THE SIMPLEST NORMAL FORM FOR THE
SINGULARITY OF A PURE IMAGINARY PAIR
AND A ZERO EIGENVALUE

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Abstract. This paper is concerned with further reduction of the conventional normal forms of differential equations. The attention is focused on the case when the Jacobian of a system evaluated at an equilibrium has a pair of purely imaginary eigenvalues and a simple zero. Explicit, recursive formulas are derived, which can be used to compute the coefficients of the simplest normal form and the associated nonlinear transformation up to any order. It is shown that unlike conventional normal forms, the simplest normal form is unique and invariant for a fixed “form”, and different “forms” of simplest normal forms are equivalent in the sense of simplicity. The explicit formulas have been implemented using the computer algebra system Maple, and an example is given to demonstrate the efficiency of the computer software.

Keywords. Differential equation, Hopf-zero bifurcation, conventional normal form, simplest normal form, symbolic computation, Maple.

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1 Introduction

Normal form theory plays an important role in the study of differential equations related to complex behavior patterns such as bifurcation and instability [5, 9, 10, 13]. It provides a convenient tool to compute a simple form of the original differential equations. The basic idea of normal form theory is employing successive, near identity nonlinear transformations to eliminate the so-called non-resonant nonlinear terms, and the terms called resonant which cannot be eliminated are remained in normal forms. Many research results related to this area may be found, for example, in the references [4, 5, 8].

It is well known that in general normal forms are not uniquely defined. For example, consider a codimension two example governed by the following differential equations:
\[
\begin{align*}
\dot{x}_1 &= -x_2 - (x_2 - x_1 + x_3)^2 \\
\dot{x}_2 &= x_1 - (x_3 - x_2 + x_5)^2 \\
\dot{x}_3 &= (3x_3 - x_2 - x_4)^2 \\
\dot{x}_4 &= -x_4 + (x_3 - x_5)^2 \\
\dot{x}_5 &= -x_5 + x_6 + (x_3 - x_4)^2 \\
\dot{x}_6 &= -x_5 - x_6 + (x_2 + x_5)^2.
\end{align*}
\] (1)

It is clear that system (1) has an equilibrium at the origin \( x = 0 \). The eigenvalues of the Jacobian matrix of the system evaluated at the equilibrium are \( \pm i, 0, -1 \) and \( -1 \pm i \) \((i^2 = -1)\). The normal form of system (1) can be described on a 3-dimensional center manifold. The dynamic behaviour of such a general codimension two system like bifurcations and stabilities near the origin has been extensively studied by many researchers (e.g., see [10, 12, 22]). The attention in this paper is, however, focused on the study of normal forms.

The normal form of system (1) obtained by using the Poincaré-Birkhoff normal form theory with the aid of Maple [3] up to the 5th-order was given in cylindrical coordinates as follows:

\[
\begin{align*}
\dot{r} &= 3r - 114r^3 + \frac{49}{2}r^2 + \frac{2325}{10}r^3 + \frac{674}{4}r^3 + 692z^2 - 6917z^2 - 149437r^3 - \frac{56795}{20}r^4 \\
\dot{\theta} &= 1 + 2z - \frac{67}{40}r^2 - \frac{27}{4}r^2 + \frac{53}{100}r^2 + \frac{405}{2}r^2 z^3 - 3992513r^4 - \frac{473347}{1000}r^2 z^2 \\
&\quad + \frac{36343}{20}z^4 - 397104361707r^4 z + 3612601r^2 z^3 - 12638063z^5, \\
\dot{z} &= \frac{1}{2}r^2 + 9z^2 + 5^2z^3 + \frac{1371}{160}r^4 - 212r^2z^2 + 111z^4 - \frac{80683}{800}r^4z \\
&\quad - \frac{2277}{8}r^2z^3 - \frac{22107}{2}z^5.
\end{align*}
\] (2)

A different normal form obtained via a perturbation approach using the Maple program given in [18, 19] up to the 5th-order is:

\[
\begin{align*}
\dot{r} &= 3r - \frac{11}{40}r^3 - \frac{49}{2}r^2 - \frac{147}{50}r^3 + 136r^3 + \frac{903271}{48000}r^5 - \frac{1301177}{2000}r^3 + 42873r^4 \\
\dot{\theta} &= 1 + 2z - \frac{57}{40}r^2 - 9r^2 z^3 + 2869r^2 z^2 - 2317267r^4 - \frac{2241r^2z^2}{125} \\
&\quad + \frac{177461}{40}z^4 + \frac{304061282000}{123456}r^4 z - 821447r^2 z^3 - 4711345z^5 \\
\dot{z} &= \frac{1}{2}r^2 + 9z^2 + 46r^2 z + 18r^3 - \frac{257}{32}r^4 - 3r^2 z^2 + 111z^4 - \frac{295}{4}r^4 z \\
&\quad - 1961r^2 z^3 - \frac{22107}{2}z^5.
\end{align*}
\] (3)

Note that the above two normal forms are different though they have the same “form”, i.e., the two normal forms have the same terms but their coefficients may be different. Thus, naturally, some questions arise: Are the two normal forms correct? If they are correct, then how many different normal forms can system (1) have? Does a general form exist which includes all possible different normal forms of the system? Does there exist the “simplest” form among the different normal forms, and how to find it?

The first question whether the two different normal forms (2) and (3) are correct is related to verifying a given normal form associated with a nonlinear
transformation regardless of the method used. A verification scheme developed early (e.g., see [18, 19]) can be employed here: Substitute the nonlinear transformation with the aid of the normal form back to the original differential equations and simplify the resulting equations. If the residue starts from 6th-order terms, then the normal form and its corresponding nonlinear transformation are correct. Otherwise they must be wrong! This approach has been used via Maple to show that the two normal forms (2) and (3) and their associated nonlinear transformations are correct! In fact, all the results presented in this paper have been verified using this approach via Maple.

To answer “how many different normal forms can system (1) exist?” and “Does a general form exist which includes all possible different normal forms?”, it is enough to show that system (1) indeed has infinite number of normal forms which are similar to those given by equations (2) and (3). This can be achieved by using a general nonlinear transformation and then appropriately determine the coefficients. In fact, a similar example has been studied in detail and a general form including all possible different normal forms up to the 5th-order has been obtained [20].

The answer to the question: “Does there exist the simplest form among the different normal forms?” is yes, not only to the particular example (1) but also to a general $n$-dimensional system with the same singularity as that of system (1). In fact, from the viewpoint of uniqueness theory and invariant theory, further reductions on several cases of conventional normal forms have been discussed. Ushiki [15] introduced the method of infinitesimal deformation associated with Lie Bracket to obtain simpler forms than conventional normal forms (CNF). Baider and Churchill [1] developed grading functions based on Lie algebra to define the first, the second, etc., order normal forms. They considered the singularity of one pair of purely imaginary eigenvalue, and obtained the simplest “form”. This case was recently re-investigated from the computational point of view, and explicit formulas were obtained for computing the coefficients of the simplest normal form (SNF) and the nonlinear transformation (NT) [21]. Later, Baider and Sanders [2] applied Lie algebra to study the further reduction of normal forms of Bogdanov-Takens singularity (a double zero eigenvalue). They classified the case as three sub-cases and solved two of them. The remaining one was solved recently by Kokubu et al. [11] and Wang et al. [16]. Sanders and Meer [14] also used the same method to study the Hamiltonian 1:2 resonant case and showed that the second order normal form is “unique”. Here, the “unique” means that the normal form is the “minimum” or the “simplest”. More references can be found from the tutorial articles [6, 7].

In the CNF theory, by saying that “normal forms are not unique” it usually implies at least one of the two cases: (1) For the same system, its normal forms may be different; (2) Even for the same form, the CNFs may be different. For example, a system associated with Bogdanov-Takens singularity may typically have two forms [10]; while the normal forms of Hopf bifurcation have the same form with only odd order terms presented, but their coeffi-
cients may be different [21]. Thus, certain methods such as adjoint operator 
approach [8] were developed to determine a “unique” normal form. But such 
a “unique” normal form is not the “simplest” form. In order to avoid possible 
confusions, we will use “simplest normal form” rather than “unique normal 
form” in this paper.

In this paper, a method is presented for efficiently computing SNFs of 
differential equations. Here, by the “simplest” we mean that if the SNF is 
obtained, then it cannot be further simplified by any other NTs. Our method 
does not use Lie Algebra, but is just based on NTs. The key step in the com-
putation is to find an appropriate pattern of NT so that the resulting normal 
form is the simplest. The main result is summarized in a theorem and the 
proof is based on matrix theory and algebraic computations, which can thus 
provide a direct guideline for developing computer software. Indeed, Maple 
programs have been developed directly following the proof. The explicit for-
mulas of the SNFs for Hopf and generalized Hopf bifurcations up to any order 
have been obtained [21]. For example, the SNF of Hopf bifurcation is given 
by (in polar coordinates) [21]

\[
\begin{align*}
\hat{R} &= a_1 R^3 + a_2 R^5 \\
\hat{\Theta} &= 1 + b_1 R^2
\end{align*}
\] 

if \( a_1 \neq 0 \). Similar results have also been obtained for generalized Hopf 
bifurcations. It has been shown that at most only two terms are presented in 
the amplitude equation of the SNF up to an arbitrary order. This indicates 
that a CNF may be greatly further simplified.

This paper is concerned with the SNF of systems whose Jacobian matrices 
evaluated at an equilibrium include a pair of purely imaginary eigenvalues and 
a simple zero (Hopf-zero bifurcation). It is shown that a generic SNF exists, 
and a detailed procedure is given for computing the SNF and associated 
NT. A user-friendly computer software written in Maple has been developed, 
which can be easily executed on a computer system without any interaction.

A theorem for the SNF of Hopf-zero bifurcation and its proof are given 
in Section 2. Section 3 outlines the symbolic computation procedure using 
Maple, and an example is given in Section 4 to demonstrate the applicability 
of the theorem and the efficiency of the computer software. Conclusions 
are drawn in Section 5. The Maple source code for computing the SNF of 
Hopf-zero bifurcation is listed in Appendix for reference.

2 Main Result –

SNF of Hopf-Zero Bifurcation

In this section, we’ll use matrix theory and the method of mathematical 
induction to prove a main theorem. The theorem gives the SNF, up to an 
arbitrary order, for the following general system:
\[
\dot{x} = Jx + F(x), \quad x \in \mathbb{R}^n,
\]  
(5)

where function \( F \) and its first derivative vanishes at the origin \( 0 \), and the Jacobian matrix \( J \) is given by

\[
J = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & A
\end{bmatrix},
\]  
(6)

where \( A \) is an \((n-3) \times (n-3)\) hyperbolic matrix (i.e., all eigenvalues of \( A \) have non-zero real parts). Assume that by the normal form theory, a CNF of system (5) is found up to an arbitrary order \( n \) in the form (e.g., see [10]):

\[
\dot{r} = r \left( a_{101} z + \sum_{i=1}^{m_1} \sum_{j=0}^{i} a_{12(i-j)} 2_j r^{2(i-j)} z^{2j} + \sum_{i=1}^{m_2} \sum_{j=0}^{i} a_{21(i-j)} 2_j r^{2(i-j)} z^{2j+1} \right)
\]

\[
\dot{\theta} = 1 + a_{201} z + \sum_{i=1}^{m_3} \sum_{j=0}^{i} a_{22(i-j)} 2_j r^{2(i-j)} z^{2j} + \sum_{i=1}^{m_2} \sum_{j=0}^{i} a_{22(i-j)} 2_j r^{2(i-j)} z^{2j+1}
\]

\[
\dot{z} = \sum_{i=1}^{m_3} \sum_{j=0}^{i} a_{32(i-j)} 2_j r^{2(i-j)} z^{2j} + \sum_{i=1}^{m_1} \sum_{j=0}^{i} a_{32(i-j)} 2_j r^{2(i-j)} z^{2j+1}
\]  
(7)

where \( m_1 = m_2 + 1 = m_3 = \frac{1}{2}(n-1) \) when \( n \) is an odd integer; and \( m_1 = m_2 = m_3 - 1 = \frac{n}{2} - 1 \) when \( n \) is an even integer. The coefficients \( a_{ijk} \)'s can be found explicitly in terms of the derivatives of function \( F \) evaluated at the origin. A method for computing the CNF of system (5) can be found in the reference [3] and the Maple programs can be downloaded from the web site: http://pyga.apmaths.uwo.ca/\texttt{pyga/pub/software}. The names of the source code file and input file are \texttt{program4} and \texttt{input4}, respectively.

**Theorem.** The simplest normal form of system (5) up to an arbitrary order \( n \) is given by

\[
\begin{align*}
\tilde{R} &= R(b_{101} Z + b_{120} R^2 + b_{140} R^4) \\
\tilde{\theta} &= 1 + b_{201} Z + b_{220} R^2 + \sum_{i=2}^{m_3} b_{202i} Z^{2i} \\
\tilde{Z} &= b_{302} Z^2 + b_{303} Z^3 + \sum_{i=2}^{m_3} b_{32i0} R^{2i} + \sum_{i=3}^{m_1} b_{32i1} R^{2i} Z
\end{align*}
\]  
(8)

if \( a_{101}/a_{302} \) is not an algebraic number. Here, \( m_1 = m_3 = \frac{1}{2}(n-1) \) when \( n \) is odd, and \( m_1 = m_3 - 1 = \frac{n}{2} - 1 \) when \( n \) is even. In particular, \( a_{101} \) and \( a_{302} \) should satisfy the following conditions:
2 (m_1 - i) a_{1i01} + (2i - 1) a_{302} \neq 0 \text{ for } i = 1, \ldots, k; \quad k = 1, \ldots, m_1 \\
(m_1 - i) a_{1i01} + (i - 1) a_{302} \neq 0 \text{ for } i = 0, \ldots, k; \quad k = 0, \ldots, m_1 \quad (9)

when \( n \) is odd; and

2 (m_1 + 1 - i) a_{1i01} + (2i - 3) a_{302} \neq 0 \text{ for } i = 1, \ldots, k; \quad k = 1, \ldots, m_1 \\
(m_1 - i) a_{1i01} + i a_{302} \neq 0 \text{ for } i = 0, \ldots, k; \quad k = 0, \ldots, m_1 \quad (10)

when \( n \) is even. Note that in equation (8) the coefficients \( b_{i01} = a_{i01}, \quad b_{201} = a_{2i0}, \quad b_{320} = a_{302}, \quad b_{202} = a_{302}, \) and others are expressed explicitly in terms of \( a_{ijk} \)’s.

Proof. The main part of the proof is to show that the CNF (7) of system (5) can be further reduced to the SNF given by equation (8). First we assume that a CNF, given by equation (7), has been obtained from equation (5) by using any method of normal form theory [10]. In order to find explicit NT between the original system (5) and the SNF (8), we need to transform the cylindrical form (7) into Cartesian coordinates, which can be written as

\[
\begin{align*}
\dot{y}_1 &= -y_2 + y_1 \left[ a_{101} y_3 + \sum_{i=1}^{m_1} \sum_{j=0}^{i} a_{12(i-j)2j} S_{1y} + \sum_{i=1}^{m_2} \sum_{j=0}^{i} a_{12(i-j)2j+1} S_{2y} \right] \\
\dot{y}_2 &= y_1 + y_2 \left[ a_{201} y_3 + \sum_{i=1}^{m_1} \sum_{j=0}^{i} a_{22(i-j)2j} S_{1y} + \sum_{i=1}^{m_2} \sum_{j=0}^{i} a_{22(i-j)2j+1} S_{2y} \right] \\
\dot{y}_3 &= \sum_{i=1}^{m_1} \sum_{j=0}^{i} a_{32(i-j)2j} S_{1y} + \sum_{i=1}^{m_1} \sum_{j=0}^{i} a_{32(i-j)2j+1} S_{2y} 
\end{align*}
\]

where \( S_{1y} = (y_1^2 + y_2^2)^{i-j} y_3^{2j}, \quad S_{2y} = (y_1^2 + y_2^2)^{i-j} y_3^{2j+1} \) and the transformation

\[
y_1 = r \cos \theta, \quad y_2 = r \sin \theta, \quad y_3 = z
\]

has been used.

Next, based on equation (11), we shall derive the SNF and the associated NT. The basic idea of the procedure is to eliminate the terms in equation (11), order by order, as many as possible using an appropriate NT. The method of mathematical induction will be used in the proof. The procedure can be divided into three steps: The first step is to show that the main
theorem (or the SNF (8)) is true for \( n = 2, 3, 4 \). Usually the method of mathematical induction only takes one value for the first step. But here we take three values because: (i) As it will be seen that the 2nd-order NT does not simplify the normal form and some coefficients are remained to be determined in simplifying higher order normal forms. (ii) The pattern of the SNF is different when \( n \) is odd from that when \( n \) is even. This is why we need to first prove for \( n: 2 \rightarrow 3 \) and then for \( n: 3 \rightarrow 4 \). The second step is to prove that the main theorem is true up to the \( n \)th order under the assumption that the theorem is true up to the \((n-1)\)th order. Again, one should distinct the cases when \( n \) is odd and when \( n \) is even. The third step, based on the first and second steps, is to conclude that the main theorem (or the SNF (8)) is true for an arbitrary order \( n \). At each order of the procedure, the basic calculation is to substitute a general NT and an assumed SNF, which is at least as simple as the CNF, to system (11) and then balance the coefficients of the same order terms in the resulting equation. This results in a set of linear algebraic equations for the coefficients of the NT and the SNF. Finally, solve the linear equations such that as many as possible coefficients of the NT used to eliminate the coefficients of the assumed SNF as many as possible. The details are given below.

1st Step. We start from the lowest order: \( n = 2 \), and want to see if any of the 2nd-order terms in equation (11) can be eliminated by using the following general 2nd-order NT:

\[
y_i = u_i + \sum_{j+k+l=2} c_{ijkl} u_1^j u_2^k u_3^l \quad (i = 1, 2, 3; \; j, k, l \geq 0).
\] (13)

It should be noted that under any transformations, a new normal form should be at least as simple as the CNF (11). Thus, we may assume that upon applying the NT (13), the normal form up to the 2nd-order is in the form:

\[
\dot{u}_1 &= -u_2 + b_{101} u_1 u_3 - b_{201} u_2 u_3 \\
\dot{u}_2 &= u_1 + b_{101} u_2 u_3 + b_{201} u_1 u_3 \\
\dot{u}_3 &= b_{320} (u_1^2 + u_2^2) + b_{302} u_3^2
\] (14)

which is actually the equation (11) truncated at the 3rd-order terms.

Next, we want to find whether one can set all or part of \( b_{101}, b_{201}, b_{320} \) and \( b_{302} \) zero by suitably choosing the coefficients \( c_{ijkl} \)'s. To find the equations for determining the \( c_{ijkl} \)'s, substituting equations (13) and (14) into equation (11), and then balancing the 2nd-order terms on both sides of the resulting equations yields the following eighteen linear algebraic equations for the eighteen unknown \( c_{ijkl} \)'s, which can be grouped as three sets of decoupled equations, written in the matrix from:
\[
\begin{bmatrix}
1 & -1 & 2 \\
-2 & 1 & 2 \\
-1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
c_{1200} \\
c_{2110} \\
c_{1020} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} 
\tag{15}
\]

\[
\begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & -2 \\
-1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
c_{2200} \\
c_{1110} \\
c_{2030} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} 
\tag{16}
\]

and

\[
\begin{bmatrix}
1 & 2 & -2 \\
-1 & -1 & 1 \\
-1 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
c_{3110} \\
c_{3020} \\
c_{3200} \\
\end{bmatrix}
= \begin{bmatrix}
b_{320} - a_{320} \\
b_{320} - a_{320} \\
b_{320} - a_{320} \\
\end{bmatrix} 
\tag{17}
\]

where a black space in the matrices indicates a zero entry. It is obvious to see from equations (15),(16) and (17) that

\[
c_{1200} = c_{2110} = c_{1020} = c_{1002} = 0 \\
c_{2200} = c_{1110} = c_{2030} = c_{2002} = 0 \\
c_{3110} = c_{3101} = c_{3011} = 0 
\tag{18}
\]

\[
c_{2011} = c_{1101} \\
c_{2101} = - c_{1011} \\
c_{3020} = c_{3200} 
\tag{19}
\]

and

\[
b_{01} = a_{101} \\
b_{201} = a_{201} \\
b_{320} = a_{320} \\
b_{302} = a_{302} 
\tag{20}
\]

Therefore, the NT (13) is given by

\[
y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} c_{1101} u_1 u_3 + c_{1011} u_2^3 \\ c_{1101} u_2 u_3 - c_{1011} u_1 u_3 \\ c_{3200} (u_1^2 + u_2^2) + c_{3002} u_3^2 \end{bmatrix} \equiv u + NT_2 
\tag{21}
\]
where \( NT_2 \) represents the 2nd-order terms of the NT. The SNF up to the 2nd-order then becomes

\[
\dot{\mathbf{u}} = \begin{pmatrix}
-\mathbf{u}_2 \\
\mathbf{u}_3
\end{pmatrix} = \begin{pmatrix}
\mathbf{u}_1 \\
0
\end{pmatrix} + \begin{pmatrix}
 \begin{pmatrix}
 a_{101} \mathbf{u}_1 \mathbf{u}_3 - a_{201} \mathbf{u}_2 \mathbf{u}_3 \\
 a_{101} \mathbf{u}_2 \mathbf{u}_3 + a_{201} \mathbf{u}_1 \mathbf{u}_3
\end{pmatrix} +
 a_{320} (\mathbf{u}_1^2 + \mathbf{u}_2^2) + a_{302} \mathbf{u}_3^2
\end{pmatrix} \equiv LF + NF_2 \quad (22)
\]

where \( LF \) and \( NF_2 \) denote the linear and 2nd-order terms of the SNF, respectively. Equation (22) suggests that no matter what NTs one may choose, the 2nd-order terms in the CNF (11) or (7) cannot be eliminated, and therefore, the normal form up to the 2nd-order cannot be further simplified. It can be seen from equation (21) that the coefficients \( c_{1101}, c_{1011}, c_{3200} \) and \( c_{3002} \) (which does not appear in the 2nd-order equations) can be chosen arbitrarily. The simplest choice for these coefficients is, of course, \( c_{1101} = c_{1011} = c_{3200} = c_{3002} = 0 \), which leads to the identity transformation \( y_i = \mathbf{u}_i \) \((i = 1, 2, 3)\) implying that nothing has been performed. However, as it will be seen, these coefficients may be used to simplify higher order normal forms, and thus should not be determined at this step. This is the key idea used for further simplifying a CNF.

When \( n = 3 \), similarly we’ll try to eliminate the third order terms in the normal form (11) by using the following 3rd-order NT:

\[
y_i = \mathbf{u}_i + NT_{2i} + \sum_{j+k+l=3} c_{ijkl} \mathbf{u}_1^j \mathbf{u}_2^k \mathbf{u}_3^l \quad (i = 1, 2, 3; j, k, l \geq 0). \quad (23)
\]

Assume that the 3rd-order terms in the normal form are the same as that given, in general, by the corresponding part of the CNF, and then add these terms to the previously obtained (2nd-order) SNF, \( NF_2 \), to construct a form up to the 3rd-order terms as follows:

\[
\dot{\mathbf{u}} = LF + NF_2 + \begin{pmatrix}
 u_1 \left[ b_{120} S_u + b_{102} u_2^2 \right] - u_2 \left[ b_{220} S_u + b_{202} u_2^2 \right] \\
 u_2 \left[ b_{120} S_u + b_{102} u_2^2 \right] + u_1 \left[ b_{220} S_u + b_{202} u_2^2 \right] \\
 b_{321} S_u \mathbf{u}_3 + b_{303} \mathbf{u}_3^3
\end{pmatrix} \quad (24)
\]

where \( S_u = u_1^2 + u_2^2 \).

Now following the procedure used in obtaining the 2nd-order SNF, one can find thirty linear algebraic equations for solving the thirty unknown 3rd-order coefficients. Again, the thirty equations can be divided into three decoupled groups as listed below:
\[
\begin{bmatrix}
1 & -1 & 2 \\
-3 & 1 & -1 \\
2 & 1 & -3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]
(25)

\[
\begin{bmatrix}
1 & -1 & 2 \\
-1 & 1 & 1 \\
2 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
c_3 \\
c_4 \\
c_5
\end{bmatrix}
=
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]
(26)

\[
\begin{bmatrix}
1 & 2 & 3 \\
-2 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
c_{3210} \\
c_{3212} \\
c_{3300}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
(27)

where
\[
C_1 = b_{220} - a_{220} - a_{320} c_{1011} - a_{201} c_{3200}
\]
\[
C_2 = b_{202} - a_{202} - a_{302} c_{1011} - a_{201} c_{3002}
\]
\[
C_3 = b_{120} - a_{120} + a_{320} c_{1101} - a_{101} c_{3200}
\]
\[
C_4 = b_{102} - a_{102} + a_{302} c_{1101} - a_{101} c_{3002}
\]
\[
C_5 = b_{321} - a_{321} + 2 (a_{320} c_{3002} + a_{101} c_{3200} - a_{320} c_{1101} - a_{302} c_{3200}).
\]
(28)

It is easy to solve equations (25), (26) and (27). However, in order to show the patterns of the equations and solutions which may be used in the next
Simplest Normal Form for Hopf-Zero Bifurcation

step of the proof, we list the solutions and the key equations obtained from
the above equations below:

\[
\begin{align*}
c_{2111} &= c_{1201} = c_{1021} = c_{1003} = 0 \\
c_{1111} &= c_{2201} = c_{2021} = c_{2003} = 0 \\
c_{3210} &= c_{3120} = c_{3030} = c_{3300} = c_{3111} = c_{3102} = c_{3012} = 0
\end{align*}
\]  

(29)

\[
\begin{align*}
c_{2300} &= -c_{2120} = c_{1030} = c_{1210} \\
c_{1300} &= c_{2210} = c_{2030} = c_{1120} \\
c_{2102} &= -c_{1012} \\
c_{2012} &= c_{1102} \\
c_{3021} &= c_{3201}
\end{align*}
\]

(30)

and the key equations are

\[
\begin{align*}
b_{303} &= a_{303} \\
b_{102} - a_{101} c_{3002} + a_{302} c_{1101} &= a_{102} \\
b_{321} + 2 a_{320} c_{3002} - 2 a_{320} c_{1101} + 2 (a_{101} - a_{302}) c_{3200} &= a_{321} \\
b_{120} + a_{320} c_{1101} - a_{101} c_{3200} &= a_{120} \\
b_{202} - a_{201} c_{3002} - a_{302} c_{1101} &= a_{202} \\
b_{220} - a_{201} c_{3200} - a_{320} c_{1011} &= a_{220}.
\end{align*}
\]

(31)

It is seen from equation (31) that four coefficients \( c_{3002}, c_{1101}, c_{3200}, c_{1011} \) and six \( b \)'s: \( b_{303}, b_{102}, b_{321}, b_{120}, b_{202}, b_{220} \) are involved. What we would like to obtain from these equations is to eliminate the \( b \)'s as many as possible by appropriately choosing the coefficients \( c \)'s. Note that the first equation of (31) determines \( b_{303} \), while the last two equations are coupled with other equations by the coefficient \( a_{201} \) only. If we want to find solutions which are independent of the coefficient \( a_{201} \) (for example, in case \( a_{201} = 0 \)), then only one of the \( b_{202} \) and \( b_{220} \) can be set zero since only one other coefficient \( c_{1011} \) is involved in the two equations. This becomes clear later that in the generic case when \( a_{101} \neq 0 \) and \( a_{302} \neq 0 \), then the condition \( a_{201} \neq 0 \) is not required. This is similar to Hopf bifurcation (see equation (4).) The remaining three equations have three coefficients \( c \)'s and three \( b \)'s. However, if the three coefficients \( c \)'s are considered as the variables of the three linear equations, they are not independent (which can be easily verified by checking that the rank of the coefficient matrix is 2), so only two of the three coefficients \( c \)'s can be used to eliminate the \( b \)'s, and thus one of the three \( b \)'s must be retained non-zero. Hence, equation (31) gives six possibilities for choosing the non-zero \( b \)'s at this order, indicating that the SNF is not unique. However, the SNF is unique subject to such choices. In other words, given a choice or given a fixed “form” of the SNF, one can have only one simplest form. This is true even different CNFs are used. It will be seen that two CNFs of system (1), given by equations (2) and (3), indeed lead to an identical SNF using an identical NT.
Here, we would like to choose $b_{120}$ and $b_{220}$ as the non-zero coefficients. The reasons are as follows: Since the CNF (11) is associated with the singularities of a pair of purely imaginary eigenvalues and a simple zero, so the resulting SNF should be able to recover the case of Hopf bifurcation if the part associated with the simple zero eigenvalue is neglected. To achieve this, $b_{120}$ and $b_{220}$ must be chosen non-zero. In fact, we can show this by considering the third equation of (11). This equation implies that the codimension-2 system is reduced to a case of Hopf bifurcation if the coefficients $a_{32i0} = 0$ for $i = 1, 2, 3, \ldots$. Now under these conditions, it is easy to see from equation (31) that $b_{220} = a_{220}$ and one of $b_{120}$ and $b_{321}$ should be chosen non-zero. However, it can be shown from the 4th-order equations that $b_{120}$ must equal $a_{120}$ and then $b_{321}$ is determined from equation (31) as $b_{321} = a_{321}$. Similarly, one can show by using the 6th-order equations that $b_{40} = a_{140}$ and then $b_{341}$ is determined from the 5th-order equations as non-zero. From the 5th-order equations, another $b$'s (e.g., $b_{221}$ or $b_{322}$) should be chosen non-zero. Starting from the 7th-order equations, the process is back to normal, that is, $b_{160}$ etc. can be set zero, which agrees with the SNF of Hopf bifurcation (see equation (4)).

To see the pattern of equation (31) more clearly we rewrite the equation in the matrix form

$$
\begin{bmatrix}
M_{11} & M_{12}
\end{bmatrix}
\begin{pmatrix}
v_{11} \\
v_{12}
\end{pmatrix} = \alpha_1
$$

where

$$
M_{11} =
\begin{bmatrix}
1 & & & & \\
& 1 & & & \\
& & 1 & & \\
& & & 1 & & \\
& & & & 1 &
\end{bmatrix}
$$

$$
M_{12} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
-a_{101} & a_{302} & 0 & 0 & 0 \\
2a_{320} & -2a_{320} & 2(a_{101} - a_{320}) & 0 & 0 \\
0 & a_{320} & -a_{101} & 0 & 0 \\
-a_{201} & 0 & 0 & -a_{302} & -a_{320}
\end{bmatrix}
$$

$$
v_{11} = (b_{303}, b_{102}, b_{321}, b_{120}, b_{202}, b_{220})^T
$$

$$
v_{12} = (c_{3002}, c_{1101}, c_{3200}, c_{1011})^T
$$

$$
\alpha_1 = (a_{303}, a_{102}, a_{321}, a_{120}, a_{202}, a_{220})^T.
$$
It can be observed from equation (33) that at most four of six $b$’s can be eliminated if $a_{101} \neq 0$ and at least one of $a_{320}$ and $a_{302}$ is non-zero. Since, in general, the procedure for the case $a_{302} \neq 0$ is same as the case $a_{320} \neq 0$. So for definite, we choose $a_{302}$ and suppose $a_{302} \neq 0$ in the following analysis. The case for $a_{320} \neq 0$ can be similarly derived. Thus, we have

\[
\begin{align*}
 b_{303} &= a_{303} \\
 b_{202} &= 0 \\
 b_{321} &= 0 \\
 b_{102} &= 0 \\
 c_{3002} &= - \frac{a_{102} - a_{302} c_{101}}{a_{101}} \\
 c_{1011} &= - \frac{a_{202} + a_{201} c_{3002}}{a_{302}} \\
 c_{3200} &= - \frac{a_{321} + 2 a_{320} (c_{1101} - c_{3002})}{2 (a_{101} - a_{302})} \\
 b_{120} &= a_{120} - a_{320} c_{1101} + a_{101} c_{3200} \\
 b_{220} &= a_{220} + a_{201} c_{3200} + a_{320} c_{1101}.
\end{align*}
\]

(34)

Consequently, the NT (23) becomes

\[
y = u + NT_2 + \left( u_1 \left[ c_{1120} S_u + c_{1102} u_2^2 \right] + u_2 \left[ c_{1210} S_u + c_{1012} u_3^2 \right] \right) \\
\equiv u + NT_2 + NT_3
\]

(35)

where $S_u = u_1^2 + u_2^2$, and the SNF up to the 3rd-order terms is thus given by

\[
\begin{align*}
 \dot{u} &= LF + NF_2 + \left( b_{120} u_1 (u_1^2 + u_2^2) - b_{220} u_2 (u_1^2 + u_2^2) \right) \\
 & \quad + \left( b_{120} u_2 (u_1^2 + u_2^2) + b_{220} u_1 (u_1^2 + u_2^2) \right) \\
 & \quad + \left( c_{3201} u_3^2 \right) \\
 \equiv LF + NF_2 + NF_3,
\end{align*}
\]

(36)

where the notations $NT_3$ and $NF_3$ have the similar meanings as $NT_2$ and $NF_2$. It is noted from equations (35) and (36) that the only second order coefficient which has not been used in this order ($n = 3$) is $c_{1101}$. Whether this coefficient can be used in further simplifying higher order normal forms will be clear in the next order ($n = 4$).

When $n = 4$, following the procedure used in obtaining the 3rd-order NT, we have the pattern of the 4th-order NT:
\[
y = u + NT_2 + NT_3 + \left( \begin{array}{c}
u_1 u_3 S_{1u} + u_2 u_3 S_{2u} \\
u_2 u_3 S_{1u} - u_1 u_3 S_{2u} \\
c_{3400} S_u + c_{3004} u_1^4 + c_{3202} S_u u_1^2 \\
\end{array} \right)
\]

\[
y \equiv u + NT_2 + NT_3 + NT_4 \quad (37)
\]

where \( S_{1u} = c_{1121} S_u + c_{1103} u_1^2 \) and \( S_{2u} = c_{1211} S_u + c_{1013} u_1^3 \).

Similarly, assume that the 4th-order terms in the SNF is given, in general, as the same as the corresponding part of the CNF, and then add them to the previously obtained (2nd and 3rd-order) SNFs to construct a form up to the 4th-order terms:

\[
\hat{u} = LF + NF_2 + NF_3 + \left( u_1 u_3 \left[ b_{211} S_u + b_{103} u_1^2 \right] - u_2 u_3 \left[ b_{221} S_u + b_{203} u_1^3 \right] \right) + u_1 u_3 \left[ b_{221} S_u + b_{203} u_1^3 \right]
\]

\[
+ b_{340} S_u^2 + b_{322} S_u u_1^2 + b_{304} u_1^4 \quad (38)
\]

It has been found that although the undetermined second order coefficient \( c_{1101} \) is involved in the NT (37) (in \( NT_2 \)), it does not appear in the final key algebraic equations obtained from the 4th order. This suggests that this coefficient cannot be used to eliminate \( b's \) from the 4th order normal form.

If we continue to let this undetermined coefficients be carried to higher order equations, then it can be shown that it even does not appear in the 5th order equations. Therefore, we may set

\[
c_{1101} = 0, \quad (39)
\]

which will simply the analysis and computations in higher order equations \((n \geq 5)\). Thus, following the procedure used in obtaining the 3rd-order SNF, one can find seven linear algebraic equations which involve seven \( b's \) and six remained 3rd-order coefficients, given in the following matrix form

\[
\left[ \begin{array}{cc}
M_{21} & M_{22}
\end{array} \right] \left( \begin{array}{c}
v_{21} \\
v_{22}
\end{array} \right) = \alpha_2 \quad (40)
\]

where

\[
M_{21} = \left[ \begin{array}{cccc|c}
1 & 1 & \frac{1}{2} & 1 & \frac{1}{2} \\
& 1 & & 1 & \\
& & & 1 & \\
& & & & 1 \\
& & & & \\
& & & & 1
\end{array} \right]
\]
Simplest Normal Form for Hopf-Zero Bifurcation

\[
M_{22} = \begin{bmatrix}
a_{302} & -a_{101} & 2a_{302} & 2a_{101} - a_{302} & 2a_{101} - 2a_{320} \\
-4a_{302} & -2a_{320} & -a_{101} & -2a_{320} & -2a_{320} \\
3a_{320} & -2a_{320} & -a_{101} & -2a_{320} \\
a_{302} & -a_{302} & -a_{320} & -a_{320} & -a_{320} \\
a_{302} & -a_{302} & -2a_{320} & -2a_{101} & -2a_{101}
\end{bmatrix}
\]

\[
\mathbf{v}_2 = (b_{304}, b_{103}, b_{322}, b_{121}, b_{340} | b_{203}, b_{221})^T
\]

\[
\mathbf{v}_2 = (c_{3003}, c_{1102}, c_{3201}, c_{1120} | c_{1012}, c_{1210})^T
\]

\[
\mathbf{\alpha}_2 = (a_{304} + *, a_{103} + *, a_{322} + *, a_{121} + *, a_{340} + *, a_{203} + *, a_{221} + *)^T
\]

where an asterisk * represents the terms involving the coefficients which are either known or have been solved from the previous order equations. The idea of the simplification procedure is to use the coefficients \( c_{3003}, c_{1102}, c_{3201}, c_{1120}, c_{1012} \) and \( c_{1210} \) to eliminate as many as possible \( b \)'s. Thus, the best choice is that we can uniquely determine the six coefficients to eliminate six of the seven \( b \)'s because the maximum rank of the second sub-matrix of the coefficient matrix of equation (41) may reach 6. In order to guarantee the existence of solutions for the six coefficients, it requires

\[
a_{302} \neq 0, \quad a_{101} \neq 0 \quad \text{and} \quad 2a_{101} - a_{302} \neq 0.
\]

Any one of the seven \( b \)'s may be chosen nonzero and determined from equation (40), which is remained in the SNF. For convenience, we choose \( b_{340} \) as this coefficient. Hence, the SNF up to the 4th-order becomes

\[
\dot{u} = LF + NF_2 + NF_3 + \left( \begin{array}{c} 0 \\ 0 \\ b_{340} (u^2_1 + u^2_2) \end{array} \right) \equiv LF + NF_2 + NF_3 + NF_4.
\]

2nd Step. Having finished the first step in the method of mathematical induction, the next step is to prove that the SNF (8) is also true under the assumption that the equation (8) is true up to the \((n-1)\)th order. It is noted from the above analysis that the patterns of the SNF and the associated NT are different between the odd \((n = 3)\) and even \((n = 4)\) order. So we first consider the case when \( n \) is odd and then the case when \( n \) is even.

(A) \( n \) is odd. The NT up to the \( n \)th-order can be written in the form

\[
y_k = u_k + NT_{2k} + NT_{3k} + \cdots + NT_{(n-1)k} + \sum_{i=0}^{n-i} \sum_{j=0}^{n-i} c_{ik} \left( \begin{array}{c} u_i \\ u_j \end{array} \right) u_i^j u_3^{n-i-j}
\]

for \( k = 1, 2, 3 \). Again note in equation (44) that the \( n \)th-order terms are, similarly as what we did for the case \( n = 3 \), assumed in the general form of a NT.
The SNF up to the $n$th-order is given by
\[
\begin{align*}
\dot{u}_1 &= -u_2 + NF_{21} + NF_{31} + \cdots + NF_{(n-1)1} \\
&\quad + u_1 \left[ b_{12m10} S_u^{m1} + b_{12(m1-1)2} S_u^{m1-1} u_3^2 + \cdots + b_{102m1} u_3^{2m1} \right] \\
&\quad - u_2 \left[ b_{22m10} S_u^{m1} + b_{22(m1-1)2} S_u^{m1-1} u_3^2 + \cdots + b_{202m1} u_3^{2m1} \right]
\end{align*}
\]
\[
\begin{align*}
\dot{u}_2 &= u_1 + NF_{22} + NF_{32} + \cdots + NF_{(n-1)2} \\
&\quad + u_2 \left[ b_{12m10} S_u^{m1} + b_{12(m1-1)2} S_u^{m1-1} u_3^2 + \cdots + b_{102m1} u_3^{2m1} \right] \\
&\quad + u_3 \left[ b_{22m10} S_u^{m1} + b_{22(m1-1)2} S_u^{m1-1} u_3^2 + \cdots + b_{202m1} u_3^{2m1} \right]
\end{align*}
\]
\[
\begin{align*}
\dot{u}_3 &= NF_{23} + NF_{33} + \cdots + NF_{(n-1)3} \\
&\quad + b_{32m11} S_u^{m1} u_3 + b_{32(m1-1)3} S_u^{m1-1} u_3^2 + \cdots + b_{302m1+1} u_3^{2m1+1} \tag{45}
\end{align*}
\]
where the $n$th-order terms are assumed in the same form of the CNF.

Now differentiating the first two equations of (44) with respect to time $t$ and substituting the resulting equation into equation (11) with the aid of equation (45), and then balancing the coefficients of the $n$th-order terms results in two equations that involve even powers of $v_3$ only on the right-hand side’s expressions. Thus, for the left-hand side of the two equations, when the last subscript $n-i-j$ of $c$’s is an odd number, i.e., when $n-i-j = 2k+1$ for $k = 0, 1, \ldots, m_1$, the following coefficients must vanish:
\[
c_{1i} j (n-i-j) = c_{2i} j (n-i-j) = 0. \tag{46}
\]
Also note that the indices $i$ and $j$ appearing in (46) must satisfy $i + j = n - (2k + 1) = 2(m_1 - k)$ for each $k$, where $n = 2m_1 + 1$ has been used.

When $n - i - j = 2k$, an even number, first consider $k = m_1$. A similar algebraic manipulation leads to
\[
\begin{align*}
\begin{bmatrix}
1 & 1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
c_{0102m1} \\
c_{1102m1}
\end{bmatrix}
&= \begin{bmatrix}
a_{10} 2m1 - b_{10} 2m1 + a_{0101300} 2m1 - (2m2+1) a_{302} 2m1 0 + F_{1m1} \\
a_{10} 2m1 - b_{10} 2m1 + a_{0101300} 2m1 - (2m2+1) a_{302} 2m1 0 + F_{1m1}
\end{bmatrix} \tag{47}
\end{align*}
\]
and
\[
\begin{align*}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
c_{1102m1} \\
c_{2102m1}
\end{bmatrix}
&= \begin{bmatrix}
a_{20} 2m1 - b_{20} 2m1 + a_{2010300} 2m1 + (2m2+1) a_{302} 2m1 0 + F_{2m1} \\
a_{20} 2m1 - b_{20} 2m1 + a_{2010300} 2m1 + (2m2+1) a_{302} 2m1 0 + F_{2m1}
\end{bmatrix} \tag{48}
\end{align*}
\]
where $F_{im1} (i = 1, 2)$ represent the sum of terms which have the known coefficients solved from the previous steps. Solving equations (47) and (48) yields
\[
\begin{align*}
c_{101} 2m_1 &= -c_{101} 2m_1 \\
c_{201} 2m_1 &= c_{110} 2m_1 \\
b_{102m_1} - a_{101} c_{3002} 2m_1 + (2m_2+1) a_{302} c_{1012m_2+1} &= a_{102m_1} + F_1 m_1 \\
b_{202m_1} - a_{201} c_{3002} 2m_1 - (2m_2+1) a_{302} c_{1012m_2+1} &= a_{202m_1} + F_2 m_1. \quad (49)
\end{align*}
\]

Now consider \( n - i - j = 2k \) for \( k = 0, 1, \ldots, m_1 - 1 \). Following the procedure used in solving the case \( k = m_1 \), we can find the equations written in the matrix form:

\[
\begin{bmatrix}
1 & 1 \\
2\alpha + 1 & 1 & -2 \\
\vdots & \vdots & \vdots \\
-2 & 1 & 2\alpha + 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
c_{2 \alpha+1 \ 0 \ 2k} \\
c_{2 \alpha 1 \ 2k} \\
\vdots \\
c_{2 \alpha 1 \ 2k} \\
c_{1 0 \ 2\alpha+1 \ 2k}
\end{bmatrix} = V_1 \quad (50)
\]

and

\[
\begin{bmatrix}
1 & -1 \\
-(2\alpha+1) & 1 & 2 \\
\vdots & \vdots & \vdots \\
2 & 1 & -(2\alpha+1) \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
c_{1 2\alpha+1 \ 0 \ 2k} \\
c_{2 \alpha 1 \ 2k} \\
\vdots \\
c_{1 1 \ 2\alpha 2k} \\
c_{2 0 \ 2\alpha+1 \ 2k}
\end{bmatrix} = V_2 \quad (51)
\]

where \( \alpha = m_1 - k \), \( V_1 \) and \( V_2 \) are given by

\[
V_{i-5} = \left(C_i, C_i, (\alpha_i^1)C_i, (\alpha_i^1)C_i, \cdots (\alpha_i^1)C_i, (\alpha_i^1)C_i, C_i, C_i\right)^T \quad (52)
\]

in which \( i = 6, 7 \) and

\[
\begin{align*}
C_6 &= a_{12\alpha 2k} - b_{12\alpha 2k} + a_{101} c_{32\alpha 0 2k} - (2k+1) a_{320} c_{112(\alpha-1) \ 2k+1} \\
&\quad - [2\alpha a_{101} + (2k-1) a_{320}] c_{112\alpha 2k-1} + G_{1k} \\
C_7 &= a_{22\alpha 2k} - b_{22\alpha 2k} + a_{201} c_{32\alpha 0 2k} + (2k+1) a_{320} c_{12(\alpha-1) \ 1 2k+1} \\
&\quad + [2\alpha a_{101} + (2k-1) a_{320}] c_{12\alpha 0 2k-1} + G_{2k}. \quad (53)
\end{align*}
\]

Note that \( G_{ik}, i = 1, 2, 3 \) \((G_{3k} \text{ appears in equation (57)})\) represent the sum of the terms which have the known coefficients solved from the previous steps. Also note that the notation given in equation (52) actually indicates the law of binomial expansion.

To solve equations (50) and (51), first it is noted that the determinant of the coefficient matrices of the two equations are zero, which implies that

\[
C_6 = C_7 = 0. \quad (54)
\]
Having established that $C_0 = 0$, it is not difficult to solve equation (50) to obtain

$$
\begin{pmatrix}
\frac{c_1}{2} a + 1 & 0 & 2k \\
\frac{c_1}{2} a - 1 & 2 & 2k \\
\frac{c_1}{2} a - 3 & 4 & 2k \\
\frac{c_1}{2} a - 5 & 6 & 2k \\
\vdots & \vdots & \vdots \\
\frac{c_1}{2} a - (2a - 3) & 2 & 2k \\
\frac{c_1}{2} a - (2a - 2) & 2 & 2k \\
\frac{c_1}{2} a - (2a - 1) & 2 & 2k \\
\frac{c_1}{2} a + 1 & 2 & 2k \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
\alpha \\
\alpha \\
\vdots \\
\alpha \\
\alpha \\
1 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
\frac{c_2}{2} a + 1 & 2k \\
\frac{c_2}{2} a - 1 & 2 & 2k \\
\frac{c_2}{2} a - 3 & 4 & 2k \\
\frac{c_2}{2} a - 5 & 6 & 2k \\
\vdots \\
\frac{c_2}{2} a - (2a - 3) & 2 & 2k \\
\frac{c_2}{2} a - (2a - 2) & 2 & 2k \\
\frac{c_2}{2} a - (2a - 1) & 2 & 2k \\
\frac{c_2}{2} a + 1 & 2 & 2k \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
\alpha \\
\alpha \\
\vdots \\
\alpha \\
1 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\alpha \\
\alpha \\
\vdots \\
\alpha \\
1 \\
1 \\
\end{pmatrix}
\left(\begin{array}{c}
1 \\
1 \\
\vdots \\
1 \\
\end{array}\right),
$$

where the coefficients involved in the vector of right hand side again follow the law of binomial expansion. Similarly, solving equation (51) results in

$$
\begin{pmatrix}
\frac{c_1}{2} a + 1 & 2k \\
\frac{c_1}{2} a - 1 & 3 & 2k \\
\frac{c_1}{2} a - 2 & 5 & 2k \\
\frac{c_1}{2} a - 3 & 7 & 2k \\
\vdots & \vdots & \vdots \\
\frac{c_1}{2} a - (2a - 5) & 2k \\
\frac{c_1}{2} a - (2a - 4) & 2k \\
\frac{c_1}{2} a - (2a - 3) & 2k \\
\frac{c_1}{2} a - (2a - 1) & 2k \\
\frac{c_1}{2} a + 1 & 2k \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
\alpha \\
\alpha \\
\vdots \\
\alpha \\
1 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
\frac{c_2}{2} a + 1 & 2k \\
\frac{c_2}{2} a - 1 & 2 & 2k \\
\frac{c_2}{2} a - 3 & 4 & 2k \\
\frac{c_2}{2} a - 5 & 6 & 2k \\
\vdots \\
\frac{c_2}{2} a - (2a - 3) & 2 & 2k \\
\frac{c_2}{2} a - (2a - 2) & 2 & 2k \\
\frac{c_2}{2} a - (2a - 1) & 2 & 2k \\
\frac{c_2}{2} a + 1 & 2 & 2k \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
\alpha \\
\alpha \\
\vdots \\
\alpha \\
1 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\alpha \\
\alpha \\
\vdots \\
\alpha \\
1 \\
1 \\
\end{pmatrix}
\left(\begin{array}{c}
1 \\
1 \\
\vdots \\
1 \\
\end{array}\right),
$$

Finally, a similar treatment of the third equation of (44) yields $c_{3,2} i (n-i-j) = 0$ when $n-i-j = 2k$. Similarly, when $n-i-j = 2k+1$, first compute the simplest case $k = m_1$ to obtain

$$
b_{3,0} 2m_1 + 2 (m_1 - 1) a_{302} c_{3,0} 0 2m_1 = a_{30} 2m_1 + G_3 2m_1 + 1,
$$

and then consider $n-i-j = 2k+1$ for $k = 0, 1, \cdots, m_1 - 1$ to find

$$
\begin{pmatrix}
1 \\
-(2a-1) \\
3 \\
\vdots \\
2a \\
-1 \\
\end{pmatrix}
\begin{pmatrix}
2 \\
3 \\
\vdots \\
2a \\
\end{pmatrix}
= \begin{pmatrix}
\frac{c_3}{2} a + 1 & 2k+1 \\
\frac{c_3}{2} a - 1 & 2k+1 \\
\frac{c_3}{2} a - 3 & 2k+1 \\
\vdots \\
\frac{c_3}{2} a + 1 & 2k+1 \\
\end{pmatrix}
V_3
$$

where

$$
V_3 = \left(\begin{array}{c}
C_8, 0 \left(\begin{array}{c}
\alpha \\
1
\end{array}\right) C_8, 0, \cdots, 0, \left(\begin{array}{c}
\alpha \\
1
\end{array}\right) C_8, 0, C_8
\end{array}\right)^T,
$$

$$
C_8 = a_{30} 2a 2k+1 - b_{3,0} 2a 2k+1 + 2 \left[(1-k) a_{302} - \alpha a_{101}\right] c_{3,0} 0 2k
+ 2 \left[a_{320} c_{1,1} 2(a-1) 2k+1 - (k+1) a_{320} c_{3,0} (a-1) 0 2(k+1)\right].
$$

(59)
Solving equation (58) produces

\[ c_3 2^{\alpha-1} j 2^k = c_3 2^{\alpha-3} j 2^k = \cdots = c_3 1 2^{\alpha-1} j 2^k = 0, \quad (60) \]

\[ C_8 = 0, \quad (61) \]

and

\[
\begin{pmatrix}
  c_3 2^{\alpha-1} j 2^k \\
  c_3 2^{(\alpha-1)} j 2^k \\
  c_3 2^{(\alpha-2)} j 2^k \\
  \vdots \\
  c_3 2^{(\alpha-2)} j 2^k \\
  c_3 2^{(\alpha-1)} j 2^k \\
  c_3 2^{\alpha-1} j 2^k
\end{pmatrix} = \begin{pmatrix}
  1 \\
  \alpha \\
  \alpha (\alpha - 1)/2 \\
  \vdots \\
  \alpha (\alpha - 1)/2 \\
  \alpha \\
  1
\end{pmatrix} \cdot (62)
\]

So far, we have obtained the pattern of NT when \( n \) is an odd number, given by

\[
y_1 = u_1 + u_1 u_2 \sum_{i=0}^{m_2} \sum_{j=0}^{i} c_{112^{(i-j)} 2^j} S_{3u} + u_2 u_3 \sum_{i=0}^{m_2} \sum_{j=0}^{i} c_{12^{(i-j)} 2^j} S_{3u}
\]

\[ + u_1 \sum_{i=0}^{m_1} \sum_{j=0}^{i} c_{112^{(i-j)} 2^j} S_{3u} + u_2 \sum_{i=0}^{m_1} \sum_{j=0}^{i} c_{12^{(i-j)} 2^j} S_{3u} \]

\[
y_2 = u_2 + u_2 u_3 \sum_{i=0}^{m_2} \sum_{j=0}^{i} c_{112^{(i-j)} 2^j} S_{3u} - u_1 u_3 \sum_{i=0}^{m_2} \sum_{j=0}^{i} c_{12^{(i-j)} 2^j} S_{3u}
\]

\[ + u_2 \sum_{i=0}^{m_1} \sum_{j=0}^{i} c_{112^{(i-j)} 2^j} S_{3u} - u_2 \sum_{i=0}^{m_1} \sum_{j=0}^{i} c_{12^{(i-j)} 2^j} S_{3u} \]

\[
y_3 = u_3 + \sum_{i=1}^{m_3} \sum_{j=0}^{i} c_{32^{(i-j)} 2^j} S_{3u} + u_3 \sum_{i=1}^{m_3} \sum_{j=0}^{i} c_{32^{(i-j)} 2^j} S_{3u}. \quad (63) \]

where \( S_{3u} = (u_1^2 + u_2^2)^{i-j} u_3^{2j} \).

Next, consider the key equations (based on equations (54), (57) and (61)) relevant to the undetermined \( b \)'s, which can be written in the matrix form

\[
\begin{bmatrix}
  M_{31} & M_{32}
\end{bmatrix} \begin{bmatrix}
  v_{31} \\
  v_{32}
\end{bmatrix} = \alpha_3 \quad (64)
\]

where \( \alpha_3 \) is known, and

\[
M_{31} = \begin{bmatrix}
  E_1 & O \\
  O & E_2
\end{bmatrix}, \quad M_{32} = \begin{bmatrix}
  B_1 & O \\
  -B_3 & -B_2
\end{bmatrix}. \quad (65)
\]
Note: $E_1$ and $E_2$ are, respectively, $2(m_1+1)$ and $(m_1+1)$ identity matrices, while $B_1$, $B_2$ and $B_3$ are $(m_1+1) \times (m_1+1)$, $(m_1+1) \times m_1$ and $(m_1+1)\times (2m_1+1)$ matrices, given by

$$B_1 = \begin{bmatrix} 2(m_1-1)a_{302} & -a_{101} & (2m_1-1)a_{302} \\ 2m_1a_{320} & -2a_{320} & B_1^{(1)} \\ (2m_1-1)a_{320} & -a_{101} & B_1^{(2)} \\ \cdots & \cdots & \cdots \\ 3a_{320} & -a_{101} & B_1^{(3)} \\ 2a_{320} & -2a_{320} & B_1^{(4)} \\ a_{320} & -a_{101} & \end{bmatrix} \quad (66)$$

$$B_2 = \begin{bmatrix} (2m_1-1)a_{302} & B_2^{(1)} \\ (2m_1-1)a_{320} & (2m_1-3)a_{320} & B_2^{(2)} \\ \cdots & \cdots & \cdots \\ 3a_{320} & B_2^{(3)} \\ a_{320} & \end{bmatrix} \quad (67)$$

where

$$B_1^{(1)} = 2a_{101} + (2m_1 - 4) a_{302}$$
$$B_2^{(1)} = B_1^{(2)} = 2a_{101} + (2m_1 - 3) a_{302}$$
$$B_1^{(3)} = 2(m_1 - 1) a_{101} + a_{302}$$
$$B_1^{(4)} = 2m_1 a_{101} - 2a_{302}$$
$$B_2^{(2)} = 4a_{101} + (2m_1 - 5) a_{302}$$
$$B_2^{(3)} = 2(m_1 - 1) a_{101} + a_{302}$$

and

$$B_3 = \begin{bmatrix} -a_{201} & 0 & -a_{201} \\ 0 & -a_{201} & 0 & -a_{201} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \end{bmatrix} \quad (68)$$

respectively, and

$$\psi_31 = (b_{3s2m_1+1}, b_{102m_1}, \cdots, b_{32m_1+1}, b_{12m_1}, b_{202m_1}, b_{222(m_1-1)}, \cdots, b_{222m_1+1})^T$$
$$\psi_32 = (c_{3s2m_1+1}, c_{1102m_1}, \cdots, c_{1122(m_1-1)}, c_{32m_1+1}, c_{11012m_1}, c_{1212m_1}, \cdots, c_{1222(m_1-1)})^T.$$
It should be noted that similar to the case $n = 3$, equation (64) involves $(3m_1 + 1)$'s and $3(m_1 + 1)$'s. We wish to eliminate the $b$'s as many as possible by appropriately choosing the coefficients $c$'s. It is clear that at most $(3m_1 + 1)$'s can be eliminated if the rank of $M_1 = (M_{31} \ M_{32})$ is $(3m_1 + 1)$, which is the largest rank that $M_1$ may have. In order for $M_1$ to have the rank $(3m_1 + 1)$, the conditions given in equation (9) must be satisfied. Moreover, from the construction of equation (64), we may choose

$$b_{1 \ 2(m_1 - i) \ 2i} = 0 \quad (i = 0, 1, \ldots, m_1),$$

$$b_{3 \ 2(m_1 - i) \ 2i+1} = \begin{cases} 0 & \text{if } i \neq 0, \\ \neq 0 & \text{if } i = 0, \end{cases}$$

$$b_{2 \ 2(m_1 - i) \ 2i} = \begin{cases} 0 & \text{if } i \neq m_1, \\ \neq 0 & \text{if } i = m_1, \end{cases} \quad (69)$$

except for $m_1 = 1, 2$. For the two cases when $m_1 = 1, 2$, one may choose $b_{1k} \neq 0$ instead of $b_{3k} \neq 0$, $k = 2, 4$. Thus, the $(3m_1 + 1)$'s can be uniquely determined from equation (64).

Summarizing the above results indicates that the main theorem is true when $n$ is an odd number. Next, we shall prove the case when $n$ is an even number.

(B) $n$ is even. In this case, the NT up to the $n$th-order can be written in the form:

$$y_k = u_k + NT_{2k} + NT_{3k} + \cdots + NT_{(n-1)k} + \sum_{i=0}^{n} \sum_{j=0}^{n-i} c_{k+i \ j \ \ |n-i-j|} u_1^i u_2^j u_3^{n-i-j} \quad (70)$$

for $k = 1, 2, 3$. It should be noted that although the form of equation (70) is identical to equation (44), they are different in that (70) ended with an even term while (44) ended with an odd term. The SNF up to the $n$th-order can be written as

$$\dot{u}_1 = -u_2 + NF_{21} + NF_{31} + \cdots + NF_{(n-1)1}$$

$$\dot{u}_2 = u_1 + NF_{22} + NF_{32} + \cdots + NF_{(n-1)2}$$

$$\dot{u}_3 = NF_{23} + NF_{33} + \cdots + NF_{(n-1)3}$$

$$+ b_{2m_1 \ 0} S_{u}^{m_1} + b_{2(m_1 - 2) \ 2} S_{u}^{m_1-1} u_3^2 + \cdots + b_{30 \ 2m_3} u_3^{2m_3} \quad (71)$$

where the $n$th-order terms are in the same form of the CNF.
Similarly, we can obtain the following key equation

$$\begin{bmatrix} M_{41} & M_{42} \end{bmatrix} \begin{bmatrix} \psi_{41} \\ \psi_{42} \end{bmatrix} = \alpha_4$$  \hspace{1cm} (72)

where the vector $\alpha_4$ is known, and

$$M_{41} = \begin{bmatrix} E_3 & O \\ O & E_4 \end{bmatrix}, \quad M_{42} = \begin{bmatrix} B_4 & O \\ -B_6 & -B_5 \end{bmatrix}$$  \hspace{1cm} (73)

where $E_3$ and $E_4$ are $2(m_1+1) \times 1$ and $(m_1+1)$ identity matrices, respectively. $B_4, B_5$ and $B_6$ are $(2m_1+3) \times (2m_1+2), (m_1+1) \times (m_1+1)$ and $(m_1+1) \times (2m_1+2)$ matrices, given by

$$B_4 = \begin{bmatrix} (2m_1-1)a_{302} & -a_{101} & 2m_1a_{302} \\ (2m_1+1)a_{320} & -2a_{320} & B_4^{(2)} \\ 2m_1a_{320} & -a_{101} & B_4^{(1)} \\ \vdots & \vdots & \vdots \\ 3a_{320} & -2a_{320} & B_4^{(2)} \\ 2a_{320} & a_{320} & a_{101} \end{bmatrix}$$

$$B_5 = \begin{bmatrix} 2m_1a_{302} & B_5^{(1)} \\ 2m_1a_{320} & 2(m_1-1)a_{320} & B_5^{(2)} \\ \vdots & \vdots \end{bmatrix}$$

$$B_6 = \begin{bmatrix} -a_{201} & 0 & -a_{201} \\ 0 & -a_{201} & 0 \\ \vdots & \vdots & \vdots \\ 0 & -a_{201} & 0 \end{bmatrix}$$  \hspace{1cm} (74)

where

$$B_4^{(1)} = 2a_{101} + (2m_1-2)a_{302}$$

$$B_4^{(2)} = 2m_1a_{101} - a_{302}$$

$$B_5^{(1)} = 2a_{101} + 2(m_1-1)a_{302}$$

$$B_5^{(2)} = 4a_{101} + 2(m_1-2)a_{302}$$

and

$$\psi_{41} = [b_{302}(2m_1+1); b_{11} 2m_1+1; \cdots; b_{12} 2(m_1+1); b_{22} 2m_1+1; \cdots; b_{22} 2m_1+1]^T$$

$$\psi_{42} = [c_{302} 2m_1+1; c_{11} 2m_1+1; \cdots; c_{32} 2m_1+1; c_{11} 2m_1+1; c_{12} 2m_1+1; \cdots; c_{12} 2m_1+1]^T.$$
It can be seen that equation (72) involves \((3m_1 + 3)\) c's and \((3m_1 + 4)\) b's, and that \(M_2\) is an \((2m_1 + 3) \times (2m_1 + 2)\) matrix. So the maximum number of b's can be eliminated by appropriately choosing the coefficient c's is \((3m_1 + 3)\) if the matrix \(M_2 = (M_{41} \ M_{42})\) has full rank \((3m_1 + 3)\). This can be achieved if the conditions given in equation (10) are satisfied. According to the construction of the equation (72), we may choose

\[
\begin{align*}
    b_1 \ 2^{(m_1-i)} \ 2^{i+1} &= 0 & \text{for } i = 0, 1, \ldots, m_1. \\
    b_2 \ 2^{(m_1-i)} \ 2^{i+1} &= 0 & \text{for } i = 0, 1, \ldots, m_1. \\
    b_3 \ 2^{(m_1-i)} \ 2^{i} & \begin{cases} 
        = 0 & \text{if } i \not= 0, \\
        \neq 0 & \text{if } i = 0. 
    \end{cases}
\end{align*}
\]

(75)

Then all of the c's can be uniquely determined from equation (72).

3rd Step. Finally, based on the 1st and 2nd steps, we can conclude that the SNF (8) is true for an arbitrary order \(n\).

This completes the proof.

Remark: Ushiki [15] used Lie Algebra to discuss further reduction of the normal form of Hopf-zero bifurcation, but the obtained SNF was only up to the 5th order [15, 17].

3 Outline of Symbolic Computation

The detailed procedure and formulas given in the proof of the main theorem can be directly applied to develop symbolic computation programs. We have used Maple to develop a software package which can be conveniently used for computing the SNF and associated NT of Hopf-zero bifurcation. The program only requires a minimum preparation for an input file from a user, without any interaction. The Maple program is outlined below. (The Maple source code is listed in Appendix.)

(a) Read a prepared input file. The input file indicates the upper boundary order of the computation of the SNF, Order, and gives the coefficients of the original differential equations (i.e., the CNF of the system), \(a_{ijkl}\).

(b) Set the solution of \(b_{101}, b_{201}, b_{220}, b_{302}\) for the initial step \((i = 2)\), which are given by equation (20), and set \(c_{1101} = 0\).

(c) For a sub-order \(i (3 \leq i \leq \text{Order})\), recursively compute the coefficients of the SNF and the corresponding NT.

(i) Create the variable \(T_{jk}\) which is used to eliminate the terms higher than the given Order in the NT so that computation effort can be reduced greatly.

(ii) Set the general NT \(h_j, j = 1, 2, 3\) using the undetermined coefficients \(c_{jklm}\) via the variable \(T_{jk}\).
(iii) Create the variable $t_pj$ which is used to eliminate the terms higher than the given Order in the SNF so that computation is simplified.

(iv) Set the general SNF $dy_j$, $j = 1, 2, 3$ using the undetermined coefficients $b_{jkl}$ through the variable $t_pj$.

(vi) Create the variable $H_{jk}$ and $H_{jkl}$ which are used to eliminate the terms higher than the given Order in the nonlinear functions $f_j$ so that computation time can be reduced.

(vii) Set the general nonlinear functions $f_j$, $j = 1, 2, 3$ using $H_{jk}$ and $H_{jkl}$.

(viii) Substitute the NT $h_j$ and the SNF $dy_j$ into the differential equations, and pick up the expression of the particular sub-order $i$, $Res_{jk}$.

(ix) From $Res_{jk}$ obtained above, find the expressions of the coefficients for each trinomial $y^i_1 y^m_2 y^p_3$.

(x) By balancing the coefficient of each trinomial, solve the coefficients of the $i$th order NT, $c_{j,k,l,m}$.

(xi) Solve the coefficients of the $i$th order SNF, $b_{j,k,l}$.

(d) Write the SNF and NT into the output file “Nform”.

4 An Example

In this section we shall use the example given by equation (1) to demonstrate the application of the theory and Maple program developed in the previous sections. The Maple program can be used to compute the coefficients of the SNF (8) and the associated NT from the CNF (11) up to any order.

As has been mentioned in the introduction, different CNFs of a system generate an identical SNF by using a same NT. To demonstrate this, we list below two CNFs of system (1) up to the 9th-order which are obtained using two different approaches: One used Poincaré normal form theory [3] and the other used a perturbation theory [19]. The two CNFs are, respectively, given by

\[
\dot{r} = 3r_z + \frac{9}{40} r^3 - \frac{49}{2} r z^2 + \frac{1239}{100} r^3 z + \frac{674}{4} r z^3 - \frac{691277}{136000} r^5 - \frac{1494327}{2000} r^3 z^2 \\
- \frac{56729}{20} r^4 z + \frac{7223581331}{4624000} r^5 z + \frac{101688239}{2000} r^3 z^3 - \frac{12480307}{400} r z^5 \\
- \frac{10492721137718}{932476200678691} r^2 z^2 + \frac{932476200678691}{40000} r^3 z^2 - \frac{61073523699}{40000} r^3 z^4 + \frac{807869647}{2000} r z^6 \\
+ \frac{117343986034664721139}{3986112128482869} r^7 z - \frac{3986112128482869}{34143200000} r^7 z^3 + \frac{7003528659703}{1200000} r^3 z^5 \\
\frac{7504882976000000}{34143200000} 999063412907 r z^7 - \frac{24330370678306965297561}{999063412907} r z^7 + \frac{1845721999852686720541343}{999063412907} r z^7 \\
+ \frac{149069315125329699961}{72162144000000} r^2 z^4 + \frac{1558103589493351}{36000000} r^3 z^4 - \frac{8872021498973}{100000} r z^8 \
\]
\[
\dot{\theta} = 1 + 2z - \frac{47}{10} r^2 - \frac{47}{10} z^2 + \frac{225}{10} r^2 z + \frac{405}{10} z^3 - \frac{499593}{10} r^4 - \frac{4734}{10} r^2 z^2 + \frac{3644}{10} z^4
- \frac{39104}{10} r^2 z^3 + \frac{361296}{10} r^3 z^3 - \frac{1263663}{10} r^4 z^5 + \frac{273618360}{10} r^5 z^5 + \frac{273618360}{10} r^6 z^6
+ \frac{23582400}{10} r^5 z^5 + \frac{298470050}{10} r^4 z^4 + \frac{208746180}{10} r^4 z^3 + \frac{26361382335045}{10} r^6 z^6
- \frac{128007657066310}{10} r^4 z^3 + \frac{235932862163}{10} r^2 z^5 + \frac{65825504350}{10} z^7
- \frac{282988800000}{10} + \frac{10946075070727208305671}{10} r^5 z^7 + \frac{1565741769000}{10} r^5 z^8
- 61397241545729000000000000000000000000000000000
\]

\[
\ddot{\dot{\theta}} = \frac{1}{2} r^2 + \frac{47}{10} r^2 - \frac{47}{10} z^2 + \frac{225}{10} r^2 z + \frac{405}{10} z^3 - \frac{499593}{10} r^4 - \frac{4734}{10} r^2 z^2 + \frac{3644}{10} z^4
- \frac{39104}{10} r^2 z^3 + \frac{361296}{10} r^3 z^3 - \frac{1263663}{10} r^4 z^5 + \frac{273618360}{10} r^5 z^5 + \frac{273618360}{10} r^6 z^6
+ \frac{23582400}{10} r^5 z^5 + \frac{298470050}{10} r^4 z^4 + \frac{208746180}{10} r^4 z^3 + \frac{26361382335045}{10} r^6 z^6
- \frac{128007657066310}{10} r^4 z^3 + \frac{235932862163}{10} r^2 z^5 + \frac{65825504350}{10} z^7
- \frac{282988800000}{10} + \frac{10946075070727208305671}{10} r^5 z^7 + \frac{1565741769000}{10} r^5 z^8
- 61397241545729000000000000000000000000000000000
\]

It is seen from the above two equations that most of the coefficients are different.
Executing the Maple program (for computing the SNF, listed in Appendix) for equations (76) and (77) yields the following identical SNF up to the 9th-order:

\[
\begin{align*}
\dot{R} &= R (3 Z + \frac{91}{80} R^2 + \frac{17390128913}{82620000} R^4), \\
\dot{\Theta} &= 1 + 2 Z - \frac{7}{6} R^2 + \frac{835435429}{2000} Z^4 - \frac{103828415943456810161}{1379754000000000} Z^6 \\
&\quad + \frac{5411802839883981136711604324796311}{2104727085270886806000000} Z^8 \\
\dot{Z} &= 9 Z^2 + 18 Z^3 + \frac{1209312}{101126680000} R^4 - \frac{3723285238613}{10407335000491702103576000000} R^6 \\
&\quad - \frac{43462761875526855647}{11033902988000000000} R^8 \\
&\quad - \frac{773335418067005647044448079082782997}{73243543159688396515200000000000} R^8 Z.
\end{align*}
\] (78)

This indeed shows that a CNF can be further greatly simplified. It is well known that CNFs are not unique and different even for a fixed “form”, like the two forms given in equations (76) and (77). However, it is amazing to note that even with the two different CNFs, our formula yields the exact same SNF. In other words, for a given system, no matter how many different CNFs we may have, the SNF is unique for a fixed “form”. Also note that letting \( Z = 0 \) in the first two equations of (78) leads to the simplest normal form of Hopf bifurcation [21], while setting \( R = 0 \) in the third equation of (78) yields the SNF associated with a single zero eigenvalue.

The Maple source code (given in Appendix) and the input file for this example are available from the web site: http://pyu1.apmaths.uwo.ca/~pyu/pub/software. The names of the Maple source code and the input file are programa2 and programa2, respectively. A reader can download these programs and execute them to verify the above results. Also, readers can apply the source code to solve their own problems by simply preparing an input file.

5 Conclusions

A theorem has been established for computing the generic SNF of Hopf-zero bifurcation. It has been shown that a normal form obtained using the CNF theory is not the simplest form. In the proof of the theorem, explicit, recursive formulas have been derived which can be directly implemented on a computer algebra system. User-friendly symbolic computation programs written in Maple have been developed. An example is presented to show that different CNFs of a system lead to an identical SNF through the same formulas (coded in the Maple program). This implies that: (1) CNFs obtained by using different approaches are inherently related; (2) The SNF is unique and invariant for a fixed “form”.

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Appendix: Maple Source Program

The source code written in the symbolic computer language Maple for computing the SNF and the corresponding NT for a system associated with Hopf-zero bifurcation is given in the appendix. The Maple source code is also available from the web site:

http://pyg1.apmaths.uwo.ca/~pyg/pub/software (file name program52).

```maple
read input;

b[1,0,1] := a[1,0,1];
b[2,0,1] := a[2,0,1];
b[3,2,0] := a[3,2,0];
b[3,0,2] := a[3,0,2];
c[1,1,0,1] := 0;
for i from 3 to Order do
  evold := modp(i,2);
  if evold=1 then
    m1 := (i-1)/2;
    m2 := m1 - 1;
    m3 := m1 + 1;
  else
    m1 := i/2 - 1;
    m2 := m1;
    m3 := m1 + 1;
  fi:
  for j from 0 to m3 do
    if j=0 and j<=m2 then
      T[1,2*j+1] := T[1,2*j+1] + c[1,1,2*(j-k),2*k+1]*temp:
      T[2,2*j+1] := T[2,2*j+1] + c[1,2*(j-k),1,2*k+1]*temp:
    fi:
    if j=m1 and j<=m1 then
      T[3,2*j] := T[3,2*j] + c[1,1,2*(j-k),2*k+1]*temp:
      T[4,2*j] := T[4,2*j] + c[1,2*(j-k),1,2*k+1]*temp:
      T[5,2*j+1] := T[5,2*j+1] + c[3,2*(j-k),0,2*k+1]*temp:
    fi:
    if j=m1 and j<=m3 then
      T[6,2*j] := T[6,2*j] + c[3,2*(j-k),0,2*k+1]*temp:
    fi:
  od:
  for j from 1 to 3 do
    h[j] := y[j];
  od:
  for j from 1 to m3 do
  od:
```

for \( j \) from 1 to \( m1 \) do
\[
\begin{align*}
\end{align*}
\]
end for

for \( j \) from 1 to 6 do
\[
\begin{align*}
\text{tp}[j] &:= 0; \\
\end{align*}
\]
end for

for \( j \) from 1 to \( m3 \) do
for \( k \) from 0 to \( j \) do
\[
\begin{align*}
\text{temp} &:= (y[1]*2+y[2]*2-y[3]*2-k)*y[3]*2; \\
\text{if } j=1 &\text{ then} \\
\text{tp}[1] &:= \text{tp}[1] + b[1,2*(j-k),2*2]; \\
\text{tp}[2] &:= \text{tp}[2] + b[2,2*(j-k),2*2]; \\
\text{tp}[6] &:= \text{tp}[6] + b[3,2*(j-k),2*2]; \\
\text{fi}; \\
\text{if } k=0 &\text{ and } j<=2 \text{ then} \\
\text{tp}[3] &:= \text{tp}[3] + b[1,2*(j-k),2*2]; \\
\text{tp}[4] &:= \text{tp}[4] + b[2,2*(j-k),2*2]; \\
\text{fi}; \\
\text{if } j<=3 &\text{ then} \\
\text{tp}[5] &:= \text{tp}[5] + b[3,2*(j-k),2*2]; \\
\text{fi}; \\
\end{align*}
\]
end for

\[
\begin{align*}
\text{dy}[1] &:= y[1]*\text{tp}[1] + y[3]*\text{tp}[3] - y[2]*\text{tp}[2] + y[3]*\text{tp}[4]; \\
\text{dy}[2] &:= y[2]*\text{tp}[1] + y[3]*\text{tp}[3] + y[1]*\text{tp}[2] + y[3]*\text{tp}[4]; \\
\text{dy}[3] &:= \text{tp}[5] + y[3]*\text{tp}[6]; \\
\end{align*}
\]
end for

for \( k \) from 0 to \( j \) do
if \( k=0 \) then
\[
\begin{align*}
\text{h1}[j,k] &:= (h[1]^2+h[2]^2)^{-j}; \\
\text{h2}[k] &:= 1; \\
\text{h1}[j,k] &:= \text{subs}(y[1]=\text{eps}*y[1],y[2]=\text{eps}*y[2],y[3]=\text{eps}*y[3],\text{h1}[j,k]) \\
\text{h1}[i,j,k] &:= \text{h1}[i,j,k] + \text{subs}(\text{eps}=0, \text{diff}(\text{h1}[j,k],\text{eps}1)/1!); \\
\end{align*}
\]
end if
else if \( k=1 \) then
\[
\begin{align*}
\text{h1}[j,k] &:= 1; \\
\text{s2}[k] &:= y[3]^2*(2*2); \\
\text{s2}[k] &:= \text{subs}(y[1]=\text{eps}*y[1],y[2]=\text{eps}*y[2],y[3]=\text{eps}*y[3],\text{s2}[k]); \\
\text{h2}[k] &:= 0; \\
\text{h2}[k] &:= \text{h2}[k] + \text{subs}(\text{eps}=0, \text{diff}(\text{s2}[k],\text{eps}1)/1!); \\
\end{align*}
\]
else
\[
\begin{align*}
\text{h1}[j,k] &:= (h[1]^2+h[2]^2)^{-j-1} \\
\text{s2}[k] &:= h[3]^2*(2*2); \\
\text{s2}[k] &:= \text{subs}(y[1]=\text{eps}*y[1],y[2]=\text{eps}*y[2],y[3]=\text{eps}*y[3],\text{s2}[k]); \\
\text{h2}[k] &:= 0; \\
\text{h2}[k] &:= \text{h2}[k] + \text{subs}(\text{eps}=0, \text{diff}(\text{s2}[k],\text{eps}1)/1!); \\
\end{align*}
\]
fi
end if
for \( j \) from 1 to 6 do
\[
\begin{align*}
\text{tp}[j] &:= 0; \\
\end{align*}
\]
end for

\[
\begin{align*}
\text{tp}[2] &:= 1; \\
\text{tp}[3] &:= b[1,0,1]; \\
\end{align*}
\]
for j from 1 to \( m3 \) do
   for k from 0 to j do
      temp := \( H[i,j,k] \times H[2][k] \)
      if \( j \leq m1 \) then
         \( tp[1] := tp[1] + a[1,2,(j-k),2*kl]*temp; \)
      fi;
      if \( m2 < 0 \) and \( j \leq m2 \) then
      fi;
      if \( j \leq m3 \) then
      fi;
   od;
   od;
   for i from 1 to 3 do
      \( f[j] := \text{subs}(y[1] = \text{eps}*y[1], y[2] = \text{eps}*y[2], y[3] = \text{eps}*y[3], f[j]); \)
      \( f[j] := \text{subs}(\text{eps}=0, \text{diff}(f[j], \text{eps}^k)/k!); \)
   od;
   \( \text{Res}[j] := 0; \)
   for k from 1 to 3 do
      \( \text{Res}[j][k] := \text{res}[j][k] + \text{diff}(h[j1,y[k]]*dy[k]); \)
   od;
   \( \text{res}[j] := \text{subs}(y[1] = \text{eps}*y[1], y[2] = \text{eps}*y[2], y[3] = \text{eps}*y[3], \text{res}[j]); \)
   for k from 1 to i do
      \( \text{Res}[j[k]] := \text{subs}(\text{eps}=0, \text{diff}(\text{res}[j], \text{eps}^k)/k!); \)
   od;
   \( \text{Res}[j,1] := \text{Res}[j,1] - f[j]; \)
   \( \text{res}[j] := 0; \)
   od;
   for j from 1 to 3 do
      for k from 1 to i do
         for m from 0 to k-1 do
            \( p := k-l-m; \)
            \( \text{cof}[j1,l,m,p] := \text{diff}(\text{Res}[j[k], y[1], y[2], y[3], m]; p); \)
            \( \text{cof}[j1,l,m,p] := \text{simplify}(\text{subs}(y[1]=0, y[2]=0, y[3]=0, \text{cof}[j1,l,m,p]/l!/m!/p!)); \)
         od;
      od;
   od;
   if \( \text{evold}=1 \) then
      if \( \text{mi}=1 \) then
         \( b[1,0,2] := 0; \)
         \( b[2,0,2] := 0; \)
         \( b[3,0,2] := 0; \)
         \( b[1,0,3] := a[3,0,3]; \)
         \( c[1,0,1,1] := \text{solve}(\text{cof}[1,0,1,2], c[1,0,1,1]); \)
         \( c[3,0,0,2] := \text{solve}(\text{cof}[1,0,1,2], c[3,0,0,2]); \)
         \( c[3,2,0,0] := \text{solve}(\text{cof}[1,0,1,2], c[3,2,0,0]); \)
         \( b[1,2,0] := \text{solve}(\text{cof}[1,1,2,0], b[1,2,0]); \)
         \( b[2,2,0] := \text{solve}(\text{cof}[1,2,0,0], b[2,2,0]); \)
      elseif \( \text{mi}=2 \) then
         \( b[1,2,2] := 0; \)
         \( b[1,0,4] := 0; \)
         \( b[2,4,0] := 0; \)
         \( b[2,2,2] := 0; \)
      else
         for j from 0 to 2 do
            \( b[3,2*(2-j),2*j+1] := 0; \)
         od;
      od;
   od;
for j from m1 by -1 to 1 do
  k := 2*j(m1-j):
  c[3, k, 0, 2*j] := solve(\( c[3, k, 0, 2*j+1], c[3, k, 0, 2*j] \)):
  c[1, 1, k, 2*j] := solve(\( c[1, 1, k, 2*j+1], c[1, 1, k, 2*j] \)):
od;

\[
\begin{align*}
&c[3, 2*m1, 0, 0] := solve(\( c[3, 2*m1, 0, 1], c[3, 2*m1, 0, 0] \)));
&b[1, 2*m1, 0] := solve(\( c[1, 2*m1+1, 0, 0], b[1, 2*m1, 0] \)));
&c[1, 2, 1, 1] := solve(\( c[1, 2, 1, 2*(m1-1)], c[1, 2, 1, 1] \)));
&c[1, 0, 1, 2*m1-1] := solve(\( c[1, 2, 1, 2*(m1-1)], c[1, 0, 1, 2*m1-1] \)));
&b[2, 0, 2*m1] := solve(\( c[1, 0, 1, 2*m1], b[2, 0, 2*m1] \)));
\end{align*}
\]

else
  for j from 0 to m1 do
    b[1, 2*m1-j, 2*j] := 0;
  od;

  for j from 1 to m1 do
    b[3, 2*(m1-j), 2*j+1] := 0;
  od;

  for j from m1 by -1 to 1 do
    k := 2*(m1-j):
    c[3, k, 0, 2*j] := solve(\( c[3, k, 0, 2*j+1], c[3, k, 0, 2*j] \)):
    c[1, 1, k, 2*j] := solve(\( c[1, 1, k, 2*j+1], c[1, 1, k, 2*j] \)):
  od;

\[
\begin{align*}
&c[3, 2*m1, 0, 0] := solve(\( c[3, 2*m1, 0, 1], c[3, 2*m1, 0, 0] \)));
&b[3, 2*m1, 1] := solve(\( c[3, 2*m1, 0, 1], b[3, 2*m1, 1] \)));
&for j from 0 to m1-1 do
&  b[2, 2*m1-j, 2*j] := 0;
&  c[1, 2*(m1-1), 1, 1] := solve(\( c[1, 2*(m1-1), 0, 0], c[1, 2*(m1-1), 1, 1] \)));
&for j from 1 to 2*m1 do
&  k := 2*(m1-j):
&  c[1, k, 1, 2*j-1] := solve(\( c[1, k+2, 1, 2*(j-1)], c[1, k, 1, 2*j-1] \)));
&od;
&b[2, 0, 2*m1] := solve(\( c[1, 0, 1, 2*m1], b[2, 0, 2*m1] \)));
\end{align*}
\]

fi:
else
  for j from 0 to m1 do
    b[1, 2*(m1-j), 2*j+1] := 0;
    b[2, 2*(m1-j), 2*j+1] := 0;
  od;

  for j from 1 to m3 do
    b[3, 2*(m3-j), 2*j] := 0;
  od;

  for j from m1 by -1 to 0 do
    k := 2*(m1-j):
    c[3, k, 0, 2*j+1] := solve(\( c[3, k, 0, 2*j+2], c[3, k, 0, 2*j+1] \)):
    c[1, 1, k, 2*j] := solve(\( c[1, 1, k, 2*j+1], c[1, 1, k, 2*j] \)):
    c[1, 1, k, 2*j] := solve(\( c[1, 1, k, 2*j+1], c[1, 1, k, 2*j] \)):
  od;

\[
\begin{align*}
&b[3, 2*m3, 0] := solve(\( c[3, 2*m3, 0, 0], b[3, 2*m3, 0] \)));
\end{align*}
\]

fi:
for j from 1 to 3 do
  h[j] := b[j];
od:

od:

\[
\begin{align*}
&dr := r*(b[1,0,1]*z + b[1,2,0]*r^2 + b[1,4,0]*r^4); \\
&dt := 1 + b[2,0,1]*z + b[2,2,0]*r^2; \\
&dz := b[3,0,2]*r^2 + b[3,0,3]*z^3; \\
&for j from 2 to m3 do \\
&  dt := dt + b[2,0,2]*r^2; \\
&  dz := dz + b[3,2,j]*r^2; \\
&od:
&for j from 3 to m1 do \\
&  dz := dz + b[3,2,j,1]*r^2; \\
&od:
save dr, dt, dz, h, 'Norm':
\]
References


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