(x)$. Let $H(n, m)$ denote the maximal number of limit cycles of system (1.3). The problem of finding $H(n, m)$ consists of two parts: Finding an upper bound and a lower bound of $H(n, m)$. But no upper bound has ever been found up to now. About the lower bound of $H(n, m)$, many works have been done, for example, see [2–6], [9–11], [13], [17], [23], [27]. In this paper, we will study the lower bound of $H(n, m)$ for some special values of n and m.

For system (1.3), suppose $g(x) = x(x - 1)(x - \alpha)(x - \beta)$, where $0 \le \alpha \le \beta \le 1$. Then, this system becomes

$$\dot{x} = y, \quad \dot{y} = -x(x - 1)(x - \alpha)(x - \beta) - \varepsilon f(x) y. \tag{1.4}$$

In [22], Xiao showed the bifurcation diagram and the related phase portraits of system (1.4)$|_{\varepsilon=0}$. In [24] we studied system (1.4) for $0 < \alpha = \beta < 1$ and $\frac{2}{5} < \alpha < 1, \beta = 1$ when $1 \le n \le 18$, and found that the coefficients a_4, a_9 and a_{14} have no effect on the number of limit cycles. This was identified when we study the number of limit cycles near the elementary center because the coefficients of the Melnikov function near the center can be explicitly expressed in terms of the coefficients. This raises a question: Is this generally true even for the limit cycles bifurcating not only near the center but also near the homoclinic loop and the cuspidal loop? However, this is not easy to prove because some coefficients in the expansion of the first order Melnikov function near the cuspidal loop and the homoclinic loop take approximate values. In this paper, we will give a rigorous proof for the above question without using the Melnikov function.

In [25] we considered bifurcation of limit cycles of system (1.4) for $1 \le n \le 27$ when $\alpha = \frac{2}{5}, \beta = 1$. For these particular values α and β, this system has a heteroclinic loop with a hyperbolic saddle and a nilpotent cusp. The coefficients of the first order Melnikov function near the heteroclinic loop take accurate values. We found that the coefficients a_4, a_9, a_{14}, a_{19} and a_{24} do not affect the number of limit cycles.

Now, we want to ask if all the coefficients $a_{5i+4}, i = 0, 1, 2, \dots$, with $i \leq \frac{n-4}{5}$ have no effect on the number of limit cycles of systems (1.4) for any n. If it is true, the analysis on bifurcation of limit cycles for system (1.4) will be greatly simplified.

For system $(1.3)|_{\varepsilon=0}$, we suppose that it has at least a singular point. And without loss of generality, we can assume it is at the origin. Then $g(x)$ can be written as

$$g(x) = r_m x^m + r_{m-1}x^{m-1} + \cdots + r_1 x, \tag{1.5}$$

where $r_i \in \mathbf{R}, i = 1, 2, \cdots, m, r_m \neq 0$. Further, we suppose system $(1.3)|_{\varepsilon=0}$ has at least a family of periodic orbits L_h defined by the equation $H(x, y) = h$.

In this paper, we first assume

$$m \geq 2, \quad r_1^2 + r_2^2 + \cdots + r_{m-1}^2 \neq 0. \tag{1.6}$$

We prove that the first order Melnikov function of system (1.3) has at most $n + 1 - \left[\frac{n+1}{m+1}\right]$ independent perturbation parameters, and one may assume $a_{(m+1)i+m} = 0$ for $i = 0, 1, 2, \dots, \left[\frac{n+1}{m+1}\right] - 1, n \geq m$. Applying this result to system (1.4), one can simply set $a_{5i+4} = 0, i = 0, 1, 2, \dots, \left[\frac{n+1}{5}\right] - 1$ for any $n \geq 4$.

Our main result is given in the following theorem.

Theorem 1.1. *Let (1.5) and (1.6) hold. Then for system (1.3), the first order Melnikov function $M(h, \delta)$ has at most $n + 1 - \left[\frac{n+1}{m+1}\right]$ independent perturbation parameters. Further, if $n \geq m$, we may assume $a_{(m+1)i+m} = 0, i = 0, 1, 2, \dots, \left[\frac{n+1}{m+1}\right] - 1$.*

It easily follows Theorem 1.1 to obtain the following corollary.

Corollary 1.1. *Consider system*

$$\dot{x} = y - \varepsilon \sum_{i=1}^{n} a_i x^i, \quad \dot{y} = -g(x), \tag{1.7}$$

where $g(x)$ satisfies (1.5) and (1.6). The first order Melnikov function $M(h, \delta)$ has at most $n - \left[\frac{n}{m+1}\right]$ independent perturbation parameters. And one may assume $a_{(m+1)(i+1)} = 0, i = 0, 1, 2, \dots, \left[\frac{n}{m+1}\right] - 1$ if $n \geq m$.

The proof of Theorem 1.1 will be given in Section 2.

As an application, in Section 3 we first simplify system (1.4) in the case $\frac{2}{5} < \alpha < 1, \beta = 1$ by using Theorem 1.1, and then obtain the new lower bounds of $H(n, 4)$ for $n = 19$ and 28. The result is as follows.

Theorem 1.2. *Consider system (1.4) with $\frac{2}{5} < \alpha < 1, \beta = 1$. Let $H^*(n, 4)$ denote the maximal number of limit cycles of system (1.4). We have $H^*(19, 4) \geq 17, H^*(28, 4) \geq 25$.*

For a comparison, we next introduce some existing results on the lower bound of $H(n, 4)$. Christopher and Lynch [3] obtained $H(9, 4) \geq 9$. Later, Yu and Han [27] got $H(n, 4) \geq n, n = 10, 11, \dots, 14$. In 2011, Yang and Han [24] showed that $H(n, 4) \geq n + 4 - \left[\frac{n+1}{5}\right]$, for $3 \leq n \leq 18$. Recently, Yang and Zhou [26] obtained that $H(20,4) \geq 21$ and $H(n,4) \geq n+4-\left[\frac{n+1}{5}\right]$ for $21 \leq n \leq 24$. A more general result obtained by Han and Romanovski [10] is as follows:

$$H(n,4) \geq H(n,3) \geq 2\left[\frac{n-1}{4}\right]+\left[\frac{n-1}{2}\right], \quad n \geq 3, \tag{1.8}$$

which gives some new estimations on the lower bound of $H(n,4)$ for $n=19$ and $n \geq 25$.

For system (1.4) with $\alpha=\frac{8}{15}, \beta=1$ and $19 \leq n \leq 32$, we don't find more limit cycles than the ones given in [10]. But for $n=19$ and $n=28$, by Theorem 1.2 it is easy to see that we give two new estimations on $H(n,4)$ which are the same as the newest results given in [10].

The proof of Theorem 1.2 will be given in Section 3.

2. The proof of Theorem 1.1

Consider system (1.3), where $g(x)$ satisfies (1.5) and (1.6). System (1.3) can be transformed into the following form by taking a suitable linear transformation and time scaling,

$$\dot{x}=y, \quad \dot{y}=-(\gamma x^m+\gamma_{m-1}x^{m-1}+\cdots+\gamma_1 x)-\varepsilon f(x)y, \tag{2.1}$$

where $\gamma=1$ or -1, $\gamma_i \in \mathbf{R}, i=1,2,\cdots,m-1$ and $\gamma_1^2+\gamma_2^2+\cdots+\gamma_{m-1}^2 \neq 0$. At $\varepsilon=0$, system (2.1) becomes

$$\dot{x}=y, \quad \dot{y}=-(\gamma x^m+\gamma_{m-1}x^{m-1}+\cdots+\gamma_1 x), \tag{2.2}$$

which is a Hamiltonian system, with the Hamiltonian function,

$$H(x,y)=\frac{y^2}{2}+h_2x^2+h_3x^3+\cdots+h_mx^m+\frac{\gamma}{m+1}x^{m+1},$$

where $h_j=\frac{\gamma_{j-1}}{j} \in \mathbf{R}, j=2,3,4,\ldots,m$. It is obvious that $h_2^2+h_3^2+\cdots+h_m^2 \neq 0$. Note that we have assumed system $(1.3)|_{\varepsilon=0}$ has at least a family of periodic orbits L_h with clockwise orientation defined by the equation $H(x,y)=h$. Thus, the equation $H(x,y)=h$ can be rewritten as

$$h-\frac{y^2}{2}=h_2x^2+h_3x^3+h_4x^4+\cdots+h_mx^m+\frac{\gamma}{m+1}x^{m+1}. \tag{2.3}$$

Along the orbit L_h we have $\oint_{L_h}\left(h-\frac{y^2}{2}\right)dy=0$. Therefore,

$$\oint\limits_{L_h}\left(h_2x^2+h_3x^3+h_4x^4+\cdots+h_mx^m+\frac{\gamma}{m+1}x^{m+1}\right)dy=0. \tag{2.4}$$

Applying Green's formula twice to the left-hand side of (2.4) we obtain

$$\begin{aligned}0&=\oint\limits_{L_h}\left(h_2x^2+h_3x^3+h_4x^4+\cdots+h_mx^m+\frac{\gamma}{m+1}x^{m+1}\right)dy\\&=-\iint\limits_{H\leq h}\left(2h_2x+3h_3x^2+4h_4x^3+\cdots+mh_mx^{m-1}+\gamma x^m\right)dxdy\end{aligned}$$

$$= -\oint_{L_h} \left(2h_2x + 3h_3x^2 + 4h_4x^3 + \cdots + mh_mx^{m-1} + \gamma x^m\right) ydx$$
$$= -\sum_{i=1}^{m-1}(i+1)h_{i+1}\oint_{L_h} x^i ydx - \gamma \oint_{L_h} x^m ydx,$$

which yields

$$\oint_{L_h} x^m ydx = \sum_{i=1}^{m-1} s_{0i} \oint_{L_h} x^i ydx, \tag{2.5}$$

where $s_{0i} = -\gamma(i+1)h_{i+1}$. Note that $h_2^2 + h_3^2 + \cdots + h_m^2 \neq 0$, and so it is obvious that $\sum\limits_{i=1}^{m-1} s_{0i}^2 \neq 0$.

Further, it follows from (2.3) that

$$\left(h - \frac{y^2}{2}\right)^{l+1} = \left(h_2x^2 + h_3x^3 + h_4x^4 + \cdots + h_mx^m + \frac{\gamma}{m+1}x^{m+1}\right)^{l+1},$$

where $l = 1, 2, \ldots, \left[\frac{n+1}{m+1}\right] - 1$ with $n \geq 2m+1$. Similarly, we have

$$\oint_{L_h} \left(h_2x^2 + h_3x^3 + h_4x^4 + \cdots + h_mx^m + \frac{\gamma}{m+1}x^{m+1}\right)^{l+1} dy = 0 \tag{2.6}$$

since $\oint_{L_h} \left(h - \frac{y^2}{2}\right)^{l+1} dy = 0$. Again applying Green's formula twice to the above equation we have

$$\begin{aligned}
0 &= \oint_{L_h} \left(h_2x^2 + h_3x^3 + h_4x^4 + \cdots + h_mx^m + \frac{\gamma}{m+1}x^{m+1}\right)^{l+1} dy \\
&= \oint_{L_h} \left(h_2^{l+1}x^{2(l+1)} + \sum_{i=2l+2}^{l(m+1)+m-1} h_{l,i}^* x^{i+1} + \frac{\gamma^{l+1}}{(m+1)^{l+1}}x^{(l+1)(m+1)}\right) dy \\
&= \oint_{L_h} \left(\sum_{i=2l+1}^{l(m+1)+m-1} h_{l,i}^* x^{i+1} + \frac{\gamma^{l+1}}{(m+1)^{l+1}}x^{(l+1)(m+1)}\right) dy \\
&= -\iint_{H\leq h} \left(\sum_{i=2l+1}^{l(m+1)+m-1} (i+1)h_{l,i}^* x^i + \frac{l+1}{(m+1)^l}\gamma^{l+1}x^{l(m+1)+m}\right) dxdy \\
&= -\oint_{L_h} \left(\sum_{i=2l+1}^{l(m+1)+m-1} (i+1)h_{l,i}^* x^i + \frac{l+1}{(m+1)^l}\gamma^{l+1}x^{l(m+1)+m}\right) ydx \\
&= -\sum_{i=2l+1}^{l(m+1)+m-1} (i+1)h_{l,i}^* \oint_{L_h} x^i ydx - \frac{l+1}{(m+1)^l}\gamma^{l+1}\oint_{L_h} x^{l(m+1)+m}ydx,
\end{aligned} \tag{2.7}$$

where $h_{l,i}^*$ is a polynomial in $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ for each i and $\sum\limits_{i=2l+1}^{l(m+1)+m-1} {h_{l,i}^*}^2 \neq 0$. Then it follows from (2.7) that

$$\oint_{L_h} x^{l(m+1)+m} y dx = -\frac{(m+1)^l}{(l+1)\gamma^{l+1}} \sum_{i=2l+1}^{(m+1)\,l+m-1} (i+1) h^*_{l,i} \oint_{L_h} x^i y dx. \tag{2.8}$$

Note that $\gamma = 1$ or -1. Letting $l = 1$ in (2.8) results in

$$\begin{aligned}\oint_{L_h} x^{2m+1} y dx &= -\frac{m+1}{2} \sum_{i=3}^{2m} (i+1) h^*_{1,i} \oint_{L_h} x^i y dx \\ &= -\frac{m+1}{2} \left(\sum_{\substack{i=3,\\ i\neq m}}^{2m} (i+1) h^*_{1,i} \oint_{L_h} x^i y dx + (m+1) h^*_{1,m} \oint_{L_h} x^m y dx \right).\end{aligned}$$

Then substituting (2.5) into the above equation we obtain

$$\oint_{L_h} x^{2m+1} y dx = \sum_{\substack{i=1,\\ i\neq m}}^{2m} s_{1i} \oint_{L_h} x^i y dx, \tag{2.9}$$

where s_{1i} is a polynomial in $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ for each i and $\sum\limits_{\substack{i=1,\\ i\neq m}}^{2m} s_{1i}^2 \neq 0$.

For $l = 2$, we get from (2.8) that

$$\begin{aligned}\oint_{L_h} x^{3m+2} y dx &= -\frac{(m+1)^2}{3\gamma^3} \sum_{i=5}^{3m+1} (i+1) h^*_{2,i} \oint_{L_h} x^i y dx \\ &= -\frac{(m+1)^2}{3\gamma^3} \left[\sum_{\substack{i=5,\\ i\neq m,\, 2m+1}}^{3m+1} (i+1) h^*_{2,i} \oint_{L_h} x^i y dx + (m+1) h^*_{2,m} \oint_{L_h} x^m y dx \right. \\ &\quad \left. + (2m+2) h^*_{2,2m+1} \oint_{L_h} x^{2m+1} y dx \right],\end{aligned} \tag{2.10}$$

which, together with (2.5) and (2.9) yields

$$\oint_{L_h} x^{3m+2} y dx = \sum_{\substack{i=1,\\ i\neq m,\, 2m+1}}^{3m+1} s_{2i} \oint_{L_h} x^i y dx, \tag{2.11}$$

where s_{2i} is a polynomial in $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ for each i and $\sum\limits_{\substack{i=1,\\ i\neq m,\, 2m+1}}^{3m+1} s_{2i}^2 \neq 0$.

Carrying out the above procedure for $l = 3, 4, 5, \ldots, \left[\frac{n+1}{m+1}\right] - 1$ $(n \geq 4m+3)$ we have

$$\oint_{L_h} x^{(m+1)l+m} y dx = \sum_{\substack{i=1\\ i\neq (m+1)(j-1)+m,\\ j=1,2,\cdots,l}}^{(m+1)l+m-1} s_{li} \oint_{L_h} x^i y dx, \tag{2.12}$$

where s_{li} is a polynomial in $\gamma_1, \gamma_2, \ldots, \gamma_{m-2}$ for each i and $\sum_{\substack{i=1 \\ i\neq(m+1)(j-1)+m, \\ j=1,2,\cdots,l}}^{(m+1)l+m-1} s_{li}^2 \neq 0$.

Summarizing the above results, we obtain (2.12) holds for $l = 0, 1, 2, 3, \ldots, \left[\frac{n+1}{m+1}\right] - 1$ with $n \geq m$.

Let $k = \left[\frac{n+1}{m+1}\right] - 1$. Then, for $n \geq m$ we have

$$(m+1)k + m \leq (m+1)\left(\frac{n+1}{m+1} - 1\right) + m = n.$$

Further, by (2.12) the first order Melnikov function of system (2.1) can be written as

$$\begin{aligned} M(h,\delta) &= -\oint_{L_h} \sum_{i=0}^{n} a_i x^i y dx = -\sum_{i=0}^{n} a_i \oint_{L_h} x^i y dx \\ &= -\sum_{\substack{i=(m+1)r+m, \\ r=0,1,2,\cdots,k}} a_i \oint_{L_h} x^i y dx - \sum_{\substack{i=0, \\ i\neq(m+1)r+m, \\ r=0,1,2,\cdots,k}}^{n} a_i \oint_{L_h} x^i y dx \\ &= -\sum_{\substack{i=0, \\ i\neq(m+1)r+m, \\ r=0,1,2,\cdots,k}}^{n} \bar{a}_i \oint_{L_h} x^i y dx, \end{aligned} \tag{2.13}$$

where

$$\bar{a}_i = \begin{cases} a_i, & i = 0 \text{ or } (m+1)k + m + 1 \leq i \leq n, \\ a_i + \sum\limits_{r=0}^{k} a_{(m+1)r+m} s_{ri}, & 1 \leq i \leq m-1, \\ a_i + \sum\limits_{r=j}^{k} a_{(m+1)r+m} s_{ri}, & (m+1)j \leq i \leq (m+1)j + m - 1, j = 1, 2, \cdots, k. \end{cases}$$

Obviously, for $n \geq m$, the expression in (2.13) contains at most $n-k$ independent perturbation parameters and one may assume $a_{(m+1)i+m} = 0, i = 0, 1, 2, \cdots, \left[\frac{n+1}{m+1}\right] - 1$. If $n < m$, it is obvious that system (2.1) has at most $n+1$, i.e., $n-k$ independent perturbation parameters.

The proof of Theorem 1.1 is complete.

Corollary 1.1 can be directly proved by using Theorem 1.1 since system (1.7) has the same first order Melnikov function as the following system

$$\dot{x} = y, \quad \dot{y} = -g(x) - \varepsilon \sum_{i=0}^{n-1} b_i x^i y,$$

where $b_i = (i+1)a_{i+1}$.

From the proof of Theorem 1.1, it can be seen that Theorem 1.1 also holds if $g(x) = r_m x^m, r_m \neq 0, m \geq 1$ under the assumption that system (1.5) has at least a family of periodic orbits. And so is Corollary 1.1.

3. The proof of Theorem 1.2

Consider system (1.4) with $\frac{2}{5} < \alpha < 1, \beta = 1$. It can be shown that the phase portraits of system (1.4) are qualitatively same for any values of $\frac{2}{5} < \alpha < 1$ by [22]. Thus, for simplicity, we take $\alpha = \frac{8}{15}$ under which system (1.4) can be written as

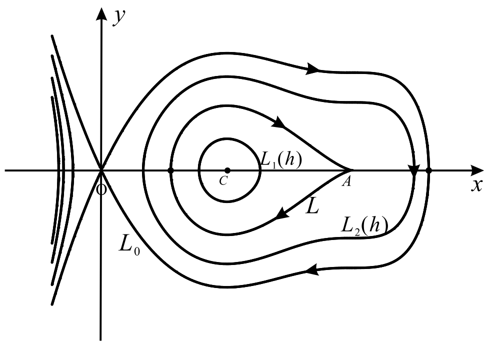

Fig. 1. The phase portrait of system (3.1)$|_{\varepsilon=0}$.

$$\dot{x} = y, \quad \dot{y} = -x(x - \frac{8}{15})(x-1)^2 - \varepsilon y f(x). \tag{3.1}$$

System (3.1)$|_{\varepsilon=0}$ has the Hamiltonian function,

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{5}x^5 - \frac{19}{30}x^4 + \frac{31}{45}x^3 - \frac{4}{15}x^2,$$

with the phrase portrait shown in Fig. 1.

It can be seen that system (3.1) has three singular points: a hyperbolic saddle $O(0,0)$, an elementary center $C(\frac{8}{15},0)$ and a cusp $A(1,0)$. Let

$$h_o \equiv H(0,0) = 0, \quad h_c \equiv H(\frac{8}{15},0) = -\frac{5888}{421875}, \quad h_a \equiv H(1,0) = -\frac{1}{90}.$$

The cuspidal loop L and the homoclinic loop L_0 are defined by the equations $H(x,y) = h_a$ and $H(x,y) = 0$, respectively. The equation $H(x,y) = h, h \in (h_c, h_a)$ (or $H(x,y) = h, h \in (h_a, 0)$) defines a family of periodic orbits $L_1(h)$ (or $L_2(h)$). The corresponding two Melnikov functions are given by

$$M_1(h,\delta) = \oint_{L_1(h)} q dx - p dy, \quad h \in (h_c, h_a),$$
$$M_2(h,\delta) = \oint_{L_2(h)} q dx - p dy, \quad h \in (h_a, 0).$$

In the following, we prove Theorem 1.2.

First, take $n = 28$. By Theorem 1.1 one may assume $a_{5i+4} = 0, i = 0,1,2,3,4$. In order to study the number of limit cycles of system (3.1) we first give the expansions with some coefficients for the first order Melnikov functions $M_1(h,\delta)$ and $M_2(h,\delta)$.

For (x,y) near the cusp point $(1,0)$, using the formulas given in [13] we can expand $M_1(h,\delta)$ for $0 < h_a - h \ll 1$ and $M_2(h,\delta)$ for $0 < h - h_a \ll 1$ as

$$\begin{aligned} M_1(h,\delta) &= c_0 + B_{00}c_1|h-h_a|^{\frac{5}{6}} + (c_2 + O(c_1))(h-h_a) + B_{10}c_3|h-h_a|^{\frac{7}{6}} \\ &\quad -\tfrac{B_{00}}{11}c_4|h-h_a|^{\frac{11}{6}} + O(|h-h_a|^2), \\ M_2(h,\delta) &= c_0 + B^*_{00}c_1(h-h_a)^{\frac{5}{6}} + (c_2 + O(c_1))(h-h_a) + B^*_{10}c_3(h-h_a)^{\frac{7}{6}} \\ &\quad +\tfrac{B^*_{00}}{11}c_4(h-h_a)^{\frac{11}{6}} + O(|h-h_a|^2), \end{aligned} \tag{3.2}$$

where $B_{00} > 0, B_{10} > 0, B^*_{00} < 0, B^*_{10} < 0$ are constants.

With the aid of Maple, we apply the formulas (3.22) and (3.23) in [24] to directly get

$$c_0 = \sum_{i=0}^{28} a_i J_i, \quad c_2 = \sum_{i=1}^{28} a_i \bar{J}_i,$$

where

$$J_i = -\oint_L x^i y dx = -\tfrac{2}{15}\int_{\frac{1}{3}}^{1} x^i(1-x)f_2(x)dx,$$
$$\bar{J}_i = -\oint_L (x^i-1)dt = -\oint_L \tfrac{x^i-1}{y}dx = -30\int_{\frac{1}{3}}^{1}\tfrac{x^i-1}{(1-x)f_2(x)}dx,$$
$$f_2(x) = \sqrt{-90x^3+105x^2-10x-5},$$

with

$$J_0 = \left(\tfrac{2}{189}s_1 - \tfrac{286}{14175}s_2\right)\sqrt{105},\quad J_1 = \left(\tfrac{38}{25515}s_1 - \tfrac{382}{76545}s_2\right)\sqrt{105},$$
$$J_2 = \left(\tfrac{656}{841995}s_1 - \tfrac{3352}{1148175}s_2\right)\sqrt{105},\cdots,$$
$$\bar{J}_1 = \tfrac{4}{7}s_1\sqrt{105},\quad \bar{J}_2 = \left(\tfrac{10}{21}s_1 + \tfrac{2}{3}s_2\right)\sqrt{105},\quad \bar{J}_3 = \left(\tfrac{8}{21}s_1 + \tfrac{32}{27}s_2\right)\sqrt{105},\cdots$$

and $s_1 = \text{EllipticK}\left(\frac{2}{7}\sqrt{7}\right), s_2 = \text{EllipticE}\left(\frac{2}{7}\sqrt{7}\right)$. To obtain c_1 and c_3 we introduce a change of variables in the form of

$$u = 1-x,\quad v = -y,$$

under which system (1.4)$|_{\beta=1}$ becomes

$$\dot{u} = v,\quad \dot{v} = (1-\alpha)u^2 + (\alpha-2)u^3 + u^4 - \varepsilon v f(1-u). \tag{3.3}$$

The Hamiltonian function of system (3.3)$|_{\varepsilon=0}$ is

$$H(u,v) = \frac{1}{2}v^2 + \frac{\alpha-1}{3}u^3 + \frac{2-\alpha}{4}u^4 - \frac{1}{5}u^5.$$

Now we can apply the programs in [13] to obtain the coefficients c_1, c_3 and c_4, given as follows:

$$\begin{aligned}
c_1 &= -2\sqrt{2}\left(\tfrac{45}{7}\right)^{\frac{1}{3}}\sum_{i=0}^{28} a_i,\\
c_3 &= \left(\tfrac{45}{7}\right)^{\frac{2}{3}}\tfrac{30\sqrt{2}}{7}\big(\tfrac{11}{15}a_0 + \tfrac{4}{15}a_1 - \tfrac{1}{5}a_2 - \tfrac{2}{3}a_3 - \tfrac{8}{5}a_5 - \tfrac{31}{15}a_6 - \tfrac{38}{15}a_7 - 3a_8 - \tfrac{59}{15}a_{10}\\
&\quad -\tfrac{22}{5}a_{11} - \tfrac{73}{15}a_{12} - \tfrac{16}{3}a_{13} - \tfrac{94}{15}a_{15} - \tfrac{101}{15}a_{16} - \tfrac{36}{5}a_{17} - \tfrac{23}{3}a_{18} - \tfrac{43}{5}a_{20} - \tfrac{136}{15}a_{21}\\
&\quad -\tfrac{143}{15}a_{22} - 10a_{23} - \tfrac{164}{15}a_{25} - \tfrac{57}{5}a_{26} - \tfrac{178}{15}a_{27} - \tfrac{37}{3}a_{28}\big),\\
c_4 &= 7^{\frac{2}{3}}45^{\frac{1}{3}}\tfrac{101250}{16807}\sqrt{2}\big(\tfrac{27181}{3375}a_0 + \tfrac{12922}{3375}a_1 + \tfrac{5131}{3375}a_2 + \tfrac{14}{27}a_3 - \tfrac{14}{3375}a_5 - \tfrac{2513}{3375}a_6 - \tfrac{8834}{3375}a_7\\
&\quad -\tfrac{4207}{675}a_8 - \tfrac{71309}{3375}a_{10} - \tfrac{113498}{3375}a_{11} - \tfrac{169799}{3375}a_{12} - \tfrac{48454}{675}a_{13} - \tfrac{443954}{3375}a_{15} - \tfrac{577283}{3375}a_{16}\\
&\quad -\tfrac{735014}{3375}a_{17} - \tfrac{183841}{675}a_{18} - \tfrac{1375199}{3375}a_{20} - \tfrac{1651118}{3375}a_{21} - \tfrac{1961729}{3375}a_{22} - \tfrac{461818}{675}a_{23}\\
&\quad -\tfrac{3122294}{3375}a_{25} - \tfrac{3592253}{3375}a_{26} - \tfrac{4107194}{3375}a_{27} - \tfrac{186767}{135}a_{28}\big).
\end{aligned}$$

Next for (x,y) near $C(\frac{8}{15},0)$, we similarly use the formulas given in [7] to write the Melnikov function $M_1(h,\delta)$ as

$$M_1(h,\delta) = \sum_{j\ge 0} b_j(\delta)(h-h_c)^{j+1},\quad 0 < h-h_c \ll 1, \tag{3.4}$$

where the formulas of b_j can be similarly obtained from [24] by executing the Maple programs in [12]. We list the formulas of b_0, and b_1 below and omit other lengthy expressions.

$$
\begin{aligned}
b_0 = & -\sqrt{30}\,\pi\Big(\tfrac{15}{14}a_0 + \tfrac{4}{7}a_1 + \tfrac{32}{105}a_2 + \tfrac{256}{1575}a_3 + \tfrac{16384}{354375}a_5 + \tfrac{131072}{5315625}a_6 + \tfrac{1048576}{79734375}a_7 \\
& + \tfrac{8388608}{1196015625}a_8 + \tfrac{536870912}{269103515625}a_{10} + \tfrac{4294967296}{4036552734375}a_{11} + \tfrac{34359738368}{60548291015625}a_{12} \\
& + \tfrac{274877906944}{908224365234375}a_{13} + \tfrac{17592186044416}{204350482177734375}a_{15} + \tfrac{140737488355328}{3065257232666015625}a_{16} \\
& + \tfrac{1125899906842624}{45978858489990234375}a_{17} + \tfrac{9007199254740992}{689682877349853515625}a_{18} + \tfrac{576460752303423488}{155178647403717041015625}a_{20} \\
& + \tfrac{4611686018427387904}{2327679711055755615234375}a_{21} + \tfrac{36893488147419103232}{34915195665836334228515625}a_{22} \\
& + \tfrac{295147905179352825856}{523727934987545013427734375}a_{23} + \tfrac{18889465931478580854784}{117838785372197628021240234375}a_{25} \\
& + \tfrac{151115727451828646838272}{1767581780582964420318603515625}a_{26} + \tfrac{1208925819614629174706176}{26513726708744466304779052734375}a_{27} \\
& + \tfrac{9671406556917033397649408}{397705900631166994571685791015625}a_{28}\Big), \\
b_1 = & -\sqrt{30}\,\pi\Big(\tfrac{2357859375}{68841472}a_0 + \tfrac{252871875}{8605184}a_1 + \tfrac{28198125}{1075648}a_2 + \tfrac{2966625}{134456}a_3 + \tfrac{218040}{16807}a_5 + \tfrac{155712}{16807}a_6 \\
& + \tfrac{537088}{84035}a_7 + \tfrac{16216064}{3781575}a_8 + \tfrac{171180032}{94539375}a_{10} + \tfrac{14644412416}{12762815625}a_{11} + \tfrac{45785022464}{63814078125}a_{12} \\
& + \tfrac{424530673664}{957211171875}a_{13} + \tfrac{35467839930368}{215372513671875}a_{15} + \tfrac{320301481066496}{3230587705078125}a_{16} + \tfrac{8618521894322176}{145376446728515625}a_{17} \\
& + \tfrac{2845536092684288}{80764692626953125}a_{18} + \tfrac{6008646327842373632}{49064550770874023437 5}a_{20} + \tfrac{175888083446954 72128}{2453227538543701171875}a_{21} \\
& + \tfrac{153824948872466661376}{36798413078155517578125}a_{22} + \tfrac{402066963212830297292 8}{16559285885169982910156 25}a_{23} \\
& + \tfrac{1007099992704172970475 52}{1241946441387748718261718 75}a_{25} + \tfrac{2607115233425518348992512}{55887589862448692321777343 75}a_{26} \\
& + \tfrac{24928192071448139671797 76}{93145983104081153869628906 25}a_{27} + \tfrac{21397042533882362463256576}{1397189746561217308044433593 75}a_{28}\Big).
\end{aligned}
$$

Now all the coefficients needed for the proof of Theorem 1.2 have been obtained. For convenience, in the remaining of the proof we first introduce the following lemma which can be obtained by [24].

Lemma 3.1. *Consider system (3.1), and assume (3.2) and (3.4) hold. If there exists a parameter δ_0 such that*

$$b_j(\delta_0) = 0, \quad j = 0, 1, \ldots, k_2 - 1, \quad b_{k_2}(\delta_0) \neq 0, \tag{3.5}$$

and

$$
\begin{gathered}
c_j(\delta_0) = 0, j = 0, 1, 2, 3, \;\; c_4(\delta_0) \neq 0, \\
\operatorname{rank} \frac{\partial(c_0, c_1, c_2, c_3, b_0, \cdots, b_{k_2-1})}{\partial(\delta_1, \delta_2, \cdots, \delta_N)} = k_2 + 4,
\end{gathered} \tag{3.6}
$$

then system (3.1) has $k_2 + k + 6$ limit cycles for some (ε, δ) near $(0, \delta_0)$, where $k = 1$ if $c_4(\delta_0)b_{k_2}(\delta_0) > 0$ and $k = 0$ if $c_4(\delta_0)b_{k_2}(\delta_0) < 0$.

Next, we solve the following equations:

$$b_j(\delta) = 0, j = 0, 1, \ldots, 18, \;\; c_j(\delta) = 0, j = 0, 1, 2, 3, \tag{3.7}$$

to obtain the solution for δ. It should be noted that all the computations are symbolic.

Let $\delta_0 = (a_0, a_1, a_2, \cdots, a_{27})$ denote the solution of (3.7). We can get

$$\operatorname{rank} \frac{\partial(b_0, b_1, b_2, \cdots, b_{18}, c_0, c_1, c_2, c_3)}{\partial(a_0, a_1, a_2, a_3, \cdots, a_{31})}\bigg|_{\delta_0} = 23,$$

and

$$\begin{aligned} b_{19}(\delta_0) &= -\tfrac{\bar{l}_0}{l_0}\tfrac{\bar{l}_1 s_1^2-\bar{l}_2 s_1 s_2+\bar{l}_3 s_2^2}{l_1 s_1^2-l_2 s_1 s_2+l_3 s_2^2}\pi\sqrt{30}\,a_{28}, \\ c_4(\delta_0) &= \tfrac{31309846957052568296392}{9314650960306341648101806640625}\tfrac{\bar{k}_1 s_1^2-\bar{k}_2 s_1 s_2+\bar{k}_3 s_2^2}{k_1 s_1^2-k_2 s_1 s_2+k_3 s_2^2}45^{\frac{1}{3}}7^{\frac{2}{3}}\sqrt{2}\,a_{28}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\bar{l}_0 = {} & 857076012434422799620740006721757444629980916134259771865799848 \\
& 4559357166290283203125, \\
l_0 = {} & 863076717231988931496802466892678558991373606566634632401719534 \\
& 6085019648, \\
\bar{l}_1 = {} & 142463458341751890288308086366668655564405512940538868197581857 \\
& 03643924796161248660150100902171716324668045550297370917521728 \\
& 1377674917770128516328073163985970 4025, \\
\bar{l}_2 = {} & 416374943172774514046953154981010330770169821145790633577082219 \\
& 27647481623176774671809192140837177381341886764576757628107782 \\
& 149568205301978810792651833233188259 00, \\
\bar{l}_3 = {} & 304232564827681964476412963220570056629790154011594466541892756 \\
& 67770959357687278279340741716346801986366079143427980773713647 \\
& 678108803987428262520236218451100439 03, \\
l_1 = {} & 169348424314546157758624476101265387867272735809361415805365899 \\
& 45825415780437498740925094894190976712400328782186602870845804 \\
& 00536283239955870364300, \\
l_2 = {} & 494950999424420509100963711131458792705352705828420388071886674 \\
& 41410096502906495283339801777118432966524791200224580033064899 \\
& 29525075240045761793500, \\
l_3 = {} & 361645661633395305492967406294543364091660821711199222817059688 \\
& 81326973190060970150940908771177138802583764185643831332915521 \\
& 36358216479270550370011, \\
\bar{k}_1 = {} & 188011399837272731239366321110617468189125848416668593659924148 9 \\
& 954234350901951644477684681250005042062655500752357337465983386 \\
& 02803467637994631526675, \\
\bar{k}_2 = {} & 549496952729627317000329957521160395262532514031851092613089191116 25 \\
& 21804354720198226288916573811122672332879654281721650178221775019 78 \\
& 636783187226830, \\
\bar{k}_3 = {} & 401500788410286271055989400025951952611877482107573084709663502830 08 \\
& 75814262512901832818439305695244088175413294682910653436821245057 45 \\
& 201570305490959,
\end{aligned}$$

$$\begin{aligned}
k_1 = {} & 16934842431454615775862447610126538786727273580936141580536589945825\\
& 41578043749874092509489419097671240032878218660287084580400536283 23\\
& 9955870364300,\\
k_2 = {} & 49495099942442050910096371113145879270535270582842038807188667441410\\
& 09650290649528333980177711843296652479120022458003306489929525075 24\\
& 0045761793500,\\
k_3 = {} & 36164566163339530549296740629454336409166082171119922281705968881326\\
& 97319006097015094090877117713880258376418564383133291552136358216 47\\
& 9270550370011.
\end{aligned}$$

From (3.8), we get

$$b_{19}(\delta_0) \neq 0, \quad c_4(\delta_0) \neq 0, \quad a_{28}c_4(\delta_0) > 0, \;\; a_{28}b_{19}(\delta_0) < 0, \tag{3.9}$$

if $a_{28} \neq 0$.

Then, by Lemma 3.1, we know that $k_2 = 19$ and $c_4(\delta_0)b_{19}(\delta_0) < 0$. Thus, system (3.1) has 25 limit cycles for some (ε, δ) near $(0, \delta_0)$, i.e., $H^*(28, 4) \geq 25$.

Similarly as before, for $n = 19$ we can find $\bar{\delta}_0 = (a_0, a_1, \cdots, a_{17})$ such that

$$\operatorname{rank} \frac{\partial(b_0, b_1, b_2, \cdots, b_{10}, c_0, c_1, c_2, c_3)}{\partial(a_0, a_1, a_2, a_3, \cdots, a_{17})}\bigg|_{\bar{\delta}_0} = 15,$$

and

$$\begin{aligned}
b_{11}(\bar{\delta}_0) &= -\tfrac{\bar{\mu}_0}{\mu_0} \tfrac{\bar{\mu}_1 s_1^2 - \bar{\mu}_2 s_1 s_2 + \bar{\mu}_3 s_2^2}{\mu_1 s_1^2 - \mu_2 s_1 s_2 + \mu_3 s_2^2} \pi \sqrt{30}\, a_{18},\\
c_4(\bar{\delta}_0) &= \tfrac{110730297608}{954024822509765625} \tfrac{\bar{\nu}_1 s_1^2 - \bar{\nu}_2 s_1 s_2 + \bar{\nu}_3 s_2^2}{\nu_1 s_1^2 - \nu_2 s_1 s_2 + \nu_3 s_2^2} 45^{\frac{1}{3}} 7^{\frac{2}{3}} \sqrt{2}\, a_{18},
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
\bar{\mu}_0 &= 33306323073978122500251629389822483062744140625,\\
\mu_0 &= 40721642681997003402589351854133658569932 8,\\
\bar{\mu}_1 &= 8390307287983706036862909464924044611874106791755139454987660 11\\
& \quad 81905377158320450,\\
\bar{\mu}_2 &= 2452215499657443198256804214395871635931509107173963083000316 85\\
& \quad 041616706908482325,\\
\bar{\mu}_3 &= 1791758230764878397409925908454571155587031235309802056471224 66\\
& \quad 384390815817095289,\\
\mu_1 &= 9336952172514429530745822986371544938136507590072155547943245 14\\
& \quad 47257425,\\
\mu_2 &= 2728863466388749857371264576291955566248031196277755653102990 39\\
& \quad 478372750,
\end{aligned}$$

$$
\begin{aligned}
\mu_3 &= 199387757370565634442733119011498216057118470791980043455952568\\
&\quad 975719511,\\
\bar{\nu}_1 &= 192744009346447066561792521935881833608981466659642558958735275\\
&\quad 289062552325,\\
\bar{\nu}_2 &= 563328835472493043656268492264904349240798366984646434631934512\\
&\quad 927350085320,\\
\bar{\nu}_3 &= 411607315255638258851678076045835734705112024856070879134988139\\
&\quad 219879456361,\\
\nu_1 &= 933695217251442953074582298637154493813650759007215554794324514\\
&\quad 47257425,\\
\nu_2 &= 272886346638874985737126457629195556624803119627775565310299039\\
&\quad 478372750,\\
\nu_3 &= 199387757370565634442733119011498216057118470791980043455952568\\
&\quad 975719511.
\end{aligned}
$$

If $a_{18} \neq 0$, from (3.10) we get

$$
b_{11}(\bar{\delta}_0) \neq 0, \quad c_4(\bar{\delta}_0) \neq 0, \quad a_{18}c_4(\bar{\delta}_0) < 0, \;\; a_{18}b_{11}(\bar{\delta}_0) > 0.
$$

Then, by Lemma 3.1, system (3.1) has 17 limit cycles for some (ε, δ) near $(0, \bar{\delta}_0)$, i.e., $H^*(19, 4) \geq 17$. This completes the proof of Theorem 1.2.

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