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Nine limit cycles around a singular point by perturbing a cubic Hamiltonian system with a nilpotent center

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ABSTRACT

In this paper, we study bifurcation of limit cycles in planar cubic near-Hamiltonian systems with a nilpotent center. We use normal form theory to compute the generalized Lyapunov constants and show that there exist at least 9 limit cycles around the nilpotent center. This is a new lower bound on the number of limit cycles in planar cubic near-Hamiltonian systems with a nilpotent center.

1. Introduction

It is well known that dynamical systems can exhibit self-sustained oscillations, called limit cycles, which may appear in almost all fields of science and engineering such as physics, mechanics, electronics, ecology, economy, biology, finance etc. Developing theory and methodology for solving limit cycle problems is not only theoretically significant, but also practically important. The phenomenon of limit cycles was first discovered and introduced by Poincaré who developed the break-through qualitative method, the Poincaré Map [1], to determine the existence of limit cycles, which is still the most basic tool for studying stability and bifurcation of periodic orbits. Later, many quantitative methodologies were developed to approximate the solution of limit cycles, in particular bifurcating from Hopf critical points, for example, see [2] and reference therein. Recently, with the aid of computer algebra systems such as Mathematica, Maple, symbolic algorithms and programs have been developed to overcome the computational complexity in the analysis of bifurcation of limit cycles, for example, see [3] in which many practical problems are presented and solved by using limit cycles theory and normal form theory. Very recently, bifurcation of multiple limit cycles in disease models has attracted attention of researchers in this field, since such bifurcation can cause complex biological behavior like bistable states which may involve equilibria and periodic motions. For example, in a simple 2-dimensional in-host model of HIV, developed in [4–6], besides studying the interesting phenomenon–viral blips, bifurcation of two limit cycles from a Hopf critical point has been found in [6]. These two limit cycles enclosing a stable equilibrium with the outer cycle stable indeed show that depending upon different initial conditions, the system trajectories can either converge to the disease-free equilibrium or to a stable periodic motion of disease.

The development of limit cycles theory is closely related to the well-known Hilbert's 16th problem, one of the 23 mathematical problems proposed by Hilbert at the Second International Congress of Mathematics in 1900 [7]. A modern version of this problem was later formulated by Smale, chosen as one of his 18 most challenging mathematical problems for the 21st century [8]. To be more specific, consider the following planar differential system:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{1.1}$$

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where $P_n(x, y)$ and $Q_n(x, y)$ represent nth-degree polynomials in x and y. The second part of Hilbert's 16th problem is to find an upper bound on the number of limit cycles that system (1.1) can have. This upper bound, called Hilbert number, is the function of n only denoted as $H(n)$. In early 1990's, Ilyashenko and Yakovenko [9], and Écalle [10] independently proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, four limit cycles were obtained more than 30 years ago [11,12]. This result was also proved recently for near-integrable quadratic systems [13]. However, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number, and the best result so far is $H(3) \geq 13$ [14,15]. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [16]. It should be pointed out that in real applications, many systems have dimension higher than two [3–5] and Hopf bifurcation leading to limit cycles is a common phenomenon. In such case, the system can be reduced to a 2-dimensional dynamical system by using center manifold theory (e.g., see [3,17]) to study the limit cycles bifurcation.

Later, Arnold [18] posed the weak Hilbert's 16th problem, which is closely related to the so-called near-Hamiltonian system [19]:

$$\dot{x} = H_y(x, y) + \varepsilon\, p_n(x, y), \qquad \dot{y} = -H_x(x, y) + \varepsilon\, q_n(x, y), \tag{1.2}$$

where $H(x, y)$, $p_n(x, y)$ and $q_n(x, y)$ are all polynomial functions in x and y, and $0 < \varepsilon \ll 1$ represents a small perturbation. Then, the problem on bifurcation of limit cycles for such a system can be transferred to studying the zeros of the Abelian function or the (first-order) Melnikov function, given in the form of

$$M(h, \delta) = \oint_{H(x,y)=h} q_n(x, y)\, dx - p_n(x, y)\, dy, \tag{1.3}$$

where $H(x, y) = h$ for $h \in (h_1, h_2)$ defines a closed orbit, and δ represents the parameters (or coefficients) involved in the polynomial functions $p_n(x, y)$ and $q_n(x, y)$.

If the problem is restricted to the vicinity of an isolated fixed point, which is either an elementary center or an elementary focus, then it is equivalent to study generalized Hopf bifurcations. This problem is usually called local bifurcation of limit cycles, and the number of bifurcating small-amplitude limit cycles is denoted by $M(n)$. The best-known result is $M(2) = 3$, which was obtained by Bautin in 1952 [20]. For $n = 3$, a number of results have been obtained. Around an elemental focus, James and Lloyd [21] considered a particular class of cubic systems to obtain 8 limit cycles in 1991, and the systems were reinvestigated couple of years later by Ning et al. [22] to find another solution of 8 limit cycles. Yu and Corless [23] constructed a cubic system and combined symbolic and numerical computations to show 9 limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [24]. Another cubic system was also recently constructed by Lloyd and Pearson [25] to show 9 limit cycles with purely symbolic computation.

On the other hand, around a center, there are also a few results obtained in the past two decades. In 1995, Żołądek [26] first proposed a rational Darboux integral, and claimed the existence of 11 small-amplitude limit cycles around a center. After more than ten years, another two cubic systems are constructed to show 11 limit cycles [27,28]. Recently, based on the system given in [27], 12 small-amplitude limit cycles around a singular point has been proved [29], which is perdhaps the maximal numver which can be obtained for cubic integrable polynomial systems. The system considered in [26] was reinvestigated by Yu and Han [30] using the method of focus value computation, and only 9 limit cycles were obtained. More recently, Tian and Yu proved the Żołądek's example can indeed have only 9 limit cycles [31].

For the local bifurcation problem associated with a singularity of focus, Lyapunov constants are needed to solve the center-focus problem and determine the number and stability of bifurcating limit cycles. There are mainly three approaches which are widely used to compute the Lyapunov constants: the method of normal forms [3,32,33], the method of Poincaré return map or focus value method [34,35], and the method of Lyapunov function [36,37]. Other approaches can be found, for example, in [3]. Since in this paper we apply the method of normal forms to study bifurcation of limit cycles, in the following we briefly describe how this method is used to determine the number of bifurcating limit cycles. Without loss of generality, suppose that system (1.1) has a singularity of focus at the origin, that is, $(x, y) = (0, 0)$ is an equilibrium of system (1.1) and that the Jacobian of the system evaluated at this equilibrium is in the real Jordan canonical form,

$$J = \begin{bmatrix} 0 & \omega_c \\ -\omega_c & 0 \end{bmatrix}.$$

Then, by using the method of normal forms with the aid of computer algebra systems (e.g., see [3,33,38,39]) we can obtain the normal form of the system, given in polar coordinates, as

$$\begin{aligned} \dot{r} &= r\left(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k} + \cdots\right), \\ \dot{\theta} &= \omega_c + \tau_0 + \tau_1 r^2 + \tau_2 r^4 + \cdots + \tau_k r^{2k} + \cdots, \end{aligned} \tag{1.4}$$

where r and θ represent the amplitude and phase of motion, respectively. v_k $(k = 0, 1, 2, \ldots)$ is called the kth-order focus value. v_0 and τ_0 are obtained from linear analysis. The first equation of (1.4) can be used for studying bifurcation of limit cycles and stability of bifurcating limit cycles, while the second equation can be used to determine the frequency of bifurcating periodic motion. Moreover, the coefficients τ_j's can be used to determine the order (or critical periods) of a center (i.e., when $v_j = 0,\ j \geq 0$). These focus values are equivalent to Lyapunov constants, L_j, in the sense that

$$v_j = 0,\ j = 0, 1, \cdots, k-1,\ v_k \neq 0 \iff L_j = 0,\ j = 0, 1, \ldots, k-1,\ L_k \neq 0,$$

where v_k and L_k are either equal or different by a positive constant multiplier.

Having obtained the focus values or the Lyapunov constants for a given dynamical system, we use these quantities to determine the number of bifurcating limit cycles. The basic idea of finding k small-amplitude limit cycles due to Hopf bifurcation is as follows: First, find the conditions such that $v_0 = v_1 = \cdots = v_{k-1} = 0$ (note that $v_0 = 0$ is automatically satisfied at the critical point), but $v_k \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. The procedure contains two main steps in finding multiple limit cycles: Computing the focus values (i.e., computing the normal form) or the Lyapunov constants, and solving multivariate coupled nonlinear polynomial equations: $v_0 = v_1 = \cdots = v_{k-1} = 0$. The following Lemma gives sufficient conditions for proving the existence of k small-amplitude limit cycles. (The proofs can be found in [40].)

Lemma 1.1. *Suppose that the focus values depend on k parameters, a_j, $j = 1, 2, \ldots, k$, expressed as*

$$v_j = v_j(a_1, a_2, \ldots, a_k), \quad j = 0, 1, \ldots, k-1, \tag{1.5}$$

satisfying

$$v_j(a_{1c}, \ldots, a_{kc}) = 0, \quad j = 0, 1, \ldots, k-1, \quad v_k(a_{1c}, \ldots, a_{kc}) \neq 0,$$
$$\text{and} \ \ \text{rank}\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(a_1, a_2, \ldots, a_k)}\right]_{(a_1,\ldots,a_k)=(a_{1c},\ldots,a_{kc})} = k. \tag{1.6}$$

Then, for any given $a^* > 0$, there exist $a_1, a_2, \ldots, a_k$ and $\delta > 0$ with $|a_j - a_{jc}| < a^*$, $j = 1, 2, \ldots, k$ such that the equation $\dot{r} = 0$ has exactly k real positive roots r (i.e., system (1.1) has exactly k limit cycles) in a δ-ball with the center at the origin.

To study bifurcation of limit cycles for the near-Hamiltonian system (1.2), the Melnikov function method is usually applied [3,19]. Suppose the origin of system (1.3) is a center and $M(0, \delta) = 0$. Then, the problem becomes a study of Hopf bifurcation, and the first-order Melnikov function (1.4) can be expanded as

$$M(h, \delta) = h\,(\mu_0 + \mu_1 h + \mu_2 h^2 + \cdots) \quad \text{for} \quad 0 < h \ll 1, \tag{1.7}$$

where the coefficients μ_j's can be used to determine the small-amplitude limit cycles due to the perturbing terms. When $M(H, \delta) \equiv 0$, one has to find higher-order Melnikov functions to study bifurcation of limit cycles. The computation of higher-order Melnikov functions is much more complex than that of first-order Melnikov function.

The method of normal forms can be also applied to consider Hopf bifurcation in system (1.2). In this case, the first equation of the normal form (1.4) can be written as

$$\dot{r} = r\Big[V_0(\varepsilon) + V_1(\varepsilon)\, r^2 + V_2(\varepsilon)\, r^4 + \cdots + V_k(\varepsilon)\, r^{2k} + \cdots\Big], \tag{1.8}$$

where

$$V_j(\varepsilon) = V_{0j} + \varepsilon\, V_{1j} + \varepsilon^2\, V_{2j} + \cdots, \quad j = 0, 1, 2, \ldots, \tag{1.9}$$

where $V_{0j} = 0$, $j = 0, 1, 2, \ldots$ since system (1.2) is a Hamiltonian system when $\varepsilon = 0$.

Remark 1.1.

(1) V_{1j}, $j = 0, 1, 2, \ldots$ are equivalent to the coefficients of the first-order Melnikov function, μ_j, $j = 0, 1, 2, \ldots$, and $V_{kj}, k \geq 2$ are equivalent to the kth-order Melnikov functions.
(2) The computation of the coefficients V_{1j} has more or less complexity of computing the coefficients μ_j in the first-order Melnikov function. However, the computation of the coefficients in higher-order Melnikov functions is much more involved than the computation of the higher-order coefficients $V_{kj}, k \geq 2$.
(3) In particular, when the original system is not a Hamiltonian system but an integrable system, then even computing the coefficients of the first-order Melnikov function is much more involved than the computation of using the method of normal forms, since the former needs to find an integrating factor while the latter does not.
(4) However, the method of normal forms is restricted to Hopf bifurcation, while the Melnikov function method can also be applied to study bifurcation of limit cycles from homoclinic/heteroclinic orbits or any closed orbits.

The above mentioned three methods: the normal form method, the focus value method and Lyapunov function method, have also been used to study the center-focus problem for nilpotent critical points, see for example [41–46]. Regarding the use of the methods for generating limit cycles from the nilpotent critical point, the method of normal forms was only recently applied to compute the so-called *generalized Lyapunov constants* which can be used to find the lower bound of cyclicity [47]. As we know, it is more difficult to distinguish focus from center when the singular point is degenerate. Andreev [48] studied analytic systems having a nilpotent singular point at the origin and obtained the local phase portraits, which however does not distinguish focus from center. Takens [49] developed a normal form theory for nilpotent center of foci, and later, Moussu [43] found the C^∞ normal form for analytic nilpotent centers. Further, Berthier and Moussu [50] investigated the reversibility of nilpotent centers. Teixeria and Yang [51] used a convenient normal form to consider the relationship between reversibility and the center-focus problem, and studied the reversibility of certain types of polynomial vector fields. Recently, Han et al. [52] studied polynomial Hamiltonian systems with a nilpotent singular point, and used

Melnikov function method to obtain necessary and sufficient conditions for determining the number of limit cycles bifurcating in quadratic and cubic Hamiltonian systems with a nilpotent singular point which may be a center, a cusp or a saddle. In a later article [53], Han and Romanovski developed a Melnikov function method to invetigate limit cycle bifurcations in analytic planar systems.

In [47], many examples are given to demonstrate the application of normal forms to determine the number of bifurcating limit cycles in planar dynamical system with a nilpotent critical point. The main goal of this paper is to generalize the normal form method to consider bifurcation of limit cycles in near-Hamiltonian systems. In fact, according to the third point of Remark 1.1, our method can be applied to consider near-integrable systems (i.e., integrable systems with perturbations) without computing the integrating factor. In the next section, we present some basic formulations and preliminary results which are needed in proving our main result in Section 3. Finally, conclusion is drawn in Section 4.

2. Mathematical formulation and preliminary results

In this section, we present some basic formulas and preliminary results which will be used in the next section. Consider the following differential system:

$$\begin{aligned} \dot{x} &= -y + F_1(x, y), \\ \dot{y} &= F_2(x, y), \end{aligned} \tag{2.1}$$

where F_1 and F_2 are analytic functions with power series beginning from second order. As long as the limit cycles bifurcation is considered near the origin, the system (2.1) with a nilpotent center at the origin is more difficult to analyze than the general system (1.1) with an element center or focus at the origin, since the conventional normal form of Hopf bifurcation [3,17] can be directly applied to the latter but not the former. In fact, there exist conventional normal forms for system (2.1) associated with Bogdanov–Takens bifurcation (i.e., the linearized system contains a double-zero eigenvalue at the origin) [3,17], which is however not able to be directly applied to study bifurcation of limit cycles near the origin. Therefore, a modified normal form of system (2.1) needs to be developed to study bifurcation of limit cycles near the origin. In real applications, many physical systems involve a number of parameters and can thus have higher co-dimensional singularity such as Bogdanov–Takens bifurcation (which is characterized by a double-zero eigenvalue at a critical point, leading to a nilpotent center), and thus it is interesting and important to explore the periodic solutions near such a critical point. For example, in the 2-dimensional HIV model [6], a critical point with B–T bifurcation is identified for certain parameter values and thus the system can be put in the form of system (2.1) in the vicinity of the critical point. Limit cycles due to Hopf bifurcation have been obtained near this critical point and even multiple limit cycles can be found if more parameters are treated as bifurcation parameters. Moreover, homoclinic orbits are identified near this degenerate singular point [6].

To mathematically analyze bifurcation of limit cycles for system (2.1) near the origin, we first present the following result [41,47,48], which can be used to determine the monodromy of the origin of system (2.1).

Lemma 2.1 (Theorem 2.1 in [47]). *Assume that the origin of system (2.1) is an isolated singularity. Define two functions $f(x)$ and $\phi(x)$ as*

$$\begin{aligned} f(x) &= F_2(x, Y(x)) = a x^{\alpha} + O(x^{\alpha+1}), \quad a \neq 0, \ \alpha \geq 2, \\ \phi(x) &= \frac{\partial F_1(x, Y(x))}{\partial x} + \frac{\partial F_2(x, Y(x))}{\partial y}, \end{aligned}$$

where $y = Y(x)$ is the solution of the equation, $-y + F_1(x, y) = 0$, passing through the origin $(0,\ 0)$. Write $\phi(x) = bx^{\beta} + O(x^{\beta+1})$, $b \neq 0$ and $\beta \geq 1$, or $\phi(x) \equiv 0$. Then, the origin of system (2.1) is monodromic if and only if $a > 0$, $\alpha = 2n - 1$ ($n \geq 1$) being an odd number, and one of the following three conditions holds:

(i) $\beta > n - 1$;
(ii) $\beta = n - 1$, *and* $b^2 - 4an < 0$;
(iii) $\phi \equiv 0$.

Under the above conditions, we apply the near-identity state transformation and time scaling:

$$\begin{aligned} x &= u + \sum_{i+j=2}^{k} h_{1ij} u^i v^j, \\ y &= v + \sum_{i+j=2}^{k} h_{2ij} u^i v^j, \\ \tau &= \left(1 + \sum_{i+j=2}^{k} h_{3ij} u^i v^j\right) t, \end{aligned} \tag{2.2}$$

into system (2.1) to yield the following normal form by choosing appropriate coefficients h_{1ij}, h_{2ij} and h_{3ij}:

$$\begin{aligned}\frac{du}{d\tau} &= -v + O(\|(u,v)\|^{k+1}),\\ \frac{dv}{d\tau} &= u^{2n-1} + v\sum_{j\ge\beta}^{k-1} B_j u^j + O(\|(u,v)\|^{k+1}),\end{aligned} \tag{2.3}$$

where B_j is called the jth-order generalized Lyapunov constant. Based on the simplest normal form theory associated with B–T bifurcation (e.g., see [57–59]), we have developed an algorithm with explicit recursive formulas for computing B_j, with a computationally efficient Maple program which can be easily implemented in a computer using Maple. This method can also be applied to consider the near-Hamiltonian system discussed blow.

It has been noted that Liu and Li [54] have developed a different method to compute the so-called *quasi Lyapunov constants*, which are equivalent to the generalized Lyapunov constants. However, this method is only applicable for cubic systems.

Next, we consider the near-Hamiltonian systems, described by

$$\begin{aligned}\dot{x} &= \frac{\partial H(x,y,\boldsymbol{a})}{\partial y} + \varepsilon P(x,y,\boldsymbol{\delta}),\\ \dot{y} &= -\frac{\partial H(x,y,\boldsymbol{a})}{\partial x} + \varepsilon Q(x,y,\boldsymbol{\delta}),\end{aligned} \tag{2.4}$$

where $H(x, y, \boldsymbol{a})$ is an nth-degree real polynomial in x and y and P, Q are mth-degree of polynomials in x and y, and $\boldsymbol{a} = (a_1, \ldots, a_{k_1}) \in R^{k_1}$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{k_2}) \in R^{k_2}$ are vector parameters, and $0 < \varepsilon \ll 1$ is a small perturbation parameter. Thus, the perturbed system (2.4) contains a total of $k_1 + k_2$ parameters. The function $H(x, y, \boldsymbol{a})$ is called the Hamiltonian of system (2.4). The origin of the system is an equilibrium and assumed to be a nilpotent center. When $\varepsilon = 0$, system (2.4) becomes a Hamiltonian system:

$$\dot{x} = \frac{\partial H(x,y,\boldsymbol{a})}{\partial y}, \quad \dot{y} = -\frac{\partial H(x,y,\boldsymbol{a})}{\partial x}. \tag{2.5}$$

It should be noted that system (2.4) is a subset of system (2.1). When system (2.1) satisfies the property (2.5) for certain values of the system coefficients, perturbing these coefficients can lead to system (2.1). This often happens in real applications.

The monodromy of the origin of system (2.5) has been studied in [55]. Suppose $H(x, y, \boldsymbol{a})$ is given by

$$H(x,y) = \frac{1}{2}y^2 + \sum_{3\le i+j\le 4} a_{ij}x^i y^j. \tag{2.6}$$

Then, a simple approach based on the Hamiltonian function, given in [55], is used to prove the following lemma.

Lemma 2.2 (Theorem 2.1 in [55])**.** *The cubic Hamiltonian system (2.5) with (2.6) has a nilpotent center at the origin if and only if one of the following three conditions holds:*

(1) $a_{30} = 0,\ a_{40} > \frac{1}{2}a_{21}^2$;
(2) $a_{30} = 0,\ a_{40} = \frac{1}{2}a_{21}^2 \ne 0,\ a_{31} = a_{12}a_{21},\ a_{22} > \frac{1}{2}a_{12}^2 + a_{03}a_{21}$;
(3) $a_{30} = 0,\ a_{40} = \frac{1}{2}a_{21}^2 \ne 0,\ a_{31} = a_{12}a_{21},\ a_{22} = \frac{1}{2}a_{12}^2 + a_{03}a_{21},\ a_{13} = a_{03}a_{12},\ a_{04} > \frac{1}{2}a_{03}^2$.

Since a Hamiltonian system is a special type of general dynamical systems, the above conditions can be obtained directly by using Lemma 2.1, but the proof for Lemma 2.1 is more involved than that of Hamiltonian system [56].

Now system (2.4) contains parameters (or constants) not only in the perturbation terms, but also in the Hamiltonian functions, we need to generalize Lemma 1.1. To achieve this, suppose now the Melnikov function for system (2.4) is given as

$$M(h,\boldsymbol{a},\boldsymbol{\delta}) = h\sum_{j\ge 0}\mu_j(\boldsymbol{a},\boldsymbol{\delta})\,h^j. \tag{2.7}$$

Then we have the following result (e.g., see [56,60]).

Lemma 2.3. *Consider the near-Hamiltonian system (2.4). Suppose there exist $\boldsymbol{a}_0$ and $\boldsymbol{\delta}_0$ such that*

$$\begin{aligned}&\mu_j(\boldsymbol{a}_0,\boldsymbol{\delta}_0) = 0, \quad j = 0,1,\ldots,k-1, \quad \text{but} \quad \mu_k(\boldsymbol{a}_0,\boldsymbol{\delta}_0) \ne 0,\\ &\text{and} \ \det\left[\frac{\partial(\mu_0,\mu_1,\ldots,\mu_{k-1})}{\partial(a_1,a_2,\ldots,a_{l_1},\delta_1,\delta_2,\ldots,\delta_{l_2})}\right]_{\boldsymbol{a}=\boldsymbol{a}_0} \ne 0, \quad (l_1+l_2=k),\end{aligned} \tag{2.8}$$

where $k \le k_1 + k_2$. Then, system (2.4)) can have k small-amplitude limit cycles around the origin for some $(\varepsilon, \boldsymbol{a}, \boldsymbol{\delta})$ near $(0, \boldsymbol{a}_0, \boldsymbol{\delta}_0)$.

Lemma 2.3 implies that if system (2.4) contains a total of $k = k_1 + k_2$ *independent* parameters, the system can have maximal k small-amplitude limit cycles around the origin.

In this paper, we consider a special case of system (2.4) when $n = 4$ and $m = 3$, and suppose the 4th-degree polynomial Hamiltonian function be given by

$$\begin{aligned} H(x, y, \boldsymbol{a}) = & \frac{1}{2} y^2 + a_0 x^3 + a_1 x^2 y + a_2 xy^2 + a_3 y^3 \\ & + a_8 x^4 + a_4 x^3 y + a_5 x^2 y^2 + a_6 xy^3 + a_7 y^4, \end{aligned} \tag{2.9}$$

where $a_i \in \boldsymbol{a}$ are constant coefficients, and P and Q can be assumed in the general forms:

$$\begin{aligned} P(x, y, \boldsymbol{\delta}) = & \, a_{10} x - a_{01} y + a_{20} x^2 - a_{11} xy + a_{02} y^2 \\ & + a_{30} x^3 - a_{21} x^2 y + a_{12} x^2 y - a_{03} y^3, \\ Q(x, y, \boldsymbol{\delta}) = & - b_{10} x + b_{01} y - b_{20} x^2 + b_{11} xy - b_{02} y^2 \\ & - b_{30} x^3 + b_{21} x^2 y - b_{12} xy^2 + b_{03} y^3, \end{aligned} \tag{2.10}$$

where $a_{ij}, b_{ij} \in \boldsymbol{\delta}$ are constant coefficients. Since system (2.4) is a near-Hamiltonian system, i.e., system $(2.4)|_{\varepsilon=0}$ is a Hamiltonian system, we may further assume $P = 0$ and $b_{30} = 0$. Thus, the cubic system considered in this paper becomes

$$\begin{aligned} \dot{x} &= y + a_1 x^2 + 2a_2 xy + 3a_3 y^2 + a_4 x^3 + 2a_5 x^2 y + 3a_6 xy^2 + 4a_7 y^3, \\ \dot{y} &= -3a_0 x^2 - 2a_1 xy - a_2 y^2 - 4a_8 x^3 - 3a_4 x^2 y - 2a_5 xy^2 - a_6 y^3 + \varepsilon\, Q(x, y, \delta). \end{aligned} \tag{2.11}$$

Moreover, in order to be consistent with our formulas, we assume that the Jacobian matrix of system (2.4) evaluated at the origin is in the form of

$$J = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}. \tag{2.12}$$

Hence, introducing the transformation $x \to x, \ y \to -y$ into (2.11) we obtain

$$\begin{aligned} \dot{x} &= -y + a_1 x^2 - 2a_2 xy + 3a_3 y^2 + a_4 x^3 - 2a_5 x^2 y + 3a_6 xy^2 - 4a_7 y^3, \\ \dot{y} &= 3a_0 x^2 - 2a_1 xy + a_2 y^2 + 4a_8 x^3 - 3a_4 x^2 y + 2a_5 xy^2 - a_6 y^3 \\ &\quad + \varepsilon (b_{10} x + b_{01} y + b_{20} x^2 + b_{11} xy + b_{02} y^2 + b_{21} x^2 y + b_{12} xy^2 + b_{03} y^3). \end{aligned} \tag{2.13}$$

In order for the origin of the perturbed system (2.13) to become a nilpotent center, it requires that

$$a_0 = 0, \quad b_{10} = b_{01} = b_{20} = 0, \quad a_8 > 0. \tag{2.14}$$

Further, without loss of generality, we may assume $a_2 \neq 0$, and then introducing the scaling:

$$x \to \frac{1}{a_2} x \quad y \to \frac{\sqrt{a_8}}{a_2^2} y \quad t \to \frac{a_2}{\sqrt{a_8}} t, \tag{2.15}$$

into (2.13) together with (2.14) yields the final system to be considered in this paper:

$$\begin{aligned} \dot{x} &= -y + a_1 x^2 - 2 xy + 3 a_3 y^2 + a_4 x^3 - 2 a_5 x^2 y + 3 a_6 xy^2 - 4 a_7 y^3, \\ \dot{y} &= 4 x^3 - 2 a_1 xy + y^2 - 3 a_4 x^2 y + 2 a_5 xy^2 - a_6 y^3 \\ &\quad + \varepsilon\, (b_{11} xy + b_{02} y^2 + b_{21} x^2 y + b_{12} xy^2 + b_{03} y^3), \end{aligned} \tag{2.16}$$

where the rescaled parameters still use the same notations a_j, a_{ij} and b_{ij} for simplicity.

3. Main result

In this section, we present our main result of this paper. To demonstrate the efficiency of our method, we first reinvestigate an example, which was studied in [55] to obtain 5 limit cycles. The equations for this example are given by

$$\begin{aligned} \dot{x} &= y + 2 xy + 3 y^2 + 2 x^2 y + \varepsilon P, \\ \dot{y} &= - 4 x^3 - y^2 - 2 xy^2 + \varepsilon Q, \end{aligned} \tag{3.1}$$

where P and Q are given in (2.10). The Melnikov function method was used in [55] to show the existence of 5 limit cycles. In the following, we will apply our method to this example to show that our computation is simpler.

Lemma 3.1. *[55] For the near-Hamiltonian system (3.1) with a nilpotent center at the origin, there exist at least 5 small-amplitude limit cycles around the origin.*

Proof. In order to put system (3.1) in our format, we use the transformation $x \to x, \ y \to -y$ to obtain

$$\begin{aligned} \dot{x} = & -y - 2xy + 3y^2 - 2x^2 y + \varepsilon\, (a_{10} x + a_{01} y + a_{20} x^2 + a_{11} xy + a_{02} y^2 \\ & + a_{30} x^3 + a_{21} x^2 y + a_{12} x^2 y + a_{03} y^3), \end{aligned}$$

$$\dot{y} = 4x^3 + y^2 + 2xy^2 + \varepsilon\,(b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3), \tag{3.2}$$

We could set $P = 0$ in (3.2), but we leave it there to show the general case consistent with that used in [55], and will find that there are only 5 independent perturbation parameters which can be used to obtain 5 limit cycles. The linear analysis of (3.2) shows that the zero order generalized Lyapunov constant is given by

$$B_0 = \frac{1}{2}\,\varepsilon\,(a_{10} + b_{01}).$$

The system satisfies $n = 2$, $a = 4 + \varepsilon\, b_{30} > 0$ for $0 < \varepsilon \ll 1$, $b = \varepsilon\,(2a_{20} + b_{11}) > 0$ requiring $2a_{20} + b_{11} > 0$, and $\beta = 1 = n - 1$, yielding $b^2 - 4an = -32 - 8\varepsilon b_{30} + \varepsilon^2(2a_{20} + b_{11})^2 < 0$ for $0 < \varepsilon \ll 1$. Thus, all the required conditions are satisfied.

To compute higher-order generalized Lyapunov constants, we need the perturbed system (3.2) to have a nilpotent center at the origin, which requires $a_{10} = b_{01} = b_{10} = 0$, and $b_{20} = 0$. Then, executing our Maple program yields the ε-order generalized Lyapunov constants as follows:

$$\begin{aligned}
B_2 &= c_{20} - c_{10},\\
B_4 &= \frac{1}{2}\big(c_{20} + 8\,c_{02} + 24\,c_{01} + c_{10}\big),\\
B_6 &= -\frac{1}{8}\big(673\,c_{20} - 48\,c_{02} + 160\,c_{11} - 400\,c_{01} - 197\,c_{10}\big),\\
B_8 &= \frac{1}{80}\big(10021\,c_{20} - 72\,c_{02} - 3360\,c_{11} + 1704\,c_{01} - 26959\,c_{10}\big),\\
B_{10} &= -\frac{1}{896}\,\big(15544003\,c_{20} - 1362976\,c_{02} + 864576\,c_{11}\\
&\quad - 5689440\,c_{01} - 13079611\,c_{10}\big),\\
B_{12} &= \frac{1}{3840}\,\big(30518153\,c_{20} - 98702616\,c_{02} - 46061760\,c_{11}\\
&\quad - 229917768\,c_{01} - 226882847\,c_{10}\big),
\end{aligned}$$

where

$$\begin{aligned}
c_{00} &= a_{10} + b_{01}, & c_{10} &= 2a_{20} + b_{11}, & c_{01} &= a_{11} + 2b_{02},\\
c_{20} &= 3a_{30} + b_{21}, & c_{11} &= 2(a_{21} + b_{12}), & c_{02} &= a_{12} + 3b_{03}.
\end{aligned}$$

Now solving $B_2 = B_4 = B_6 = B_8 = 0$ for c_{20}, c_{02}, c_{11} and c_{01} we obtain

$$c_{20} = c_{10}, \quad c_{02} = \frac{133}{24}\,c_{10}, \quad c_{11} = -\frac{221}{36}\,c_{10}, \quad c_{01} = -\frac{139}{72}\,c_{10},$$

and then B_{10} and B_{12} become

$$B_{10} = -\frac{1967}{3}\,c_{10} < 0, \quad B_{12} = -\frac{26101}{6}\,c_{10} < 0.$$

Moreover, for the above solution we have

$$\det\left[\frac{\partial\,(B_2, B_4, B_6, B_8)}{\partial\,(c_{20}, c_{02}, c_{11}, c_{01})}\right] = 3456,$$

showing that we can perturb c_{20}, c_{02}, c_{11} and c_{01} to obtain 4 small-amplitude limit cycles. Moreover, performing a linear perturbation on c_{10} such that $B_0B_2 < 0$ and $|B_0| \ll |B_2|$, yields the 5th small-amplitude limit cycle. □

It is easy to see that we could set $P = 0$ (i.e., setting all $a_{ij} = 0$) at the beginning of the proof, which would make the computations even be further simplified.

Now we return to system (2.16), which has 6 parameters: a_1, a_3, a_4, a_5, a_6, a_7 in the Hamiltonian function H, and still have 5 parameters $b_{11}, b_{02}, b_{21}, b_{12}, b_{03}$ in the perturbation function Q. So we may apply Lemma 2.3 to get more limit cycles by using the additional parameters in the H function, but computation will become much more involved. There are a total of 11 parameters in the system, which means we may obtain 11 small-amplitude limit cycles near the origin. Further a linear perturbation on the parameter b_{10} or b_{01} results in one more limit cycle, and so it seems that we may obtain 12 limit cycles. However, it has been found from solving the generalized Lyapunov constants equations that the 5 perturbation parameters are not independent, one of which can be set to equal 1. In other words, one of the perturbation parameters, for example, b_{11} can be treated as a free parameter. Therefore, system (2.16) may have 11 small-amplitude limit cycles around the origin.

However, due to the difficulty in computing the generalized Lyapunov constants, we assume $a_1 = 0$ in the computation of the generalized Lyapunov constants. But even we can obtain the generalized Lyapunov constants B_2, B_4, …, B_{18}, we still have difficulty to solve the 9 equations, $B_k = 0$, $k = 2, 4, \ldots, 18$, for the 9 parameters: a_3, a_4, a_5, a_6, a_7 and $b_{02}, b_{21}, b_{12}, b_{03}$.

Hence, we further set $a_4 = 0$ and then solve the 8 polynomial equations to get 8 limit cycles, and finally use a linear perturbation to get one more limit cycle.

Our main result is summarized in the following theorem.

Theorem 3.1. *For the near-Hamiltonian system (2.13) with $a_1 = a_4 = 0$, whose unperturbed system $(2.13)_{\varepsilon=0}$ has a nilpotent center at the origin, there exist perturbation parameter values of b_{ij} such that at least 9 small-amplitude limit cycles can bifurcate from the origin.*

Proof. It is easy to obtain from the linear part of system (2.13) that the (ε-order) zero order generalized Lyapunov constant is given by

$$B_0 = \frac{1}{2}\, b_{01}. \tag{3.3}$$

Letting $B_0 = 0$ yields $b_{01} = 0$. Also, set $b_{01} = b_{20} = 0$ so that the origin of the perturbed system is a nilpotent center. Hence, we can apply the method of normal forms and the Maple program we developed to system (2.16) to obtain higher-order Lyapunov constants, as given below.

$$\begin{aligned}
B_2 &= b_{21} - b_{11},\\
B_4 &= \frac{1}{4}\big[48b_{03} + 8a_4 b_{12} + 24(4a_3 - a_4)\, b_{02} + (12a_3a_4 - 5a_4^2 + 8a_5 - 4)\, b_{11}\big],\\
B_6 &= \frac{1}{16}\big[8(a_4^3 - 4a_4a_5 - 40a_3 + 10a_4 + 8a_6)\, b_{12} + 8(12a_3a_4^2 - 13a_4^3\\
&\quad - 32a_3a_5 + 48a_4a_5 + 160a_3 - 40a_4 - 80a_6)\, b_{02} - (31a_4^4 - 52a_3a_4^3\\
&\quad - 136a_4^2a_5 + 192a_3a_4a_5 + 960a_3^2 + 84a_4^2 + 64a_5^2 - 480a_3a_4\\
&\quad - 320a_3a_6 + 192a_4a_6 - 64a_5 - 256a_7 + 16)\, b_{11}\big],
\end{aligned}$$

$$\begin{aligned}
B_8 &= \frac{3}{64(a_4^3 - 4a_4a_5 - 40a_3 + 10a_4 + 8a_6)}\big[(96a_3a_4^7 - 72a_4^8 - 896a_3a_4^5a_5 + 736a_4^6a_5\\
&\quad - 1536a_3^2a_4^4 + 1536a_3a_4^5 + 2816a_3a_4^4a_6 + 2048a_3a_4^3a_5^2 + 160a_4^6\\
&\quad - 1472a_4^5a_6 - 2304a_4^4a_5^2 + 8192a_3^2a_4^2a_5 - 9216a_3a_4^3a_5 - 8192a_3a_4^3a_7\\
&\quad - 12288a_3a_4^2a_5a_6 - 1792a_4^4a_5 + 2048a_4^4a_7 + 8192a_4^3a_5a_6 + 2048a_4^2a_5^3\\
&\quad - 21504a_3^2a_4^2 - 81920a_3^2a_4a_6 + 16384a_3^2a_5^2 + 3584a_3a_4^3 + 47104a_3a_4^2a_6\\
&\quad + 32768a_3a_4a_5a_7 + 16384a_3a_4a_6^2 + 448a_4^4 + 512a_4^3a_6 + 6144a_4^2a_5^2\\
&\quad - 8192a_4^2a_5a_7 - 7168a_4^2a_6^2 - 8192a_4a_5^2a_6 + 57344a_3^2a_5 + 327680a_3^2a_7\\
&\quad - 163840a_3a_4a_7 - 16384a_3a_5a_6 - 65536a_3a_6a_7 - 3584a_4^2a_5\\
&\quad + 20480a_4^2a_7 - 24576a_4a_5a_6 + 16384a_4a_6a_7 + 8192a_5a_6^2 - 57344a_3a_6\\
&\quad + 14336a_4a_6 + 28672a_6^2)\, b_{02} + (36a_3a_4^8 - 27a_4^9 - 368a_3a_4^6a_5 + 300a_4^7a_5\\
&\quad - 768a_3^2a_4^5 + 896a_3a_4^6 + 736a_3a_4^5a_6 + 1152a_3a_4^4a_5^2 - 24a_4^7 - 568a_4^6a_6\\
&\quad - 1088a_4^5a_5^2 + 4864a_3^2a_4^3a_5 - 6400a_3a_4^4a_5 - 1024a_3a_4^4a_7 - 4096a_3a_4^3a_5a_6\\
&\quad - 1024a_3a_4^2a_5^3 - 48a_4^5a_5 + 256a_4^5a_7 + 3968a_4^4a_5a_6 + 1280a_4^3a_5^3\\
&\quad - 17024a_3^2a_4^3 - 14848a_3^2a_4^2a_6 - 4096a_3^2a_4a_5^2 + 5888a_3a_4^4 + 17920a_3a_4^3a_6\\
&\quad + 8704a_3a_4^2a_5^2 + 4096a_3a_4^2a_5a_7 + 3584a_3a_4^2a_6^2 + 4096a_3a_4a_5^2a_6 - 216a_4^5\\
&\quad - 1376a_4^4a_6 + 1536a_4^3a_5^2 - 1024a_4^3a_5a_7 - 3584a_4^3a_6^2 - 6656a_4^2a_5^2a_6\\
&\quad + 64512a_3^2a_4a_5 + 40960a_3^2a_4a_7 + 12288a_3^2a_5a_6 - 19456a_3a_4^2a_5\\
&\quad - 14336a_3a_4^2a_7 - 49152a_3a_4a_5a_6 - 8192a_3a_4a_6a_7 + 4096a_3a_5^3\\
&\quad - 4096a_3a_5a_6^2 - 448a_4^3a_5 - 1024a_4^3a_7 + 1792a_4^2a_5a_6 + 2048a_4^2a_6a_7\\
&\quad - 3072a_4a_5^3 + 11264a_4a_5a_6^2 + 71680a_3^3 - 53760a_3^2a_4 - 114688a_3^2a_6\\
&\quad + 10752a_3a_4^2 + 44032a_3a_4a_6 - 10240a_3a_5^2 - 16384a_3a_5a_7 + 45056a_3a_6^2\\
&\quad - 320a_4^3 - 2304a_4^2a_6 + 4608a_4a_5^2 + 12288a_4a_5a_7 - 3584a_4a_6^2\\
&\quad + 4096a_5^2a_6 - 6144a_6^3 + 7168a_3a_5 + 24576a_3a_7 - 2304a_4a_5 - 6144a_4a_7\\
&\quad - 4096a_5a_6 - 16384a_6a_7 - 1536a_3 + 384a_4 + 1024a_6)\, b_{11}\big],\\
B_{10} &= \cdots,\\
&\ \ \vdots
\end{aligned}$$

where $B_{2j} = 0,\ j = 1, 2, \ldots, k-1$ have been used in computing $B_{2(j+1)}$. Note that all these generalized Lyapunov constants are linear functions of the perturbation parameters: $b_{11}, b_{21}, b_{03}, b_{12}$ and b_{02}.

Now solving $B_2 = 0$ for b_{21} we have $b_{21} = b_{11}$, solving $B_4 = 0$ for b_{03} we obtain

$$b_{03} = -\frac{1}{96}\Big[16a_4 b_{12} + 48(4a_3 - a_4)\, b_{02} + 2(12a_3a_4 - 5a_4^2 + 8a_5 - 4)\, b_{11}\Big], \tag{3.4}$$

and solving $B_6 = 0$ for b_{12} yields

$$\begin{aligned} b_{12} = {} & \frac{-1}{128(a_4^3 - 4a_4a_5 - 40a_3 + 10a_4 + 8a_6)}\Big[64(12a_3a_4^2 - 13a_4^3 - 32a_3a_5 \\ & + 48a_4a_5 + 160a_3 - 40a_4 - 80a_6)\, b_{02} - 8(31a_4^4 - 52a_3a_4^3 \\ & - 136a_4^2a_5 + 192a_3a_4a_5 + 960a_3^2 + 84a_4^2 + 64a_5^2 - 480a_3a_4 \\ & - 320a_3a_6 + 192a_4a_6 - 64a_5 - 256a_7 + 16)\, b_{11}\Big]. \end{aligned} \tag{3.5}$$

Then, solving b_{02} from the equation $B_8 = 0$ we obtain $b_{02} = \frac{b_{02n}}{b_{02d}}\, b_{11}$, where b_{02n} and b_{02d} are functions of the 5 parameters, a_3, a_4, a_5, a_6 and a_7, and so all the solutions b_{21}, b_{03}, b_{12} are given in the form of $\frac{b_{ijn}}{b_{ijd}}\, b_{11}$, where b_{ijn} and b_{ijd} are functions of the 5 parameters, a_3, a_4, a_5, a_6 and a_7. Thus, we can use these solutions to simplify the expression of $B_{10}, B_{12}, \ldots, B_{18}$, which are however nonlinear functions of the parameters, a_3, a_4, a_5, a_6 and a_7. In principle, we may solve these nonlinear polynomial equations to obtain the critical values of the 5 parameters. However, it is too difficult to obtain the solutions. Thus, we set $a_4 = 0$ in these equations to obtain the simplified expressions for B_{10}, B_{12}, B_{14} and B_{16}, which are not listed here for brevity.

To solve the equations $B_{10} = B_{12} = B_{14} = B_{16} = 0$ for the parameters, a_3, a_5, a_6 and a_7, we take the numerators of these rational questions to get $B_{10n}, B_{12n}, B_{14n}, B_{16n}$ which are polynomial equations in a_3, a_5, a_6 and a_7. However, it is still even not easy to solve these coupled multivariate polynomial equations. We first use the Maple built-in command *eliminate* to eliminate a_7 from the three pairs of equations: $\{B_{10n} = B_{12n} = 0\}$, $\{B_{10n} = B_{14n} = 0\}$ and $\{B_{10n} = B_{16n} = 0\}$, yielding three solutions for a_7: $a_{7a} = a_{7a}(a_3, a_5, a_6)$, $a_{7b} = a_{7b}(a_3, a_5, a_6)$, and $a_{7c} = a_{7c}(a_3, a_5, a_6)$, as well as three resultants:

$$\mathrm{R}_{12} = \mathrm{R}_0\, \mathrm{R}_{12\mathrm{a}}, \quad \mathrm{R}_{13} = \mathrm{R}_0\, \mathrm{R}_{13\mathrm{a}}, \quad \mathrm{R}_{14} = \mathrm{R}_0\, \mathrm{R}_{14\mathrm{a}},$$

where the common factor is given by

$$\begin{aligned} \mathrm{R}_0 = {} & (a_3 - a_6)(2a_3^2a_5 - 2a_3a_6 + a_6^2)(120a_3^4a_5a_6 + 48a_3^3a_5^3 - 64a_3^3a_5a_6^2 \\ & - 8a_3^2a_5^3a_6 + 8a_3^2a_5a_6^3 + 700a_3^5 - 1260a_3^4a_6 - 84a_3^3a_5^2 + 664a_3^3a_6^2 \\ & + 60a_3^2a_5^2a_6 - 148a_3^2a_6^3 - 4a_3a_5^2a_6^2 + 12a_3a_6^4 + 28a_3^3a_5 - 42a_3^2a_5a_6 \\ & + 8a_3a_5a_6^2 + 4a_5a_6^3 - 15a_3^3 + 55a_3^2a_6 - 51a_3a_6^2 + 14a_6^3), \end{aligned}$$

and $\mathrm{R}_{12\mathrm{a}}$, $\mathrm{R}_{13\mathrm{a}}$ and $\mathrm{R}_{12\mathrm{a}}$ are lengthy polynomials in a_3, a_5 and a_6.

Next, we used the Maple built-in command *resultant* to eliminate a_5 from the two pairs of equations: $\{\mathrm{R}_{12\mathrm{a}} = \mathrm{R}_{13\mathrm{a}} = 0\}$ and $\{\mathrm{R}_{12\mathrm{a}} = \mathrm{R}_{14\mathrm{a}} = 0\}$ to obtain two resultants:

$$\begin{aligned} \mathrm{R}_{123} = {} & -913438523331814323877303020447676887284957839360 0 \\ & \times a_3^{35}\, \mathrm{R}_{123\mathrm{a}}\, \mathrm{R}_{123\mathrm{b}}, \\ \mathrm{R}_{124} = {} & 126854524609344241821270421154365059064474913106274634082711 \\ & 31806943135852099204756275200 00\, a_3^{41}\, \mathrm{R}_{124\mathrm{a}}\, \mathrm{R}_{124\mathrm{b}}, \end{aligned}$$

where $\mathrm{R}_{123\mathrm{a}}, \mathrm{R}_{123\mathrm{b}}, \mathrm{R}_{124\mathrm{a}}$ and $\mathrm{R}_{124\mathrm{b}}$ are lengthy polynomial equations in a_3 and a_6, having respectively 916, 8937, 1510 and 13182 terms.

For the next step elimination, we have four possible combinations of groups: ($\mathrm{R}_{123\mathrm{a}}$, $\mathrm{R}_{124\mathrm{a}}$), ($\mathrm{R}_{123\mathrm{a}}$, $\mathrm{R}_{124\mathrm{b}}$), ($\mathrm{R}_{123\mathrm{b}}$, $\mathrm{R}_{124\mathrm{a}}$), and ($\mathrm{R}_{123\mathrm{b}}$, $\mathrm{R}_{124\mathrm{b}}$). For example, eliminating a_6 from the equations $\mathrm{R}_{123\mathrm{a}} = \mathrm{R}_{124\mathrm{a}} = 0$ yields the resultant:

$$\mathrm{R}_{1234\mathrm{aa}} = C_{1234aa}\, a_3^{1097}\, \mathrm{R}_{1234\mathrm{aaI}}\, \mathrm{R}_{1234\mathrm{aaII}},$$

where C_{1234aa} is a big integer, and $\mathrm{R}_{1234\mathrm{aaI}}$ and $\mathrm{R}_{1234\mathrm{aaII}}$ are respectively 353th- and 97904th-degree polynomials in a_3^2. Since what we want is to prove the existence of 9 limit cycles, we will not pursue finding all solutions. Therefore, we will focus on the real positive solutions of $\mathrm{R}_{1234\mathrm{aaI}} = 0$, which in turn results in 19 positive solutions for a_3^2:

$$\begin{aligned} a_3^2 = {} & 0.0000047737\ldots, \quad 0.0043833120\ldots, \quad 0.0050516787\ldots, \\ & 0.0057391498\ldots, \quad 0.0155849899\ldots, \quad 0.0161597724\ldots, \\ & 0.0274462424\ldots, \quad 0.0371693574\ldots, \quad 0.0436129450\ldots, \\ & 0.1178983870\ldots, \quad 0.1555918035\ldots, \quad 0.1649910472\ldots, \end{aligned}$$

$$\begin{array}{lll} 0.1956514110\ldots, & 0.4483101251\ldots, & 0.4612276143\ldots, \\ 3.9699788837\ldots, & 13.0139833181\ldots, & 14.3192212987\ldots. \end{array}$$

Then, using the above solutions and the equations $\mathrm{R}_{123a} = \mathrm{R}_{124a} = 0$ to find the solutions for a_6, given below:

$$\begin{aligned} a_6 = \; & 0.0032904222\ldots, && -0.0101420304\ldots, && 0.1267047595\ldots, \\ & -0.0083406402\ldots, && 0.1922454390\ldots, && 0.2018786400\ldots, \\ & -0.0298534229\ldots, && 0.4235319232\ldots, && 0.5411579525\ldots, \\ & 2.2707745188\ldots, && -2.8071053800\ldots, && 1.2451525547\ldots, \\ & 0.2086442785\ldots, && 1.8777771810\ldots, && 2.3732374571\ldots, \\ & 11.7595031566\ldots, && 17.2346275398\ldots, && 11.1715933095\ldots. \end{aligned}$$

Next, with the above solutions (a_3, a_6), we use the equations $\mathrm{R}_{12a} = \mathrm{R}_{13a} = \mathrm{R}_{14a} = 0$ to obtain the corresponding solutions for a_5:

$$\begin{aligned} a_5 = \; & 0.2277587348\ldots, && 0.1647267417\ldots, && 0.0951642602\ldots, \\ & -0.0886453485\ldots, && 0.0599036391\ldots, && 0.0610011693\ldots, \\ & -0.1575993309\ldots, && -0.2887545838\ldots, && 0.5271077816\ldots, \\ & 3.4396716042\ldots, && -3.7364594137\ldots, && -0.8685506261\ldots, \\ & -0.3167963445\ldots, && -0.9749256575\ldots, && -2.2207448688\ldots, \\ & -7.2450730009\ldots, && -9.2187025739\ldots, && 2.9913889631\ldots. \end{aligned}$$

Finally, we need to verify the above solutions if they satisfy $a_{7a}(a_3, a_5, a_6) = a_{7b}(a_3, a_5, a_6) = a_{7c}(a_3, a_5, a_6)$. It is indeed true to give the following solutions:

$$\begin{aligned} a_7 = \; & 0.0189364482\ldots, && 0.0517693763\ldots, && 0.0189656802\ldots, \\ & -0.0051527715\ldots, && 0.0344175159\ldots, && 0.0366298619\ldots, \\ & -0.0063769221\ldots, && 0.0234700554\ldots, && 0.2279041909\ldots, \\ & 3.1588396190\ldots, && 0.1527460842\ldots, && -0.4539447259\ldots, \\ & 0.1183456948\ldots, && -0.0733264349\ldots, && -0.6036030526\ldots, \\ & -5.1642784609\ldots, && -8.6230707758\ldots, && 3.8240906488\ldots. \end{aligned}$$

Thus, since the solutions of a_3^2 give $\pm\, a_3$, we have obtained at least 36 solutions which satisfy $B_2 = B_4 = \cdots = B_{16} = 0$. For example, taking the third solution, we have

$$\begin{array}{lll} a_3 = 0.0710751623\ldots, & a_5 = 0.0951642602\ldots, & \\ a_6 = 0.1267047595\ldots, & a_7 = 0.0189656807\ldots, & a_4 = 0, \\ b_{03} = 0.0483231993\ldots b_{11}, & b_{12} = -0.2221842342\ldots b_{11}, & \\ b_{02} = 0.1347124879\ldots b_{11}, & b_{21} = b_{11}, & \end{array}$$

for which

$$\begin{aligned} & B_2 = B_4 = B_6 = B_8 = B_{10} = B_{12} = B_{16} = 0, \\ & B_{18} = 0.0065604251\ldots\, b_{11}, \end{aligned}$$

where the free parameter b_{11} can be set as $b_{11} = 1$, as expected. Moreover, at the above critical parameter values, we obtain

$$\det\left[\frac{\partial(B_2, B_4, B_6, B_8, B_{10}, B_{12}, B_{14}, B_{16})}{\partial(b_{21}, b_{03}, b_{12}, b_{02}, a_3, a_5, a_6, a_7)}\right] = -\,0.3604666515\ldots \times 10^4\, b_{11}^4 \neq 0.$$

Therefore, proper perturbations on $b_{21}, b_{03}, b_{12}, b_{02}, a_3, a_5, a_6$ and a_7 can be taken to find 8 small-amplitude limit cycles. Finally, we perturb b_{01} such that $B_0 B_2 < 0$ and $|B_0| \ll |B_2|$ to have one more small-amplitude limit cycle, leading to a total of 9 limit cycles around the nilpotent center – the origin.

The proof of Theorem 3.1 is complete. □

Remark 3.1. It should be noted that all the computations in the above proof are symbolic, except in the last step when solving the polynomial equation $\mathrm{R}_{1234aa1} = 0$ with integer coefficients the Maple built-in command “fsolve” is used to find the solutions of a_3^2 with accuracy of 1000 digits. In fact, we have tried to apply the *Regular Chain* method (e.g., see [24]), which was recently developed on the basis of triangular decomposition to solve multi-variate polynomial equations and has been implemented in Maple, to solve the four polynomial equations $B_{10n} = B_{12n} = B_{14n} = B_{16n} = 0$ for a_3, a_5, a_6 and a_7. However, it failed to get any results form a fast computer with higher Ram memory. That is why we turned to use the procedure described above to solve these polynomial equations. In fact, at the last step, with 1000 digits accuracy, at the critical values, the exact values of B_i's and the determinant are given as follows (in which $b_{11} = 1$ for simplicity):

$$B_2 = 0, \quad B_4 = 0.4 \times 10^{-999}, \quad B_6 = -0.104 \times 10^{-998}, \quad B_8 = -0.266 \times 10^{-997},$$

$$
\begin{aligned}
&B_{10} = -0.458062 \times 10^{-994}, \quad B_{12} = -0.4001502 \times 10^{-993},\\
&B_{14} = -0.4028116935380759371 \times 10^{-981}, \quad B_{16} = -0.3807411080171892 \times 10^{-980},\\
&B_{18} = 0.0065604251 \ldots \neq 0, \quad \det = -0.3604666515 \ldots \times 10^{4} \neq 0.
\end{aligned}
$$

This clearly indicates that we can appropriately perturb the (almost zero) generalized Lyapunov constants B_{2k}, $k = 1, 2, \ldots, 8$ such that $B_{2k}B_{2(k+1)} < 0$ to prove the existence of 9 limit cycles by Sturm's theorem. An alternative to symbolically prove the existence of real solutions of the polynomial equation $\mathrm{R}_{1234aa1} = 0$ is to use the "interval computation". One may apply the Maple built-in command "realroot" to solve the polynomial equation $\mathrm{R}_{1234aa1} = 0$ to obtain (with the accuracy, say, 10^{-10} for the interval length, which can be changed to any small numbers as one wishes)

$$
\begin{aligned}
\Bigg[&\left[\frac{20503}{4294967296}, \frac{82013}{17179869184}\right], &&\left[\frac{9413091}{2147483648}, \frac{75304729}{17179869184}\right], &&\left[\frac{86787179}{17179869184}, \frac{21696795}{4294967296}\right],\\
&\left[\frac{49298921}{8589934592}, \frac{98597843}{17179869184}\right], &&\left[\frac{33468511}{2147483648}, \frac{267748089}{17179869184}\right], &&\left[\frac{34702847}{2147483648}, \frac{277622777}{17179869184}\right],\\
&\left[\frac{235761427}{8589934592}, \frac{471522855}{17179869184}\right], &&\left[\frac{319282349}{8589934592}, \frac{638564699}{17179869184}\right], &&\left[\frac{374632345}{8589934592}, \frac{749264691}{17179869184}\right],\\
&\left[\frac{2025478865}{17179869184}, \frac{1012739433}{8589934592}\right], &&\left[\frac{1336523415}{8589934592}, \frac{2673046831}{17179869184}\right], &&\left[\frac{44289447}{268435456}, \frac{2834524609}{17179869184}\right],\\
&\left[\frac{3361265647}{17179869184}, \frac{210079103}{1073741824}\right], &&\left[\frac{962738663}{2147483648}, \frac{7701909305}{17179869184}\right], &&\left[\frac{3961915039}{8589934592}, \frac{7923830079}{17179869184}\right],\\
&\left[\frac{34101858943}{8589934592}, \frac{68203717887}{17179869184}\right], &&\left[\frac{27947316371}{2147483648}, \frac{223578530969}{17179869184}\right], &&\left[\frac{246002348729}{17179869184}, \frac{123001174365}{8589934592}\right]\Bigg],
\end{aligned}
$$

which indeed clearly shows the existence of the 18 real solutions. However, using the numerical expressions is straightforward and more clear.

4. Conclusion

In this paper, we have shown that planar cubic near-Hamiltonian systems can have at least 9 limit cycles around a nilpotent center. Normal form theory has been used to compute the generalized Lyapunov constants, and then to determine the number of bifurcating limit cycles from Hopf bifurcation. It has demonstrated the computational efficiency of this method, which can be applied to consider bifurcation of limit cycles in other dynamical systems. Future work includes finding 10 or 11 small-amplitude limit cycles for planar cubic near-Hamiltonian systems with a nilpotent center.

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