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Linear feedback control, adaptive feedback control and their combination for chaos (lag) synchronization of LC chaotic systems

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Abstract

In this paper, we study chaos (lag) synchronization of a new LC chaotic system, which can exhibit not only a two-scroll attractor but also two double-scroll attractors for different parameter values, via three types of state feedback controls: (i) linear feedback control; (ii) adaptive feedback control; and (iii) a combination of linear feedback and adaptive feedback controls. As a consequence, ten families of new feedback control laws are designed to obtain global chaos lag synchronization for $\tau < 0$ and global chaos synchronization for $\tau = 0$ of the LC system. Numerical simulations are used to illustrate these theoretical results. Each family of these obtained feedback control laws, including two linear (adaptive) functions or one linear function and one adaptive function, is added to two equations of the LC system. This is simpler than the known synchronization controllers, which apply controllers to all equations of the LC system. Moreover, based on the obtained results of the LC system, we also derive the control laws for chaos (lag) synchronization of another new type of chaotic system.

1. Introduction

In 1963, Lorenz simplified the Navior–Stokes equations of modelling weather forecasting and discovered sensitivity dependence on initial conditions in a set of three ordinary nonlinear differential equations [1]. Li and Yorke [2] first presented the name ''chaos'' in the sense of ''period three implies chaos''. Chaos embodies three important principles: extreme sensitivity to initial conditions, cause and effect being not proportional, as well as nonlinearity. Chaos exists in many disciplines of sciences and engineering such as atmosphere, mechanics, electronics, biology, chemistry, stock market, etc. Since the discovery of the Lorenz system, more chaotic (hyperchaotic) systems have been constructed such as Rössler system, hyperchaotic Rössler system, Chua's circuit, Hénon attractor, Logistic map, Chen system, generalized Lorenz system, hyperchaotic MCK circuit, hyperchaotic Chen system, etc. [3–11]. Nowadays, it is perhaps not difficult

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to construct a new chaotic (hyperchaotic) system. Recently, Liu and Chen [12] presented the following chaotic LC system, which contains three ordinary differential equations with three cross-product nonlinear terms, described by

$$\begin{cases} \dot{x} = ax + yz, \\ \dot{y} = -by - xz, \\ \dot{z} = -cz - xy, \end{cases} \tag{1.1}$$

where dot denotes differentiation with respect to time t, $a > 0$, $b > 0$, $c > 0$ and $b + c > a$. System (1.1) is symmetrical and dispersive. It admits five equilibria, but does not have Hopf and pitch fork bifurcations. It can exhibit not only a two-scroll attractor but also two double-scroll attractors for different parameter values.

Chaos control and chaos synchronization have received a significant attention in the last few years, since Pecora and Carroll [14] presented the principal of chaos synchronization to synchronize two identical chaotic systems with different initial values and realized that synchronized chaos may be used for secure communication. In 1990, Ott, Grebogi and Yorke [15], on the other hand, developed the so-called OGY method for controlling chaos. Many methods have been developed to investigate chaos control and synchronization of chaotic (hyperchaotic) attractors such as linear feedback control, adaptive control, backstepping design control, fuzzy control, implosive control, nonlinear feedback control, time-delayed feedback control, etc. [10,11,16–30].

Recently, Liu and Chen [13] further analyzed the dynamical behaviors of system (1.1). Based on more general quadratic Lyapunov function, Sun[28] studied the stabilization and synchronization of system (1.1) via an impulsive control with varying impulsive intervals. More recently, Luo et al. [29] studied chaos synchronization of two identical LC systems in the form of (1.1) using nonlinear feedback control, adaptive control and adaptive sliding mode type of control. Yassen [30] considered chaos control and synchronization of another equivalent form of system (1.1) under the simple transformation $(x \to x, y \to y, z \to -z)$ using three linear feedback control functions. It should be pointed out that Luo et al. [29] used the following adaptive control:

$$u_1 = -k_1e_1 - 0.5z_1e_2 - 0.5y_1e_3, \quad u_2 = -k_2e_2 + x_1e_3, \quad u_3 = -k_3e_3 + x_1e_2,$$

where $\dot{k}_1 = \gamma_1 e_1^2, \dot{k}_2 = \gamma_2 e_2^2, \dot{k}_3 = \gamma_3 e_3^2$, and γ_i's are non-zero constants, and that Yassen [30] applied the linear feedback control, given by

$$u_1 = k_1e_1, \quad u_2 = k_2e_2, \quad u_3 = k_3e_3, \quad (k_1k_2k_3 \neq 0).$$

In fact, system (1.1) has 'good' symmetric property. In this paper, we will present *simpler control laws* for globally exponential (lag) synchronization of the LC system (1.1) via three different methods: (i) linear feedback control with only two linear functions; (ii) adaptive feedback control with only two adaptive functions; and (iii) combination of (i) and (ii) with one linear function and one adaptive function.

The rest of this paper is arranged as follows. In Section 2, we give Lyapunov stability criteria for globally exponential (lag) synchronization of n-dimensional chaotic systems. Then we use the theory and linear feedback controls to obtain a number of different linear feedback controllers, each of which contains only two linear control functions, to investigate globally exponential (lag) synchronization of the LC system (1.1). In Section 3, we present two types of adaptive control laws, which also includes only two functions to demonstrate global chaos (lag) synchronization of (1.1). In Section 4, we obtain four families of combined control laws with one linear feedback control function and one adaptive feedback control function to study global chaos (lag) synchronization of (1.1). Numerical simulation results are presented to illustrate the analytical predictions in Section 5. In Section 6, based on the obtained results of the LC system, we derive the control laws for chaos (lag) synchronization of another new type of chaotic system, studied recently by Lü et al. [31]. Finally, conclusions are given in Section 7.

2. Linear feedback control laws

We first recall the definition and lemma for the globally exponential (lag) synchronization of chaotic systems. Consider the drive system:

$$\dot{\mathbf{x}}_d = \mathbf{F}(t, \mathbf{x}_d), \tag{2.1}$$

and response system:

$$\dot{\mathbf{y}}_r = \mathbf{F}(t, \mathbf{y}_r) + \mathbf{u}, \tag{2.2}$$

where the subscripts "d" and "r" stand for the drive system and response system, respectively, $\mathbf{x}_d = (x_{1d}, x_{2d}, \ldots, x_{nd})^{\mathrm{T}}$, $\mathbf{y}_r = (y_{1r}, y_{2r}, \ldots, y_{nr})^{\mathrm{T}}, \mathbf{F}$: $R_+ \times R^n \to R^n$, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ is a vector function of time t and the state variables $(x_{id}, y_{id}, x_{ir}, y_{ir})$.

Let the error state be $\mathbf{e}(t) = (e_1(t), e_2(t), \ldots, e_n(t))^{\mathrm{T}} = (x_{1d}(t) - y_{1r}(t-\tau), x_{2d}(t) - y_{2r}(t-\tau), \ldots, x_{nd}(t) - y_{nr}(t-\tau))^{\mathrm{T}}$ ($\tau \leqslant 0$). Then the error dynamical system on $\mathbf{e}(t)$ is given by

$$\dot{\mathbf{e}}(t) = \mathbf{F}(t, \mathbf{x}_d(t)) - \mathbf{F}(t-\tau, \mathbf{y}_r(t-\tau)) - \mathbf{U}, \tag{2.3}$$

where $\mathbf{U} = \mathbf{u}|_{t=t-\tau}$.

Definition 1. For arbitrary given initial points, $(x_{1d}(t), x_{2d}(t), \ldots, x_{nd}(t))$ and $(y_{1r}(t), y_{2r}(t), \ldots, y_{nr}(t)) \in R^n$, $t \in [0, -\tau]$, of the drive system (2.1) and the response system (2.2), respectively, if the solution of the error system (2.3) has the estimation $\sum_{i=1}^n e_i^2(t) \leqslant K(e(t_0))\exp(-\alpha(t-t_0))$, where $K(e(t_0)) > 0$ is a constant depending on the initial value $e(t_0)$, while $\alpha > 0$ is a constant independent of $e(t_0)$, then the zero solution of the error system (2.3) is said to be globally, exponentially stable, and thus the drive-response systems (2.1) and (2.2) are (i) globally exponentially lag synchronized for $\tau < 0$; and (ii) globally exponentially synchronized for $\tau = 0$.

Lemma 1. *The zero solution of the error dynamical system* (2.3) *is globally, exponentially stable, i.e.,* (i) *the drive-response systems* (2.1) *and* (2.2) *are globally exponentially lag synchronized for* $\tau < 0$; *and* (ii) *globally exponentially synchronized for* $\tau = 0$, *if there exists a positive definite quadratic polynomial* $V = (e_1\ e_2\ \cdots\ e_n)P(e_1\ e_2\ \cdots\ e_n)^{\mathrm{T}}$ *such that* $\frac{\mathrm{d}V}{\mathrm{d}t} = -(e_1\ e_2\ \cdots\ e_n)Q(e_1\ e_2\ \cdots\ e_n)^{\mathrm{T}}$. *Moreover, the following negative Lyapunov exponent estimation for the error dynamical system* (2.3) *holds*:

$$\sum_{i=1}^n e_i^2(t) \leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \sum_{i=1}^n e_i^2(t_0) \exp\left[-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_0)\right],$$

where $P = P^{\mathrm{T}} \in R^{n\times n}$ *and* $Q = Q^{\mathrm{T}} \in R^{n\times n}$ *are both positive definite matrices,* $\lambda_{\max}(P)$ *and* $\lambda_{\min}(P)$ *stand for the minimum and maximum eigenvalues of the matrix* P, *respectively, and* $\lambda_{\min}(Q)$ *denotes the minimum eigenvalue of the matrix* Q.

Now, consider the LC chaotic system (1.1) as a drive system:

$$\begin{cases} \dot{x}_d = ax_d + y_d z_d, \\ \dot{y}_d = -by_d - x_d z_d, \\ \dot{z}_d = -cz_d - x_d y_d, \end{cases} \tag{2.4}$$

and the system related to (2.4) with feedback controllers u_i ($i = 1, 2, 3$), given by

$$\begin{cases} \dot{x}_r = ax_r + y_r z_r + u_1, \\ \dot{y}_r = -by_r - x_r z_r + u_2, \\ \dot{z}_r = -cz_r - x_r y_r + u_3, \end{cases} \tag{2.5}$$

as a response system, where u_i's are unknown functions of $(x_d, y_d, z_d, x_r, y_r, z_r)$.

Let the error state be $\mathbf{e}(t) = (e_x(t), e_y(t), e_z(t))^{\mathrm{T}} = [x_d(t) - x_r(t-\tau), y_d(t) - y_r(t-\tau), z_d(t) - z_r(t-\tau)]^{\mathrm{T}}$, where $\tau \leqslant 0$. Then from (2.4) and (2.5), we obtain the error dynamical system:

$$\begin{cases} \dot{e}_x(t) = ae_x(t) + y_d(t)z_d(t) - y_r(t-\tau)z_r(t-\tau) - U_1, \\ \dot{e}_y(t) = -be_y(t) - x_d(t)z_d(t) + x_r(t-\tau)z_r(t-\tau) - U_2, \\ \dot{e}_z(t) = -ce_z(t) - x_d(t)y_d(t) + x_r(t-\tau)y_r(t-\tau) - U_3, \end{cases} \tag{2.6}$$

where $U_i = u_i|_{t=t-\tau}$ ($i = 1, 2, 3$).

Before presenting theorems, in the following we first list different types of decompositions of the nonlinear terms in (2.6):

$$y_d(t)z_d(t) - y_r(t-\tau)z_r(t-\tau) = y_d(t)e_z(t) + z_r(t-\tau)e_y(t), \tag{2.6a}$$

$$y_d(t)z_d(t) - y_r(t-\tau)z_r(t-\tau) = y_r(t-\tau)e_z(t) + z_d(t)e_y(t), \tag{2.6b}$$

$$x_d(t)z_d(t) - x_r(t-\tau)z_r(t-\tau) = x_d(t)e_z(t) + z_r(t-\tau)e_x(t), \tag{2.6c}$$

$$x_d(t)z_d(t) - x_r(t-\tau)z_r(t-\tau) = x_r(t-\tau)e_z(t) + z_d(t)e_x(t), \tag{2.6d}$$

$$x_d(t)y_d(t) - x_r(t-\tau)y_r(t-\tau) = x_d(t)e_y(t) + y_r(t-\tau)e_x(t), \tag{2.6e}$$

$$x_d(t)y_d(t) - x_r(t-\tau)y_r(t-\tau) = x_r(t-\tau)e_y(t) + y_d(t)e_x(t). \tag{2.6f}$$

By using these different types of decompositions (2.6a)–(2.6f), we can obtain a number of linear feedback control laws with two linear functions. Therefore, we have the following theorem.

Theorem 1. *For the given drive system (2.4), the response system (2.5) with unknown controllers and the corresponding error dynamical system (2.6), suppose that* $M_{x_d}, M_{x_r}, M_{y_d}, M_{y_r}, M_{z_d}$ *and* M_{z_r} *are the upper bounds of the state variables* $|x_d(t)|$, $|x_r(t-\tau)|$, $|y_d(t)|$, $|y_r(t-\tau)|$, $|z_d(t)|$ *and* $|z_r(t-\tau)|$, *respectively. If one of the following families of linear feedback control laws* ($U_i = u_i|_{t=t-\tau}$ $(i=1,2,3)$) *is chosen for the response system (2.5):*

(i) $u_1 = k_1 e_x(t)$, $u_2 = k_2 e_y(t)$, $u_3 = 0$, *where* $k_1 > a, k_2 > \min\left[\frac{(M_{x_d}+M_{x_r})^2}{4c}, \frac{(M_{z_d}+M_{z_r})^2}{4(k_1-a)} + \frac{\min(M_{x_r}^2, M_{x_d}^2)}{c}\right] - b$;

(ii) $u_1 = k_1 e_x(t)$, $u_2 = 0$, $u_3 = k_3 e_z(t)$, *where* $k_1 > a, k_3 > \min\left[\frac{(M_{x_d}+M_{x_r})^2}{4b}, \frac{(M_{y_d}+M_{y_r})^2}{4(k_1-a)} + \frac{\min(M_{x_r}^2, M_{x_d}^2)}{b}\right] - c$;

(iii) $u_1 = k_1 e_x(t)$, $u_2 = k_2 e_y(t)$, $u_3 = 0$, *where* $k_2 > \frac{4(k_1-a)\min(M_{x_r}^2, M_{x_d}^2)}{4c(k_1-a)-(M_{y_d}+M_{y_r})^2} - b, k_1 > \frac{(M_{y_d}+M_{y_r})^2}{4c} + a$;

(iv) $u_1 = k_1 e_x(t)$, $u_2 = 0$, $u_3 = k_3 e_z(t)$, *where* $k_3 > \frac{4(k_1-a)\min(M_{x_r}^2, M_{x_d}^2)}{4b(k_1-a)-(M_{z_d}+M_{z_r})^2} - c, k_1 > \frac{(M_{z_d}+M_{z_r})^2}{4b} + a$,

then the zero solution of the error system (2.6) is globally, exponentially stable, and thus the drive-response systems (2.4) and (2.5) are (i) *globally exponentially lag synchronized for* $\tau < 0$; *and* (ii) *globally exponentially synchronized for* $\tau = 0$.

Proof

(i) For this case, we choose the positive definite Lyapunov function:

$$V(t) = \frac{1}{2}\left(e_x^2(t) + e_y^2(t) + e_z^2(t)\right), \tag{2.7}$$

from which we know that $P = \mathrm{diag}[0.5, 0.5, 0.5]$ and $\lambda_{\min} = \lambda_{\max} = 0.5$. Differentiating both sides of (2.7) with respect to time t along the trajectory of system (2.6) and using (2.6a)–(2.6f) yields

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= e_x(t)\dot{e}_x(t) + e_y(t)\dot{e}_y(t) + e_z(t)\dot{e}_z(t)\\
&= ae_x^2(t) + y_d(t)z_d(t)e_x(t) - y_r(t-\tau)z_r(t-\tau)e_x(t) - k_1e_x^2(t) - be_y^2(t)\\
&\quad - x_d(t)z_d(t)e_y(t) + x_r(t-\tau)z_r(t-\tau)e_y(t) - k_2e_y^2(t) - ce_z^2(t) - x_d(t)y_d(t)e_z(t) + x_r(t-\tau)y_r(t-\tau)e_z(t)\\
&= \begin{cases}
ae_x^2(t) + y_d(t)e_z(t)e_x(t) + z_r(t-\tau)e_y(t)e_x(t) - k_1e_x^2(t) - be_y^2(t)\\
\quad -x_d(t)e_z(t)e_y(t) - z_r(t-\tau)e_x(t)e_y(t) - k_2e_y^2(t) - ce_z^2(t)\\
\quad -y_d(t)e_x(t)e_z(t) - x_r(t-\tau)e_y(t)e_z(t),\\
ae_x^2(t) + y_d(t)e_z(t)e_x(t) + z_r(t-\tau)e_y(t)e_x(t) - k_1e_x^2(t) - be_y^2(t)\\
\quad -z_d(t)e_x(t)e_y(t) - x_r(t-\tau)e_z(t)e_y(t) - k_2e_y^2(t) - ce_z^2(t)\\
\quad -y_d(t)e_x(t)e_z(t) - x_r(t-\tau)e_y(t)e_z(t),\\
ae_x^2(t) + z_d(t)e_y(t)e_x(t) + y_r(t-\tau)e_z(t)e_x(t) - k_1e_x^2(t) - be_y^2(t)\\
\quad -x_d(t)e_z(t)e_y(t) - z_r(t-\tau)e_x(t)e_y(t) - k_2e_y^2(t) - ce_z^2(t)\\
\quad -x_d(t)e_y(t)e_z(t) - y_r(t-\tau)e_x(t)e_z(t),
\end{cases}\\
&= \begin{cases}
(a-k_1)e_x^2(t) - (k_2+b)e_y^2(t) - ce_z^2(t) - [x_d(t) + x_r(t-\tau)]e_y(t)e_z(t),\\
(a-k_1)e_x^2(t) - (k_2+b)e_y^2(t) - ce_z^2(t) - [z_d(t) - z_r(t-\tau)]e_x(t)e_y(t)\\
\quad -2x_r(t-\tau)e_y(t)e_z(t),\\
(a-k_1)e_x^2(t) - (k_2+b)e_y^2(t) - ce_z^2(t) + [z_d(t) - z_r(t-\tau)]e_x(t)e_y(t)\\
\quad -2x_d(t)e_y(t)e_z(t),
\end{cases}\\
&\leqslant \begin{cases}
-(k_1-a)e_x^2(t) - (k_2+b)e_y^2(t) - ce_z^2(t) + (M_{x_d} + M_{x_r})|e_y(t)||e_z(t)|,\\
-(k_1-a)e_x^2(t) - (k_2+b)e_y^2(t) - ce_z^2(t) + (M_{z_d} + M_{z_r})|e_x(t)||e_y(t)|\\
\quad +2M_{x_r}|e_y(t)||e_z(t)|,\\
-(k_1-a)e_x^2(t) - (k_2+b)e_y^2(t) - ce_z^2(t) + (M_{z_d} + M_{z_r})|e_x(t)||e_y(t)|\\
\quad +2M_{x_d}|e_y(t)||e_z(t)|,
\end{cases}\\
&= \begin{cases}
-(|e_x(t)||e_y(t)||e_z(t)|)Q_1(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}},\\
-(|e_x(t)||e_y(t)||e_z(t)|)Q_2(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}},\\
-(|e_x(t)||e_y(t)||e_z(t)|)Q_3(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}},
\end{cases}
\end{aligned} \tag{2.8}$$

where $Q_i = Q_i^{\mathrm{T}}$ $(i = 1, 2, 3)$ are given by

$$Q_1 = \begin{bmatrix} k_1 - a & 0 & 0 \\ 0 & k_2 + b & -\frac{1}{2}(M_{x_d} + M_{x_r}) \\ 0 & -\frac{1}{2}(M_{x_d} + M_{x_r}) & c \end{bmatrix}, \tag{2.9a}$$

$$Q_2 = \begin{bmatrix} k_1 - a & -\frac{1}{2}(M_{z_d} + M_{z_r}) & 0 \\ -\frac{1}{2}(M_{z_d} + M_{z_r}) & k_2 + b & -M_{x_r} \\ 0 & -M_{x_r} & c \end{bmatrix}, \tag{2.9b}$$

$$Q_3 = \begin{bmatrix} k_1 - a & -\frac{1}{2}(M_{z_d} + M_{z_r}) & 0 \\ -\frac{1}{2}(M_{z_d} + M_{z_r}) & k_2 + b & -M_{x_d} \\ 0 & -M_{x_d} & c \end{bmatrix}. \tag{2.9c}$$

It is easy to see that the zero solution of the error system (8) is globally, exponentially stable if the symmetric matrices Q_i's are positive definite, which implies that the following conditions hold:

$$\text{for } Q_1: \begin{cases} k_1 - a > 0, \\ (k_1 - a)(k_1 + b) > 0, \\ (k_1 - a)[c(k_2 + b) - \frac{1}{4}(M_{x_d} + M_{x_r})^2] > 0; \end{cases} \tag{2.10a}$$

$$\text{for } Q_2: \begin{cases} k_1 - a > 0, \\ (k_1 - a)(k_2 + b) - \frac{1}{4}(M_{z_d} + M_{z_r})^2 > 0, \\ c[(k_1 - a)(k_2 + b) - \frac{1}{4}(M_{z_d} + M_{z_r})^2] - (k_1 - a)M_{x_r}^2 > 0; \end{cases} \tag{2.10b}$$

$$\text{for } Q_3: \begin{cases} k_1 - a > 0, \\ (k_1 - a)(k_2 + b) - \frac{1}{4}(M_{z_d} + M_{z_r})^2 > 0, \\ c[(k_1 - a)(k_2 + b) - \frac{1}{4}(M_{z_d} + M_{z_r})^2] - (k_1 - a)M_{x_d}^2 > 0; \end{cases} \tag{2.10c}$$

which leads to that

$$\begin{aligned} &\text{for } Q_1: \ k_1 > a, \quad k_2 > \frac{1}{4c}(M_{x_d} + M_{x_r})^2 - b; \\ &\text{for } Q_2: \ k_1 > a, \quad k_2 > \frac{(M_{z_d} + M_{z_r})^2}{4(k_1 - a)} + \frac{M_{x_r}^2}{c} - b; \\ &\text{for } Q_3: \ k_1 > a, \quad k_2 > \frac{(M_{z_d} + M_{z_r})^2}{4(k_1 - a)} + \frac{M_{x_d}^2}{c} - b. \end{aligned} \tag{2.11}$$

Then using Lemma 1, we have the exponential estimation for the subcase Q_1:

$$\begin{aligned} e_x^2(t) + e_y^2(t) + e_z^2(t) &\leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}[e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)]e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_0)} \\ &= [e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)]e^{-\min[2(k_1-a), k_2+b+c-\sqrt{(k_2+b-c)^2+(M_{x_d}+M_{x_r})^2}](t-t_0)}. \end{aligned}$$

Similarly, we can obtain the exponential estimations for the subcases Q_2 and Q_3, which are similar to that of Q_1, and thus their expressions are omitted here. This completes the proof of Case (i).

(ii) In this case, we may choose the same Lyapunov function (2.7) to obtain

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t}=e_x(t)\dot{e}_x(t)+e_y(t)\dot{e}_y(t)+e_z(t)\dot{e}_z(t)$$

$$=\begin{cases} ae_x^2(t)+y_d(t)e_z(t)e_x(t)+z_r(t-\tau)e_y(t)e_x(t)-k_1e_x^2(t)-be_y^2(t) \\ \quad -x_d(t)e_z(t)e_y(t)-z_r(t-\tau)e_x(t)e_y(t)-ce_z^2(t)-k_3e_z^2(t) \\ \quad -y_d(t)e_x(t)e_z(t)-x_r(t-\tau)e_y(t)e_z(t), \\ ae_x^2(t)+y_d(t)e_z(t)e_x(t)+z_r(t-\tau)e_y(t)e_x(t)-k_1e_x^2(t)-be_y^2(t) \\ \quad -z_d(t)e_x(t)e_y(t)-x_r(t-\tau)e_z(t)e_y(t)-ce_z^2(t)-k_3e_z^2(t) \\ \quad -y_d(t)e_x(t)e_z(t)-x_r(t-\tau)e_y(t)e_z(t), \\ ae_x^2(t)+z_d(t)e_y(t)e_x(t)+y_r(t-\tau)e_z(t)e_x(t)-k_1e_x^2(t)-be_y^2(t) \\ \quad -x_d(t)e_z(t)e_y(t)-z_r(t-\tau)e_x(t)e_y(t)-ce_z^2(t)-k_3e_z^2(t) \\ \quad -x_d(t)e_y(t)e_z(t)-y_r(t-\tau)e_x(t)e_z(t), \end{cases}$$

$$=\begin{cases} (a-k_1)e_x^2(t)-be_y^2(t)-(k_3+c)e_z^2(t)-[x_d(t)+x_r(t-\tau)]e_y(t)e_z(t), \\ (a-k_1)e_x^2(t)-be_y^2(t)-(k_3+c)e_z^2(t)+[-y_d(t)+y_r(t-\tau)]e_x(t)e_z(t) \\ \quad -2x_r(t-\tau)e_y(t)e_z(t), \\ (a-k_1)e_x^2(t)-be_y^2(t)-(k_3+c)e_z^2(t)+[y_d(t)-y_r(t-\tau)]e_x(t)e_z(t) \\ \quad -2x_d(t)e_y(t)e_z(t), \end{cases}$$

$$\leqslant\begin{cases} -(k_1-a)e_x^2(t)-e_y^2(t)-(k_3+c)e_z^2(t)+(M_{x_d}+M_{x_r})|e_y(t)||e_z(t)|, \\ -(k_1-a)e_x^2(t)-be_y^2(t)-(k_3+c)e_z^2(t)+(M_{y_d}+M_{y_r})|e_x(t)||e_z(t)| \\ \quad +2M_{x_r}|e_y(t)||e_z(t)|, \\ -(k_1-a)e_x^2(t)-e_y^2(t)-(k_3+c)e_z^2(t)+(M_{y_d}+M_{y_r})|e_x(t)||e_z(t)| \\ \quad +2M_{x_d}|e_y(t)||e_z(t)|, \end{cases}$$

$$=\begin{cases} -(|e_x(t)||e_y(t)||e_z(t)|)Q_1(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}}, \\ -(|e_x(t)||e_y(t)||e_z(t)|)Q_2(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}}, \\ -(|e_x(t)||e_y(t)||e_z(t)|)Q_3(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}}, \end{cases} \tag{2.12}$$

where $Q_i=Q_i^{\mathrm{T}}$ $(i=1,2,3)$ are given by

$$Q_1=\begin{bmatrix} k_1-a & 0 & 0 \\ 0 & b & -\frac{1}{2}(M_{x_d}+M_{x_r}) \\ 0 & -\frac{1}{2}(M_{x_d}+M_{x_r}) & k_3+c \end{bmatrix}, \tag{2.13a}$$

$$Q_2=\begin{bmatrix} k_1-a & 0 & -\frac{1}{2}(M_{y_d}+M_{y_r}) \\ 0 & b & -M_{x_r} \\ -\frac{1}{2}(M_{y_d}+M_{y_r}) & -M_{x_r} & k_3+c \end{bmatrix}, \tag{2.13b}$$

$$Q_3=\begin{bmatrix} k_1-a & 0 & -\frac{1}{2}(M_{y_d}+M_{y_r}) \\ 0 & b & -M_{x_d} \\ -\frac{1}{2}(M_{y_d}+M_{y_r}) & -M_{x_d} & k_3+c \end{bmatrix}. \tag{2.13c}$$

It is clear that the zero solution of the error system (8) is globally, exponentially stable if the symmetric matrices Q_i's are positive definite, implying that the following conditions are satisfied, i.e.,

$$\text{for } Q_1: \begin{cases} k_1 - a > 0, \\ (k_1 - a)b > 0, \\ (k_1 - a)[b(k_3 + c) - \frac{1}{4}(M_{x_d} + M_{x_r})^2] > 0; \end{cases} \tag{2.14a}$$

$$\text{for } Q_2: \begin{cases} k_1 - a > 0, \\ (k_1 - a)b > 0, \\ (k_1 - a)[(k_3 + c)b - M_{x_r}^2] - \frac{1}{4}b(M_{y_d} + M_{y_r})^2 > 0; \end{cases} \tag{2.14b}$$

$$\text{for } Q_3: \begin{cases} k_1 - a > 0, \\ (k_1 - a)b > 0, \\ (k_1 - a)[(k_3 + c)b - M_{x_d}^2] - \frac{1}{4}b(M_{y_d} + M_{y_r})^2 > 0; \end{cases} \tag{2.14c}$$

which lead to that

$$\begin{aligned} &\text{for } Q_1: \ k_1 > a, \quad k_3 > \frac{1}{4b}(M_{x_d} + M_{x_r})^2 - c; \\ &\text{for } Q_2: \ k_1 > a, \quad k_3 > \frac{(M_{y_d} + M_{y_r})^2}{4(k_1 - a)} + \frac{M_{x_r}^2}{b} - c; \\ &\text{for } Q_3: \ k_1 > a, \quad k_3 > \frac{(M_{y_d} + M_{y_r})^2}{4(k_1 - a)} + \frac{M_{x_d}^2}{b} - c; \end{aligned} \tag{2.15}$$

from which we have

$$k_1 > a, \quad k_3 > \min\left[\frac{1}{4b}(M_{x_d} + M_{x_r})^2 - c, \frac{(M_{y_d} + M_{y_r})^2}{4(k_1 - a)} + \frac{M_{x_r}^2}{b} - c, \frac{(M_{y_d} + M_{y_r})^2}{4(k_1 - a)} + \frac{M_{x_d}^2}{b} - c\right].$$

This completes the proof for Case (ii).

(iii) For this case we again choose the same Lyapunov function (2.7). Thus, differentiating V with respect to time t along the solution of (1.1) yields

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= e_x(t)\dot{e}_x(t) + e_y(t)\dot{e}_y(t) + e_z(t)\dot{e}_z(t) \\ &= \begin{cases} ae_x^2(t) + z_d(t)e_y(t)e_x(t) + y_r(t-\tau)e_z(t)e_x(t) - k_1e_x^2(t) - be_y^2(t) \\ \quad - z_d(t)e_x(t)e_y(t) - x_r(t-\tau)e_z(t)e_y(t) - k_2e_y^2(t) - ce_z^2(t) \\ \quad - y_d(t)e_x(t)e_z(t) - x_r(t-\tau)e_y(t)e_z(t), \\ ae_x^2(t) + y_d(t)e_z(t)e_x(t) + z_r(t-\tau)e_y(t)e_x(t) - k_1e_x^2(t) - be_y^2(t) \\ \quad - x_d(t)e_z(t)e_y(t) - z_r(t-\tau)e_x(t)e_y(t) - k_2e_y^2(t) - ce_z^2(t) \\ \quad - x_d(t)e_y(t)e_z(t) - y_r(t-\tau)e_x(t)e_z(t), \end{cases} \\ &= \begin{cases} (a - k_1)e_x^2(t) - (k_2 + b)e_y^2(t) - ce_z^2(t) + [-y_d(t) + y_r(t-\tau)]e_x(t)e_z(t) \\ \quad - 2x_r(t-\tau)e_y(t)e_z(t), \\ (a - k_1)e_x^2(t) - (k_2 + b)e_y^2(t) - ce_z^2(t) + [y_d(t) - y_r(t-\tau)]e_x(t)e_z(t) \\ \quad - 2x_d(t)e_y(t)e_z(t), \end{cases} \\ &\leqslant \begin{cases} -(k_1 - a)e_x^2(t) - (k_2 + b)e_y^2(t) - ce_z^2(t) + (M_{y_d} + M_{y_r})|e_x(t)||e_z(t)| \\ \quad + 2M_{x_r}|e_y(t)||e_z(t)|, \\ -(k_1 - a)e_x^2(t) - (k_2 + b)e_y^2(t) - ce_z^2(t) + (M_{y_d} + M_{y_r})|e_x(t)||e_z(t)| \\ \quad + 2M_{x_d}|e_y(t)||e_z(t)|, \end{cases} \\ &= \begin{cases} -(|e_x(t)||e_y(t)||e_z(t)|)Q_1(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}}, \\ -(|e_x(t)||e_y(t)||e_z(t)|)Q_2(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}}, \end{cases} \end{aligned} \tag{2.16}$$

where the symmetric matrices Q_1 and Q_2 are defined as

$$Q_1 = \begin{bmatrix} k_1 - a & 0 & -\frac{1}{2}(M_{y_d} + M_{y_r}) \\ 0 & k_2 + b & -M_{x_r} \\ -\frac{1}{2}(M_{y_d} + M_{y_r}) & -M_{x_r} & c \end{bmatrix}, \tag{2.17a}$$

$$Q_2 = \begin{bmatrix} k_1 - a & 0 & -\frac{1}{2}(M_{y_d} + M_{y_r}) \\ 0 & k_2 + b & -M_{x_d} \\ -\frac{1}{2}(M_{y_d} + M_{y_r}) & -M_{x_d} & c \end{bmatrix}. \tag{2.17b}$$

In order for the zero solution of the error system (2.6) being globally, exponentially stable, the symmetric matrices Q_1 and Q_2 should be positive definite, which implies that the following conditions hold:

$$\text{for } Q_1: \quad \begin{cases} k_1 - a > 0, \\ (k_1 - a)(k_2 + b) > 0, \\ (k_1 - a)[c(k_2 + b) - M_{x_r}^2] - \frac{1}{4}(k_2 + b)(M_{z_d} + M_{z_r})^2 > 0; \end{cases} \tag{2.18a}$$

$$\text{for } Q_2: \quad \begin{cases} k_1 - a > 0, \\ (k_1 - a)(k_2 + b) > 0, \\ (k_1 - a)[c(k_2 + b) - M_{x_d}^2] - \frac{1}{4}(k_2 + b)(M_{z_d} + M_{z_r})^2 > 0; \end{cases} \tag{2.18b}$$

which yield that

$$\begin{aligned} &\text{for } Q_1: \quad k_1 > \frac{1}{4c}(M_{y_d} + M_{y_r})^2 + a, \quad k_2 > \frac{4(k_1 - a)M_{x_r}^2}{4c(k_1 - a) - (M_{y_d} + M_{y_r})^2} - b; \\ &\text{for } Q_2: \quad k_1 > \frac{1}{4c}(M_{y_d} + M_{y_r})^2 + a, \quad k_2 > \frac{4(k_1 - a)M_{x_d}^2}{4c(k_1 - a) - (M_{y_d} + M_{y_r})^2} - b; \end{aligned} \tag{2.19}$$

from which we know that the conclusion for Case (iii) is true.

(iv) In this case, we use the same Lyapunov function (2.7), and obtain

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= e_x(t)\dot{e}_x(t) + e_y(t)\dot{e}_y(t) + e_z(t)\dot{e}_z(t) \\ &= \begin{cases} (a - k_1)e_x^2(t) - be_y^2(t) - (k_3 + c)e_z^2(t) + [-z_d(t) + z_r(t - \tau)]e_x(t)e_y(t) \\ \quad - 2x_r(t - \tau)e_y(t)e_z(t), \\ (a - k_1)e_x^2(t) - be_y^2(t) - (k_3 + c)e_z^2(t) + [z_d(t) - z_r(t - \tau)]e_x(t)e_y(t) \\ \quad - 2x_d(t)e_y(t)e_z(t), \end{cases} \\ &\leqslant \begin{cases} -(k_1 - a)e_x^2(t) - be_y^2(t) - (k_3 + c)e_z^2(t) + (M_{z_d} + M_{z_r})|e_x(t)||e_y(t)| \\ \quad + 2M_{x_r}|e_y(t)||e_z(t)|, \\ -(k_1 - a)e_x^2(t) - be_y^2(t) - (k_3 + c)e_z^2(t) + (M_{z_d} + M_{z_r})|e_x(t)||e_y(t)| \\ \quad + 2M_{x_r}|e_y(t)||e_z(t)|, \end{cases} \\ &= \begin{cases} -(|e_x(t)||e_y(t)||e_z(t)|)Q_1(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}}, \\ -(|e_x(t)||e_y(t)||e_z(t)|)Q_2(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}}, \end{cases} \end{aligned} \tag{2.20}$$

where the symmetric matrices Q_1 and Q_2 are defined as

$$Q_1 = \begin{bmatrix} k_1 - a & -\frac{1}{2}(M_{z_d} + M_{z_r}) & 0 \\ -\frac{1}{2}(M_{z_d} + M_{z_r}) & b & -M_{x_r} \\ 0 & -M_{x_r} & k_3 + c \end{bmatrix}, \tag{2.21a}$$

$$Q_2 = \begin{bmatrix} k_1 - a & -\frac{1}{2}(M_{z_d} + M_{z_r}) & 0 \\ -\frac{1}{2}(M_{z_d} + M_{z_r}) & b & -M_{x_d} \\ 0 & -M_{x_d} & k_3 + c \end{bmatrix}. \tag{2.21b}$$

Then the zero solution of the error system (2.6) is globally, exponentially stable if the symmetric matrices Q_1 and Q_2 are positive definite, i.e., the following conditions hold:

$$\text{for } Q_1: \begin{cases} k_1 - a > 0, \\ (k_1 - a)b - \frac{1}{4}(M_{z_d} + M_{z_r})^2 > 0, \\ (k_3 + c)[(k_1 - a)b - \frac{1}{4}(M_{z_d} + M_{z_r})^2] - (k_1 - a)M_{x_r}^2 > 0; \end{cases} \tag{2.22a}$$

$$\text{for } Q_2: \begin{cases} k_1 - a > 0, \\ (k_1 - a)b - \frac{1}{4}(M_{z_d} + M_{z_r})^2 > 0, \\ (k_3 + c)[(k_1 - a)b - \frac{1}{4}(M_{z_d} + M_{z_r})^2] - (k_1 - a)M_{x_d}^2 > 0; \end{cases} \tag{2.22b}$$

which lead to that

$$\begin{aligned} &\text{for } Q_1: \ k_1 > \frac{1}{4b(M_{z_d} + M_{z_r})^2} + a, \quad k_3 > \frac{4(k_1 - a)M_{x_r}^2}{4(k_1 - a)b - (M_{z_d} + M_{z_r})^2} - c; \\ &\text{for } Q_2: \ k_1 > \frac{1}{4b(M_{z_d} + M_{z_r})^2} + a, \quad k_3 > \frac{4(k_1 - a)M_{x_d}^2}{4(k_1 - a)b - (M_{z_d} + M_{z_r})^2} - c. \end{aligned} \tag{2.23}$$

Therefore, we have

$$\begin{cases} k_1 > \dfrac{1}{4b(M_{z_d} + M_{z_r})^2} + a, \\ k_3 > \min\left[\dfrac{4(k_1 - a)M_{x_r}^2}{4(k_1 - a)b - (M_{z_d} + M_{z_r})^2} - c, \dfrac{4(k_1 - a)M_{x_d}^2}{4(k_1 - a)b - (M_{z_d} + M_{z_r})^2} - c\right]. \end{cases}$$

The proof for Theorem 1 is complete. □

Remarks

(a) Four families of linear feedback control laws are derived to demonstrate that (i) globally exponential lag synchronization for $\tau < 0$; and (ii) globally exponential synchronization for $\tau = 0$ occur between the drive system (2.4) and response system (2.5). It is seen that these linear feedback controllers with only two linear functions are simpler than those known linear feedback controllers with three linear functions [30]. Certainly, they are simpler than nonlinear feedback controllers given in [29].

(b) In the four cases, we may even further simplify the controllers by using only one linear function. For example, let $k_2 = 0$ and $k_3 = 0$. Then, the four linear feedback control laws in Theorem 1 are reduced to the following three types of forms:

$$\begin{aligned} &u_1 = k_1 e_x(t), \quad k_1 > a, \quad bc > \frac{1}{4}(M_{x_d} + M_{x_r})^2, \\ &u_1 = k_1 e_x(t), \quad k_1 > \min\left[\frac{c(M_{z_d} + M_{z_r})^2}{4(bc - M_{x_r}^2)} + a, \frac{b(M_{y_d} + M_{y_r})^2}{4(bc - M_{x_r}^2)} + a\right], \quad bc > M_{x_r}^2, \\ &u_1 = k_1 e_x(t), \quad k_1 > \min\left[\frac{c(M_{z_d} + M_{z_r})^2}{4(bc - M_{x_d}^2)} + a, \frac{b(M_{y_d} + M_{y_r})^2}{4(bc - M_{x_d}^2)} + a\right], \quad bc > M_{x_d}^2. \end{aligned} \tag{2.24}$$

However, it should be noted that it might be difficult to obtain the parameter values which satisfy the conditions: $bc > \frac{1}{4}(M_{x_d} + M_{x_r})^2$, or $bc > M_{x_r}^2$, or $bc > M_{x_d}^2$, since M_{x_d} and M_{x_r} are functions of b and c. Even when these constraints are satisfied, the values chosen for b and c may not yield chaos of system (1.1).

3. Adaptive feedback control laws

Theorem 2. *For the given drive system (2.4), the response system (2.5), and the corresponding error system (2.6), if one of the following two families of adaptive controllers is chosen for the response system (2.5):*

(i) $u_1 = K_1(t)e_x(t)$, $u_2 = K_2(t)e_y(t)$, $u_3 = 0$, *and* $\dot{K}_1(t) = c_1 e_x^2(t), \dot{K}_2(t) = c_2 e_y^2(t)$, *where* $c_1 > 0$, $c_2 > 0$ *and* $K_1(0) = K_2(0) = 0$;

(ii) $u_1 = K_1(t)e_x(t)$, $u_2 = 0$, $u_3 = K_3(t)e_y(t)$, *and* $\dot{K}_1(t) = c_1 e_x^2(t), \dot{K}_3(t) = c_3 e_z^2(t)$, *where* $c_1 > 0$, $c_3 > 0$ *and* $K_1(0) = K_3(0) = 0$;

then the zero solution of the error system (2.6) *is globally stable*, *and thus the two systems* (2.4) *and* (2.5) *are* (i) *globally lag synchronized for* $\tau < 0$; *and* (ii) *globally synchronized for* $\tau = 0$.

Proof

(i) In this case, we choose the following positive definite Lyapunov function:

$$V(t) = \frac{1}{2}\left[e_x^2(t) + e_y^2(t) + e_z^2(t) + \frac{1}{c_1}(K_1(t) - K_1^*)^2 + \frac{1}{c_2}(K_2(t) - K_2^*)^2\right], \tag{3.1}$$

where K_1^* and K_2^* are constants to be determined. Thus, we have

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= e_x(t)\dot{e}_x(t) + e_y(t)\dot{e}_y(t) + e_z(t)\dot{e}_z(t) + \frac{1}{c_1}(K_1 - K_1^*)\dot{K}_1(t) + \frac{1}{c_2}(K_2 - K_2^*)\dot{K}_2(t)\\ &= ae_x^2(t) + y_d(t)e_z(t)e_x(t) + z_r(t-\tau)e_y(t)e_x(t) - K_1e_x^2(t) - be_y^2(t) - K_2e_y^2(t) - x_d(t)e_z(t)e_y(t)\\ &\quad - z_r(t-\tau)e_x(t)e_y(t) - ce_z^2(t) - y_d(t)e_x(t)e_z(t) - x_r(t-\tau)e_y(t)e_z(t) + (K_1 - K_1^*)e_x^2(t) + (K_2 - K_2^*)e_y^2(t)\\ &= (a - K_1^*)e_x^2(t) - (K_2^* + b)e_y^2(t) - ce_z^2(t) - [x_d(t) + x_r(t-\tau)]e_y(t)e_z(t)\\ &\leqslant -(K_1^* - a)e_x^2(t) - (K_2^* + b)e_y^2(t) - ce_z^2(t) + (M_{x_d} + M_{x_r})|e_y(t)||e_z(t)|\\ &= -(|e_x(t)||e_y(t)||e_z(t)|)Q(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}},\end{aligned}$$

where $Q = Q^{\mathrm{T}}$ is given by

$$Q = \begin{bmatrix} K_1^* - a & 0 & 0\\ 0 & K_2^* + b & -\frac{1}{2}(M_{x_d} + M_{x_r})\\ 0 & -\frac{1}{2}(M_{x_d} + M_{x_r}) & c\end{bmatrix}. \tag{3.2}$$

It is easy to see that when $K_1^* > a$ and $K_2^* > \frac{1}{4c}(M_{x_d} + M_{x_r})^2 - b$, the symmetric matrix Q is positive definite, and thus the zero solution of the error system (2.6) is globally stable.

(ii) For this case, similar to Case (i), we choose the following positive definite Lyapunov function:

$$V(t) = \frac{1}{2}\left[e_x^2(t) + e_y^2(t) + e_z^2(t) + \frac{1}{c_1}(K_1(t) - K_1^*)^2 + \frac{1}{c_3}(K_3(t) - K_3^*)^2\right], \tag{3.3}$$

where K_1^* and K_3^* are constants to be determined. Then we obtain

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= e_x(t)\dot{e}_x(t) + e_y(t)\dot{e}_y(t) + e_z(t)\dot{e}_z(t) + \frac{1}{c_1}(K_1 - K_1^*)\dot{K}_1(t) + \frac{1}{c_3}(K_3 - K_3^*)\dot{K}_3(t)\\ &= ae_x^2(t) + y_d(t)e_z(t)e_x(t) + z_r(t-\tau)e_y(t)e_x(t) - K_1e_x^2(t) - be_y^2(t) - x_d(t)e_z(t)e_y(t) - z_r(t-\tau)e_x(t)e_y(t)\\ &\quad - K_3e_z^2(t) - ce_z^2(t) - y_d(t)e_x(t)e_z(t) - x_r(t-\tau)e_y(t)e_z(t) + (K_1 - K_1^*)e_x^2(t) + (K_3 - K_3^*)e_z^2(t)\\ &= (a - K_1^*)e_x^2(t) - be_y^2(t) - (K_3^* + c)e_z^2(t) - [x_d(t) + x_r(t-\tau)]e_y(t)e_z(t)\\ &\leqslant -(K_1^* - a)e_x^2(t) - be_y^2(t) - (K_3^* + c)e_z^2(t) + (M_{x_d} + M_{x_r})|e_y(t)||e_z(t)|\\ &= -(|e_x(t)||e_y(t)||e_z(t)|)Q(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}},\end{aligned}$$

where $Q = Q^{\mathrm{T}}$ is given by

$$Q = \begin{bmatrix} K_1^* - a & 0 & 0\\ 0 & b & -\frac{1}{2}(M_{x_d} + M_{x_r})\\ 0 & -\frac{1}{2}(M_{x_d} + M_{x_r}) & K_3^* + c\end{bmatrix}.$$

Therefore, when $K_1^* > a$ and $K_3^* > \frac{1}{4b}(M_{x_d} + M_{x_r})^2 - c$, the symmetric matrix Q is positive definite, and thus the zero solution of the error system (2.6) is globally stable. This completes the proof of Theorem 2. □

4. The combination of linear and adaptive feedback control laws

Theorem 3. *For the drive system* (2.4), *the response system* (2.5), *and the corresponding error system* (2.6), *suppose that* $M_{x_d}, M_{x_r}, M_{y_d}, M_{y_r}, M_{z_d}$ *and* M_{z_r} *are the upper bounds of* $|x_d(t)|$, $|x_r(t-\tau)|$, $|y_d(t)|$, $|y_r(t-\tau)|$, $|z_d(t)|$ *and* $|z_r(t-\tau)|$, *respectively, and if one of the following controllers is chosen for the response system* (2.5):

(i) $u_1 = k_1 e_x(t)$, $u_2 = K_2(t)e_y(t)$, $u_3 = 0$, *where* $k_1 > a$, $\dot K_2(t) = c_2 e_y^2(t)$, *with* $c_2 > 0$ *and* $K_2(0) = 0$;
(ii) $u_1 = k_1 e_x(t)$, $u_2 = 0$, $u_3 = K_3(t)e_z(t)$, *where* $k_1 > a$, $\dot K_3(t) = c_3 e_z^2(t)$, *with* $c_3 > 0$ *and* $K_3(0) = 0$;
(iii) $u_1 = K_1(t)e_x(t)$, $u_2 = k_2 e_y(t)$, $u_3 = 0$, *where* $K_1(t)$ *satisfies* $\dot K_1(t) = c_1 e_x^2(t), k_2 > \min\left[\frac{(M_{x_d}+M_{x_r})^2}{4c}, \frac{(M_{z_d}+M_{z_r})^2}{4(k_1-a)} + \frac{\min(M_{x_r}^2, M_{x_d}^2)}{c}, \frac{4(k_1-a)\min(M_{x_r}^2, M_{x_d}^2)}{4c(k_1-a)-(M_{y_d}+M_{y_r})^2}\right] - b$, *with* $c_1 > 0$ *and* $K_1(0) = 0$;
(iv) $u_1 = K_1(t)e_x(t)$, $u_2 = 0$, $u_3 = k_3 e_z(t)$, *where* $K_1(t)$ *satisfies* $\dot K_1(t) = c_1 e_x^2(t), k_3 > \min\left[\frac{(M_{x_d}+M_{x_r})^2}{4b}, \frac{(M_{y_d}+M_{y_r})^2}{4(k_1-a)} + \frac{\min(M_{x_r}^2, M_{x_d}^2)}{b}, \frac{4(k_1-a)\min(M_{x_r}^2, M_{x_d}^2)}{4b(k_1-a)-(M_{z_d}+M_{z_r})^2}\right] - c$, *with* $c_1 > 0$ *and* $K_1(0) = 0$;

then the zero solution of the error system (2.6) *is globally stable, and thus the two systems* (2.4) *and* (2.5) *are* (i) *globally lag synchronized for* $\tau < 0$; *and* (ii) *globally synchronized for* $\tau = 0$.

Proof. (i) In this case, we choose the following positive definite Lyapunov function:

$$V(t) = \frac{1}{2}\left[e_x^2(t) + e_y^2(t) + e_z^2(t) + \frac{1}{c_2}(K_2(t) - K_2^*)^2\right], \tag{4.1}$$

where K_2^* is a constant to be determined. Thus, we have

$$\begin{aligned}
\frac{dV(t)}{dt} &= e_x(t)\dot e_x(t) + e_y(t)\dot e_y(t) + e_z(t)\dot e_z(t) + \frac{1}{c_2}(K_2 - K_2^*)\dot K_2(t) \\
&= a e_x^2(t) + y_d(t)e_z(t)e_x(t) + z_r(t-\tau)e_y(t)e_x(t) - k_1 e_x^2(t) - b e_y^2(t) - K_2(t)e_y^2(t) - x_d(t)e_z(t)e_y(t) \\
&\quad - z_r(t-\tau)e_x(t)e_y(t) - c e_z^2(t) - y_d(t)e_x(t)e_z(t) - x_r(t-\tau)e_y(t)e_z(t) + (K_2 - K_2^*)e_y^2(t) \\
&= (a - k_1)e_x^2(t) - (K_2^* + b)e_y^2(t) - c e_z^2(t) - [x_d(t) + x_r(t-\tau)]e_y(t)e_z(t) \\
&\leqslant -(k_1 - a)e_x^2(t) - (K_2^* + b)e_y^2(t) - c e_z^2(t) + (M_{x_d} + M_{x_r})|e_y(t)||e_z(t)| \\
&= -(|e_x(t)||e_y(t)||e_z(t)|)Q(|e_x(t)||e_y(t)||e_z(t)|)^{\mathrm{T}},
\end{aligned}$$

where $Q = Q^{\mathrm{T}}$ is given by

$$Q = \begin{bmatrix} k_1 - a & 0 & 0 \\ 0 & K_2^* + b & -\frac{1}{2}(M_{x_d} + M_{x_r}) \\ 0 & -\frac{1}{2}(M_{x_d} + M_{x_r}) & c \end{bmatrix}. \tag{4.2}$$

It is easy to see that when $k_1 > a$ and $K_2^* > \frac{1}{4c}(M_{x_d} + M_{x_r})^2 - b$, the symmetric matrix Q is positive definite, and thus the zero solution of the error system (2.6) is globally stable.

Similarly, we can prove for Cases (ii)–(iv), and omit the details here. This finishes the proof. □

5. Numerical simulation results

In the section, we will verify the control laws presented in the previous sections via numerical simulations. We take the parameter values as $a = 0.4$, $b = 12$, $c = 5$ in system (1.1). Here we restrict to the case $\tau = 0$.

For the controller given in Case (i) of Theorem 2, the initial values are chosen as $(x_d, y_d, z_d) = (0.2, 0.1, 0.3)$ for the drive system (2.4) and $(x_r, y_r, z_r) = (-0.1, 0.4, -0.8)$ for the response system (2.5). Fig. 1(a)–(c) show the time histories of the error variables e_x, e_y and e_z under the linear feedback controller with $k_1 = 0.9$ and $k_2 = 248$. Since the conditions given in the controllers derived from the Lyapunov function are sufficient, not necessary, we therefore may take smaller values for parameters k_1 and k_2. Fig. 1(d) displays the time history of the error state e_x under the linear feedback controller with $k_1 = 0.9$, $k_2 = 20$. For a fixed value of k_1, the smaller value of k_2 results in slower convergence of e_x to zero, as expected (see Fig. 1(a) and (d)).

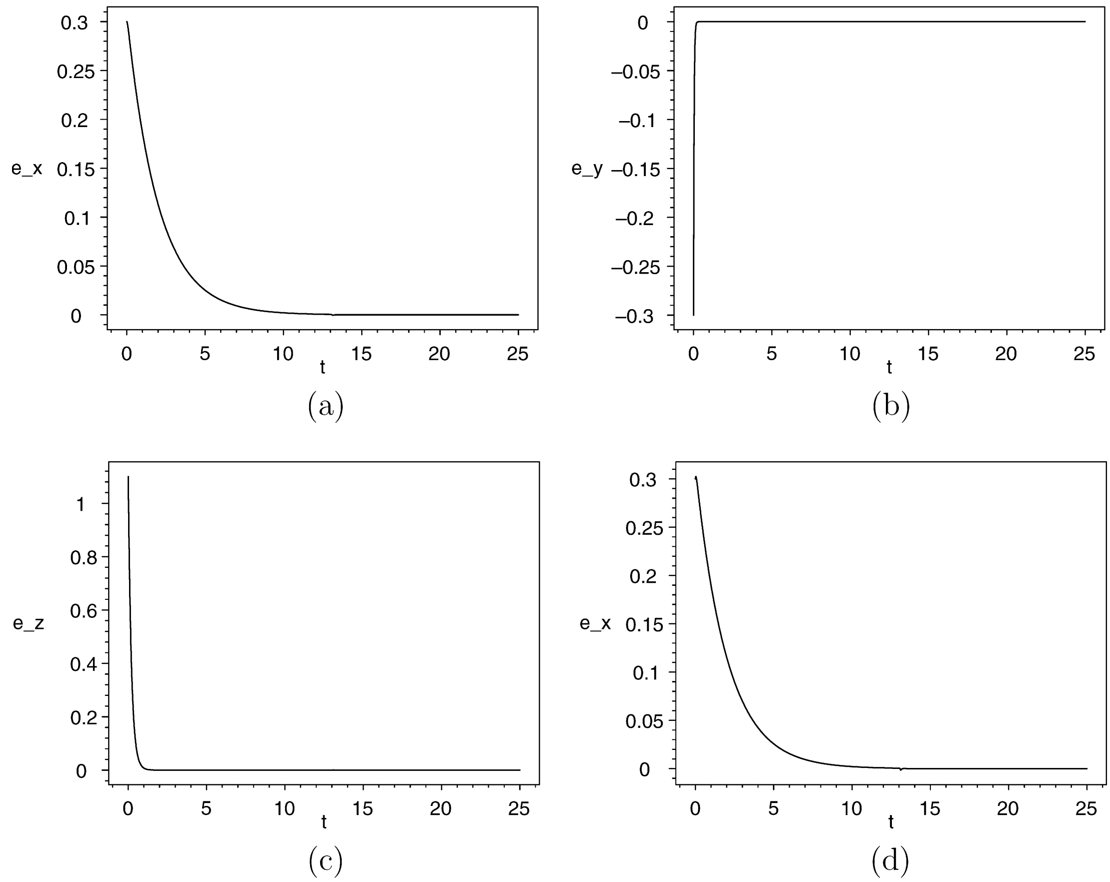

Fig. 1. Simulated results for the LC chaotic system (1.1) with parameter values $a = 0.4$, $b = 12$, $c = 5$: (a)–(c) the time histories of the error signals e_x, e_y and e_z under the linear feedback controller given in Case (i) of Theorem 1 with $k_1 = 0.9$, $k_2 = 248$; (d) the time history of the error signals e_x under the linear feedback controller given in Case (i) of Theorem 1 with $k_1 = 0.9$, $k_2 = 20$.

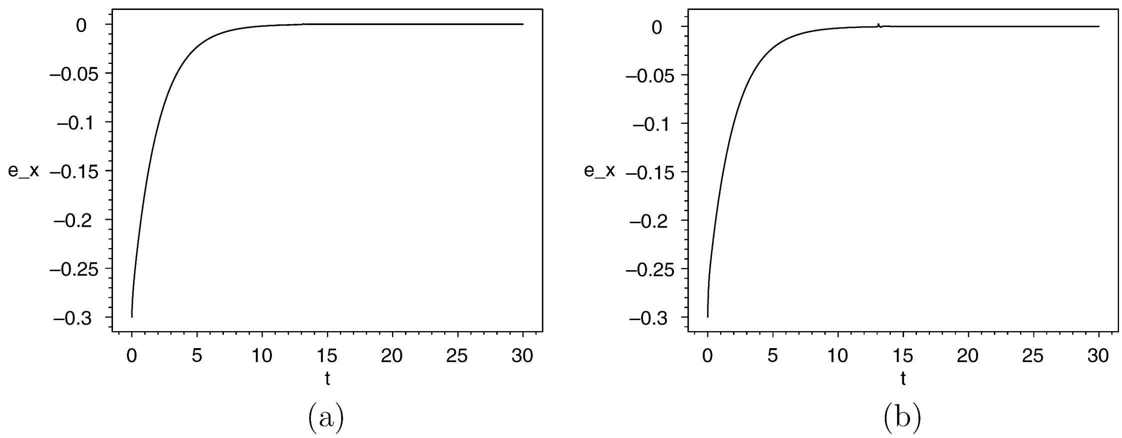

Fig. 2. Simulated time history of the error signal e_x for the LC system (1.1) under the linear feedback controller in Case (ii) of Theorem 1 with (a) $k_1 = 0.9$, $k_3 = 103$; (b) $k_1 = 0.9$, $k_3 = 9$.

For the controller given in Case (ii) of Theorem 1, we choose the initial values $(x_d, y_d, z_d) = (0.2, 0.1, 0.3)$ for the drive system (2.4), and $(x_r, y_r, z_r) = (0.5, -0.6, 0.9)$ for the response system (2.5). Fig. 2(a) shows the time history of the error state e_x under the controller for Case (ii) with $k_1 = 0.9$, $k_3 = 103$. Similar to Case (i), we may also take smaller values for parameter k_3. For example, the result for $k_3 = 9$ is shown in Fig. 2(b), again indicating a slower convergence.

For the adaptive controller given in Case (i) of Theorem 2, we choose the initial values $(x_d, y_d, z_d) = (0.2, 0.1, 0.3)$ for the drive system (2.4), and $(x_r, y_r, z_r) = (2.5, -9, 1)$ for the response system (2.5). We fix the parameter value $c_2 = 0.9$ and

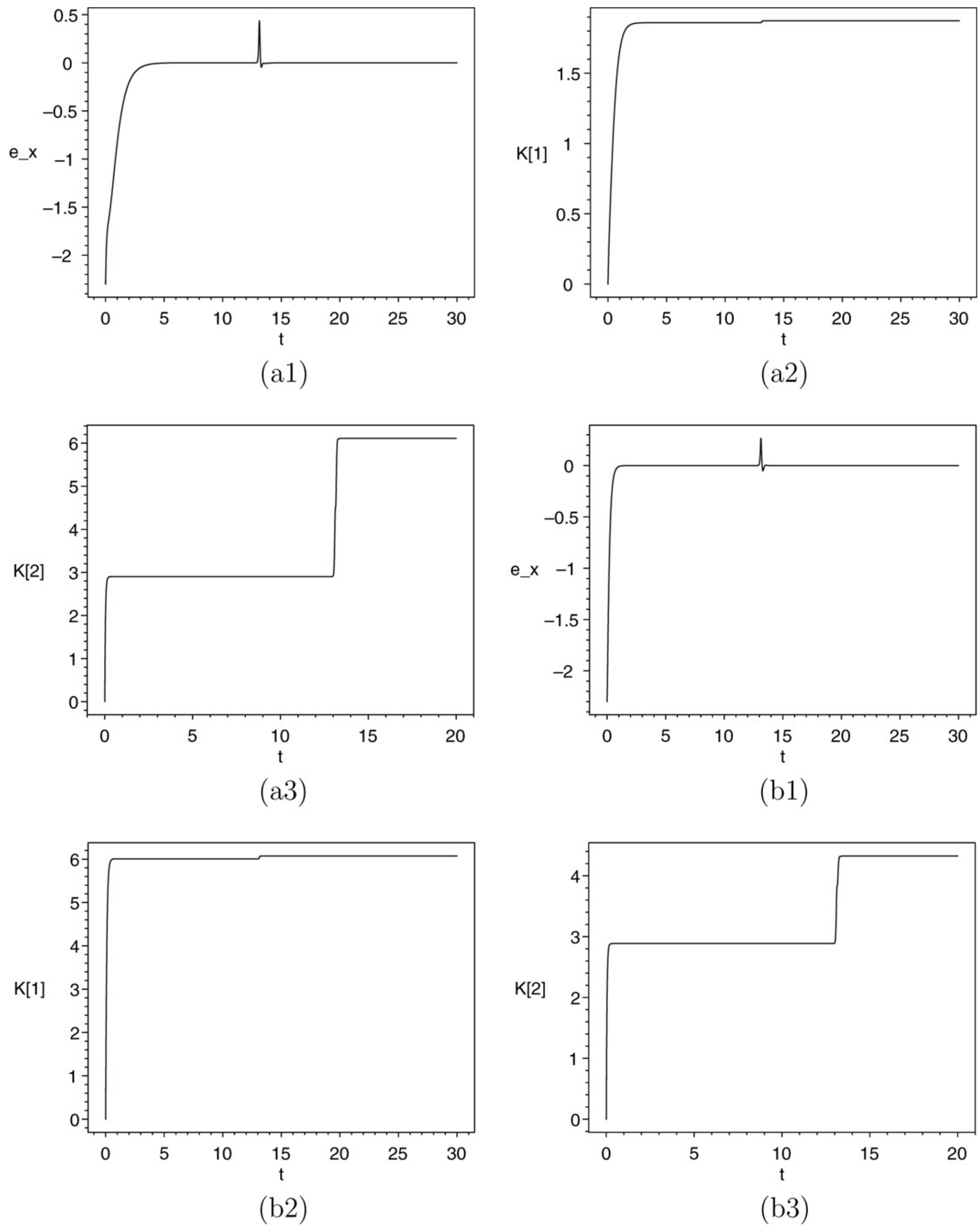

Fig. 3. Simulated results for the LC chaotic system (1.1) under the adaptive feedback control laws given in Case (i) of Theorem 2: (a1)–(d) the time histories of the error signal e_x and the control functions $K_1(t)$, $K_2(t)$ with $c_1 = 0.8$, $c_2 = 0.9$ for (a1)–(a3); $c_1 = 10.8$, $c_2 = 0.9$ for (b1)–(b3); $c_1 = 20.8$, $c_2 = 0.9$ for (c1)–(c3); and $c_1 = 0.8$, $c_2 = 2.8$ for (d).

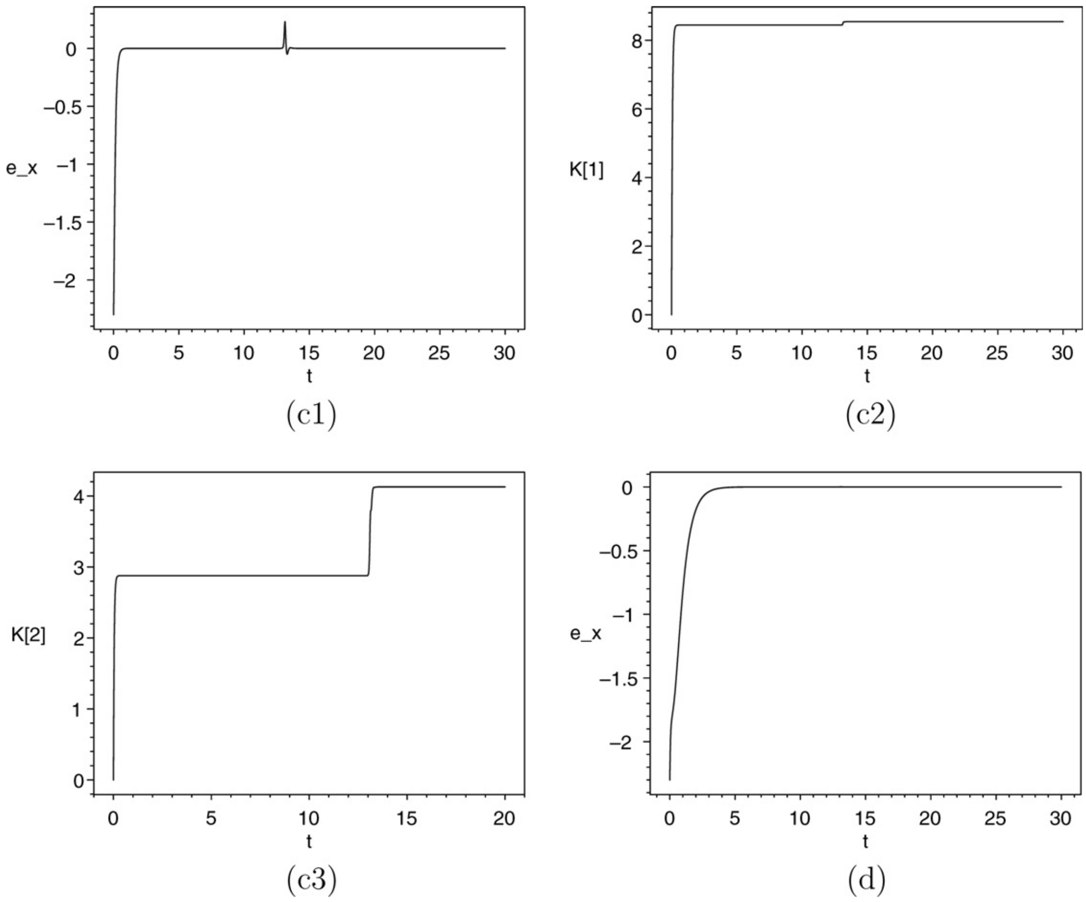

Fig. 3 (*continued*)

let c_1 vary from 0.8 to 20.8. The results are depicted in Fig. 3(a1)–(c3). It is again shown that a larger value of c_1 provides a faster convergence. Further, it is noted that there exist faster changes for the error signal e_x around $t = 13$, as shown in Fig. 3(a1)–(c1). These dramatic changes are resulted from the sudden change of the function $K_2(t)$ near $t = 13$, as seen in Fig. 3(a3), (b3) and (c3). However, when we increase c_2 to, say, $c_2 = 2.8$, these sudden jumpings become smooth, see Fig. 3(d).

For the combination of linear and adaptive feedback control laws given in Case (i) of Theorem 3, we choose the initial values $(x_d, y_d, z_d) = (0.2, 0.1, 0.3)$ for the drive system (2.4), and $(x_r, y_r, z_r) = (-0.1, 6, -0.8)$ for the response system (2.5). We fix the parameter $k_1 = 0.9$ and let c_2 change from 2.6 to 12.6. The simulation results are shown in Fig. 4(a1), (b1) and (c1). Although these results show a similar trend as that of Fig. 3, they look smoother due to the combination of feedback control laws.

6. Application

As an application, in this section, we shall apply the control laws obtained in the preceding section to consider another new chaotic system. Recently, Lü et al. [31] presented a chaotic system, described by

$$\begin{cases} \dot{x} = \frac{\alpha\beta}{\alpha+\beta}x + yz + \gamma, \\ \dot{y} = -\alpha y - xz, \\ \dot{z} = -\beta z - xy, \end{cases} \tag{6.1}$$

where $\alpha > 0$, $\beta > 0$, $|\gamma| < 19.2$. When $\gamma = 0$, system (6.1) is reduced to a special case of system (1.1).

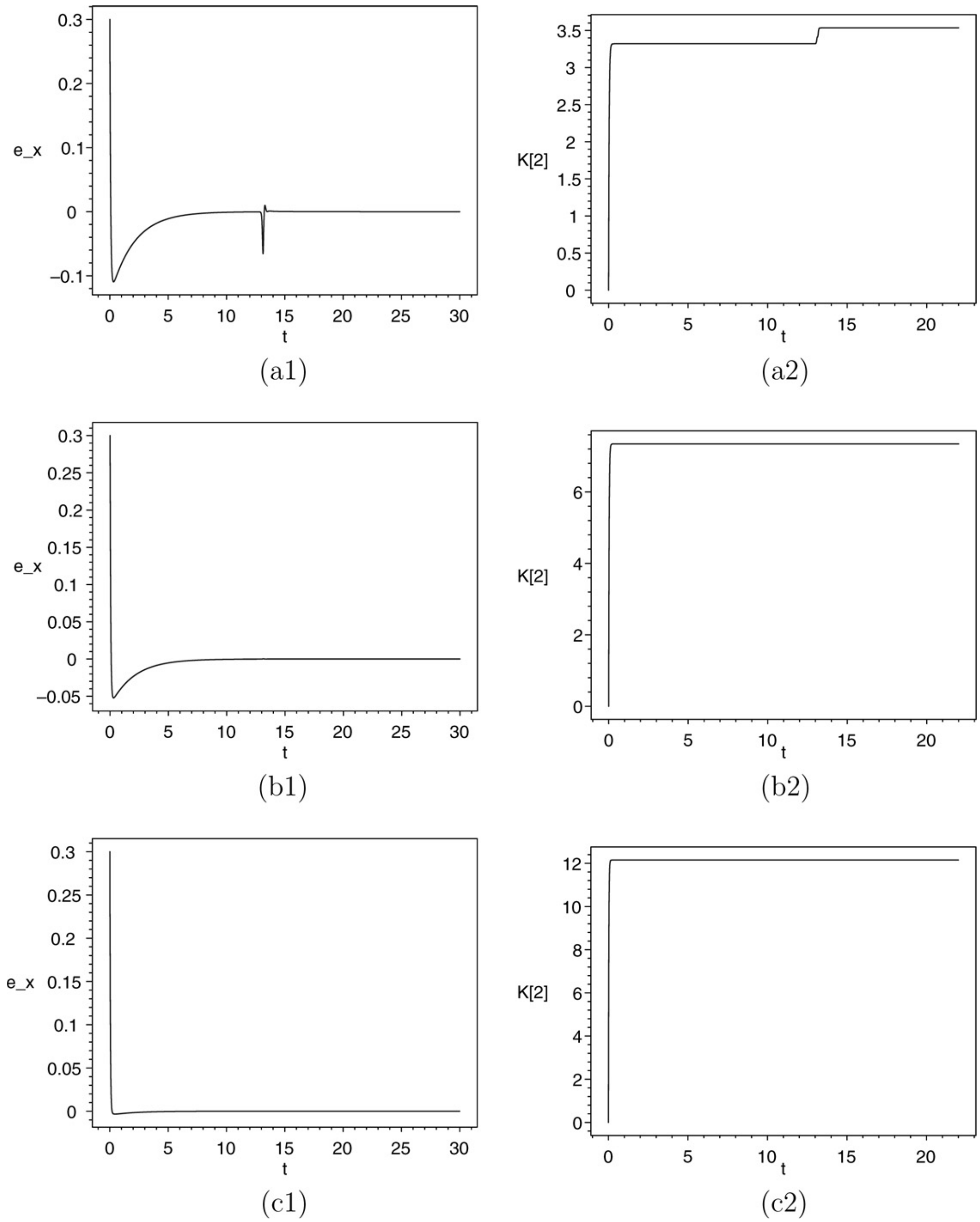

Fig. 4. Simulated results for the LC chaotic system (1.1) under the combination of linear and adaptive feedback control laws given in Case (i) of Theorem 3: (a1)–(c2) the time histories of the error signal e_x and the control function $K_2(t)$ with $k_1 = 0.9$, $c_2 = 2.6$ for (a1)–(a2); $k_1 = 0.9$, $c_2 = 6.6$ for (b1)–(b2); and $k_1 = 0.9$, $c_2 = 12.6$ for (c1)–(c2).

Consider the chaotic system (6.1) as a drive system:

$$\begin{cases} \dot{x}_d = \frac{\alpha\beta}{\alpha+\beta} x_d + y_d z_d + \gamma, \\ \dot{y}_d = -\alpha y_d - x_d z_d, \\ \dot{z}_d = -\beta z_d - x_d y_d, \end{cases} \tag{6.2}$$

and the controller system, given by

$$\begin{cases} \dot{x}_r = \frac{\alpha\beta}{\alpha+\beta}x_r + y_r z_r + \gamma + u_1, \\ \dot{y}_r = -\alpha y_r - x_r z_r + u_2, \\ \dot{z}_r = -\beta z_r - x_r y_r + u_3, \end{cases} \tag{6.3}$$

as a response system, where u_i's are functions of $(x_d, y_d, z_d, x_r, y_r, z_r)$.

Let the error state be $\mathbf{e}(t) = (e_x(t), e_y(t), e_z(t))^{\mathrm{T}} = [x_d(t) - x_r(t-\tau), y_d(t) - y_r(t-\tau), z_d(t) - z_r(t-\tau)]^{\mathrm{T}}$, where $\tau \geqslant 0$. Then from (6.2) and (6.3), we obtain the error dynamical system:

$$\begin{cases} \dot{e}_x(t) = \frac{\alpha\beta}{\alpha+\beta}e_x(t) + y_d(t)z_d(t) - y_r(t-\tau)z_r(t-\tau) - U_1, \\ \dot{e}_y(t) = -\alpha e_y(t) - x_d(t)z_d(t) + x_r(t-\tau)z_r(t-\tau) - U_2, \\ \dot{e}_z(t) = -\beta e_z(t) - x_d(t)y_d(t) + x_r(t-\tau)y_r(t-\tau) - U_3, \end{cases} \tag{6.3}$$

where $U_i = u_i|_{t=t-\tau}$ $(i = 1,2,3)$.

Let $\frac{\alpha\beta}{\alpha+\beta} = a$, $\alpha = b$, $\beta = c$. Then the error dynamical system (6.3) becomes (2.6). Therefore, we can directly apply the control laws in Theorems 1–3 to study the synchronization of systems (6.1) and (6.2). The details are omitted here.

7. Conclusions

In this paper, based on Lyapunov stability theorem, we have studied chaos (lag) synchronization of the LC chaotic system via three types of state feedback controls: (i) linear feedback control; (ii) adaptive feedback control; and (iii) a combination of linear feedback and adaptive feedback controls. As a result, a number of new feedback control laws have been designed to obtain global chaos lag synchronization for $\tau < 0$ and global chaos synchronization for $\tau = 0$ for the LC system. Each family of these obtained feedback control laws, including two linear (adaptive) functions or one linear function and one adaptive function, has been added to only two equations of the LC system. This is simpler than the known synchronization controllers, which apply controllers to all equations of the LC system. We have used numerical simulations to demonstrate the obtained theoretical results. It should be pointed out that the controllers derived from Lyapunov stability theorem are only sufficient conditions, not necessary ones. It has been shown that smaller control parameter values in the controllers may be also effective in synchronizing two same type of systems (1.1). Moreover, as the special cases of those obtained controllers, we have also derived the corresponding control laws for chaos (lag) synchronization of another new chaotic system.

Acknowledgements

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