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GLOBALLY EXPONENTIAL HYPERCHAOS (LAG) SYNCHRONIZATION IN A FAMILY OF MODIFIED HYPERCHAOTIC RÖSSLER SYSTEMS

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In this paper, we consider a new family of modified hyperchaotic Rössler systems, recently studied by Nikolov and Clodong using proper nonlinear feedback controllers. Particular attention is given to (i) globally exponential lag synchronization (GELS) for $\tau > 0$; and (ii) globally exponential synchronization (GES) for $\tau = 0$. As a representative example, one system of the family of modified hyperchaotic Rössler systems is particularly studied, and Lyapunov stability criteria for the GELS and GES are derived via eight families of proper nonlinear feedback controllers. Moreover, we also present some nonlinear feedback control laws for other modified hyperchaotic Rössler systems. Numerical simulations are used to illustrate the theoretical results.

Keywords: Modified hyperchaotic Rössler system; GES; GELS; feedback control; Lyapunov function.

1. Introduction

Chaos synchronization has played an important role in chaotic systems and attracted much attention due to its potential applications in secure communication, since Pecora and Carroll [1990] showed that chaos synchronization could be indeed realized in coupled chaotic systems. Up to now, many types of chaos synchronizations have been presented such as complete synchronization [Pecora & Carroll, 1990], lag synchronization [Rosenblum *et al.*, 1997], generalized synchronization [Rul'kov *et al.*, 1995], phase synchronization [Rosenblum *et al.*, 1996; Kolumban *et al.*, 1998], and other types of synchronizations. Moreover, many stability theories and schemes have been developed for chaos (hyperchaos) synchronization and control (e.g. see [Liao, 1993, Chen & Dong, 1998; Boccaletti, 2002; Liao, 2002; Chen & Lü, 2003; Liao & Yu, 2005, 2006; Yan, 2005a, 2005b; Yu & Liao, 2006]).

Since the first hyperchaotic system was presented by Rössler [1979], which was usually classified as a chaotic system with more than one positive Lyapunov exponent, some hyperchaotic systems have been proposed such as the fourth-order hyperchaotic circuit [Matsumoto *et al.*, 1986], the hyperchaotic Chen system [Li *et al.*, 2005; Yan, 2005c], etc. Recently, Nikolov and Clodong [2004] added a linear state feedback to the well-known hyperchaotic Rössler system such that a family of modified hyperchaotic Rössler systems

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was obtained. To our knowledge, hyperchaotic synchronization of this new family of modified hyperchaotic Rössler systems has not been studied. More recently, Liao and Yu [2005, 2006] constructed appropriate Lyapunov functions to study chaos synchronization of six classical Rössler systems, and derived explicit nonlinear feedback control laws to globally synchronize two identical Rössler systems. In this paper, we will extend the method to study globally exponential (lag) synchronization between two identical, modified hyperchaotic Rössler systems.

The rest of this paper is arranged as follows. In Sec. 2, we give Lyapunov stability criteria for globally exponential hyperchaos (lag) synchronization of n-dimensional chaotic systems. Then, we use the theory to obtain a number of different feedback controllers to investigate globally exponential hyperchaos (lag) synchronization of the family of modified hyperchaotic Rössler systems. Numerical simulation results are presented to illustrate the analytical predictions in Sec. 3. Finally, the conclusion is given in Sec. 4.

2. The Family of Modified Hyperchaotic Rössler Systems

The first well-known hyperchaotic Rössler system was introduced by Rössler in 1979, described by

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay + w, \\ \dot{z} = b + xz, \\ \dot{w} = -cz + dw, \end{cases} \tag{1}$$

where the parameters a, b, c and d are chosen as $a = 0.25, b = 3, c = 0.5$ and $d = 0.05$, for which system (1) exhibits a hyperchaotic attractor. Recently, Nikolov and Clodong [2004] added a linear state feedback in the form of $b_1x(t) + b_2y(t) + b_3z(t) + b_4w(t)$ to the third equation of (1) such that a family of modified hyperchaotic Rössler systems was obtained in the form:

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay + w, \\ \dot{z} = b + b_1x + b_2y + b_3z + b_4w + xz, \\ \dot{w} = -cz + dw. \end{cases} \tag{2}$$

System (2) includes fifteen chaotic and hyperchaotic systems for different parameter values of b_i $(i = 1, 2, 3, 4)$. Nikolov and Clodong [2004] investigated how the global dynamics could be altered in a desired direction. More recently, Nikolov and Clodong [2006] have confirmed that the transition of hyperchaos-chaos-hyperchaos depends on the change of the sign of roots of the corresponding characteristic equation, and that the prediction time is a more reliable predictor of the evolution than the information dimension, demonstrated in the following eight categories:

(I) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_1 = 1,\ b_3 \in [-5, -0.01],\ b_2^2 + b_4^2 = 0;$
(II) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_3 \in [0.01, 0.9],\ b_2^2 + b_1^2 + b_4^2 = 0;$
(III) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_1 \in [0.01, 3.7],\ b_2^2 + b_3^2 + b_4^2 = 0;$
(IV) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_2 = -0.3,\ b_3 \in [-1, -0.01],\ b_1^2 + b_4^2 = 0;$
(V) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_2 = b_3 = -0.3,\ b_1 \in [0.01, 4.5],\ b_4 = 0,\ a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_1 = 0.5,\ b_2 = -0.2,\ b_3 \in [-2.5, -0.01],\ b_4 = 0;$
(VI) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_1 = 0.3,\ b_4 = 0.2,\ b_3 \in [-2.5, -0.01],\ b_2 = 0;$
(VII) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_1 = 0,\ b_2 = -0.1,\ b_4 = 0.2,\ b_3 \in [-1.2, -0.01];$
(VIII) $a = 0.25,\ b = 3,\ c = 0.5,\ d = 0.05,\ b_1 = -0.1,\ b_3 = b_4 = 0.1,\ b_1 \in [0.01, 1.5].$

Before giving a complete study for system (2), we present the definition of synchronization and a lemma. Consider the drive system:

$$\dot{\mathbf{x}}_d = \mathbf{F}(t, \mathbf{x}_d), \tag{3}$$

and the response system:

$$\dot{\mathbf{y}}_r = \mathbf{F}(t, \mathbf{y}_r) + \mathbf{u}, \tag{4}$$

where the subscripts "d" and "r" denote the drive system and response system respectively, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$, $\mathbf{F} : \mathrm{R}_+ \times \mathrm{R}^{\mathrm{n}} \to \mathrm{R}^{\mathrm{n}}$, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ is a linear or nonlinear vector function of $(t, x_{id}, y_{id}, x_{ir}, y_{ir})$.

Let the error state be $\mathbf{e}(t) = (e_1(t), e_2(t), \ldots, e_n(t))^{\mathrm{T}} = (x_{1d}(t - \tau) - y_{1r}(t),\ x_{2d}(t - \tau) - y_{2r}(t), \ldots, x_{nd}(t-\tau) - y_{nr}(t))^{\mathrm{T}}$, where $\tau \geq 0$. Then, we have the error dynamical system:

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t - \tau) - \dot{\mathbf{y}}(t) \\ &= \mathbf{F}(t - \tau, \mathbf{x}_d(t - \tau)) - \mathbf{F}(t, \mathbf{y}_r(t)) - \mathbf{u}. \end{aligned} \tag{5}$$

Definition 1. For an arbitrary given initial point, $(x_{1d}(t), x_{2d}(t), \ldots, x_{nd}(t)) \in \mathrm{R}^{\mathrm{n}}$, of the drive system (3) and the corresponding initial point, $(y_{1r}(t), y_{2r}(t), \ldots, y_{nr}(t)) \in \mathrm{R}^{\mathrm{n}}$, $t \in [-\tau, 0]$, of the response system (4), if the solution of the error system (5) has the estimation $\sum_{i=1}^{n} e_i^2(t) \le K(e(t_0))\exp(-\alpha(t-t_0))$, where $K(e(t_0)) > 0$ is a constant depending on the initial value $e(t_0)$, while $\alpha > 0$ is a constant independent of $e(t_0)$, then the zero solution of the error system (5) is said globally, exponentially stable; thus (i) globally exponential lag synchronization for $\tau > 0$, and (ii) globally exponential synchronization for $\tau = 0$ occur between the drive system (3) and the response system (4).

Lemma 1. *For the given error system* (5), *its zero solution is globally, exponentially stable, i.e.* (*i*) *globally exponential lag synchronization for* $\tau > 0$; *and* (*ii*) *globally exponential synchronization for* $\tau = 0$ *occur between the drive system* (3) *and the response system* (4), *if there exists a positive definite quadratic polynomial*

$$V = (e_1 \quad e_2 \quad \cdots \quad e_n)P(e_1 \quad e_2 \quad \cdots \quad e_n)^{\mathrm{T}},$$

such that

$$\frac{dV}{dt} = -(e_1 \quad e_2 \quad \cdots \quad e_n)Q(e_1 \quad e_2 \quad \cdots \quad e_n)^{\mathrm{T}},$$

where $P = P^{\mathrm{T}} \in \mathrm{R}^{\mathrm{n\times n}}$ *and* $Q = Q^{\mathrm{T}} \in \mathrm{R}^{\mathrm{n\times n}}$ *are both positive definite matrixes. Moreover, the following negative Lyapunov exponent estimation for the error system* (5) *holds*:

$$\sum_{i=1}^{n} e_i^2(t) \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \sum_{i=1}^{n} e_i^2(t_0) \times \exp\left[-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_0)\right],$$

where $\lambda_{\max}(P)$ *and* $\lambda_{\min}(P)$ *stand for the minimum and maximum eigenvalues of matrix* P, *respectively, and* $\lambda_{\min}(Q)$ *denotes the minimum eigenvalue of* Q.

The proof of Lemma 1 is straightforward and thus omitted.

In the following, we apply Lemma 1 to consider globally exponential (lag) synchronization between two identical, modified hyperchaotic Rössler systems (2) for different parameter values b_i ($i = 1, 2, 3, 4$).

Case I. In this case, $b_1 = 1$, $b_3 \in [-5, -0.01]$, and $b_2^2 + b_4^2 = 0$. We consider (2) as a drive system

$$\begin{cases} \dot{x}_d = -y_d - z_d, \\ \dot{y}_d = x_d + ay_d + w_d, \\ \dot{z}_d = b + b_1x_d + b_3z_d + x_dz_d, \\ \dot{w}_d = -cz_d + dw_d, \end{cases} \tag{6}$$

and suppose that the response system related to the drive system (6) with feedback controllers u_i ($i = 1, 2, 3, 4$) is given by

$$\begin{cases} \dot{x}_r = -y_r - z_r + u_1, \\ \dot{y}_r = x_r + ay_r + w_r + u_2, \\ \dot{z}_r = b + b_1x_r + b_3z_r + x_rz_r + u_3, \\ \dot{w}_r = -cz_r + dw_r + u_4, \end{cases} \tag{7}$$

where the parameters are chosen as $a = 0.25$, $b = 3$, $c = 0.5$, $d = 0.05$, $b_1 = 1$ and $b_3 \in [-5, -0.01]$, and u_i's are linear or nonlinear functions of (x_d, $y_d, z_d, w_d, x_r, y_r, z_r, w_r$).

Let the error state be $\mathbf{e}(t) = (e_x(t), e_y(t), e_z(t), e_w(t))^{\mathrm{T}} = [x_d(t-\tau) - x_r(t), y_d(t-\tau) - y_r(t), z_d(t-\tau) - z_r(t), w_d(t-\tau) - w_r(t)]^{\mathrm{T}}$, where $\tau \ge 0$. Then, the error system can be written as

$$\begin{cases} \dot{e}_x(t) = -e_y(t) - e_z(t) - u_1, \\ \dot{e}_y(t) = e_x(t) + ae_y(t) + e_w(t) - u_2, \\ \dot{e}_z(t) = b_1e_x(t) + b_3e_z(t) + x_d(t-\tau)z_d(t-\tau) \\ \qquad\qquad - x_r(t)z_r(t) - u_3, \\ \dot{e}_w(t) = -ce_z(t) + de_w(t) - u_4. \end{cases} \tag{8}$$

We have the following theorem.

Theorem 1. *For the modified hyperchaotic Rössler system* (6), *if one of the following feedback controllers* u_i ($i = 1, 2, 3, 4$) *is chosen for the response system* (7):

$$(A) \quad \begin{cases} u_1 = k_1e_x(t) + \dfrac{1}{b_1}z_r(t)e_z(t), \\ u_2 = k_2e_y(t), \\ u_3 = x_d(t-\tau)e_z(t), \\ u_4 = k_4e_w(t) + e_y(t) - ce_z(t); \end{cases}$$

$$(B) \quad \begin{cases} u_1 = k_1e_x(t) + \dfrac{1}{b_1}z_r(t)e_z(t), \\ u_2 = k_2e_y(t), \\ u_3 = -b_1ce_w(t) + x_d(t-\tau)e_z(t), \\ u_4 = k_4e_w(t) + e_y(t); \end{cases}$$

$$(C) \quad \begin{cases} u_1 = k_1e_x(t) + \dfrac{1}{b_1}z_r(t)e_z(t), \\ u_2 = k_2e_y(t) + e_w(t), \\ u_3 = x_d(t-\tau)e_z(t), \\ u_4 = k_4e_w(t) - ce_z(t); \end{cases}$$

$$(D)\quad \begin{cases} u_1 = k_1 e_x(t) + \dfrac{1}{b_1} z_r(t) e_z(t), \\ u_2 = k_2 e_y(t) + e_w(t), \\ u_3 = -b_1 c e_w(t) + x_d(t-\tau) e_z(t), \\ u_4 = k_4 e_w(t); \end{cases}$$

$$(E)\quad \begin{cases} u_1 = k_1 e_x(t) + \dfrac{1}{b_1} z_d(t-\tau) e_z(t), \\ u_2 = k_2 e_y(t), \\ u_3 = x_r(t) e_z(t), \\ u_4 = k_4 e_w(t) + e_y(t) - c e_z(t); \end{cases}$$

$$(F)\quad \begin{cases} u_1 = k_1 e_x(t) + \dfrac{1}{b_1} z_d(t-\tau) e_z(t), \\ u_2 = k_2 e_y(t), \\ u_3 = -b_1 c e_w(t) + x_r(t) e_z(t), \\ u_4 = k_4 e_w(t) + e_y(t); \end{cases}$$

$$(G)\quad \begin{cases} u_1 = k_1 e_x(t) + \dfrac{1}{b_1} z_d(t-\tau) e_z(t), \\ u_2 = k_2 e_y(t) + e_w(t), \\ u_3 = x_r(t) e_z(t), \\ u_4 = k_4 e_w(t) - c e_z(t); \end{cases}$$

$$(H)\quad \begin{cases} u_1 = k_1 e_x(t) + \dfrac{1}{b_1} z_d(t-\tau) e_z(t), \\ u_2 = k_2 e_y(t) + e_w(t), \\ u_3 = -b_1 c e_w(t) + x_r(t) e_z(t), \\ u_4 = k_4 e_w(t); \end{cases}$$

where $k_1 > 0$, $k_2 > a$ and $k_4 > d$, then the zero solution of the error system (8) is globally, exponentially stable, and thus (i) globally exponential lag synchronization for $\tau > 0$, and (ii) globally exponential synchronization for $\tau = 0$ occur between the drive system (6) and the response system (7).

Proof. Consider the controller (A) and choose the following positive definite, quadratic form of Lyapunov function:

$$V(t) = \frac{1}{2}\left[e_x^2(t) + e_y^2(t) + \frac{1}{b_1} e_z^2(t) + e_w^2(t)\right], \tag{9}$$

which implies that $P = \mathrm{diag}(1/2, 1/2, 1/(2b_1), 1/2)$ and thus $\lambda_{\min}(P) = \min\{1/2, 1/(2b_1)\}, \lambda_{\max}(P) = \max\{1/2, 1/(2b_1)\}$. Differentiating $V(t)$ with respect to time t along the trajectory of system (8) yields

$$\begin{aligned} \left.\frac{dV(t)}{dt}\right|_{(8)} &= e_x(t)\dot{e}_x(t) + e_y(t)\dot{e}_y(t) \\ &\quad + \frac{1}{b_1} e_z(t)\dot{e}_z(t) + e_w(t)\dot{e}_w(t) \\ &= -e_x(t)e_y(t) - e_x(t)e_z(t) - k_1 e_x^2(t) \\ &\quad - \frac{1}{b_1} e_x(t)e_z(t)z_r(t) + e_x(t)e_y(t) \\ &\quad + a e_y^2(t) + e_y(t)e_w(t) - k_2 e_y^2(t) \\ &\quad + e_x(t)e_z(t) + \frac{1}{b_1} x_d(t-\tau) e_z^2(t) \\ &\quad + \frac{1}{b_1} z_r(t)e_x(t)e_z(t) - \frac{1}{b_1} x_d(t-\tau) e_z^2(t) \\ &\quad + \frac{b_3}{b_1} e_z^2(t) - c e_z(t)e_w(t) \\ &\quad + d e_w^2(t) - e_y(t)e_w(t) \\ &\quad + c e_z(t)e_w(t) - k_4 e_w^2(t) \\ &\quad - (e_x(t) \;\; e_y(t) \;\; e_z(t) \;\; e_w(t)) \\ &\quad \times Q(e_x(t)e_y(t) \;\; e_z(t) \;\; e_w(t))^{\mathrm{T}}. \end{aligned} \tag{10}$$

where $Q = \mathrm{diag}(k_1, k_2 - a, -(b_3/b_1), k_4 - d)$.

Then using Lemma 1, we have the exponential estimation:

$$\begin{aligned} & e_x^2(t) + e_y^2(t) + \frac{1}{b_1} e_z^2(t) + e_w^2(t) \\ &\quad \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left[e_x^2(t_0) + e_y^2(t_0) + \frac{1}{b_1} e_z^2(t_0)\right. \\ &\qquad \left. + \; e_w^2(t_0)\right] e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_0)} \\ &\quad = \frac{\max\left\{\dfrac{1}{2}, \dfrac{1}{2b_1}\right\}}{\min\left\{\dfrac{1}{2}, \dfrac{1}{2b_1}\right\}}\left[e_x^2(t_0) + e_y^2(t_0) + \frac{1}{b_1} e_z^2(t_0)\right. \\ &\qquad \left. + \; e_w^2(t_0)\right] e^{-\frac{\min\{k_1, k_2 - a, -b_3/b_1, k_4 - d\}}{\max\{\frac{1}{2}, \frac{1}{2b_1}\}}(t-t_0)}, \end{aligned} \tag{11}$$

which implies that the conclusion is true.

Similarly, for other controllers (B)–(H), we can still use the same Lyapunov function (9) to obtain the same estimation, as given in (11). ■

Remarks

(1) The eight nonlinear feedback controllers can be used to simultaneously obtain (i) globally exponential lag synchronization for $\tau > 0$, and (ii) globally exponential synchronization for $\tau = 0$ between the drive system (6) and the response system (7).

(2) Although the above-obtained feedback controllers are nonlinear, they are simpler than those of so-called "natural control" controllers:

$$\begin{cases} u_1 = -e_x(t) - e_y(t) - e_z(t), \\ u_2 = e_x(t) + (a+1)e_y(t) + e_w(t), \\ u_3 = [b_1 + z_r(t)]e_x(t) + [b_3 + 1 + x_d(t-\tau)]e_z(t), \\ u_4 = -ce_z(t) + (d+1)e_w(t), \end{cases}$$

which are derived by using $\mathbf{u} = \mathbf{Ae} + \mathbf{F}(t-\tau, x_d(t-\tau)) - \mathbf{F}(t, y_r(t))$ with a simple stable matrix $\mathbf{A} = \mathrm{diag}(-1,-1,-1,-1)$ for the drive system (6) and the response system (7).

Following the discussion of Case I, we can establish similar Lemmas to investigate globally exponential hyperchaos (lag) synchronization for other types of modified hyperchaotic Rössler systems (2) by choosing proper positive definite Lyapunov functions. Here, we omit the detailed derivations but list the control laws u_i $(i = 1, 2, 3, 4)$ for the different modified hyperchaotic Rössler systems in Tables 1–7, where the notation "LF" stands for the "Lyapuonv function".

Table 1. Feedback control laws for Case II of system (2): $b_3 \in [0.01, 0.9]$, $b_1^2 + b_2^2 + b_4^2 = 0$, and $k_1 > 0$, $k_2 > a$, $k_3 > b_3$, $k_4 > d$.

Case II	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_3 z + xz,\ \dot{w} = -cz + dw.$	
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + e_z^2(t) + e_w^2(t)]$	
A	$u_1 = k_1 e_x(t) + [z_r(t) - 1]e_z(t)$, $u_3 = k_3 e_z(t) + x_d(t-\tau)e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t) - ce_z(t)$.
B	$u_1 = k_1 e_x(t) + [z_r(t) - 1]e_z(t)$, $u_3 = k_3 e_z(t) - ce_w(t) + x_d(t-\tau)e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
C	$u_1 = k_1 e_x(t) + [z_r(t) - 1]e_z(t)$, $u_3 = k_3 e_z(t) + x_d(t-\tau)e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t) - ce_z(t)$.
D	$u_1 = k_1 e_x(t) + [z_r(t) - 1]e_z(t)$, $u_3 = k_3 e_z(t) - ce_w(t) + x_d(t-\tau)e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
E	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1]e_z(t)$, $u_3 = k_3 e_z(t) + x_r(t)e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t) - ce_z(t)$.
F	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1]e_z(t)$, $u_3 = k_3 e_z(t) - ce_w(t) + x_r(t)e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
G	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1]e_z(t)$, $u_3 = k_3 e_z(t) + x_r(t)e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t) - ce_z(t)$.
H	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1]e_z(t)$, $u_3 = k_3 e_z(t) - ce_w(t) + x_r(t)e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.

Table 2. Feedback control laws for Case III of system (2): $b_1 \in [0.01, 3.7]$, $b_2^2+b_3^2+b_4^2 = 0$, and $k_1 > 0$, $k_2 > a$, $k_3 > 0$, $k_4 > d$.

Case III	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_1 x + xz,\ \dot{w} = -cz + dw.$	
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + \frac{1}{b_1}e_z^2(t) + e_w^2(t)]$	
A	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_r(t) e_z(t)$, $u_3 = k_3 e_z(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t) - c e_z(t)$.
B	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_r(t) e_z(t)$, $u_3 = k_3 e_z(t) - b_1 c e_w(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
C	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_r(t) e_z(t)$, $u_3 = k_3 e_z(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t) - c e_z(t)$.
D	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_r(t) e_z(t)$, $u_3 = k_3 e_z(t) - b_1 c e_w(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
E	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = k_3 e_z(t) + x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t) - c e_z(t)$.
F	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = k_3 e_z(t) - b_1 c e_w(t) + x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
G	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = k_3 e_z(t) + x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t) - c e_z(t)$.
H	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = k_3 e_z(t) - b_1 c e_w(t) + x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.

Table 3. Feedback control laws for Case IV of system (2): $b_2 = -0.3$, $b_3 \in [-1, -0.01]$, $b_1^2 + b_4^2 = 0$, and $k_1 > 0$, $k_2 > a$, $k_4 > d$.

Case IV	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_2 y + b_3 z + xz,\ \dot{w} = -cz + dw.$	
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + e_z^2(t) + e_w^2(t)]$	
A	$u_1 = k_1 e_x(t) + [z_r(t) - 1] e_z(t)$, $u_3 = x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t)$, $u_4 = k_4 e_w(t) + e_y(t) - c e_z(t)$.
B	$u_1 = k_1 e_x(t) + [z_r(t) - 1] e_z(t)$, $u_3 = -c e_w(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
C	$u_1 = k_1 e_x(t) + [z_r(t) - 1] e_z(t)$, $u_3 = x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t) - c e_z(t)$.
D	$u_1 = k_1 e_x(t) + [z_r(t) - 1] e_z(t)$, $u_3 = -c e_w(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
E	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1] e_z(t)$, $u_3 = x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t) - c e_z(t)$.
F	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1] e_z(t)$, $u_3 = -c e_w(t) + x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.

Table 3. (*Continued*)

Case IV	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_2y + b_3z + xz,\ \dot{w} = -cz + dw.$
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + e_z^2(t) + e_w^2(t)]$
G	$u_1 = k_1e_x(t) + [z_d(t-\tau) - 1]e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t)$, $u_3 = x_r(t)e_z(t)$, $u_4 = k_4e_w(t) + e_y(t) - ce_z(t)$.
H	$u_1 = k_1e_x(t) + [z_d(t-\tau) - 1]e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t)$, $u_3 = -ce_w(t) + x_r(t)e_z(t)$, $u_4 = k_4e_w(t) + e_y(t)$.

Table 4. Feedback control laws for Case V of system (2): $b_2 = b_3 = -0.3$, $b_1 \in [0.01, 4.5]$, $b_4 = 0$, and $k_1 > 0$, $k_2 > a$, $k_4 > d$.

Case V	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_1x + b_2y + b_3z + xz,\ \dot{w} = -cz + dw.$
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + \frac{1}{b_1}e_z^2(t) + e_w^2(t)]$
A	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_r(t)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t)$, $u_3 = x_d(t-\tau)e_z(t)$, $u_4 = k_4e_w(t) + e_y(t) - ce_z(t)$.
B	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_r(t)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t)$, $u_3 = -b_1ce_w(t) + x_d(t-\tau)e_z(t)$, $u_4 = k_4e_w(t) + e_y(t)$.
C	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_r(t)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t) + e_w(t)$, $u_3 = x_d(t-\tau)e_z(t)$, $u_4 = k_4e_w(t) - ce_z(t)$.
D	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_r(t)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t) + e_w(t)$, $u_3 = -b_1ce_w(t) + x_d(t-\tau)e_z(t)$, $u_4 = k_4e_w(t)$.
E	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_d(t-\tau)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t)$, $u_3 = x_r(t)e_z(t)$, $u_4 = k_4e_w(t) + e_y(t) - ce_z(t)$.
F	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_d(t-\tau)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t)$, $u_3 = -b_1ce_w(t) + x_r(t)e_z(t)$, $u_4 = k_4e_w(t) + e_y(t)$.
G	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_d(t-\tau)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t) + e_w(t)$, $u_3 = x_r(t)e_z(t)$, $u_4 = k_4e_w(t) - ce_z(t)$.
H	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_d(t-\tau)e_z(t)$, $u_2 = k_2e_y(t) + b_2e_z(t) + e_w(t)$, $u_3 = -b_1ce_w(t) + x_r(t)e_z(t)$, $u_4 = k_4e_w(t)$.

Table 5. Feedback control laws for Case VI of system (2): $b_1 = 0.3, b_4 = 0.2$, $b_3 \in [-2.5, -0.01]$, $b_2 = 0$, and $k_1 > 0$, $k_2 > a$, $k_4 > d$.

Case VI	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_1x + b_3z + b_4w + xz,\ \dot{w} = -cz + dw.$
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + \frac{1}{b_1}e_z^2(t) + e_w^2(t)]$
A	$u_1 = k_1e_x(t) + \frac{1}{b_1}z_r(t)e_z(t)$, $u_2 = k_2e_y(t)$, $u_3 = x_d(t-\tau)e_z(t)$, $u_4 = k_4e_w(t) + e_y(t)$.

Table 5. (*Continued*)

Case VI	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_1 x + b_3 z + b_4 w + xz,\ \dot{w} = -cz + dw.$	
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + \frac{1}{b_1} e_z^2(t) + e_w^2(t)]$	
B	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_r(t) e_z(t)$, $u_3 = x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
C	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
D	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.

Table 6. Feedback control laws for Case VII of system (2): $b_1 = 0$, $b_2 = -0.1$, $b_4 = 0.2$, $b_3 \in [-1.2, -0.01]$, and $k_1 > 0$, $k_2 > a$, $k_4 > d$.

Case VII	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_2 y + b_3 z + b_4 w + xz,\ \dot{w} = -cz + dw$	
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + e_z^2(t) + e_w^2(t)]$	
A	$u_1 = k_1 e_x(t) + [z_r(t) - 1] e_z(t)$, $u_3 = x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
B	$u_1 = k_1 e_x(t) + [z_r(t) - 1] e_z(t)$, $u_3 = x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
C	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1] e_z(t)$, $u_3 = x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
D	$u_1 = k_1 e_x(t) + [z_d(t-\tau) - 1] e_z(t)$, $u_3 = x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.

Table 7. Feedback control laws for Case VIII of system (2): $b_1 = -0.1$, $b_3 = b_4 = 0.1$, $b_1 \in [0.01, 1.5]$, and $k_1 > 0$, $k_2 > a$, $k_3 > b_3$, $k_4 > d$.

Case VIII	$\dot{x} = -y - z,\ \dot{y} = x + ay + w,\ \dot{z} = b + b_1 x + b_2 y + b_3 z + b_4 w + xz,\ \dot{w} = -cz + dw.$	
LF	$V(t) = \frac{1}{2}[e_x^2(t) + e_y^2(t) + \frac{1}{b_1} e_z^2(t) + e_w^2(t)]$	
A	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_r(t) e_z(t)$, $u_3 = k_3 e_z(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
B	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_r(t) e_z(t)$, $u_3 = k_3 e_z(t) + x_d(t-\tau) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.
C	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = k_3 e_z(t) + x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t)$, $u_4 = k_4 e_w(t) + e_y(t)$.
D	$u_1 = k_1 e_x(t) + \frac{1}{b_1} z_d(t-\tau) e_z(t)$, $u_3 = k_3 e_z(t) + x_r(t) e_z(t)$,	$u_2 = k_2 e_y(t) + b_2 e_z(t) + e_w(t)$, $u_4 = k_4 e_w(t)$.

3. Numerical Simulation Results

In this section, we use numerical simulation to demonstrate the controllers obtained in the previous section for different modified hyperchaotic Rössler systems. In particular, we consider $\tau = 0.5$ for Cases I–IV, while $\tau = 0$ for Cases V–VIII.

For Case I with $\tau = 0.5$, the parameters are taken as $a = 0.25$, $b = 3$, $c = 0.5$, $d = 0.05$, $b_1 = 1$, and $b_3 = -2$. The initial points of the drive system and the response system related to the modified hyperchaotic Rössler system are chosen as $(x_d(t), y_d(t), z_d(t), w_d(t)) = (-40, 1, -3, 20)$, $t \in [-0.5, 0]$ and $(x_r(0), y_r(0), z_r(0), w_r(0)) = (-36, 2, 0, 19)$, respectively. Figures 1(a) and 1(b) display the projections of the modified hyperchaotic Rössler attractor on the x_d-y_d-z_d and the x_d-y_d-w_d spaces, respectively. Figures 1(c)–1(f) show respectively the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $k_1 = 0.4$, $k_2 = 0.8$, $k_4 = 0.5$. It is clearly seen from the errors that the drive system and the response system are exponentially synchronized. Other arbitrary initial points have also been chosen to show the global convergence.

For Cases II–IV with $\tau = 0.5$ and Cases V–VIII with $\tau = 0$, we list in Table 8 the parameter values b_i $(i = 1, 2, 3, 4)$, the initial points of the drive system and the response system related to the modified hyperchaotic Rössler systems, as well as the control parameter values for controller (A) presented in Tables 1–7. Figures 2(a)–2(d) $(j = 2, 3, \ldots, 8)$ show respectively the time histories of the error signals e_x, e_y, e_z and e_w under the corresponding controller (A). All the results given in these figures indicate the similar trend as that of Fig. 1, the drive system and the response system are synchronized with the negative exponential congregent rate by using the controller (A).

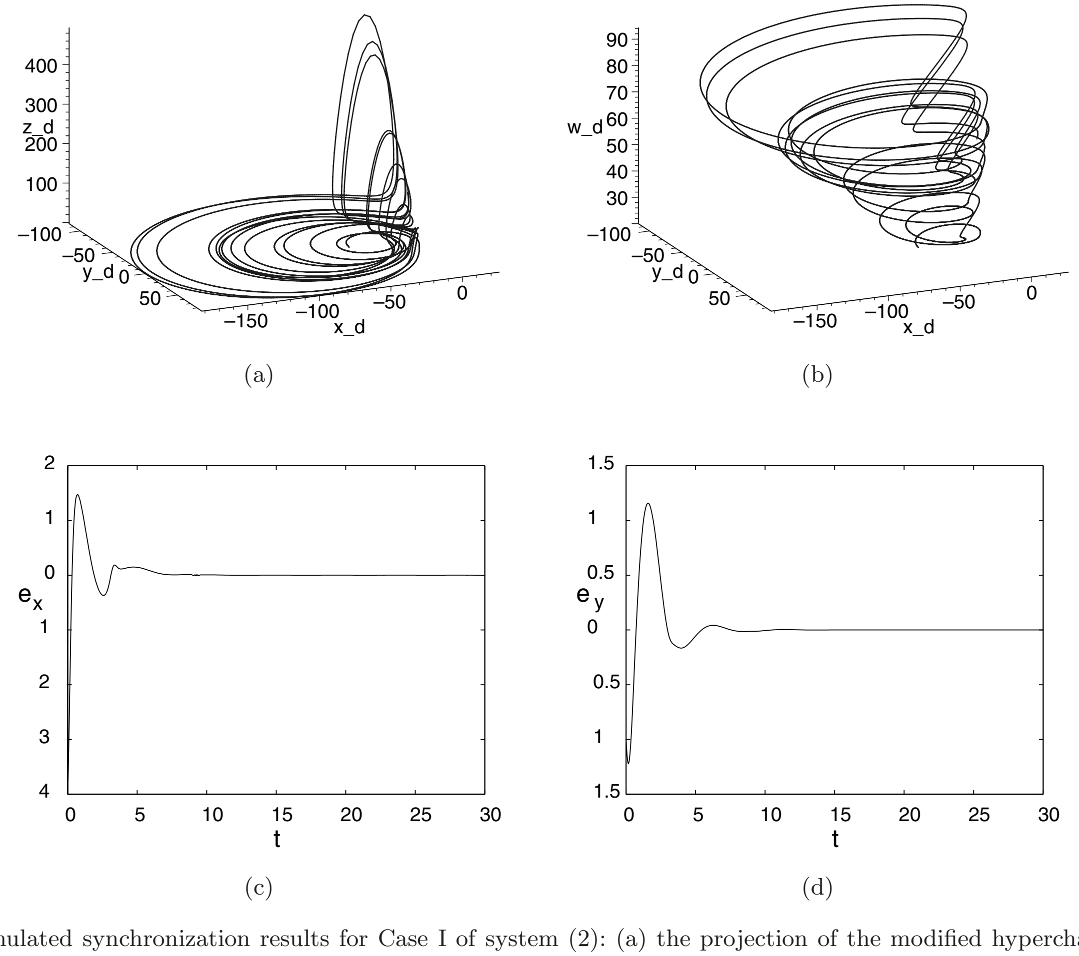
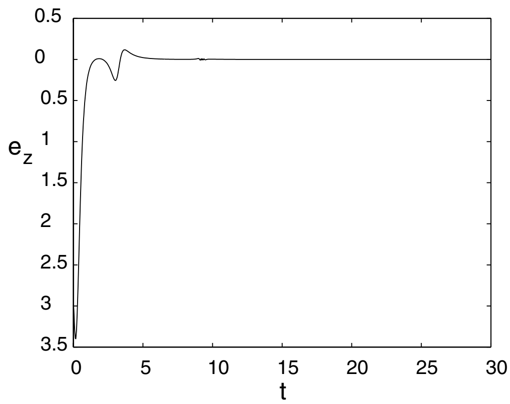
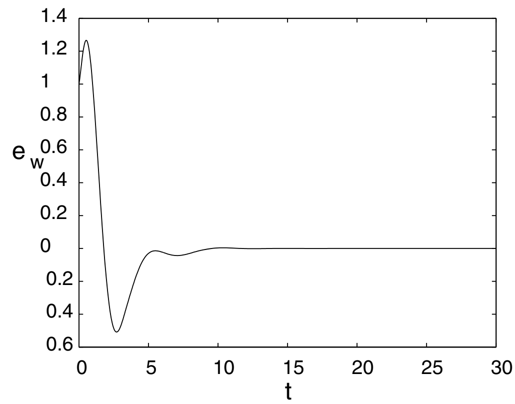


Fig. 1. Simulated synchronization results for Case I of system (2): (a) the projection of the modified hyperchaotic Rössler attractor on the x_d-y_d-z_d space; (b) the projection of the modified hyperchaotic Rössler attractor on the x_d-y_d-w_d space; (c)–(f) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0.5$, $k_1 = 0.4$, $k_2 = 0.8$, $k_4 = 0.5$.

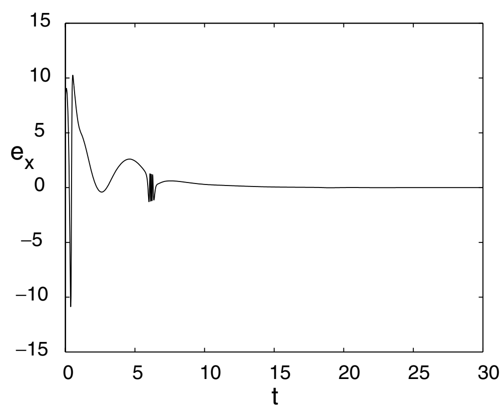


(e)

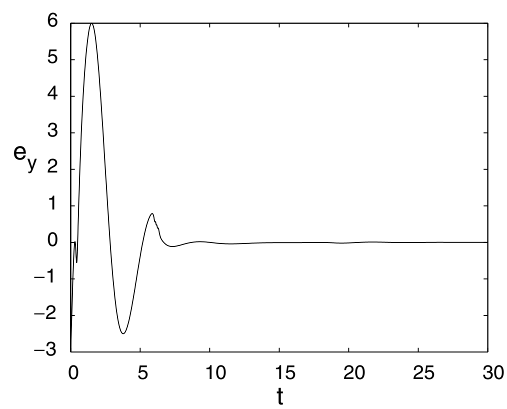


(f)

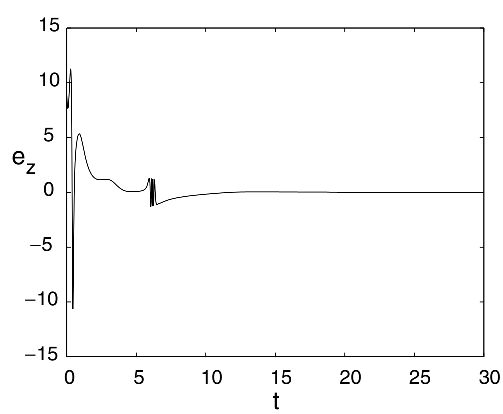
Fig. 1. (*Continued*)



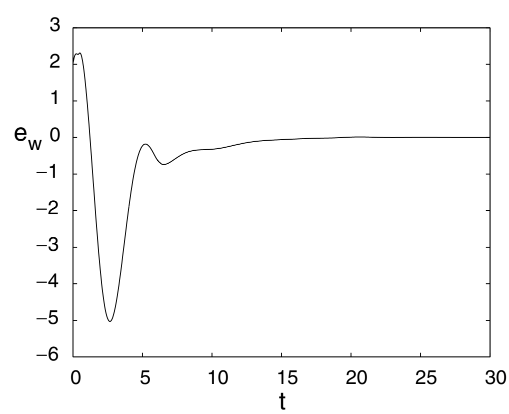
(a)



(b)



(c)



(d)

Fig. 2. Simulated synchronization results for Case II of system (2): (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0.5$, $k_1 = 0.5$, $k_2 = 0.6$, $k_3 = 0.3$, $k_4 = 0.25$.

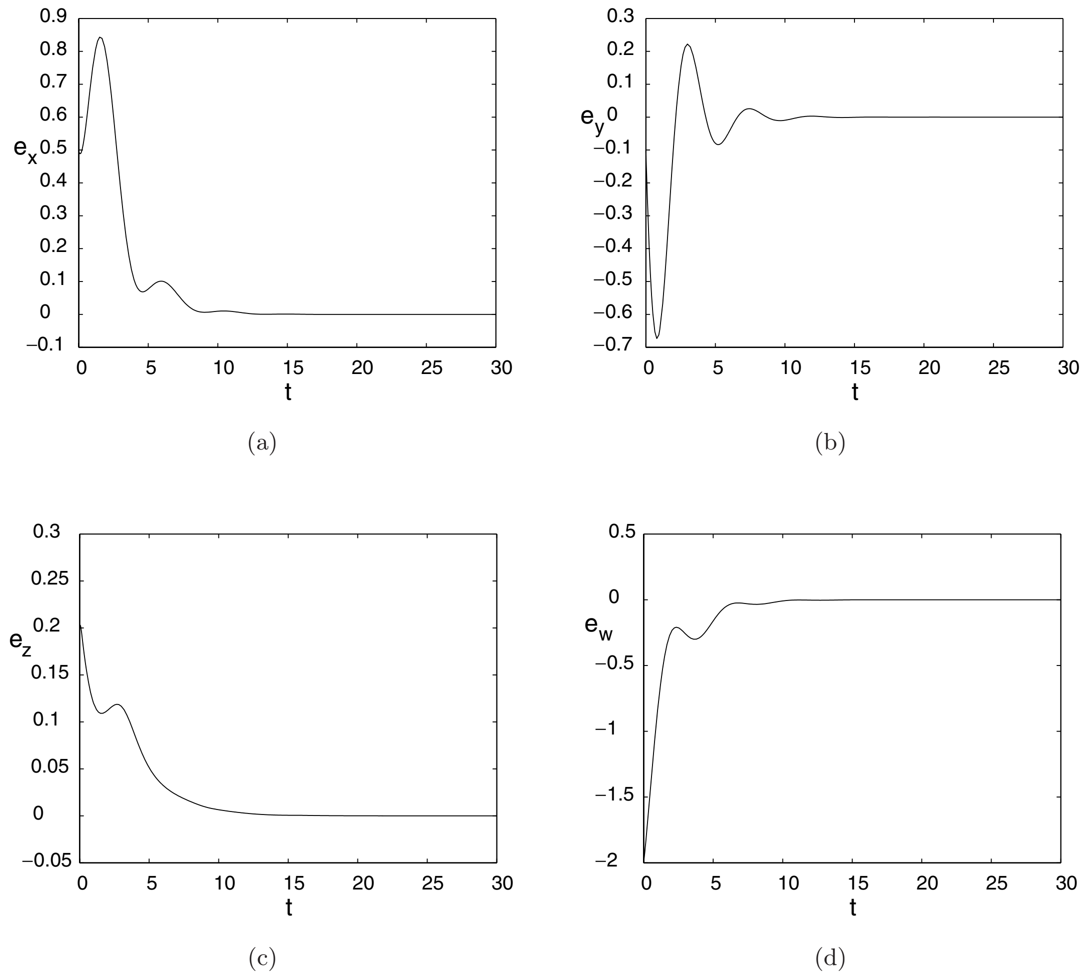


Fig. 3. Simulated synchronization results for Case III of system (2): (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0.5$, $k_1 = 0.4$, $k_2 = 0.8$, $k_3 = 0.6$, $k_4 = 0.5$.

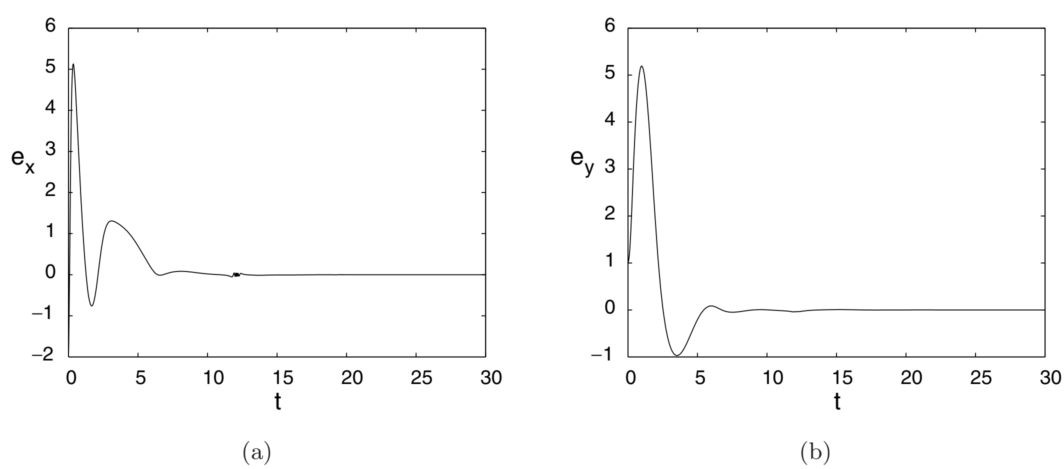
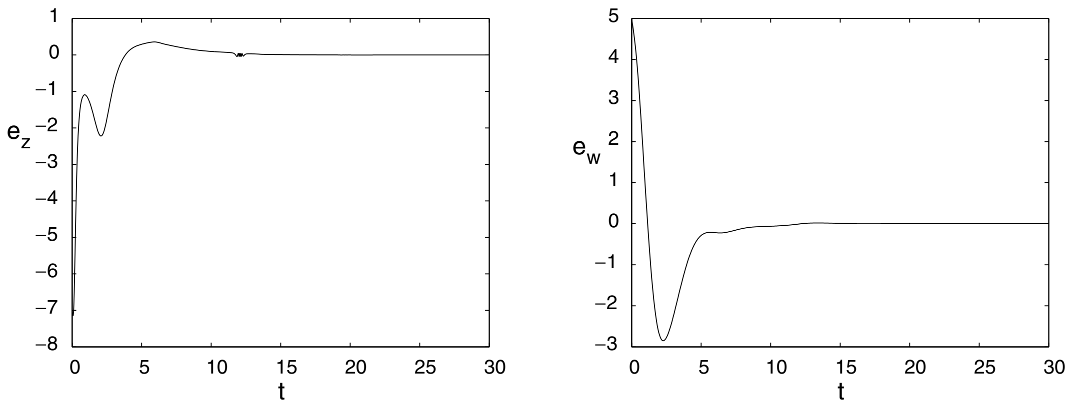
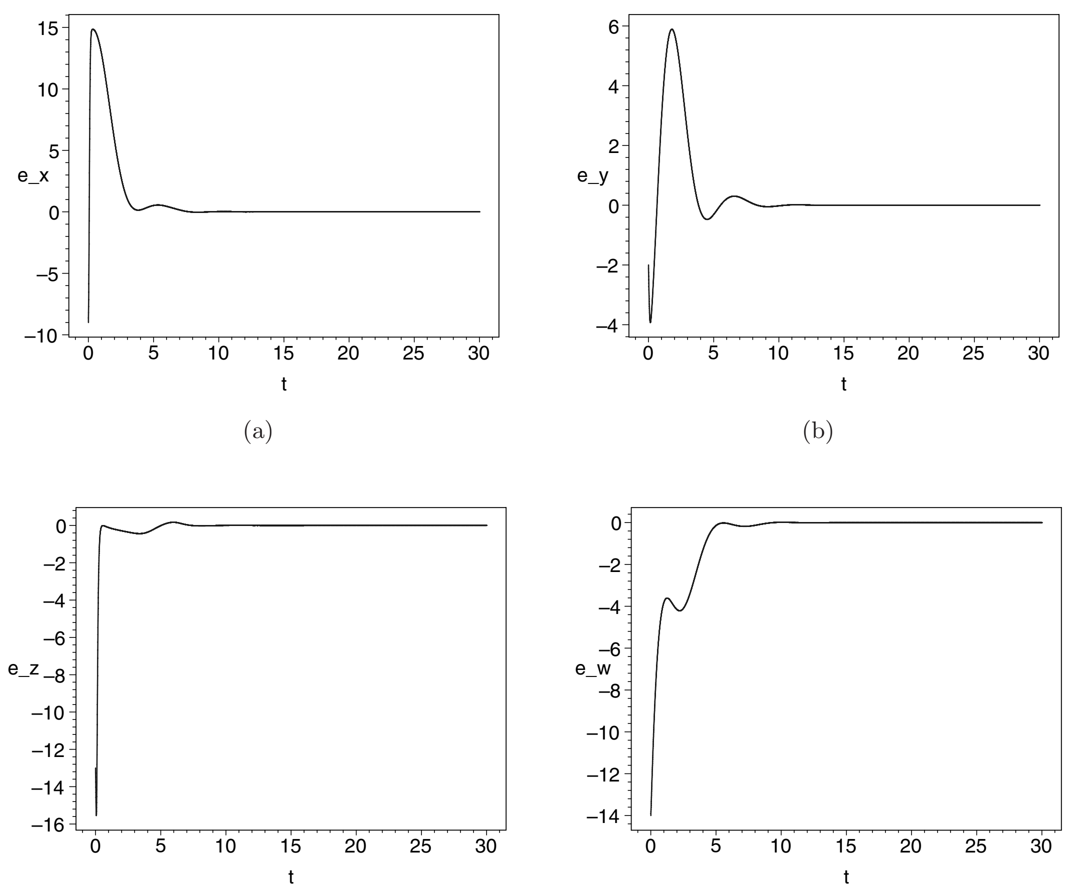


Fig. 4. Simulated synchronization results for Case IV of system (2): (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0.5$, $k_1 = 1$, $k_2 = 0.6$, $k_4 = 0.3$.



(c) (d)

Fig. 4. (*Continued*)



(a) (b)

(c) (d)

Fig. 5. Simulated synchronization results for Case V of system (2): (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0$, $k_1 = 0.2$, $k_2 = 0.8$, $k_4 = 1.3$.

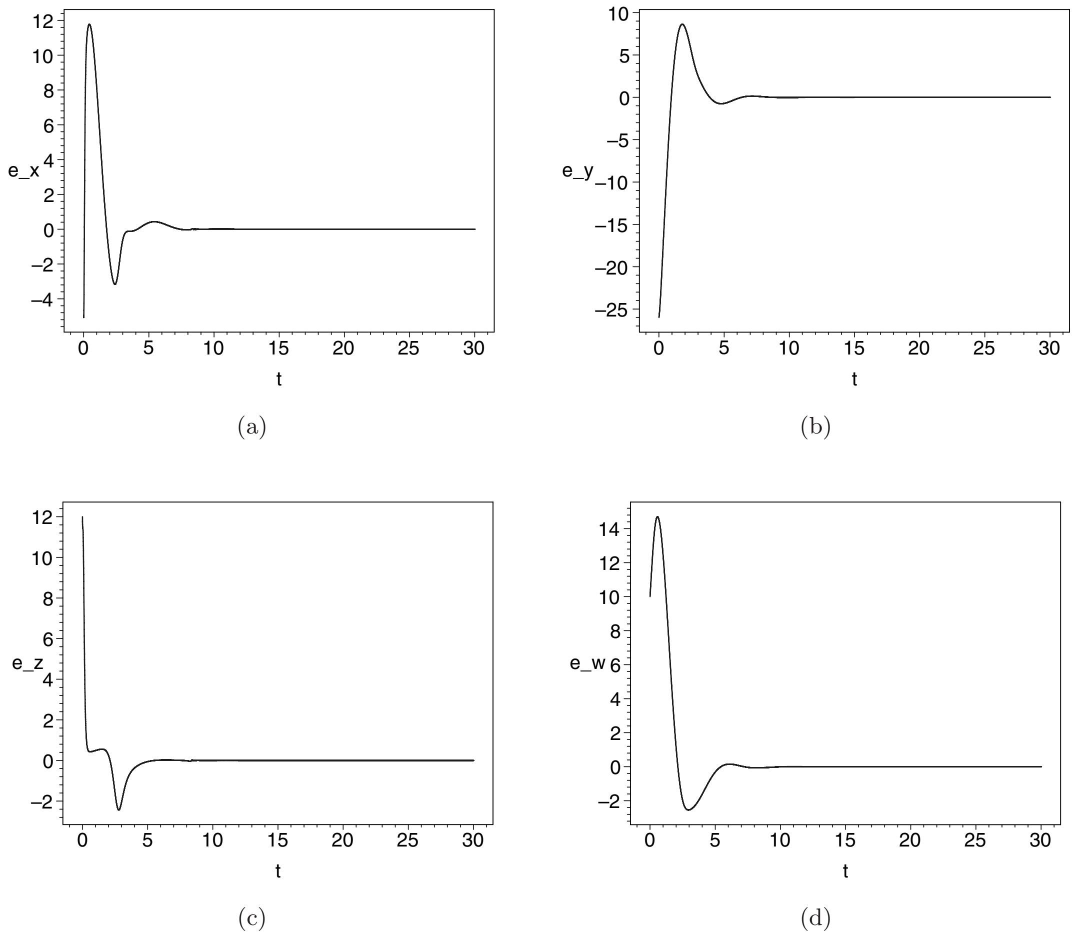


Fig. 6. Simulated synchronization results for Case VI of system (2): (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0$, $k_1 = 1.2$, $k_2 = 0.6$, $k_4 = 0.7$.

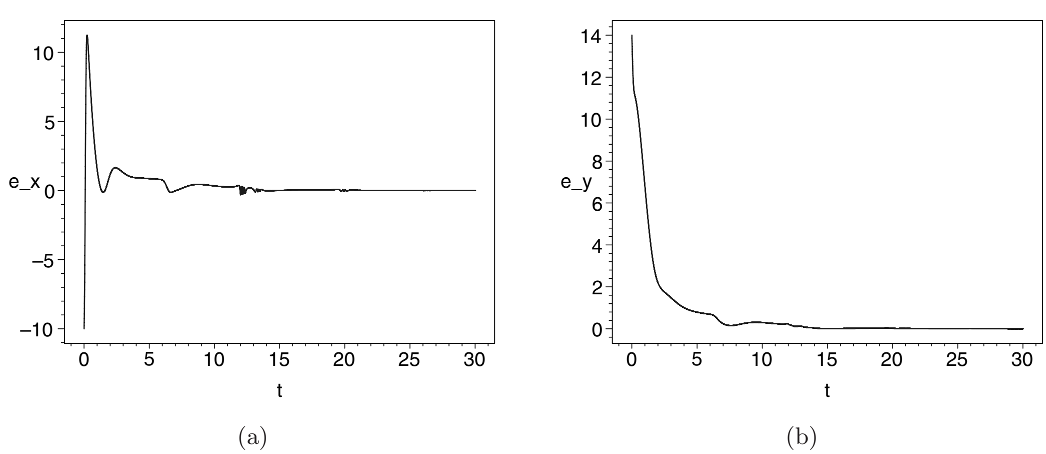
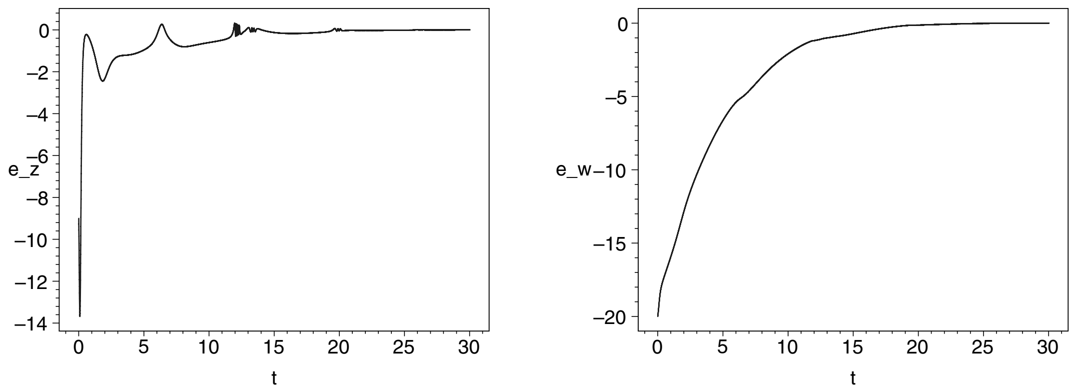
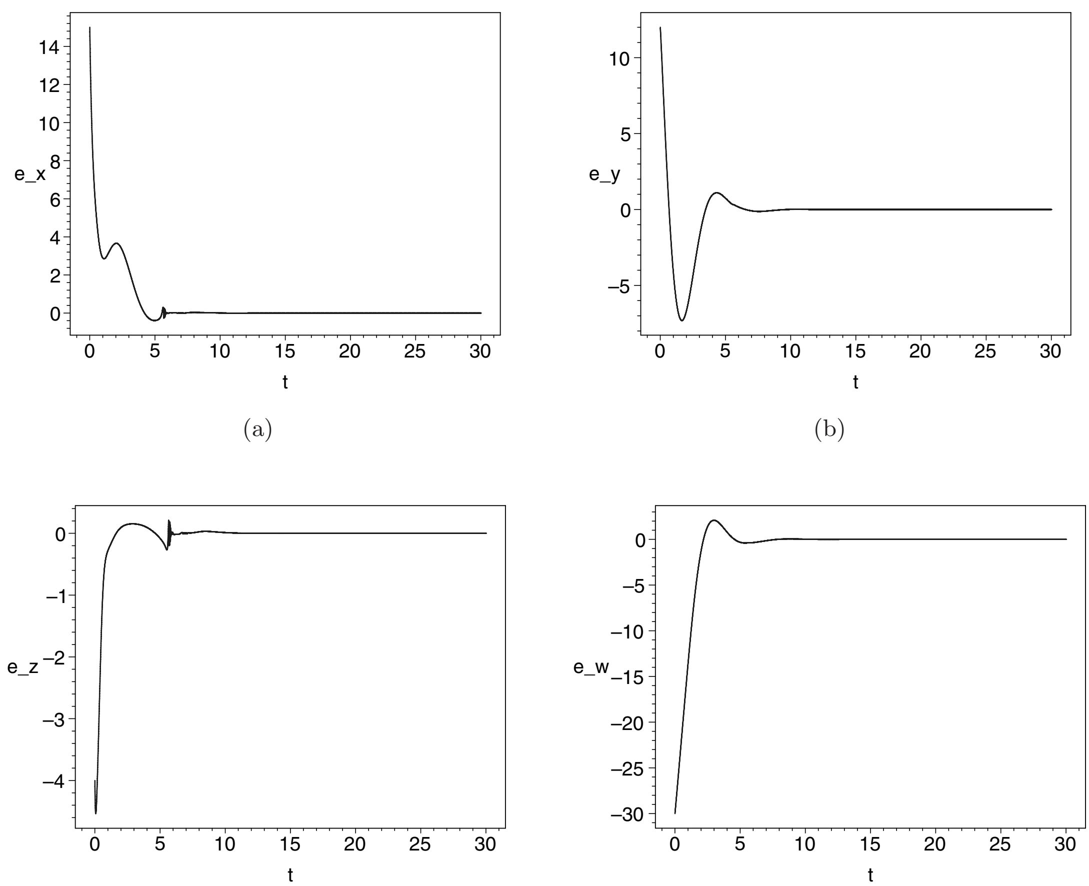


Fig. 7. Simulated synchronization results for Case VII of system (2): (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0$, $k_1 = 0.8$, $k_2 = 1.6$, $k_4 = 0.2$.



(c) (d)

Fig. 7. (*Continued*)



(a) (b)

(c) (d)

Fig. 8. Simulated synchronization results for Case VIII of system (2): (a)–(d) the time histories of the error signals e_x, e_y, e_z and e_w under the first controller (A) with $\tau = 0$, $k_1 = 1.8$, $k_2 = 0.5$, $k_3 = 1.2$, $k_4 = 0.9$.

Table 8. The data for simulations of Case II–VIII: the parameter values b_i $(i = 1, 2, 3, 4)$, the initial points of the drive system and the response system, and the control parameter values of controller (A).

Cases	Parameters	Initial Points	Parameters in Controller (A)
II	$b_3 = 0.1$	$(x_d(t), y_d(t), z_d(t), w_d(t))$ $= (5, -2, 5, 28),\ t \in [-0.5, 0]$, $(x_r(0), y_r(0), z_r(0), w_r(0))$ $= (-3, 1, -4, 26)$	$k_1 = 0.5,\ k_2 = 0.6,$ $k_3 = 0.3,\ k_4 = 0.25$
III	$b_1 = 0.8$	$(x_d(t), y_d(t), z_d(t), w_d(t))$ $= (-20, 0, 0, 15),\ t \in [-0.5, 0]$, $(x_r(0), y_r(0), z_r(0), w_r(0))$ $= (-20.5, 0.1, -0.2, 17)$	$k_1 = 0.4,\ k_2 = 0.8,$ $k_3 = 0.6,\ k_4 = 0.5$
IV	$b_2 = -0.3,$ $b_3 = -0.3$	$(x_d(t), y_d(t), z_d(t), w_d(t))$ $= (-10, 6, -5, 20),\ t \in [-0.5, 0]$, $(x_r(0), y_r(0), z_r(0), w_r(0))$ $= (-8, 5, 2, 15)$	$k_1 = 1,\ k_2 = 0.6,$ $k_4 = 0.3$
V	$b_1 = 0.4,$ $b_2 = -0.3,$ $b_3 = -0.3$	$(x_d(0), y_d(0), z_d(0), w_d(0))$ $= (-15, 6, -8, 26)$, $(x_r(0), y_r(0), z_r(0), w_r(0))$ $= (-6, 8, 5, 40)$	$k_1 = 0.2,\ k_2 = 0.8,$ $k_4 = 1.3$
VI	$b_1 = 0.3,$ $b_3 = -2.5,$ $b_4 = 0.2$	$(x_d(0), y_d(0), z_d(0), w_d(0))$ $= (-50, -16, 18, 35)$, $(x_r(0), y_r(0), z_r(0), w_r(0))$ $= (-45, 10, 6, 25)$	$k_1 = 1.2,\ k_2 = 0.6,$ $k_4 = 0.7$
VII	$b_2 = -0.1,$ $b_3 = -0.5,$ $b_4 = 0.2$	$(x_d(0), y_d(0), z_d(0), w_d(0))$ $= (-15, 2, -4, 40)$, $(x_r(0), y_r(0), z_r(0), w_r(0))$ $= (-5, -12, 5, 60)$	$k_1 = 0.8,\ k_2 = 1.6,$ $k_4 = 0.2$
VIII	$b_1 = 0.6,$ $b_2 = -0.1,$ $b_3 = 0.1,$ $b_4 = 0.1$	$(x_d(0), y_d(0), z_d(0), w_d(0))$ $= (-1, 2, -6, 50)$, $(x_r(0), y_r(0), z_r(0), w_r(0))$ $= (-16, -10, -2, 80)$	$k_1 = 1.8,\ k_2 = 0.5,$ $k_3 = 1.2,\ k_4 = 0.9$

4. Conclusion

In this paper, based on Lyapunov stability theory, we have designed a number of nonlinear feedback controllers to simultaneously obtain (i) the globally exponential lag synchronization (GELS) for $\tau > 0$; and (ii) the globally exponential synchronization (GES) for $\tau = 0$ for a family of modified hyperchaotic Rössler systems. Numerical simulations are used to illustrate the theoretical results. This method can be extended to study other hyperchaotic systems.

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