The simplest normal forms associated with a triple zero eigenvalue of indices one and two\(^1\)

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Abstract

This paper considers the computation of the simplest normal forms of differential equations, with the particular attention on the systems whose Jacobian evaluated at an equilibrium has a triple zero eigenvalue of index one or two. The computation is based on near identity nonlinear transformations. Explicit formulas are derived for computing the simplest normal forms and the associated nonlinear transformations, which facilitates a direct implementation on computer systems using Maple.

Key words: Differential equation, Conventional normal form, The simplest normal form, Nonlinear transformation, Computer algebra

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1 Introduction

Normal form theory plays an important role in the study of differential equations related to complex behavior patterns such as bifurcation and instability [1–3]. It provides a convenient tool for computing a simple form of the original differential equations. The basic idea of the normal form theory is employing successive, near identity nonlinear transformations to obtain a simpler form. Many research results related to this area may be found from the above mentioned references.

It is well known that in general normal forms are not uniquely defined, and in fact further reductions on several cases of convention normal forms (CNF) have been discussed. Ushiki [4] introduced the method of infinitesimal deformation associated with Lie bracket to obtain simpler forms than CNF. Baidar and Churchill [5] developed grading functions based on Lie algebra to define the first, the second, etc., order normal forms. They considered Hopf type

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singularity and obtained the simplest "form". This case was recently reinvesti-
gigated from the computation point of view, and explicit formulations were
obtained for computing the coefficients of the simplest normal form (SNF) and
the nonlinear transformation [6]. The further reduction of the normal forms for
the Bodganov-Takens singularity has been studied by Baider and Sanders [7],
Kokubu et al. [8] and Wang et al. [9] using the Lie bracket method. Sanders
and Meer [10] also used the same method to study the Hamiltonian 1:2 reso-
nant case and showed that the second order normal form is "unique". Here,
the "unique" means that the normal form is "minimum" or "simplest". More
references can be found from the tutorial articles [11,12]. In the CNF theory,
by saying that "normal forms are not unique" it usually implies at least one
of the two cases: (1) For a same system, its normal forms may have different
"form"; (2) Even for a same "form", the coefficient of the CNF may be differ-
ent. For example, a system associated with Bodganov-Takens singularity may
typically have two forms [1]; while the normal forms of Hopf bifurcation have
the same form with only odd order terms presented, but their coefficients may
be different [6]. Thus, certain methods such as adjoint operator approach [13]
were developed to determine a unique normal form. But such a unique normal
form is not the simplest form, and thus we prefer to use the simplest normal
form (SNF), which means that the total number of the SNF up to certain
order cannot be further simplified by any other nonlinear transformations.

Recently, many researchers have paid attention to the computation of the SNF
and efficient computation methods using computer algebra systems have been
developed (e.g., see [6,14–18]). In this paper, a method is presented to com-
pute the SNF of differential equations associated with a triple zero eigenvalue
of index one and two. Our method is based on nonlinear transformations. The
key step in the computation is to find an appropriate pattern of nonlinear
transformations so that the resulting normal form is the simplest. The main
results are obtained using matrix theory and algebraic computations, which
can provide a direct guideline for developing computer software. In fact, based
on the explicit formulas derived in this paper, Maple has been used to develop
symbolic programs.

The general formulas for commutating the SNF of differential equations are
given in the next section. The detailed results are presented in Section 3, and
conclusions are drawn in Section 4.

2 General formulation of the SNF

Consider the general system, described by

\[ \dot{x} = Jx + f(x), \quad x \in \mathbb{R}^N, \]  

(1)
which has an equilibrium at the origin 0, and \( J \) is the Jacobian matrix and assumed, without loss of generality, in a standard Jordan canonical form. Further, it is assumed that all eigenvalues of \( J \) have zero real parts. In other words, the dynamics of system (1) is described in an \( N \)-dimensional center manifold. In general, we may write system (1) in a vector form

\[
v = [J + \{f_1(x), f_2(x), \ldots, f_N(x)\}] \theta
\]

where the differentiation operator \( \theta \) is defined as \( \theta = (\partial x_1, \partial x_2, \ldots, \partial x_N)^T \), and \( f = (f_1, f_2, \ldots, f_N)^T \) is assumed to be a nonlinear analytic function. Thus, the functions \( f \) can be expanded into the vector homogeneous polynomials of \( x \) near the equilibrium \( x = 0 \). Then we define the linear vector space \( \mathcal{H}_n \) which consists of the \( n \)-th degree homogeneous vector polynomials \( f^n(x) \). Further one may use the linear part \( v_1 = J x \) to define the homological operator \( L_n \) as

\[
L_n : \mathcal{H}_n \mapsto \mathcal{H}_n
\]

\[
U_n \in \mathcal{H}_n \mapsto L_n(U_n) = [U_n, v_1] \in \mathcal{H}_n
\]

where the operator \([U_n, v_1]\) is called Lie bracket, defined by

\[
[U_n, v_1] = Dv_1 \cdot U_n - DU_n \cdot v_1.
\]

Next, we define the space \( \mathcal{R}_n \) as the range of \( L_n \), and the complementary space of \( \mathcal{R}_n \) as \( \mathcal{K}_n = Ker (L_n) \). Thus,

\[
\mathcal{H}_n = \mathcal{R}_n \oplus \mathcal{K}_n,
\]

and we can then choose basis for the spaces \( \mathcal{R}_n \) and \( \mathcal{K}_n \). Consequently, a vector homogeneous polynomial \( f^n(x) \in \mathcal{H}_n \) can be split into two parts: one can be spanned by the basis of \( \mathcal{R}_n \) and the other by that of \( \mathcal{K}_n \).

To find the SNF and the associated nonlinear transformation, one can start from the quadratic (2nd-order) terms: For \( n = 2 \), define

\[
Y_2 = \{ \sum d_{1\alpha_2} x^{\alpha_2}, \sum d_{2\alpha_2} x^{\alpha_2}, \ldots, \sum d_{N\alpha_2} x^{\alpha_2} \}^T,
\]

\[
V_2 = f^2(x) \equiv \{ \sum p_{1\alpha_2} x^{\alpha_2}, \sum p_{2\alpha_2} x^{\alpha_2}, \ldots, \sum p_{N\alpha_2} x^{\alpha_2} \}^T,
\]

where the \( \alpha \) denotes taking summation of all possible non-negative integers \( k_1, k_2, \ldots, k_N \) such that \( k_1 + k_2 + \ldots + k_N = n \), and we say that \( \alpha \) has index \( n \), denoted by \( \alpha_n \). Then \( x^{\alpha_n} = x_1^{k_1} x_2^{k_2} \ldots x_N^{k_N} \). Note that \( Y_2 \) denotes the general second-order nonlinear transformation while \( V_2 \) represents the terms of the vector field \( f \) up to second order. Further let

\[
W = V_2 + [Y_2, v_1],
\]

\[
U_2 = v_1 + \{ \sum q_{1\alpha_2} x^{\alpha_2}, \sum q_{2\alpha_2} x^{\alpha_2}, \ldots, \sum q_{N\alpha_2} x^{\alpha_2} \}^T,
\]
where $U_2$ denotes the SNF up to second order. It should be noted that the form of $U_2$ is same as that of $V_2$ since the SNF is at least as simple as the original vector fields.

Now setting $W - U_2 = 0$, and then balancing the coefficients of the second-order terms yields a set of algebraic equations which can be used to determine the coefficients of the nonlinear transformation, $d_{ijkl}$, and the coefficients of the SNF, $q_{jkl}$. It should be pointed out that in general the number of the coefficients is larger than the number of the algebraic equations. Thus, some of the coefficients of the nonlinear transformation cannot be determined. In the CNF theory, the undetermined coefficients are set zero and therefore the nonlinear transformation is simplified. However, in order to further simplify the normal form, we should not set the undetermined coefficients zero but let them carry over to higher order equations and hope that they can be used to simplify higher order normal forms. This is the fundamental difference between the CNF and the SNF.

Similarly, we can carry out the above procedure to $n = 3, 4, \ldots$, and in general we may obtain

$$Y_n = \{ \sum d_{1\alpha} x^{\alpha}, \sum d_{2\alpha} x^{\alpha}, \ldots, \sum d_{N\alpha} x^{\alpha} \}^T,$$

$$V_n = \{ \sum p_{1\alpha} x^{\alpha}, \sum p_{2\alpha} x^{\alpha}, \ldots, \sum p_{N\alpha} x^{\alpha} \}^T,$$

$$U_n = \{ \sum q_{1\alpha} x^{\alpha}, \sum q_{2\alpha} x^{\alpha}, \ldots, \sum q_{N\alpha} x^{\alpha} \}^T,$$

$$W = [Y_n, v_1] + \sum_{i=2}^{n-1} \left\{ [Y_{n-i+1}, V_i] + DY_i \left( f^{n-i+1} - U_{k-i+1} \right) \right\}$$

$$+ \sum_{m=2}^{[\frac{n}{2}]} \frac{1}{m!} \sum_{i=m}^{n-m} D^m f^i \sum_{l_1 + l_2 + \cdots + l_m = n-(i-m)}^{2 \leq l_1, l_2, \cdots, l_m \leq n-(i-m)-2(m-1)} Y_{l_1} Y_{l_2} \cdots Y_{l_m}. \quad (8)$$

Then setting $W - U_n = 0$ results in a set of algebraic equations for solving the coefficients of the $n$th order SNF and the associated nonlinear transformation. The coefficients of the lower order nonlinear transformations which have not been determined in the previous steps are used to eliminate the coefficients of the $n$th order CNF, $p_{jkl}$, as many as possible and therefore, the SNF up to $n$th order is obtained.

The iterative formula given in equation (8) for $W$ has been obtained and proved in the reference [20]. This formula provides an equation which involves the $n$th-order terms only, and thus greatly saves computational time and computer memory.
3 The SNF for a triple zero eigenvalue

Consider the following system

$$\dot{z} = Jz + f(z), \quad z \in \mathbb{R}^M,$$

(9)

which vanishes at the origin 0, and $J$ is given by

$$J = \begin{bmatrix} J_1 \\ A \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} J_2 \\ A \end{bmatrix},$$

(10)

where

$$J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(11)

and $A \in \mathbb{R}^{(M-3) \times (M-3)}$ is hyperbolic, i.e., all eigenvalues of $A$ have non-zero real parts. $J_1$ is said to be index one while $J_2$ index two.

3.1 The SNF for a triple zero of index one

By using the normal form theory (e.g., see [1,18,19]), one may find the CNF for the triple zero singularity with index one:

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= \sum_{k=2}^{\infty} \left\{ \sum_{i=0}^{k} a_{(k-i)1} x_1^{k-i} x_2^i + x_1^i x_3 \sum_{i=0}^{J} b_{N(u-i)1} x_1^{J-i} x_3^i \right\},
\end{align*}$$

(12)

where

$$N = 1, \quad J = \frac{1}{2} (k - 1), \quad I = J, \quad \text{when} \ k \text{ is odd},$$

$$N = 2, \quad J = \frac{k}{2} - 1, \quad I = J + 1, \quad \text{when} \ k \text{ is even}. \quad \text{(13)}$$

Next, the approach described in the previous section can be used to compute the SNF. We start from the second order terms (i.e., $n = 2$), and can write $Y_2, V_2$ and $U_2$, with the aid of the linear part $v_1 = (x_1, x_2, 0)^T$ as follows:

$$Y_2 = \left( \sum_{j+k+l=2} d_{1jkl} x_1^j x_2^k x_3^l, \sum_{j+k+l=2} d_{2jkl} x_1^j x_2^k x_3^l, \sum_{j+k+l=2} d_{3jkl} x_1^j x_2^k x_3^l \right)^T,$$

$$V_2 = \left( 0, 0, a_{20} x_1^2 + a_{11} x_1 x_2 + a_{02} x_2^2 + b_{200} x_1 x_3 \right)^T,$$

$$U_2 = v_1 + \left( 0, 0, A_{20} x_1^2 + A_{11} x_1 x_2 + A_{02} x_2^2 + B_{200} x_1 x_3 \right)^T.$$

(14)

Then solving the linear algebraic equation $V_2 + [Y_2, v_1] - U_2 = 0$ yields
\[ d_{1200} = d_{1110} = d_{2200} = d_{2020} = d_{2110} = d_{2101} = 0, \]
\[ d_{3200} = d_{3020} = d_{3110} = d_{3011} = d_{3101} = 0, \]
\[ d_{1011} = d_{2002}, \quad d_{1101} = d_{3002} - 2d_{1020}, \quad d_{2011} = d_{3002}. \]
\[ A_{20} = a_{20}, \quad A_{11} = a_{11}, \quad A_{02} = a_{02}, \quad B_{200} = b_{200}. \] (15)

Equation (15) indicates that the second order terms of the CNF cannot be further simplified. Also, it is noted that the nonlinear coefficients \( d_{1020}, \; d_{2002}, \; d_{3002} \) and \( d_{1002} \) (\( d_{1002} \) does not appear in the algebraic equation) are not determined at this step, and therefore may be used in higher order equations to eliminate higher order CNF coefficients.

Next, consider \( n = 3 \), following the above step we may find the key linear algebraic equations:

\[
\begin{bmatrix}
A_{30} - a_{30} \\
A_{21} - a_{21} \\
A_{12} - a_{12} \\
A_{03} - a_{03} \\
B_{110} - b_{110} \\
B_{101} - b_{101}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
d_{3300} \\
d_{3210} \\
d_{3120} \\
d_{3111} \\
d_{3300}
\end{bmatrix}
+
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
d_{1020} \\
d_{2002} \\
d_{3002}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where the vector \((0, \; d_{3300}, \; d_{3210}, \; d_{3120}, \; d_{3111})^T\) can be expressed in terms of the second order undetermined coefficients as follows:

\[
\begin{bmatrix}
d_{3300} \\
d_{3210} \\
d_{3120} \\
d_{3111}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 \\
\ast & \ast & 0 & 0 \\
\ast & \ast & \ast & 0 \\
\ast & \ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
d_{3002} \\
d_{1020} \\
d_{2002} \\
d_{1002}
\end{bmatrix},
\] (16)

where an \( \ast \) represents a known expression. Substituting equation (16) into the key equation results in

\[
\begin{bmatrix}
A_{30} - a_{30} \\
A_{21} - a_{21} \\
A_{12} - a_{12} \\
A_{03} - a_{03} \\
B_{110} - b_{110} \\
B_{101} - b_{101}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & 0 & 0 & 0 \\
\ast & \ast & \ast & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & 0 & 0
\end{bmatrix}
\begin{bmatrix}
d_{3002} \\
d_{1020} \\
d_{2002} \\
d_{1002}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (17)

It is easy to see from equation (17) that \( A_{30} = a_{30}, \; A_{21} = a_{21}, \) and \( d_{1002} \) is not involved in the equation. Further note that the maximum rank of the matrix in equation (17) is three, and thus three of the remaining four coefficients \( A_{12}, \; A_{03}, \; B_{110} \) and \( B_{101} \) can be eliminated by appropriately choosing values.
for $d_{3002}$, $d_{1020}$ and $d_{2002}$. In this paper we choose $A_{12} = A_{03} = B_{101} = 0$, and then $B_{110}$ is uniquely determined. Of course, there exist other choices, implying that the SNF is unique only if a "form" is fixed. Therefore, three third order coefficients in the CNF have been removed.

The above procedure can be carried out to higher order equations. One can similarly determines the key equations, and then uses the lower order coefficients to eliminate the CNF coefficients as many as possible. The detailed process is omitted here but we list the SNF coefficients up to 8th order in Table 1, where the NT coefficients are the coefficients of lower order nonlinear transformations which are used to remove the higher order CNF coefficients.

<table>
<thead>
<tr>
<th>Order</th>
<th>SNF coefficients</th>
<th>NT coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$A_{20}, A_{11}, A_{02}, B_{200}$</td>
<td>None</td>
</tr>
<tr>
<td>3</td>
<td>$A_{30}, A_{21}, B_{110}$</td>
<td>$d_{3002}, d_{1020}, d_{2002}$</td>
</tr>
<tr>
<td>4</td>
<td>$A_{40}, B_{210}$</td>
<td>$d_{1002}, d_{1120}, d_{1030}, d_{3003}, d_{1021}$</td>
</tr>
<tr>
<td>5</td>
<td>$A_{50}, A_{41}, B_{120}$</td>
<td>$d_{2003}, d_{1003}, d_{1040}, d_{3022}, d_{2022}, d_{1022}$</td>
</tr>
<tr>
<td>6</td>
<td>$A_{60}, A_{51}, A_{42}, B_{220}$</td>
<td>$d_{3004}, d_{2004}, d_{1140}, d_{1064}, d_{1050}, d_{1041}$</td>
</tr>
<tr>
<td>7</td>
<td>$A_{70}, A_{61}, A_{52}, B_{130}, B_{121}$</td>
<td>$d_{3023}, d_{2023}, d_{3005}, d_{1023}, d_{2005}, d_{1005}, d_{1060}$</td>
</tr>
<tr>
<td>8</td>
<td>$A_{80}, A_{71}, A_{62}, B_{230}$</td>
<td>$d_{3042}, d_{2042}, d_{1042}, d_{3024}, d_{2024}$</td>
</tr>
</tbody>
</table>

The SNF for a triple zero of index one up to 8th order is given below:

\[
\begin{align*}
\dot{u}_1 &= u_2, \\
\dot{u}_2 &= u_3, \\
\dot{u}_3 &= a_{20} u_1^2 + a_{11} u_1 u_2 + a_{02} u_2^2 + b_{200} u_1 u_3 + a_{30} u_1^3 + a_{21} u_1^2 u_2 + B_{110} u_1^2 u_3 \\
&+ A_{40} u_1^4 + B_{210} u_1^3 u_3 + A_{50} u_1^5 + A_{41} u_1^4 u_2 + B_{120} u_1^4 u_3 + A_{60} u_1^6 + A_{51} u_1^5 u_2 \\
&+ A_{42} u_1^2 u_2^2 + B_{220} u_1 u_2^3 + A_{70} u_1^7 + A_{61} u_1^6 u_2 + A_{52} u_1^5 u_2^2 + B_{130} u_1^5 u_3 \\
&+ B_{121} u_1^4 u_3^2 + A_{80} u_1^8 + A_{71} u_1^7 u_2 + A_{62} u_1^6 u_2^2 + B_{230} u_1^6 u_3, \\
\end{align*}
\]

(18)

where

\[
\begin{align*}
B_{110} &= - (a_{12} b_{200} - 5 b_{200} b_{110} + 4 b_{110} a_{02} + 8 b_{101} a_{20})/(5 b_{200} - 4 a_{02}), \\
A_{40} &= - \frac{1}{6} (-115 b_{200} a_{03} a_{20} - 30 b_{200} a_{40} + 11 b_{200} a_{11} a_{12} + 24 a_{40} a_{02} \\
&+ 92 a_{03} a_{20} a_{02} - 42 a_{11} b_{101} a_{20} - 13 a_{11} a_{12} a_{02})/(5 b_{200} - 4 a_{02}),
\end{align*}
\]
\[ B_{210} = -\frac{1}{372} (1259 b_{200}^2 a_{20} a_{11} a_{12} - 13690 b_{200}^2 a_{03} a_{20}^2 - 2692 b_{200} a_{20} a_{11} a_{12} a_{02} \\
- 330 b_{200} a_{11} a_{03} a_{20} + 39 b_{200} a_{11} a_{12} - 4728 b_{200} a_{11} b_{101} a_{20} \\
- 780 b_{200} a_{30} a_{20} a_{12} - 90 a_{11} b_{200} a_{31} a_{20} + 23312 b_{200} a_{03} a_{20} a_{02} \\
- 1860 b_{200} b_{210} a_{20} a_{20} + 600 b_{200} a_{22} a_{20}^2 + 1416 a_{20} a_{11} a_{12} a_{02} \\
+ 3120 a_{30} a_{20} b_{101} + 264 a_{11} a_{03} a_{20} a_{02} + 936 a_{30} a_{20} a_{12} a_{02} \\
+ 4464 a_{11} b_{101} a_{20} a_{02} - 42 a_{11} a_{12} a_{02} + 1488 b_{210} a_{20} a_{02} \\
+ 72 a_{11} a_{30} a_{20} a_{02} - 108 a_{11} b_{101} a_{20} - 9888 a_{03} a_{20} a_{02} \\
- 480 a_{22} a_{20} a_{02})/a_{20}^2 (5b_{200} - 4a_{02}), \]  

(19)

3.2 The SNF for a triple zero of index two

By using normal form theory we may find the CNF for the case of index two:

\[ \dot{x}_1 = x_2, \]

\[ \dot{x}_2 = \sum_{k=2}^{\infty} \left\{ \sum_{i=0}^{k} a_{k-i,i} x_1^{k-i} x_3^i + x_2 \sum_{i=0}^{k-1} b_{k-i-1,i} x_1^{k-i-1} x_3^i \right\}, \]

\[ \dot{x}_3 = \sum_{k=2}^{\infty} \sum_{i=0}^{k} c_{k-i,i} x_1^{k-i} x_3^i. \]  

(20)

Similarly, we may follow the procedure used in obtaining the SNF for the case of index one to find the SNF for this case. First consider \( n = 2 \) and define

\[ Y_2 = \left( \sum_{j+k+l=2} d_{1jkl} x_1^j x_2^k x_3^l, \sum_{j+k+l=2} d_{2jkl} x_1^j x_2^k x_3^l, \sum_{j+k+l=2} d_{3jkl} x_1^j x_2^k x_3^l \right)^T, \]

\[ V_2 = (0, a_{20} x_1^2 + a_{11} x_1 x_3 + a_{02} x_3^2 + b_{10} x_1 x_2 + b_{01} x_2 x_3, \]

\[ c_{20} x_1^2 + c_{11} x_1 x_3 + c_{02} x_3^2)^T, \]

\[ U_2 = v_1 + (0, A_{20} x_1^2 + A_{11} x_1 x_3 + A_{02} x_3^2 + B_{10} x_1 x_2 + B_{01} x_2 x_3, \]

\[ C_{20} x_1^2 + C_{11} x_1 x_3 + C_{02} x_3^2)^T, \]  

(21)

where \( v_1 = (x_1, 0, 0)^T \) represents the linear part. Then one may solve the linear equation given by \( V_2 + [Y_2, v_1] - U_2 = 0 \) to find

\[ d_{1200} = d_{2200} = d_{2002} = d_{2101} = d_{2110} = d_{3200} = d_{3110} = d_{3101} = 0, \]

\[ d_{1110} = d_{2020}, \quad d_{2011} = d_{1101}, \quad A_{20} = a_{20}, \quad A_{11} = a_{11}, \quad A_{02} = a_{02}, \]

\[ B_{10} = b_{10}, \quad B_{01} = b_{01}, \quad C_{20} = c_{20}, \quad C_{11} = c_{11}, \quad C_{02} = c_{02}, \]  

(22)
which implies that the second order terms of the CNF cannot be further simplified. It is noted that the nonlinear coefficients \(d_{1020}, d_{1002}, d_{1110}, d_{1201}, d_{1011}, d_{3020}, d_{3002}\) and \(d_{3011}\) are not determined at this step, and may be used to eliminate higher order CNF coefficients.

Next, consider \(n = 3\), by following the similar procedure we may find the key linear algebraic equations as follows:

\[
\begin{pmatrix}
B_{20} - b_{20} \\
B_{11} - b_{11} \\
B_{02} - b_{02}
\end{pmatrix}
+ \begin{bmatrix}
3a_{20} & 0 & 0 & 4c_{20} & 0 \\
3a_{11} & 4c_{20} & -a_{11} & 3c_{11} - b_{10} & 0 \\
3a_{02} & 2c_{11} - b_{10} - 2a_{02} & 2c_{02} & -b_{01}
\end{bmatrix}
\begin{pmatrix}
d_{2020} \\
d_{1002} \\
d_{3011} \\
d_{1110} \\
d_{3002}
\end{pmatrix} = 0 \quad (23)
\]

and

\[
\begin{pmatrix}
A_{30} - a_{30} \\
A_{21} - a_{21} \\
A_{12} - a_{12} \\
A_{03} - a_{03} \\
C_{30} - c_{30} \\
C_{21} - c_{21} \\
C_{12} - c_{12} \\
C_{03} - c_{03}
\end{pmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_{20} & 0 \\
0 & -2a_{20} & 0 & 0 & -a_{11} \\
0 & -a_{11} & 0 & a_{02} & -2a_{02} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{20} & -2c_{20} & 2c_{20} \\
0 & -2c_{20} & a_{11} & -c_{11} & c_{11} \\
0 & -c_{11} & a_{02} & 0 & 0
\end{bmatrix}
\begin{pmatrix}
d_{2020} \\
d_{1002} \\
d_{3011} \\
d_{1110} \\
d_{3002}
\end{pmatrix} = 0. \quad (24)
\]

It is easy to see from equation (23) that we can use three of the \(d\)'s coefficients to eliminate the three normal form coefficients \(B_{20}, B_{11}\) and \(B_{02}\), which further simplifies the CNF. For definite, here we choose \(d_{2020}, d_{1002}\) and \(d_{3011}\), and solve them from equation (23) (noting that \(B_{20} = B_{11} = B_{02} = 0\)), and then substitute the resulting solutions into equation (24) to further eliminate two of the coefficients \(A\)'s and/or \(C\)'s by using the coefficients \(d_{1101}\) and \(d_{3002}\). However, it should be pointed that the coefficients \(A_{30}\) and \(C_{30}\) cannot be set zero, and in fact, \(A_{30} = a_{30}\) and \(C_{30} = c_{30}\). Here we may choose \(A_{21} = A_{12} = 0\) and then uniquely determine \(d_{1101}\) and \(d_{3002}\) from equation (24). It has also been noted that the coefficient \(d_{1011}\) does not appear in equations (23) and (24), and may be used later to remove some higher order CNF coefficient.

The above procedure can be repeatedly applied to higher order equations: First determines the key equations, and then uses the lower order undetermined coefficients to eliminate the higher order CNF coefficients as many as possible. Table 2 shows the SNF coefficients up to 8th order and the nonlinear transformation (NT) coefficients which are the lower order coefficients used to eliminate the higher order CNF coefficients.
Table 2. The SNF coefficients for a triple zero of index two.

<table>
<thead>
<tr>
<th>Order</th>
<th>SNF coefficients</th>
<th>NT coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$A_{20}, A_{11}, A_{02}, B_{10}, B_{01}$ $C_{20}, C_{11}, C_{02}$</td>
<td>None</td>
</tr>
<tr>
<td>3</td>
<td>$A_{30}, A_{03}$ $C_{30}, C_{21}, C_{12}, C_{03}$</td>
<td>$d_{1101}, d_{1002}, d_{2020}, d_{3002}, d_{3011}$</td>
</tr>
<tr>
<td>4</td>
<td>$A_{40}$ $C_{40}, C_{31}, C_{22}, C_{13}, C_{04}$</td>
<td>$d_{1102}, d_{1012}, d_{1111}, d_{1003}$ $d_{1020}, d_{3020}, d_{3012}, d_{3003}$</td>
</tr>
<tr>
<td>5</td>
<td>$A_{50}$ $C_{41}, C_{32}, C_{23}, C_{14}, C_{05}$</td>
<td>$d_{1004}, d_{1012}, d_{1103}, d_{1021}, d_{1120}, d_{1112}$ $d_{3021}, d_{3004}, d_{3013}, d_{3112}, d_{3030}$</td>
</tr>
<tr>
<td>6</td>
<td>$A_{60}, A_{06}, C_{61}$ $C_{42}, C_{33}, C_{24}, C_{15}, C_{06}$</td>
<td>$d_{1113}, d_{1104}, d_{1121}, d_{1013}, d_{1022}, d_{1005}$ $d_{1030}, d_{1130}, d_{3022}, d_{3031}, d_{3014}, d_{3005}$</td>
</tr>
<tr>
<td>7</td>
<td>$A_{70}, A_{43}, A_{34}, A_{25}, A_{16}$ $C_{70}, C_{61}, C_{34}, C_{25}$</td>
<td>$d_{1040}, d_{1122}, d_{1023}, d_{1014}, d_{1114}, d_{1105}, d_{1031}$ $d_{1131}, d_{1006}, d_{3006}, d_{3015}, d_{3023}, d_{3032}, d_{3040}$</td>
</tr>
<tr>
<td>8</td>
<td>$A_{80}, C_{80}, C_{71}, C_{62}, C_{53}$ $C_{44}, C_{35}, C_{26}, C_{17}, C_{08}$</td>
<td>$d_{1007}, d_{1115}, d_{1106}, d_{3016}, d_{3007}$ $d_{1123}, d_{1015}, d_{1024}, d_{1132}, d_{3024}, d_{3033}$ $d_{1041}, d_{1032}, d_{1140}, d_{3041}, d_{1050}$</td>
</tr>
</tbody>
</table>

Finally, the SNF for a triple zero of index two up to 8th order is found to be:

\[
\dot{u}_1 = u_2, \\
\dot{u}_2 = a_{20}u_1^2 + a_{11}u_1u_3 + a_{02}u_2^2 + b_{10}u_1u_2 + b_{01}u_2u_3 + A_{30}u_3^3 + A_{03}u_3^3 + A_{40}u_4^4 + A_{50}u_5^5 + A_{60}u_6^6 + A_{70}u_7^7 + A_{80}u_8^8, \\
\dot{u}_3 = C_{20}u_1^2 + C_{11}u_1u_3 + C_{02}u_2^2 + C_{30}u_3^3 + C_{21}u_1^2u_3 + C_{12}u_1u_3^2 + C_{03}u_3^3 + C_{40}u_4^4 + C_{31}u_1^3u_3 + C_{22}u_2^2u_3 + C_{13}u_1^2u_3^2 + C_{04}u_4^4 + C_{41}u_4^4u_3 + C_{32}u_3^2u_2^2 + C_{23}u_3^2u_2^2 + C_{14}u_1^4u_3 + C_{05}u_5^5 + C_{42}u_4^4u_2 + C_{33}u_3^3u_3 + C_{24}u_2^2u_3 + C_{15}u_1^4u_3^2 + C_{06}u_6^6 + C_{51}u_5^5u_3 + C_{43}u_4^4u_3 + C_{34}u_3^3u_3 + C_{25}u_2^2u_3 + C_{16}u_1^4u_3^3 + C_{07}u_7^7 + C_{61}u_6^6u_3 + C_{52}u_5^5u_2 + C_{44}u_4^4u_2 + C_{35}u_3^3u_2^3 + C_{26}u_2^2u_2^2 + C_{17}u_1^4u_2^4 + C_{08}u_8^8, \]

where the coefficients $A_{jk}$'s and $C_{jk}$'s are given in terms of the known coefficients $a_{jk}$'s, $b_{jk}$'s and $c_{jk}$'s. The detailed expressions are omitted here.

It is noted from equation (11) that the Jacobian $J_2$ involves a non-semisimple
double zero eigenvalue. The explicit SNF for the double zero eigenvalue has
been obtained in [16]. Here we may use the SNF for the triple zero of index two,
given by equation (25), to recover the SNF of the generic case for the double
zero singularity. First note in equation (20) that setting all $c$ coefficients zero
in the 3rd equation of (20), and taking the terms in the second equation of (20)
for $i = 0$ only results in the CNF for the double zero singularity. This leads
to the coefficients of equation (25): $C_{20} = C_{30} = C_{40} = C_{50} = C_{70} = C_{80} = 0$.
Then further setting $u_3 = 0$ in equation (25) yields
\[
\begin{align*}
\dot{u}_1 &= u_2, \\
\dot{u}_2 &= a_{20}u_1^2 + b_{10}u_1u_2 + A_{30}u_1^3 + A_{40}u_1^4 + A_{50}u_1^5 + A_{70}u_1^7 + A_{80}u_1^8.
\end{align*}
\tag{26}
\]
A careful examination of the SNF for the double zero singularity has shown
that the result given by equation (26) is exactly same as that obtained in [16].

3.3 Symbolic computation using Maple

The explicit formulas derived in this section can be directly implemented on
computer algebra systems. In fact, all the results obtained in this paper are
obtained using Maple. The Maple programs we have developed can be conve-
niently executed on any computer systems for computing the SNF of a given
vector field associated with the triple zero singularity of index one or two. The
software only requires a minimum preparation for an input file from a user.

4 Conclusions

A method is presented for computing the simplest normal forms of differen-
tial equations associated with triple zero singularity of index one and two. It
has been shown that this approach is an efficient way from the computation
point of view. User-friendly symbolic computation programs written in Maple
have been developed. This method can be easily extended to consider other
singularities.

References


Math., 1, 1–37.


