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Chaos, Solitons and Fractals 35 (2008) 333–345
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Hyperchaos synchronization and control on a new hyperchaotic attractor

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Accepted 14 May 2006

Abstract

In this paper, a new hyperchaotic system is obtained by introducing an additional state, and adding two nonlinear terms of the original states and one linear term of the new state to the Chen chaotic system. Particular attention is given to globally exponential hyperchaos (time-delayed) synchronization and control for this hyperchaotic system. As a consequence, several families of control laws are designed to achieve globally exponential hyperchaos (time-delayed) synchronization, and globally exponential hyperchaos (without time delay) synchronization. The principle of synchronization is used to globally and exponentially stabilize the equilibrium points of the hyperchaotic system. Numerical simulation results are presented to illustrate the theoretical predictions.

1. Introduction

Study of chaos systems has received great attention in the past several decades [1–11]. A "regular" chaotic system has one positive Lyapunov exponent. Hyperchaotic systems with more than one positive Lyapunov exponent, on the other hand, are more complex and also play a significant role in nonlinear science. Since the discovery of the hyperchaotic Rössler system [12], many hyperchaotic systems have been developed such as the hyperchaotic MCK circuit [13], the hyperchaotic Chen system [14,15], etc. In fact, it is not difficult to construct a hyperchaotic system, based on a "regular" chaotic system. For example, the hyeprchaotic Rössler system [11], given by

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay + w, \\ \dot{z} = b + xz, \\ \dot{w} = -cz + dw, \end{cases} \tag{1.1}$$

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doi:10.1016/j.chaos.2006.05.027

was obtained from the Rössler system [16]:

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + xz - cz, \end{cases} \tag{1.2}$$

by introducing a linear feedback w to the second equation and a linear feedback cz to the third equation and then adding one additional new state equation of w.

Another example is hyperchaotic Chen system [14,15], which was derived from the Chen system [17]:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz. \end{cases} \tag{1.3}$$

by adding a new equation and then introducing a linear feedback w to the first equation, resulting in

$$\begin{cases} \dot{x} = a(y - x) + w, \\ \dot{y} = dx - xz + cy, \\ \dot{z} = xy - bz, \\ \dot{w} = yz + rw. \end{cases} \tag{1.4}$$

The simulated Chen attractor is shown in Fig. 1.

The common characteristics of the hyperchaotic systems (1.1) and (1.4) are that only linear feedbacks of a new state and an old state are introduced to a known chaotic system.

In this paper, based on the Chen system (1.3), we add nonlinear terms of the known states $\{x, y, z\}$ and a linear term of the new state w to the second equation of (1.3) to obtain a new type of hyperchaotic system.

The rest of this paper is organized as follows. In Section 2, we describe the new hyperchaotic system based on the Chen chaotic system, and give brief study on the local dynamics of the hyperchaotic system. In Section 3, we present a family of control laws to achieve globally exponential hyperchaos (time-delayed, or simply "lag") synchronization, as well as globally exponential hyperchaos (without time delay) synchronization. In Section 4, a family of controllers is

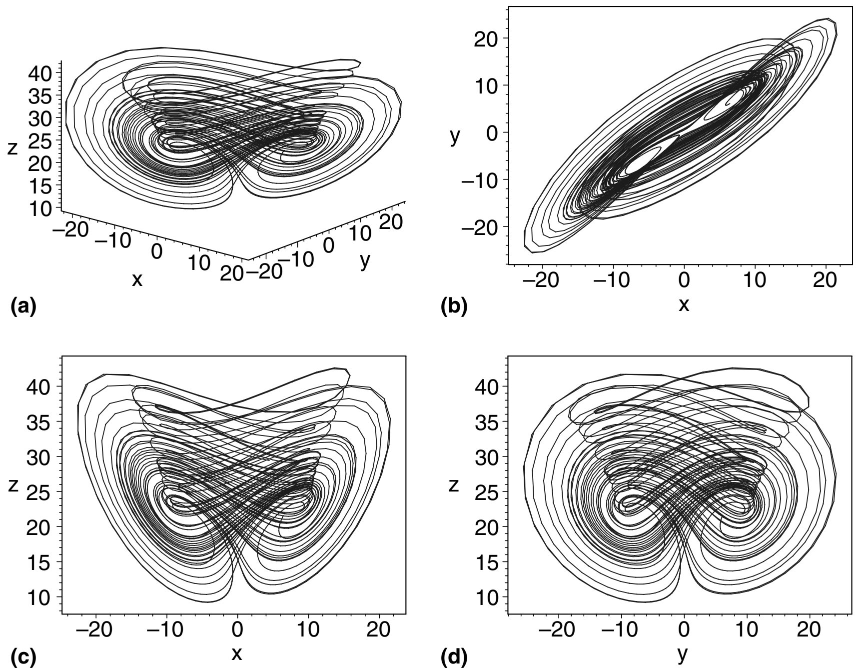

Fig. 1. Simulated phase portraits of the Chen system (1.3) with parameter values $a = 35$, $b = 3$, $c = 28$, projected (a) in the x–y–z space; (b) on the x–y plane; (c) on the x–z plane; and (d) on the y–z plane.

designed to globally and exponentially stabilize the equilibrium points of the hyperchaotic system. Numerical simulation results are given in Section 5 to illustrate the theoretical predictions. Finally conclusion is drawn in Section 6.

2. A new hyperchaotic system

To obtain a new hyperchaotic system from the Chen system (1.3), we introduce the fourth state w and then add the term $-yz + xz - w$ to the third equation of (1.3). As a consequence, a 4-dimensional nonlinear dynamical system is obtained in the form of

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = -bz + xy - yz + xz - w, \\ \dot{w} = -dw + yz - xz, \end{cases} \tag{2.1}$$

where d is a new constant.

When $a = 37$, $b = 3$, $c = 26$, $d = 38$, computation shows that system (2.1) has the following Lyapunov exponents: $\lambda_1 = 1.319$, $\lambda_2 = 0.146$, $\lambda_3 = -20.148$ and $\lambda_4 = -56.337$. The two positive Lyapunov exponents indicate that system (2.1) is hyperchaotic. Simulated results are depicted in Figs. 2 and 3. Fig. 2(a)–(d) displays the projections of the hyperchaotic attractor of system (2.1) in the x–y–z space, the x–y–w space, the x–z–w space, and the y–z–w space, respectively. To get a better view of this hyperchaotic attractor, Fig. 3(a)–(f) shows the projections of the hyperchaotic attractor (2.1) on six coordinate planes.

The hyperchaotic system (2.1) can also exhibit limit cycles. For example, when $a = 37, b = 3, d = 38, c = 33.6$, a single limit cycle is obtained, as shown in Fig. 4(a) and (b). When c is reduced to $c = 32.6$, a double limit cycle is found, which is displayed in Fig. 4(c) and (d).

Consider the special case when $w \equiv 0$. Then the subsystem of the hyperchaotic system (2.1) projected in the x–y–z space is given by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = -bz + xy - yz + xz. \end{cases} \tag{2.2}$$

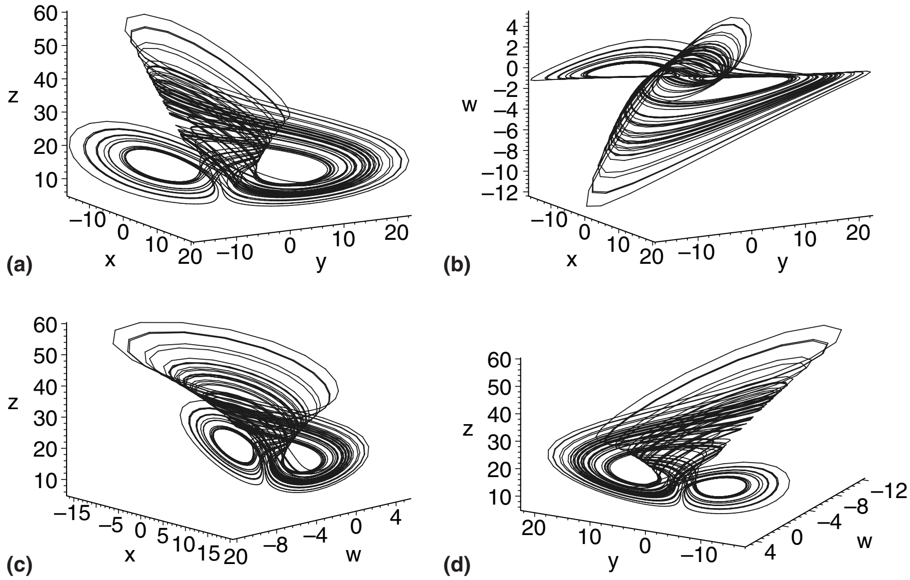

Fig. 2. Simulated phase portraits of the hyperchaotic system (2.1) with parameter values $a = 37$, $b = 3$, $c = 26$, $d = 38$, projected in (a) the x–y–z space; (b) the x–y–w space; (c) the x–z–w space; and (d) the y–z–w space.

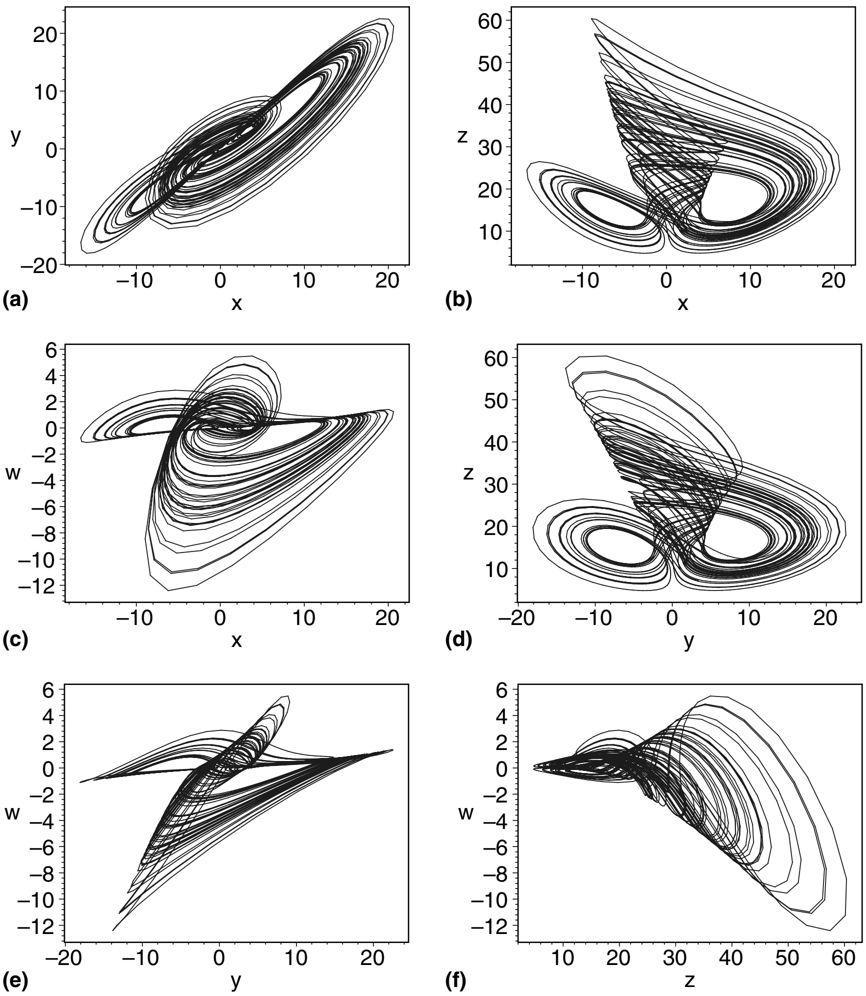

Fig. 3. Simulated phase portraits of the hyperchaotic system (2.1) with parameter values $a = 37$, $b = 3$, $c = 26$, $d = 38$, projected on (a) the x–y plane; (b) the x–z plane; (c) the x–w plane; (d) the y–z plane; (e) the y–w plane; and (f) the z–w plane.

System (2.2) has two more nonlinear terms, yz and xz, than that of the Chen system (1.3). When $a = 37, b = 3, c = 26$, a chaotic attractor is obtained, as shown in Fig. 5(a)–(d).

It can be seen from the phase portraits of the hyperchaotic system (2.1) that system (2.1) does not admit simple symmetries.

Since

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = c - (a + b + d), \tag{2.3}$$

system (2.1) is dissipative when $a + b + d > c$. Moreover, an exponential contraction rate is given by

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -(a + b + d - c)V(t), \tag{2.4}$$

which means that $V(t) = V_0 \mathrm{e}^{-(a+b+d-c)t}$.

It is easy to find the three equilibrium points of system (2.1), given by

$$E_0 = (0, 0, 0, 0,), \quad E_{\pm} = (\pm\sqrt{b(2c - a)}, \pm\sqrt{b(2c - a)}, 2c - a, 0), \tag{2.5}$$

where $E_{\pm}$ exist if $b(2c - a) > 0$.

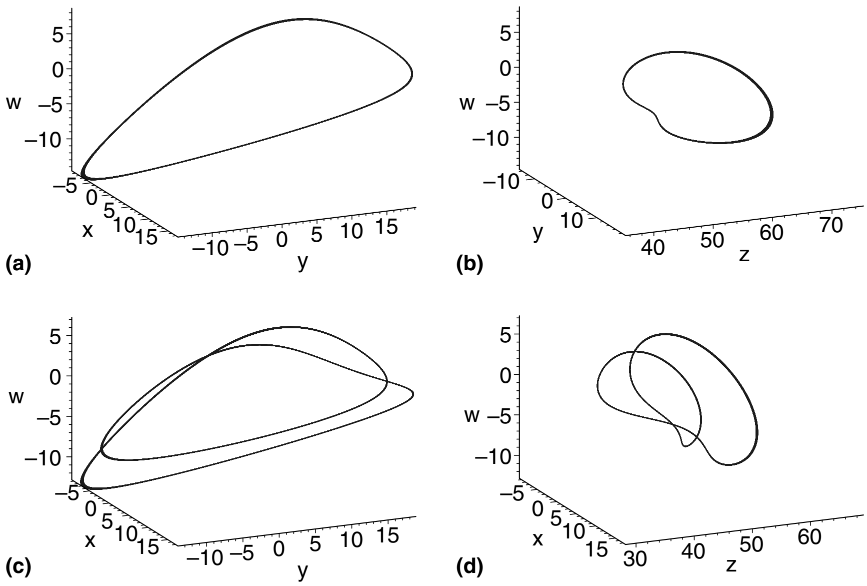

Fig. 4. Simulated limit cycles for the hyperchaotic system (2.1) with parameter values $a = 37$, $b = 3$, $d = 38$: (a) and (b) a single limit cycle when $c = 33.6$; (c) and (d) a double limit cycle when $c = 32.6$.

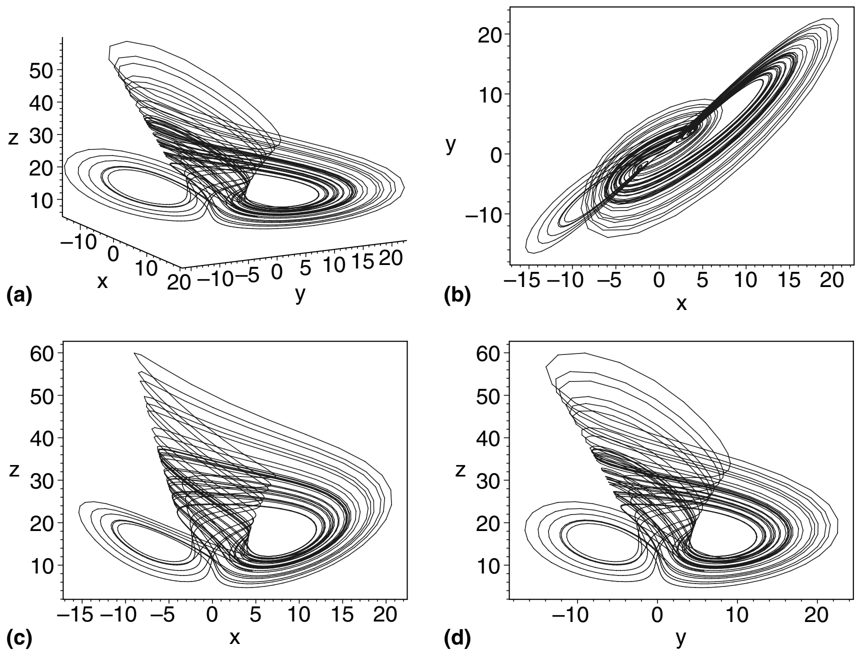

Fig. 5. Simulated phase portraits of the chaotic system (2.2) with parameter values $a = 37$, $b = 3$, $c = 26$, projected on (a) the y–z space; (b) the x–y plane; (c) the x–z plane; and (d) the y–z plane.

To determine the stability of the equilibrium point E_0, evaluating the Jacobian matrix of system (2.1) at E_0 yields

$$J|_{E_0} = \begin{bmatrix} -a & a & 0 & 0 \\ c-a & c & 0 & 0 \\ 0 & 0 & -b & -1 \\ 0 & 0 & 0 & -d \end{bmatrix}. \tag{2.6}$$

The four eigenvalues of the characteristic polynomial of the Jacobian matrix (2.6) are

$$\lambda_{1,2} = \frac{1}{2}\left[c - a \pm \sqrt{(a-c)^2 + 4a(2c-a)}\right], \quad \lambda_3 = -b, \quad \lambda_4 = -d. \tag{2.7}$$

When $a > 0$, $b > 0$, $d > 0$ and $2c > a$, we have $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, $\lambda_4 < 0$. Thus, the equilibrium E_0 is a saddle point of the hyperchaotic system (2.1).

At the equilibrium points $E_\pm$, the Jacobian matrix is

$$J|_{E_+} = \begin{bmatrix} -a & a & 0 & 0 \\ -c & c & -\sqrt{b(2c-a)} & 0 \\ \sqrt{b(2c-a)} + 2c - a & \sqrt{b(2c-a)} - 2c + a & -b & -1 \\ a - 2c & 2c - a & 0 & -d \end{bmatrix}, \tag{2.8}$$

which results in the characteristic polynomial:

$$\lambda^4 + (a - c + b + d)\lambda^3 + \left[-cd + bd + ad + bc + (a-2c)\sqrt{b(2c-a)}\right]\lambda^2 \\ + \left[(a - 2c - 2cd + d)\sqrt{b(2c-a)} - 2ba^2 + 4acb + cbd\right]\lambda - 2bda^2 + 4abcd = 0. \tag{2.9}$$

Using Routh–Hurwitz criterion, it is easy to show that when $a = 37$, $b = 3$, $c = 26$, $d = 38$, some eigenvalues of the characteristic polynomial of the Jacobian matrix (2.8) have positive real parts. Thus the equilibrium points $E_\pm$ are unstable.

3. Globally exponential hyperchaos (lag) synchronization

In this section, we first present a definition and a lemma for the globally exponential (lag) synchronization of nth-dimensional nonlinear dynamical systems.

Consider the drive system:

$$\dot{\mathbf{x}}_d = \mathbf{F}(t, \mathbf{x}_d), \tag{3.1}$$

and the corresponding response system:

$$\dot{\mathbf{y}}_r = \mathbf{F}(t, \mathbf{y}_r) + \mathbf{u}, \tag{3.2}$$

where the subscripts "d" and "r" stand for the drive system and response system, respectively, $\mathbf{x}_d = (x_{1d}, x_{2d}, \ldots, x_{nd})^{\mathrm{T}}$, $\mathbf{y}_r = (y_{1r}, y_{2r}, \ldots, y_{nr})^{\mathrm{T}}$, $\mathbf{F} : R_+ \times R^n \to R^n$, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ is a vector function of time t and the state variables $(x_{id}, y_{id}, x_{ir}, y_{ir})$.

Let the error state be

$$\mathbf{e}(t) = [e_1(t), e_2(t), \ldots, e_n(t)]^{\mathrm{T}} = [x_{1d}(t-\tau) - y_{1r}(t), x_{2d}(t-\tau) - y_{2r}(t), \ldots, x_{nd}(t-\tau) - y_{nr}(t)]^{\mathrm{T}} \quad (\tau \geqslant 0).$$

Then the error dynamics of $\mathbf{e}(t)$ is defined by

$$\dot{\mathbf{e}}(t) = \mathbf{F}(t-\tau, \mathbf{x}_d(t-\tau)) - \mathbf{F}(t, \mathbf{y}_r(t)) - \mathbf{u}, \tag{3.3}$$

Definition 1. For arbitrary given initial values, $(x_{1d}(t), x_{2d}(t), \ldots, x_{nd}(t))$ and $(y_{1r}(t), y_{2r}(t), \ldots, y_{nr}(t)) \in R^n$, $t \in [-\tau, 0]$, of the drive-response systems (3.1) and (3.2), respectively, if the solution of the error dynamical system (3.3) has the estimation $\sum_{i=1}^{n} e_i^2(t) \leqslant K(\mathbf{e}(t_0))\mathrm{e}^{-\alpha(t-t_0)}$, where $K(\mathbf{e}(t_0)) > 0$ is a constant depending on the initial value $\mathbf{e}(t_0)$, while $\alpha > 0$ is a constant independent of $\mathbf{e}(t_0)$, then the zero solution of the error system (3.3) is said to be globally and exponentially stable, and thus the drive-response systems (3.1) and (3.2) are (i) globally and exponentially (lag) synchronized for $\tau > 0$; and (ii) globally and exponentially synchronized for $\tau = 0$.

Lemma 1 [8,18]. *The zero solution of the error dynamical system (3.3) is globally and exponentially stable, i.e.,* (i) *the drive-response systems (3.1) and (3.2) are globally and exponentially* (*lag*) *synchronized for* $\tau > 0$; *and* (ii) *globally and exponentially synchronized for* $\tau = 0$, *if there exists a positive definite quadratic polynomial* $V = (e_1 e_2 \cdots e_n) P (e_1 e_2 \cdots e_n)^{\mathrm{T}}$ *such that* $\frac{\mathrm{d}V}{\mathrm{d}t} = -(e_1 e_2 \cdots e_n) Q (e_1 e_2 \cdots e_n)^{\mathrm{T}}$. *Moreover, the following negative Lyapunov exponent estimation for the error dynamical system (3.3) holds*:

$$\sum_{i=1}^{n} e_i^2(t) \leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \sum_{i=1}^{n} e_i^2(t_0) \exp\left[-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_0)\right],$$

where $P = P^T \in R^{n\times n}$ and $Q = Q^T \in R^{n\times n}$ are both positive definite matrices, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ stand for the maximal and minimal eigenvalues of the matrix P, respectively, and $\lambda_{\min}(Q)$ denotes the minimal eigenvalue of the matrix Q.

In the following, we consider the hyperchaotic system (2.1) as a drive system:

$$\begin{cases} \dot{x}_d = a(y_d - x_d), \\ \dot{y}_d = (c-a)x_d - x_d z_d + c y_d, \\ \dot{z}_d = -b z_d + x_d y_d - z_d(y_d - x_d) - w_d, \\ \dot{w}_d = -d w_d + z_d(y_d - x_d); \end{cases} \tag{3.4}$$

and the system related to (3.4) with feedback controllers u_i $(i = 1, 2, 3, 4)$, given by

$$\begin{cases} \dot{x}_r = a(y_r - x_r) + u_1, \\ \dot{y}_r = (c-a)x_r - x_r z_r + c y_r + u_2, \\ \dot{z}_r = -b z_r + x_r y_r - z_r(y_r - x_r) - w_r + u_3, \\ \dot{w}_r = -d w_r + z_r(y_r - x_r) + u_4, \end{cases} \tag{3.5}$$

as a response system, where u_i's are unknown functions of $(x_d, y_d, z_d, w_d, x_r, y_r, z_r, w_r)$.

Let the error state be

$$\mathbf{e}(t) = (e_x(t), e_y(t), e_z(t), e_w(t))^T = [x_d(t-\tau) - x_r(t), y_d(t-\tau) - y_r(t), z_d(t-\tau) - z_r(t), w_d(t-\tau) - w_r(t)]^T,$$

where $\tau \geqslant 0$. Then from (3.4) and (3.5), we obtain the error dynamical system:

$$\begin{cases} \dot{e}_x(t) = a[e_y(t) - e_x(t)] - u_1, \\ \dot{e}_y(t) = (c-a)e_x(t) + c e_y(t) - x_d(t-\tau) z_d(t-\tau) + x_r(t) z_r(t) - u_2, \\ \dot{e}_z(t) = -b e_z(t) + x_d(t-\tau) y_d(t-\tau) - x_r(t) y_r(t) + z_r(t)[y_r(t) - x_r(t)] \\ \qquad - z_d(t-\tau)[y_d(t-\tau) - x_d(t-\tau)] - e_w(t) - U_3, \\ \dot{e}_w(t) = -d e_w(t) + z_d(t-\tau)[y_d(t-\tau) - x_d(t-\tau)] - z_r(t)[y_r(t) - x_r(t)] - u_4, \end{cases} \tag{3.6}$$

For the synchronization between systems (3.4) and (3.5), we have the following theorem.

Theorem 1. *For the given hyperchaotic system (3.4), when $a > c$, $d > 0$, if one of the following families of feedback controllers u_i $(i = 1, 2, 3, 4)$ is chosen for the response system (3.5):*

$$\text{(A)} \quad \begin{cases} u_1 = \frac{a}{a-c}[-z_d(t-\tau)e_y(t) + y_d(t-\tau)e_z(t) + z_r(t)(e_z(t) - e_w(t))], \\ u_2 = k_2 e_y(t) + z_r(t)[e_w(t) - e_z(t)], \\ u_3 = k_3 e_z(t) + [y_d(t-\tau) - x_d(t-\tau) - 1]e_w(t), \\ u_4 = 0; \end{cases}$$

$$\text{(B)} \quad \begin{cases} u_1 = \frac{a}{c-a} z_r(t) e_w(t)), \\ u_2 = k_2 e_y(t) - z_d(t-\tau)e_x(t) + z_r(t)[e_w(t) - e_z(t)], \\ u_3 = k_3 e_z(t) + [z_r(t) + y_d(t-\tau)]e_x(t) + [y_d(t-\tau) - x_d(t-\tau) - 1]e_w(t), \\ u_4 = 0; \end{cases}$$

$$\text{(C)} \quad \begin{cases} u_1 = 0, \\ u_2 = k_2 e_y(t) - z_d(t-\tau)e_x(t) + z_r(t)[e_w(t) - e_z(t)], \\ u_3 = k_3 e_z(t) + [z_r(t) + y_d(t-\tau)]e_x(t) + [y_d(t-\tau) - x_d(t-\tau)]e_w(t), \\ u_4 = -z_r(t)e_x(t) - e_z(t); \end{cases}$$

$$\text{(D)} \quad \begin{cases} u_1 = 0, \\ u_2 = k_2 e_y(t) - z_d(t-\tau)e_x(t) + z_r(t)e_z(t), \\ u_3 = k_3 e_z(t) + [z_r(t) + y_d(t-\tau)]e_x(t) + [y_d(t-\tau) - x_d(t-\tau) - 1]e_w(t), \\ u_4 = z_r(t)(e_y(t) - e_x(t)); \end{cases}$$

where $k_2 > c$, $k_3 > M_{y_d} + M_{x_d} - b$, and M_{y_d} and M_{x_d} are upper bounds of the state variables $|y_d(t)|$ and $|x_d(t)|$, respectively, then the zero solution of the error dynamical system (3.6) is globally and exponentially stable, and thus (i) *globally* *exponential* (*lag*) *synchronization for* $\tau > 0$; *and* (ii) *globally exponential synchronization for* $\tau = 0$ *occur between the drive-response systems* (3.4) *and* (3.5).

Proof. Consider the controller (A) and choose the following positive definite, quadratic form of Lyapunov function:

$$V(t) = \frac{1}{2}\left[\frac{a-c}{a}e_x^2(t) + e_y^2(t) + e_z^2(t) + e_w^2(t)\right], \tag{3.7}$$

which implies that $P = \mathrm{diag}\left(\frac{a-c}{2a}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and thus $\lambda_{\min}(P) = \min\left\{\frac{1}{2}, \frac{a-c}{2a}\right\}, \lambda_{\max}(P) = \max\left\{\frac{1}{2}, \frac{a-c}{2a}\right\}$. Differentiating $V(t)$ with respect to time t along the trajectory of system (3.6) yields

$$\begin{aligned}
\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{(3.6)} &= \frac{a-c}{a}e_x(t)\dot{e}_x(t) + e_y(t)\dot{e}_y(t) + e_z(t)\dot{e}_z(t) + e_w(t)\dot{e}_w(t) \\
&= (c-a)e_x^2(t) + (a-c)e_x(t)e_y(t) + z_d(t-\tau)e_x(t)e_y(t) - y_d(t-\tau)e_x(t)e_z(t) - z_r(t)e_x(t)(e_z(t) \\
&\quad - e_w(t)) + (c-a)e_x(t)e_y(t) + ce_y^2(t) + e_x(t)e_y(t)e_z(t) - x_d(t-\tau)e_y(t)e_z(t) - z_d(t-\tau)e_xe_y(t) \\
&\quad - k_2e_y^2(t) + z_r(t)e_y(t)[e_z(t) - e_w(t)] - be_z^2(t) - e_x(t)e_y(t)e_z(t) + x_d(t-\tau)e_y(t)e_z(t) \\
&\quad + y_d(t-\tau)e_xe_z(t) + (x_d(t-\tau) - y_d(t-\tau))e_z^2(t) - z_r(t)[e_y(t) - e_x(t)]e_z(t) - e_w(t)e_z(t) - k_3e_z^2(t) \\
&\quad - de_w^2(t) - [y_d(t-\tau) - x_d(t-\tau) - 1]e_z(t)e_w(t) + (y_d(t-\tau) - x_d(t-\tau))e_ze_w(t) + z_r(t)[e_y(t) - e_x(t)]e_w(t) \\
&= (c-a)e_x^2(t) + (c-k_2)e_y^2(t) + (-b-k_3-y_d(t-\tau)+x_d(t-\tau))e_z^2(t) - de_w^2(t) \\
&\leqslant (c-a)e_x^2(t) + (c-k_2)e_y^2(t) + (-b-k_3+M_{y_d}+M_{x_d})e_z^2(t) - de_w^2(t) \\
&= -(e_x(t)e_y(t)e_z(t)e_w(t))Q(e_x(t)e_y(t)e_z(t)e_w(t))^{\mathrm{T}},
\end{aligned} \tag{3.8}$$

where

$$Q = \mathrm{diag}(a-c, k_2-c, k_3+b-M_{y_d}-M_{x_d}, d). \tag{3.9}$$

By Lemma 1, we have the following exponential estimation:

$$\begin{aligned}
\frac{a-c}{a}e_x^2(t) + e_y^2(t) + e_z^2(t) + e_w^2(t) &\leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left[\frac{a-c}{a}e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0) + e_w^2(t_0)\right]\mathrm{e}^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_0)} \\
&= \frac{\max\left\{\frac{1}{2}, \frac{a-c}{2a}\right\}}{\min\left\{\frac{1}{2}, \frac{a-c}{2a}\right\}}\left[\frac{a-c}{a}e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0) + e_w^2(t_0)\right] \\
&\quad \times \exp\left\{-\frac{\min\{a-c, k_2-c, k_3+b-M_{y_d}-M_{x_d}, d\}}{\max\left\{\frac{1}{2}, \frac{a-c}{2a}\right\}}(t-t_0)\right\},
\end{aligned} \tag{3.10}$$

which implies that the conclusion of Theorem 1 is true.

Similarly, for other controllers (B)–(D), we can use the same Lyapunov function (3.7) to obtain the same estimation given by (3.10). The details are omitted here. □

4. Hyperchaos control

In this section, we use (x^*, y^*, z^*, w^*) to denote an arbitrary equilibrium point of system (2.1). Let

$$x_c = x - x^*, \quad y_c = y - y^*, \quad z_c = z - z^*, \quad w_c = w - w^*.$$

To control the hyperchaotic system (2.1) such that all trajections converge to the equilibrium point (x^*, y^*, z^*, w^*), we consider the controlled system:

$$\begin{cases}
\dot{x}_c = a(y_c - x_c) - u_1, \\
\dot{y}_c = (c-a)x_c + cy_c - xz + x^*z^* - u_2, \\
\dot{z}_c = -bz_c + xy - x^*y^* - z(y-x) + z^*(y^* - x^*) - w_c - u_3, \\
\dot{w}_c = -dw_c + z(y-x) - z^*(y^* - x^*) - u_4,
\end{cases} \tag{4.1}$$

where u_i's are control functions to be determined.

Case 1. Consider the equilibrium point $E_0 = (0,0,0,0)$. In this case, the controlled system (4.1) becomes

$$\begin{cases} \dot{x}_c = a(y_c - x_c) - u_1, \\ \dot{y}_c = (c-a)x_c + cy_c - xz - u_2, \\ \dot{z}_c = -bz_c + xy - z(y-x) - w_c - u_3, \\ \dot{w}_c = -dw_c + z(y-x) - u_4. \end{cases} \tag{4.2}$$

Theorem 2. *When $a > c$, $d > 0$, if one of the following families of controllers*:

(i) $u_1 = 0, \quad u_2 = k_2 y_c, \quad u_3 = k_3 z_c + (y_c - x_c - 1)w_c, \quad u_4 = 0;$
(ii) $u_1 = 0, \quad u_2 = k_2 y_c, \quad u_3 = k_3 z_c + (y_c - x_c)w_c, \quad u_4 = -z_c;$
(iii) $u_1 = 0, \quad u_2 = k_2 y_c + z_c w_c, \quad u_3 = k_3 z_c - (x_c + 1)w_c, \quad u_4 = 0;$
(iv) $u_1 = 0, \quad u_2 = k_2 y_c + z_c w_c, \quad u_3 = k_3 z_c - x_c w_c, \quad u_4 = -z_c;$
(v) $u_1 = \frac{c-a}{a} z_c w_c, \quad u_2 = k_2 y_c, \quad u_3 = k_3 z_c + y_c w_c, \quad u_4 = -z_c;$
(vi) $u_1 = \frac{c-a}{a} z_c w_c, \quad u_2 = k_2 y_c + z_c w_c, \quad u_3 = k_3 z_c, \quad u_4 = -z_c;$

where $k_2 > c$, $k_3 > -b + M_y + M_x$, is chosen for the controlled system (4.2), then the zero solution of system (4.2) is globally and exponentially stable, i.e. the equilibrium point E_0 of system (2.1) is globally and exponentially stabilized.

Proof. For Case (i), we choose the positive definite Lyapuonv function:

$$V(t) = \frac{1}{2}\left[\frac{a-c}{a}x_c^2(t) + y_c^2(t) + z_c^2(t) + w_c^2(t)\right]. \tag{4.3}$$

Then we have

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t}\Big|_{(4.2)} &= \frac{a-c}{a}x_c(t)\dot{x}_c(t) + y_c(t)\dot{y}_c(t) + z_c(t)\dot{z}_c(t) + w_c(t)\dot{w}_c(t) \\ &= (c-a)x_c^2 + (a-c)x_c y_x + (c-a)x_c y_c + cy_c^2 - k_2 y_c^2 - x_c y_c z_c - bz_c^2 + x_c y_c z_c - y_c z_c^2 + x_c z_c^2 - w_c z_c \\ &\quad - k_3 z_c^2 - (y_c - x_c - 1)w_c z_c - dw_c^2 + (y_c - x_c)z_c w_c \\ &= (c-a)x_c^2 + (c-k_2)y_c^2 + (-b - y + x - k_3)z_c^2 - dw_c^2 \\ &\leqslant (c-a)x_c^2 + (c-k_2)y_c^2 + (-b - k_3 + M_y + M_x)z_c^2 - dw_c^2 \\ &= -[(a-c)x_c^2 + (k_2 - c)y_c^2 + (k_3 + b - M_y - M_x)z_c^2 + dw_c^2] < 0 \quad \text{when } x_c^2 + y_c^2 + z_c^2 + w_c^2 \neq 0. \end{aligned} \tag{4.4}$$

This completes the proof of Case (i). Other Cases (ii)–(vi) can be similarly proved. □

Case 2. Now we turn to the equilibrium points $E_\pm = (\pm\sqrt{b(2c-a)}, \pm\sqrt{b(2c-a)}], 2c-a, 0)$. For this case, the controlled system (4.1) becomes

$$\begin{cases} \dot{x}_c = a(y_c - x_c) - u_1, \\ \dot{y}_c = (c-a)x_c + cy_c - xz \pm (2c-a)\sqrt{b(2c-a)} - u_2, \\ \dot{z}_c = -bz_c + xy - b(2c-a) - z(y-x) - w_c - u_3, \\ \dot{w}_c = -dw_c + z(y-x) - u_4. \end{cases} \tag{4.5}$$

Theorem 3. *When $a > c$, $d > 0$, if one of the following families of controllers*:

$$\text{(A)} \quad \begin{cases} u_1 = \frac{a}{a-c}[-zy_c + yz_c + (2c-a)(z_c - w_c)], \\ u_2 = k_2 y_c + (2c-a)(w_c - z_c), \\ u_3 = k_3 z_c + (y - x - 1)w_c, \\ u_4 = 0; \end{cases}$$

$$
\text{(B)}\quad \begin{cases} u_1 = \frac{a(2c-a)}{c-a} w_c, \\ u_2 = k_2 y_c - z x_c + (2c-a)(w_c - z_c), \\ u_3 = k_3 z_c + [2c - a + y] x_c + (y - x - 1) w_c, \\ u_4 = 0; \end{cases}
$$

$$
\text{(C)}\quad \begin{cases} u_1 = 0, \\ u_2 = k_2 y_c - z x_c + (2c-a)(w_c - z_c), \\ u_3 = k_3 z_c + [2c - a + y] x_c + (y - x) w_c, \\ u_4 = (a - 2c) x_c - z_c; \end{cases}
$$

$$
\text{(D)}\quad \begin{cases} u_1 = 0, \\ u_2 = k_2 y_c - z x_c + (2c-a) z_c, \\ u_3 = k_3 z_c + (2c - a + y) x_c + (y - x - 1) w_c, \\ u_4 = (2c - a)(y_c - x_c); \end{cases}
$$

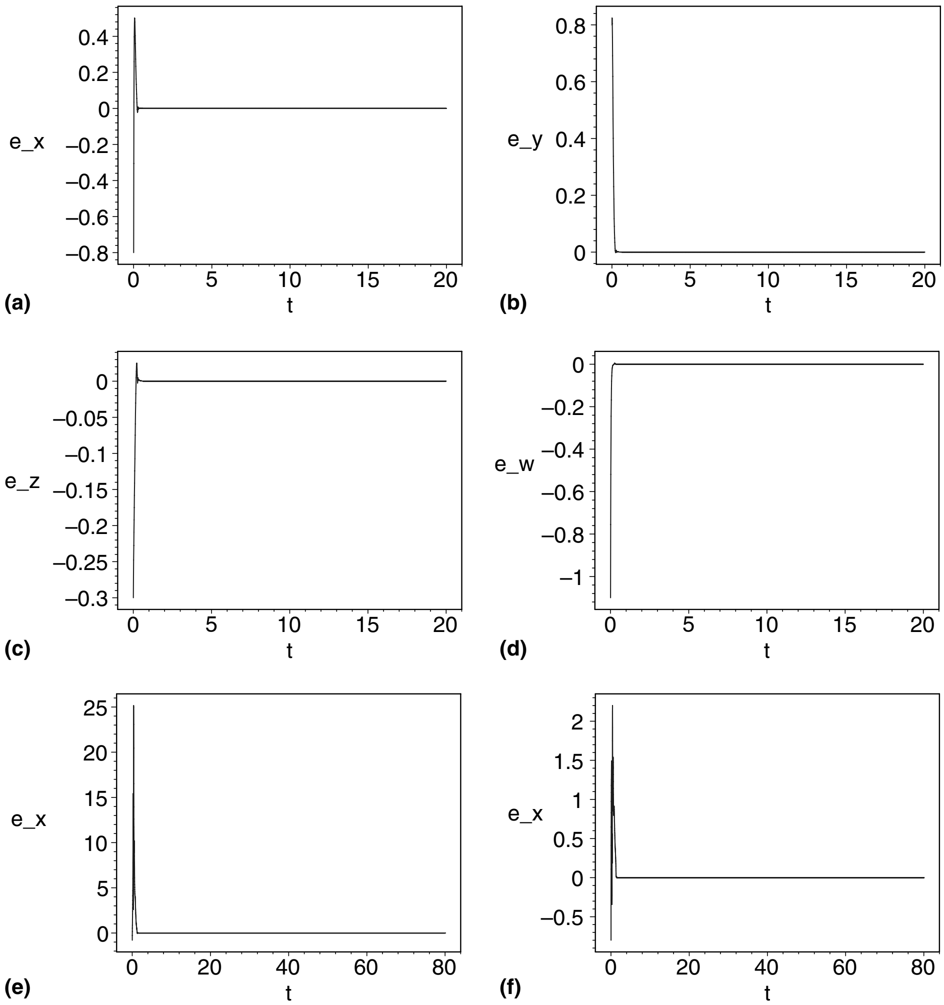

Fig. 6. Simulated results for the hyperchaotic system (2.1) with parameter values $a = 37$, $b = 3$, $c = 26$, $d = 38$: (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the controller given in Case (i) of Theorem 1 with $k_2 = 30$, $k_3 = 40$; (e) and (f) the time histories of the error signals e_x under the controller given in Case (i) of Theorem 1 with the control gains (e) $k_2 = 6$, $k_3 = 8$, and (f) $k_2 = 15$, $k_3 = 0$.

where $k_2 > c$, $k_3 > M_y + M_x - b$, and M_y and M_x are upper bounds of the state variables $|y(t)|$ and $|x(t)|$, respectively, is chosen for the controlled system (4.2), then the zero solution of system (4.2) is globally and exponentially stable, implying that the two equilibrium points $E_\pm$ of system (2.1) are globally and exponentially stabilized.

Proof. We only prove for Case (A). Other cases can be similarly proved. We choose the positive definite Lyapunov function:

$$V(t) = \frac{1}{2}\left[\frac{a-c}{a}x_c^2(t) + y_c^2(t) + z_c^2(t) + w_c^2(t)\right]. \tag{4.6}$$

Then, differentiating $V(t)$ with respect to time t along the trajectory of system (4.5) yields

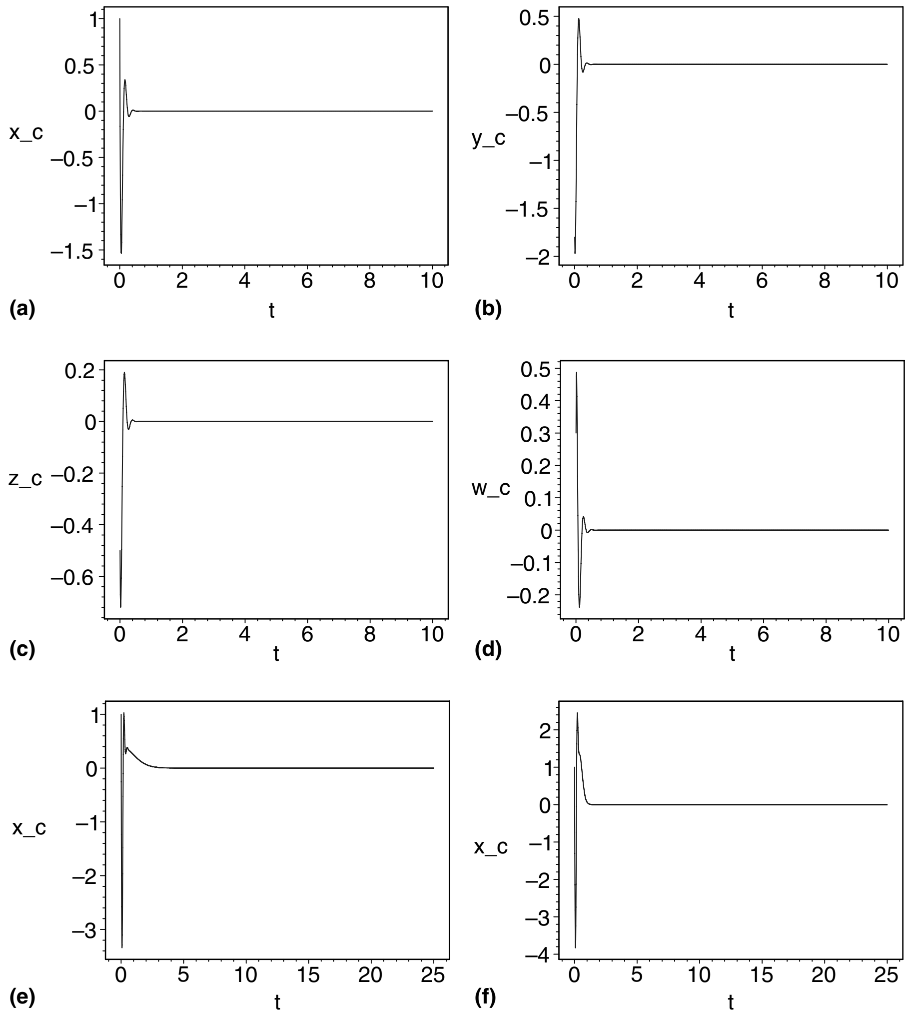

Fig. 7. Simulated results for the hyperchaotic system (2.1) with parameter values $a = 37$, $b = 3$, $c = 26$, $d = 38$: (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the controller given in Case (A) of Theorem 2 with $k_2 = 28$, $k_3 = 40$; (e) and (f) the time histories of the error signals e_x under the controller given in Case (A) of Theorem 2 with the control gains (e) $k_2 = 10$, $k_3 = 18$, and (f) $k_2 = 8$, $k_3 = 20$.

$$
\begin{aligned}
\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{(4.5)} &= \frac{a-c}{a}x_c(t)\dot{x}_c(t)+y_c(t)\dot{y}_c(t)+z_c(t)\dot{z}_c(t)+w_c(t)\dot{w}_c(t) = (c-a)x_c^2+(a-c)x_cy_c+zx_cy_c-yx_cz_c \\
&\quad -(2c-a)x_c(z_c-w_c)+(c-a)x_cy_c+cy_c^2+x_cy_cz_c-xy_cz_c-zx_cy_c-k_2y_c^2 \\
&\quad +(2c-a)y_c(z_c-w_c)-bz_c^2-x_cy_cz_c-xy_cz_c-yx_cz_c+(x-y)z_c^2-(2c-a)(y_c-x_c)z_c \\
&\quad -w_cz_c-k_3z_c^2-dw_c^2-(y-x-1)z_cw_c+(y-x)z_cw_c+(2c-a)(y_c-x_c)w_c \\
&= (c-a)x_c^2+(c-k_2)y_c^2+(-b-k_3-y+x)z_c^2-dw_c^2 \\
&\leqslant (c-a)x_c^2+(c-k_2)y_c^2+(-b-k_3+M_y+M_x)z_c^2-dw_c^2 \\
&= -(e_x(t)\ e_y(t)\ e_z(t)\ e_w(t))Q(e_x(t)\ e_y(t)\ e_z(t)\ e_w(t))^{\mathrm{T}},
\end{aligned} \tag{4.7}
$$

where

$$
Q=\mathrm{diag}(a-c,k_2-c,k_3+b-M_y-M_x,d). \tag{4.8}
$$

Since $a>c$, $k_2>c$, $k_3>M_y+M_x-b$ and $d>0$, it follows from (4.7) and (4.8) that the conclusion for Case (A) of Theorem 3 is true. □

5. Numerical simulation results

In the section, we shall use numerical simulation to verify the control laws presented in the previous sections. We take the parameter values as $a=37$, $b=3$, $c=26$, $d=38$ in system (2.1). Here we restrict to the case $\tau=0$.

First, consider hyperchaos synchronization using the controller given in Case (A) of Theorem 1. The initial values for the drive-response systems (3.4) and (3.5) are chosen as

$$
\begin{aligned}
&(x_d,y_d,z_d,w_d)=(-0.1,0.2,-0.5,0.3),\\
&(x_r,y_r,z_r,w_r)=(0.7,-0.6,-0.2,0.8).
\end{aligned}
$$

Fig. 6(a)–(d) shows the time histories of the error variables e_x, e_y, e_z and e_w under the controller (A) with the control gains chosen as $k_2=30$, $k_3=40$. Since the conditions given in the controllers derived from the Lyapunov function are sufficient, not necessary, we therefore may take smaller values for parameters k_2 and k_3. Fig. 6(e) and (f) displays the time histories of the error state e_x under the controller (A) with $k_2=6$, $k_3=8$ and $k_2=15$, $k_3=0$, respectively. Fig. 6 clearly show that the errors quickly converge to zero, implying that the drive-response systems (3.4) and (3.5) are globally and exponentially synchronized under the above chosen controls.

Next, consider hyperchaos control with the controller given in Case (A) of Theorem 2. The initial values are chosen as $(x_c,y_c,z_c,w_c)=(1,-1.8,-0.5,0.3)$ for the controlled system (4.2). Fig. 7(a)–(d) shows the time histories of the controlled variables x_c, y_c, z_c and w_c under the controller (A) with $k_2=28$ and $k_3=40$. Again due to the conditions given in the controllers being only sufficient, we may choose smaller values for parameters k_2 and k_3. For example, Fig. 7(e) and (f) display the time histories of the error state e_x under the controller (A) of Theorem 2 with $k_2=10$, $k_3=18$ and $k_2=8$, $k_3=20$, respectively. It is seen from Fig. 7 that the errors converge to zero exponentially. This indicates that the controlled system (4.2) globally and exponentially converge to the equilibrium point $E_0=(0,0,0,0)$.

6. Conclusion

In this paper, we have designed a new hyperchaotic system via the introduction of an additional state variable as well as two nonlinear terms and a linear term to the Chen system. Moreover, we have obtained a number of families of control laws to reach globally exponential hyperchaos (lag) synchronization for $\tau>0$, and globally exponential hyperchaos synchronization for $\tau=0$. We have also derived controllers to globally and exponentially stabilize the equilibrium points of the hyperchaotic system. Numerical simulation results are presented to verify the theoretical predictions.

Acknowledgements

This work was supported by the NNSF of China (No. 10401039), the NKBRF of China (No. 2004CB318000), and the NSERC of Canada (No. R2686A02).

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