Explicit Vibration Solutions of a Cable Under Complicated Loads

This paper is concerned with the dynamical analysis of a sagged cable having small equilibrium curvature and horizontal supports under both distributed and concentrated loads. The loads are applied in vertical as well as horizontal directions. Based on a free vibration analysis, a transfer matrix method is generalized for solving coupled, nonhomogeneous differential equations to obtain closed-form solutions for the natural frequencies and the associated vibration mode shapes in vertical, horizontal, and longitudinal directions. It is shown that two sets of independent mode shapes associated with two sets of independent frequencies always exist and can be obtained via an equation of one variable only. This method demonstrates its advantages in dealing with interactions of modes in different directions, complex arrangement of concentrated loads, and high-order modes oscillations.

1 Introduction

Cables are commonly employed in a wide range of practical applications in ocean and electrical engineering. The study of oscillations of suspended cables is of great importance to cable structure industry and it has attracted the attention of many researchers. For example, the galloping (a low-frequency large-amplitude oscillation) of overhead transmission lines may cause severe disruptions in the electrical supply and even a cascading collapse of a line’s supporting towers. Marine cables may experience vortex-induced oscillations which can accelerate cable fatigue and degrade the performance and positioning of attached instruments such as hydrophones. A better understanding and modeling of cable motions rely on the accurate computation of the cable’s natural frequency spectrum and vibration mode shapes.

Linear theory for bare (with no attached point loads or instruments) suspended, elastic cables with small ratio of the sag to the line’s span length was well established for either horizontal (Irvine and Caughey, 1974) or inclined (Triantafyllou, 1984) supports. Irvine (1981) also derived explicit formulas for a taut inclined cable. However, in practice, most of cables have attached point masses or concentrated loads. For example, marine cables used as electrical or optical signal transmission lines may have hundreds of attached instruments. Overhead electrical transmission lines may often be equipped with an anti-galloping control device such as detuning pendulums which are basically additional masses hanging on the line. These attached concentrated loads greatly complicate the analysis of cable motions. Therefore, numerical methods have been widely used in the study of cables with concentrated loads, in particular, in finding the cable’s natural frequencies and vibration mode shapes. The numerical solutions, however, were restricted to the low-order vibration modes of cables due to convergence limitations. Though great progress has been achieved in the study of cable dynamics (e.g., see the review article by Triantafyllou (1991)), the investigation of cables with concentrated loads has received the attention of very few researchers, among them are Irvine (1981), Sergev and Iwan (1981), and Cheng and Perkins (1992a, b).

This research was motivated by an industrial project “Modeling of Conductor Galloping.” A simple two-degree-of-freedom model (Yu et al., 1992) and a more comprehensive three-degree-of-freedom model (Yu et al., 1993) have been developed to realistically describe the dynamics oficed cables under side steady wind. These models are established with the natural frequencies and vibration mode shapes obtained using the approach given by Irvine (1981) for bare cables and therefore are only applicable for the lines with no concentrated loads. In practice, however, many power utilities, especially in North America, have employed add-on devices such as detuning pendulums to control galloping. These control devices may be treated as concentrated loads and thus, the natural frequencies and mode shapes must be recomputed for such a line. In fact, it has been found that continuing to use the natural frequencies and mode shapes of a bare cable for a line equipped with detuning pendulums yields erroneous results (Modelling, 1993).

Furthermore, it has been found that the change of the static profile (equilibrium state) of a transmission line due to wind changes the line’s natural frequencies and mode shapes and therefore, may significantly affect the dynamical analysis. Figure 1 shows a comparison of the results obtained from the three-degree-of-freedom model (Yu et al., 1993) using the data from a most documented field trial performed by Ontario Hydro (Edwards and Madeyski, 1956). In this figure, the vertical axis denotes the critical wind speed at which the cable becomes unstable (in the sense of Liapunov asymptotic stability) and galloping is initiated. The horizontal axis represents the ratio $q_{z}/q_{y}$, where $q_{z}$ is the horizontal static distributed wind load and $q_{y}$ is the vertical static distributed loads which consist of the wind load and the weights of the cable and ice. It is seen that the critical wind speed $U_{cv}$, predicted by the vertical model (assuming no static loads so that the static profile is located in the vertical plane), is quite different from the critical wind speed $U_{cv}$, obtained from the inclined model (the static profile is located in an inclined plane due to horizontal static load). For example, at $q_{z}/q_{y} = 0.1$, $U_{cv} = 3.28$ m/s, $U_{cv} = 4.23$ m/s which gives 30 percent error. However, when $q_{z}/q_{y} = 0.2$, $U_{cv} = 3.18$ m/s, but $U_{cv} \rightarrow \infty$, which implies that the inclined model predicts the line always stable while vertical model indicates the line becoming unstable at a very low wind speed. This shows that the static load, especially the horizontal component, cannot, in general, be neglected in the vibration analysis of cables subjected cross flows.

It has been noted that Irvine (1981) derived the analytical expressions for the natural frequency and mode shapes of a bare cable described by differential equations. He also obtained explicit approximate solutions to the case of a cable with vertical
multiple concentrated loads (masses) by using difference equations. The discretization approach makes the derivation and formulas simpler, and give good approximations when the number of multiple loads are not small. However, this method is only applicable for equally linked and equally concentrated loads. To obtain more accurate solutions, Sergev and Iwan (1981) employed a transfer matrix approach to derive closed form free vibration solutions for multiple attached point masses by assuming that the cable is inextensible (taut cable). Recently, Cheng and Perkins (1992a, 1992b) extended the transfer matrix method to consider elastic cables and obtained analytical formulas leading to numerically solve for only two coupled transcendental equations. One common fact noticed from these research results is that the horizontal static distributed loads are neglected so that the static profile is located in the vertical plane. Therefore, the out-plane motion is decoupled from the in-plane motion, which greatly simplifies the analysis. However, in case if the horizontal static load cannot be ignored, as has been shown in the galloping analysis, one has to find out-plane and in-plane frequencies and mode shapes by simultaneously solving coupled differential equations. Hence, a new method needs to be developed for solving the general case. It should be noted that all the cases studied before are actually special cases of the general case.

In this paper, the transfer matrix method will be modified and generalized to consider a model of elastic cables with a non-plane static profile. The natural frequencies and mode shapes will be found for the most general case; i.e., there exist static horizontal loads and multiple general concentrated loads which are not necessarily in the vertical direction. The horizontal loads are static drag forces, obtained from the measurement of aerodynamic loads. It will be shown that two sets of independent mode shapes associated with two sets of independent frequencies always exist and can be found by solving an equation of one variable only. A practical example is presented to show the applicability of the theory.

2 Equations of Motion

A sagged elastic cable, having span length $L_x$, with horizontal supports is shown in Fig. 2 where the concentrated loads are denoted by $F_i$, $i = 1, 2, \ldots, n$, each of which deviates from the vertical line by a constant angle $\theta_i$, and their locations are indicated by $x_i$ (or $s_i$) measured from the left end support of the cable. $s$ is the Lagrangian coordinate indicating a reference distance from the left end support while $\tau$ indicates a new Lagrangian coordinate under vibration.

The equations describing the motion of the cable under both distributed and concentrated loads can be derived (see Fig. 3 for the movement components along the vertical and longitudinal directions) as

$$
\left( (T + \tau)(x' + u') \right)' = m_u + m_\delta (s - s_i) \dot{u} \\
\left( (T + \tau)(y' + y') \right)' = m_y + m_\delta (s - s_i) \dot{y}
$$

$$
\left( (T + \tau)(z' + z') \right)' = m_z + m_\delta (s - s_i) \dot{z}
$$

$$
\tau(s, t) = AE \left( \frac{ds}{dt} - 1 \right) \quad \text{(Hooke’s Law)}
$$

where the prime and dot indicate differentiations with respect to $s$ and $t$ (time), respectively. $T(s)$ and $\tau(s, t)$ represent the static and dynamic tensions of the cable, respectively; $q_y$ and $q_z$ denote respectively the vertical and horizontal static loads per unit length which are assumed spatially uniform along the span. The $x(s)$, $y(s)$, and $z(s)$ are the static profile while $u(s, t)$, $y(s, t)$, and $z(s, t)$ are the dynamic movements along the longitudinal, vertical, and horizontal directions, respectively. As shown in Fig. 2, these movements are measured from the static profile. $A$ is the cross-sectional area of the cable and $E$ is

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**Fig. 1** Effect of horizontal static loads, $C$: vertical model; $\theta$: inclined model

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**Fig. 2** A sagged cable: (a) static profile; (b) dynamical motion; and (c) concentrated loads
the Young's modulus. \( \delta \) is a function satisfying \( \delta(s - s_i) = 1 \) (or 0) if \( s = s_i (s \neq s_i) \). Equations (1) and (2) are reduced to two coupled equations studied by Cheng and Perkins (1992b) by assuming \( q_c = \theta_i = 0 \), which decouples the horizontal motion from the vertical movement, and to one equation considered by Sergev and Iwan (1981) by a further assumption that the cable is inextensible. In this paper, only one assumption is required, that is, the ratio of the sag to the cable's length is small.

Before vibration analysis, the static profile described by \( x \), \( y_i \) and \( z_i \) must be determined. It needs three steps: (1) to find the static profile under the distributed loads only (i.e., \( F_i = 0 \)); (2) to compute the increase of the static tension due to the concentrated loads since usually only the tension under distributed loads is known; and (3) to determine the static profile under both distributed and concentrated loads. The detailed formulations for the three steps can be found in (Yu et al., 1995) and therefore, are not repeated here.

3 Vibration Analysis

The dynamical solutions can be obtained by free vibration analysis on the basis of Eqs. (1) and (2). The dynamic tension increase of a cable with a small ratio of the sag to the cable's length is usually very small compared to the static tension; i.e., \( \tau(s, t) < T(s) \) and furthermore, one may approximate \( \tau(s, t) \) as

\[
\tau(s, t) = \tilde{h}(t) \frac{ds}{dx} \tag{3}
\]

up to second order. Then by substituting the static profile into the second equation of (1), one can obtain the following equations for the vertical motion:

\[
\frac{\partial^2 y_2(s, t)}{\partial s^2} - \left( \frac{m}{H} \right) \frac{\partial^3 y_2(s, t)}{\partial t^3} = \left( \frac{q_c}{H^2} \right) \tilde{h}(t)
\]

\[(i = 0, 1, 2, \ldots, n)\]

\[
y_2(0, t) = y_2(L, t) = 0
\]

\[
y_2(s, t) - y_2(s, t) = 0
\]

\[
\frac{\partial y_2(s, t)}{\partial s} \frac{\partial y_2(s, t)}{\partial s} = \left( \frac{m}{H} \right) \frac{\partial^2 y_2(s, t)}{\partial t^2} - \left( \frac{P_i}{H^2} \right) \tilde{h}(t) \tag{4}
\]

where \( H \) is the horizontal component of the static tension under both the distributed and concentrated loads. Similarly, the third equation of (1) leads to the equations for the horizontal motion, which actually can be obtained from Eq. (4) by changing \( y \) to \( z \) and setting \( P_i = 0 \). Equation (2) now becomes

\[
E_{\nu} = \frac{q_c}{H} \left[ (s - \alpha_i) - \left( \frac{q_i^2}{q^2} \right) c_i \right]
\]

\[
E_{\nu} = \frac{q_i}{H} \left[ (s - \alpha_i) + \left( \frac{q_i^2}{q^2} \right) c_i \right] \tag{5}
\]

It should be noted due to Eq. (3) that the differential equation describing the longitudinal component has been dropped; i.e., the longitudinal inertia is neglected. However, the approximation of the longitudinal motion can still be obtained from Eq. (5). Next, let vibration solutions be

\[
u_i(s, t) = \sum_{k=1}^{\infty} U_i(s)e^{j\omega_k t}
\]

\[
y_2i(s, t) = \sum_{k=1}^{\infty} Y_i(s)e^{j\omega_k t}
\]

\[
z_2i(s, t) = \sum_{k=1}^{\infty} Z_i(s)e^{j\omega_k t}
\]

\[
\tilde{h}(t) = \sum_{k=1}^{\infty} \nu_k e^{j\omega_k t} \tag{6}
\]

where \( j = \sqrt{-1} \), \( \omega_k \) is a natural frequency, and \( U_i(s) \), \( Y_i(s) \), and \( Z_i(s) \) are corresponding mode shapes along the longitudinal, vertical, and horizontal directions, respectively.

Substituting Eq. (6) into Eqs. (4) and (5) yields the following ordinary differential equations for a particular frequency (mode) \( \omega_k \) where the superscripts of \( U_i(s) \), \( Y_i(s) \), and \( Z_i(s) \) and \( \nu^a \) and the subscripts of \( \omega_i \) and \( \beta_i \) have been omitted for brevity.

\[
\frac{d^2 Y_i(s)}{ds^2} + \beta^2 Y_i(s) = \left( \frac{q_c}{H^2} \right) \nu
\]

\[
Y_i(0) = Y_i(L) = 0
\]

\[
Y_i(s) - Y_{i-1}(s) = 0
\]

\[
\frac{dY_i(s)}{ds} - \frac{dY_{i-1}(s)}{ds} = -\beta \left( \frac{m}{H} \right) Y_i(s) - \left( \frac{P_i}{H^2} \right) \nu \tag{7}
\]

and

\[
\frac{\nu}{AE} \left( \frac{ds}{dx} \right)^2 = \frac{dU_i(s)}{ds} - E_{\nu} \frac{dy_i(s)}{ds} - E_{\nu} \frac{dz_i(s)}{ds} \tag{8}
\]

where \( \beta = \omega_k/(\sqrt{\nu/\omega_k} \), and a similar equation like (7) can be obtained for \( Z_i(s) \). The solution to Eq. (7) can be described by

\[
Y_i(s) = C_{\nu} \cos \beta(s - s_i) + S_{\nu} \sin \beta(s - s_i) - C_{\nu} \tag{9}
\]

where \( C_{\nu} = -q_i/\beta^3 H^2 \nu \) is a particular solution, derived from the boundary condition \( Y_i(0) = 0 \) by substituting Eq. (9) into the first equation of (7). Then the third and fourth equations of (7) lead to

\[
\begin{bmatrix}
C_{\nu} \\
S_{\nu}
\end{bmatrix} = \begin{bmatrix}
D_0 & C_{\nu} \\
S_{\nu} & C_{\nu}
\end{bmatrix} + \gamma \begin{bmatrix}
C_{\nu} \\
S_{\nu}
\end{bmatrix} \tag{10}
\]

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where
\[
[D^0] = [D_{-1}][D^0_{-1}] \text{ with } [D^0_0] = I,
\]
\[
[D_{-1}] = \begin{bmatrix}
\cos \beta_l & \sin \beta_l \\
-\sin \beta_l - \beta(m_l/m_l) \cos \beta_l & \cos \beta_l - \beta(m_l/m_l) \sin \beta_l 
\end{bmatrix}
\]  
(11)

and \( l_i = s_i - s_{i-1} \), and \( I \) is a \( 2 \times 2 \) identity matrix. Furthermore, vector \( \gamma_t \) is given by the iterative formula
\[
\gamma_t = [D_{-1}] \gamma_{t-1} + \begin{bmatrix} 0 \\ \beta(m_l/m_l) q_l + p_l \end{bmatrix}
\]
with \( \gamma_0 = 0 \).  
(12)

Now substituting Eqs. (9)-(12) into the boundary condition
\[
Y_n(L) = 0
\]
yields
\[
Q DS_{\gamma_0} = Q N_i(C_{\gamma_0}/q_l)
\]
\[
Q D = D_{\gamma_0}(1, 2) \cos \beta(L - s_i) + D_{\gamma_0}(2, 2) \sin \beta(L - s_i)
\]
\[
Q N_i = q_i - [q_i D_{\gamma_0}(1, 1) + \gamma_{n1}] \cos \beta(L - s_i) - [q_i D_{\gamma_0}(2, 1) + \gamma_{n2}] \sin \beta(L - s_i)
\]
(13)

where, \( \gamma_{n1} \) and \( \gamma_{n2} \) are components of the vector \( \gamma_n \), and \( D_{\gamma_0}(i,j) \) are elements of the matrix \([D^0]\). Similarly, the following results can be derived for the horizontal motion:
\[
Z_i(s) = C_{z_i} \cos \beta(s - s_i) + S_{z_i} \sin \beta(s - s_i) - C_{z_0}
\]
\[
Z_i(s) = C_{z_i} = q_i - [q_i D_{z_0}(1, 1) + \zeta_{i1}] \cos \beta(L - s_i) - [q_i D_{z_0}(2, 1) + \zeta_{i2}] \sin \beta(L - s_i)
\]
\[
\zeta_i = [D_{-1}] \zeta_{i-1} + \begin{bmatrix} 0 \\ \beta(m_l/m_l) q_i \end{bmatrix}
\]
with \( \zeta_0 = 0 \).  
(15)

It can be seen now from Eqs. (9)-(15) that all the variables (or solutions) are expressed in terms of \( \beta \) and \( \nu \) (or \( C_{\gamma_0} \)). However, the normalized frequency \( \beta (= \nu m_l/H \omega) \) can be found without knowing \( \nu \) (or \( C_{\gamma_0} \)). To achieve this, integrating Eq. (8) from 0 to \( L \), with the aid of Eqs. (9), (10), and (14), and the boundary conditions \( u(0) = u(L) = 0 \), yields an equation of two variables \( \beta \) and \( \nu(C_{\gamma_0}) \):
\[
0 = (BL) \left[ \left( \frac{\beta H}{q} \right)^2 - 1 \right] \sum_{i=0}^{n} SC(\beta) (q_i \gamma_i + q_{i1}) - \left( \frac{\beta H^2}{q} \right) \sum_{i=0}^{n} SC(\beta) D_{\gamma_0} \left[ q_i \begin{bmatrix} C_{\gamma_0} \\ S_{\gamma_0} \end{bmatrix} + q_i \begin{bmatrix} C_{\gamma_0} \\ S_{\gamma_0} \end{bmatrix} \right] + \beta \left( \frac{\beta H^2}{q} \right) \sum_{i=0}^{n} q_i \left( D_{\gamma_0}(1, 1) C_{\gamma_0} + D_{\gamma_0}(1, 2) S_{\gamma_0} - \left( 1 - \frac{\beta H}{q} \right) C_{\gamma_0} \right)
\]
(16)

where \( SC(\beta) = (\sin \beta l_{i+1} - \cos \beta l_{i+1}) \) is a row vector, and \( \lambda^2 \) is defined as
\[
\lambda^2 = \left( \frac{AE}{L_n^2} \right) \left( \frac{L}{H} \right)^2 q^2 \text{ where } L_n = \int_0^L \left( \frac{dx}{dt} \right)^2 \text{ dx.}
\]
(17)

The \( \lambda^2 \) is identical to the approximation of the static profile (Yu et al., 1995) if \( L \) is replaced by \( L_n \), and then \( L_n = \int_0^L \left( \frac{dx}{dt} \right)^2 \text{ dx} \), defined by Irvine (1981).

Two independent sets of free vibration solutions can now be derived as follows. First let \( Q D = 0 \), it leads to an equation of only one variable \( \beta \) (see Eqs. (11) and (13)), which can be used to solve for the unknown \( \beta \) by applying a simple numerical iteration scheme such as bisection method. Now that \( Q D = 0 \), it is immediately to observe from Eqs. (13) and (15) that \( C_{\gamma_0} = C_{\gamma_0} = \nu = 0 \); i.e., \( \tau = 0 \), while \( S_{\gamma_0} \) and \( S_{\gamma_0} \) are remained undetermined. Thus, Eq. (16) is reduced to
\[
A S_{\gamma_0} + B S_{\gamma_0} = 0
\]
(18)

where
\[
\left\{ \begin{array}{c}
A \\
B
\end{array} \right\} = \left\{ \begin{array}{c}
-\beta \sum_{i=0}^{n} D_{\gamma_0}(1, 2) P_i \\
0
\end{array} \right\}
\]
(19)

From Eq. (18), two cases should be considered:

(a) \( A = B = 0 \). This case implies that \( S_{\gamma_0} \) and \( S_{\gamma_0} \) can be chosen independently and therefore, only one independent frequency exists but it associates with two independent mode shapes. This particular frequency may be called repeated frequency. The corresponding mode shapes can be determined from Eqs. (9), (10), and (14) as
\[
Y_i(s) = S_{\gamma_0} f_i(s) \quad \text{and} \quad Z_i(s) = S_{\gamma_0} f_i(s)
\]
(20)

where
\[
f_i(s) = D_{\gamma_0}(1, 2) \cos \beta(s - s_i) + D_{\gamma_0}(2, 2) \sin \beta(s - s_i)
\]
(21)

Now, back to Eq. (6), it can be found that the vertical and horizontal motions of a particular mode are given by
\[
\begin{bmatrix} y_2(s, t) \\ z_2(s, t) \end{bmatrix} = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}
\]
(22)

where
\[
q_1(t) = S_{\gamma_0} e^{j \omega t} \quad \text{and} \quad q_2(t) = S_{\gamma_0} e^{j \omega t}
\]
(23)

which represent oscillations determined from initial conditions. This case \( \tau = 0 \) may be called antisymmetric case by generalizing the terminology used for cable analysis without concentrated loads (e.g., see Irvine, 1981). The longitudinal mode shape \( U_i(s) \) and so free vibration solution \( u_i(s, t) \) will be given later together with other cases.

(b) \( \max (|A|, |B|) \neq 0 \); i.e., at least one of \( A \) and \( B \) is nonzero. Then Eq. (18) implies that \( S_{\gamma_0} \) and \( S_{\gamma_0} \) are dependent and therefore, only one independent mode shape corresponding to the frequency exists. The final solutions of \( y_2(s, t) \) and \( z_2(s, t) \) are given by
\[
\begin{align*}
\begin{bmatrix} y_2(s, t) \\ z_2(s, t) \end{bmatrix} &= \begin{bmatrix} -Bf_i(s) \\ Af_i(s) \end{bmatrix} q_i(t) = w_1(s)q_i(t) 
\end{align*}
\] (24)

where \( i \) runs through 0 to \( n \), and \( A \) and \( B \) are described by Eq. (19); and

\[
q_i(t) = \begin{cases} 
S_{\rho_0 e^{i\omega_i 0 \tau}} & \text{if } A \neq 0, \\
S_{\rho_0 e^{i\omega_i 0 \tau}} & \text{if } A = 0 \text{ but } B \neq 0.
\end{cases}
\] (25)

Here, \( \omega_i = \beta^{(1)} L H/m \) and \( \beta^{(1)} \) is solved from equation \( QD = 0 \). The superscript \( (1) \) is used to distinguish this frequency from the second independent frequency to be found next.

For case (b), a second independent frequency and its associated mode shape must be found. They can be obtained by assuming \( QD \neq 0 \) that requires \( \nu \neq 0 \) and so \( \tau \neq 0 \), otherwise, \( C_{\rho_0} = C_{\rho_0} = S_{\rho_0} = \varphi_0 = 0 \) which yields only the zero (trivial) solution. Therefore, Eqs. (9), (10), (13), (14), and (15) produce

\[
\nu = -\left( \beta^{(2)} \right)^2 \left( \begin{array}{c} C_{\rho_0} \\ q_i \end{array} \right)
\]

\[
C_{\rho_0} = q_i \left( \begin{array}{c} C_{\rho_0} \\ q_i \end{array} \right)
\]

\[
S_{\rho_0} = \left( \begin{array}{c} QN_i \\ QD \end{array} \right) \left( \begin{array}{c} C_{\rho_0} \\ q_i \end{array} \right)
\]

\[
S_{\varphi_0} = \left( \begin{array}{c} QN_i \\ QD \end{array} \right) \left( \begin{array}{c} C_{\rho_0} \\ q_i \end{array} \right)
\] (26)

and

\[
\begin{align*}
\begin{bmatrix} Y_i(s) \\ Z_i(s) \end{bmatrix} &= \begin{bmatrix} f_i(s) \\ f_i(s) \end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
&= \begin{bmatrix} C_{\rho_0} \cos \beta^{(2)}(s - s_i) + S_{\rho_0} \sin \beta^{(2)}(s - s_i) - C_{\rho_0} \\ C_{\rho_0} \cos \beta^{(2)}(s - s_i) + S_{\rho_0} \sin \beta^{(2)}(s - s_i) - C_{\rho_0} \end{bmatrix} \\
&\quad \times \begin{bmatrix} D_0 \{ q_i, QN_i/QD \} + \gamma_i \{ C_{\rho_0} \\ q_i \} \\ D_0 \{ q_i, QN_i/QD \} + \xi_i \{ C_{\rho_0} \\ q_i \} \end{bmatrix} \\
&\quad \times \sum_{i=0}^n P_i \{ q_i, QD \} - D_0(1, 2)QN_i \\
&\quad \times \sum_{i=0}^n SC(\beta^{(2)}) \left\{ D_0 \{ q_i, QN_i + q_i QN_i \} / q_i \right\} \\
&\quad + \frac{1}{q_i} \{ q_i, \gamma_i + q_i \xi_i \} QD
\end{align*}
\] (27)

where \( i \) runs through 0 to \( n \). Next, substituting Eqs. (26) and (27) into Eq. (16) results in an equation of one variable \( \beta^{(2)} \):

\[
0 = \left( \beta^{(2)} \right) L \left[ \frac{\left( \beta^{(2)} \right)^3}{\lambda^2} - 1 \right] QD + \left( \beta^{(2)} \right)^2 \] (28)

which can be used to find the second independent frequency, \( \beta^{(2)} = \omega^{(2)} \sqrt{m/l} \), by using a numerical iteration scheme. The corresponding mode shapes are given by Eq. (27).

Consider a special case: \( n = 1 \) and \( q_i = 0 \), that is, only one vertical concentrated load is considered and static loads are neglected, then Eq. (28) reduces to

\[
0 = 2 \sin \left( \frac{\beta L}{2} \right) \left\{ \left[ \frac{1}{\lambda^2} \left( \beta L \right)^3 - \beta L \right] \right\}
\]

\[
\times \cos \left( \frac{\beta L}{2} \right) + 2 \sin \left( \frac{\beta L}{2} \right) \}
\]

\[
- \beta \left( \frac{F_1}{q_f} \right) \left[ \frac{1}{\lambda^2} \left( \beta L \right)^3 - \beta L \right] \sin (\beta x_i) \sin (L_0 - x_i)
\]

\[
+ \beta \left( \frac{F_1}{q_f} \right)^2 \sin (\beta x_i) - \sin (L_0 - x_i)
\] (29)

where the superscript \( (2) \) on \( \beta \) was dropped. If \( \Omega = \beta L, \Omega_i = \Delta \beta, \Omega_2 = \Omega(1 - d) \) and \( m = F_1/q_f L \), then, Eq. (29) can be rewritten as

\[
0 = 2 \sin \left( \frac{\Omega}{2} \right) \left\{ \left[ \frac{\Omega^3}{\lambda^2} - \Omega \right] \cos \left( \frac{\Omega}{2} \right) + 2 \sin \left( \frac{\Omega}{2} \right) \right\}
\]

\[
- \omega \Omega \left[ \frac{\Omega^3}{\lambda^2} - \Omega \right] \sin \Omega_1 \sin \Omega_2 + m \Omega^2 \sin \Omega_1 \sin \Omega_2
\] (30)

which is identical to Eq. (38) given by Cheng and Perkins (1992a).

Now, back to Eq. (6), the second independent solution of \( y_{2i}(s, t) \) and \( z_{2i}(s, t) \) can be written as

\[
\begin{align*}
\begin{bmatrix} y_{2i}(s, t) \\ z_{2i}(s, t) \end{bmatrix} &= w_2(s)q_2(t) \\
\end{align*}
\] (31)

in which \( i = 0, 1, 2, \ldots, n \), and

\[
\begin{align*}
\begin{bmatrix} w_2(s) \\ q_2(t) \end{bmatrix} &= \begin{bmatrix} f_i(s) \\ f_i(s) \end{bmatrix} \quad \text{and} \quad q_2(t) = \begin{bmatrix} C_{\rho_0} \\ q_i \end{bmatrix} e^{j\omega^{(2)} t}
\end{align*}
\] (32)

where \( f_i(s) \) and \( f_i(s) \) are given in Eq. (27). Finally, combine Eqs. (24) and (31) to yield a complete solution of \( y_{2i}(s, t) \) and \( z_{2i}(s, t) \) for case (b):

\[
\begin{align*}
\begin{bmatrix} y_{2i}(s, t) \\ z_{2i}(s, t) \end{bmatrix} &= \begin{bmatrix} -Bf_i(s) \\ Af_i(s) \end{bmatrix} \begin{bmatrix} f_i(s) \\ f_i(s) \end{bmatrix} \begin{bmatrix} q_i(t) \\ q_i(t) \end{bmatrix}
\end{align*}
\] (33)

where \( i \) runs from 0 to \( n \). This equation is similar to Eq. (22) of the antisymmetric case. This case associated with two independent frequencies may be, therefore, similarly called symmetric case.

From the above analysis, it has been found that there always exist two independent mode shapes associated with either one or two independent frequencies. To find a complete solution including all the three (longitudinal, vertical, and horizontal) direction motions, it is needed to find the approximation of mode shapes \( U_i(s) \). They can be obtained by integrating Eq. (8) with the aid of boundary conditions as well as the continuity conditions at the locations of detuning pendulums, given by

\[
\begin{align*}
u_0(0) &= u_e(L) = 0, \\
u_e(s_i) &= u(s_i)
\end{align*}
\] (34)
(a) Antisymmetric case (\( \tau = 0 \), repeated frequencies):

\[
U_{1i}(s) = \eta_i + E_n Y_i(s) - \left( \frac{q_i}{\beta H} \right) \left[ C_n \sin \beta (s - s_i) - S_n \cos \beta (s - s_i) \right]
\]

\[
\eta_0 = - \left( \frac{q_i}{\beta H} \right) S_0
\]

\[
\eta_i = \eta_{i-1} - \left( \frac{q_i}{\beta H} \right) (S_n + C_{n-1} \sin \beta l_i)
\]

\[
- S_{n-1} \cos \beta l_i + \left( \frac{P_i}{H} \right) C_0.
\] (35)

Replacing \( y \) by \( z \) and setting \( P_i = 0 \) in Eq. (35) results in \( U_{1i}(s) \).

(b) Symmetric case (\( \tau \neq 0 \), different frequencies):

(i) \( A = 0, B \neq 0 \), \( U_{1i}(s) \) is given by Eq. (35);

(ii) \( A \neq 0, B = 0 \), \( U_{1i}(s) \) equals \( U_{2i}(s) \) of Case (a);

(iii) \( A = 0, B = 0 \),

\[
U_{1i}(s) = \eta_i + (AE_n - BE_n) f_i(s) - \left( \frac{1}{\beta^{(1)} H} \right) (Aq_i - Bq_i)
\]

\[
\times \left[ C_n \sin \beta^{(1)} (s - s_i) - S_n \cos \beta^{(1)} (s - s_i) \right]
\]

\[
\eta_0 = - \left( \frac{1}{\beta^{(1)} H} \right) (Aq_i - Bq_i) S_0
\]

\[
\eta_i = \eta_{i-1} + A \left( \frac{P_i}{H} \right) C_n - \left( \frac{1}{\beta^{(1)} H} \right) (Aq_i - Bq_i)
\]

\[
\times (S_n + C_{n-1} \sin \beta^{(1)} l_i - S_{n-1} \cos \beta^{(1)} l_i)
\] (36)

and

\[
U_{2i}(s) = \nu_i - \left( \frac{q_i}{\beta^{(2)} H} \right) \left( \frac{C_0}{q_i} \right) \left[ 1 + \left( \frac{q_i C_0}{q_H} \right)^2 \right] s + \frac{1}{3} \left( \frac{q_i}{H} \right)^2 (s - \alpha_i)^3 + E_n f_i(s) + E_{2i} f_i(s)
\]

\[
- \left( \frac{q_i}{\beta^{(2)} H} \right) [C_n \sin \beta^{(2)} (s - s_i)
\]

\[
- S_n \cos \beta^{(2)} (s - s_i) - \beta^{(2)} C_{0i} \sin \beta^{(2)} (s - s_i)]
\]

\[
\nu_0 = - \left( \frac{q_i}{\beta^{(2)} q} \right)^2 \left( \frac{3AE}{\mu} \alpha_i \right) - \left( \frac{1}{\beta^{(2)} H} \right) (q_i S_0 + q H S_0)
\]

\[
\nu_i = \nu_{i-1} + \left( \frac{P_i}{H} \right) (C_n - C_{n-1})
\]

\[
+ \left( \frac{q_i}{\beta^{(2)} H} \right) \left( \frac{q_i}{q H} \right)^2 (c_i^2 - c_{i-1}^2) s_i
\]

\[
+ \frac{1}{3} \left( \frac{q_i}{H} \right)^2 ((s_i - \alpha_i)^3 - (s_i - \alpha_{i-1})^3)
\] (37)

4 An Example

In this section, the methodology and explicit formulations described in the previous sections are applied to consider an example chosen from galloping analysis of transmission lines. Low-degree-of-freedom models for galloping analysis are derived from the variational principle for given natural frequencies and vibration mode shapes. A three-degree-of-freedom model (Yu et al., 1993), described in the general form of

\[
Mq + Cq + Kq = F,
\] (38)

has been successfully applied to transmission lines without control devices. Here, \( q \) represents the generalized coordinates corresponding to the vertical, horizontal and rotational movements of the transmission line. \( M, C, \) and \( K \) are mass, damping, and stiffness matrices, respectively. Solving the nonlinear algebraic equation \( Kq = F \) yields equilibrium solutions, and their stability conditions are determined from the linearized system of (38). If an equilibrium becomes unstable for certain system parameters, then periodic, quasi-periodic, or even more complicated dynamic motions may be initiated from the equilibrium. Then one needs to find the dynamical solutions and their stability conditions. A number of dynamic case studies can be found in Yu et al. (1993).

In order to study galloping control for transmission lines equipped with detuning pendulums (DP), a software package based on the method and formulations described in the previous sections has been developed. It has been successfully applied to study galloping control design using DP for transmission lines (Modelling, 1993) and subtransmission lines (Galloping, 1994). Some results of free vibration analysis will be presented here. The transmission line considered is the Pennsylvania Power and Light’s single conductor line, S51. Galloping on this line was reported and the control device DPs was tested on this line by Ontario Hydro. Based on many years of field trials on this line and other lines, Ontario Hydro recommended a distribution for locating DPs along a span of a line for best controlling galloping. For example, if three DPs are used in one span of a line, then they should be located at \( L_x/4, 5L_x/12, \) and \( 2L_x/3 \). Accurate frequencies up to hundreds Hz and the associated mode shapes for such a span can be obtained virtually instantly through the software.

To demonstrate the free-vibration analysis, consider an important question which is often raised in galloping analysis. Since overhead transmission lines usually experience at least two types of vibrations: aeolian and galloping. Galloping is a low-frequency (0.1 to 3 Hz) large-amplitude oscillation, while aeolian vibration is a high-frequency (10 to 100 Hz) small-amplitude motion. Aeolian vibration may occur on a line almost everyday and cause the fatigue of conductors and suspensions. In practice, dampers are installed near suspensions (i.e., near the end supports) to reduce the aeolian vibration. It is interesting to note that an antigalloping control device reduces the vibrations near the midspans while an anti-aeolian vibration control device decreases the vibration amplitudes near the end of a span. So, is a galloping control device such as detuning pendulums also beneficial in controlling aeolian vibration? To help understand this question, let’s consider the following energy balance equation for the analysis of aeolian vibration (Kraus and Hagedorn, 1990):

\[
P_w = P_D + P_C
\] (39)
where $P_w$ is the power of the aerodynamic forces, $P_d$, the power dissipated in the damper, and $P_c$, the power dissipated in the conductor. Here, steady-state aeolian vibration solutions are assumed. The power imparted to the conductor by the aerodynamic forces is estimated on the basis of wind tunnel experiments. The power dissipation in conductor can be expressed as a function of the frequency and the amplitude of the vibration. Then the energy balance Eq. (39) can be used to predict approximately the maximum vibration amplitude or bending strain as a function of the vibration frequency. This shows the importance of finding accurate natural frequencies and vibration mode shapes for the study of aeolian vibration. Now consider again the line S51 with three DPs. The results shown in Figs. 4 and 5 give a comparison of 41st mode shapes (frequency 20.07 Hz) with the 42nd mode shapes (frequency 20.15 Hz). Each of these two figures shows the movement components along the longitudinal, vertical, and horizontal directions. Each direction consists of two independent mode shapes: one of them is the primary vertical (in-plane) mode and the other is the primary horizontal (out-plane) mode, denoted by dotted line and solid line, respectively.

First, it has been noted that in both cases $A \neq 0$ and $B \neq 0$, indicating both cases being symmetric cases (see Eq. (18)). Secondly, it is noted from these two figures that variation of the positions of detuning pendulums greatly changes the mode shapes, implying that it is possible to achieve better control of galloping by adjusting the positions of detuning pendulums. However, it can be seen that the galloping motion is controlled better in some parts of the span than other parts. (In the case of no control devices, the Y-mode and Z-mode shapes are just normal sine waves.) For the 41st mode, the amplitude of the motion near the left end support is much larger than that of other parts; while for the 42nd mode, the amplitudes of the motion are smaller near both end supports. In particular, the 41st mode changes its shapes sharply at the three positions of detuning pendulums while the 42nd mode changes its shapes significantly at the left and right locations of the detuning pendulums. For the U-mode shapes, the same trend becomes clear if one compares Fig. 4(a) and Fig. 5(a) with Fig. 6 which shows the 41st and 42nd U-mode shapes of the transmission line without detuning pendulums. It is seen from Fig. 6 that the 41st mode is antisymmetric and the 42nd mode is symmetric, but both the amplitudes increase towards the end supports. This explains why the U-mode shapes shown in Fig. 4(a) and Fig. 5(a) have a small portion of relatively large motion around the position of 75 m of the span. The same trend can be found in high-frequency motions: mode shapes change significantly from a frequency to an adjacent one.

The above observations seem to indicate that for this specific arrangement of DPs designed by Ontario Hydro, galloping control device (DP) is also beneficial for controlling the aeolian vibration for certain frequencies and mode shapes. How to find the "best" design for the distribution of DPs, which is not only good for controlling the galloping but also beneficial to controlling the aeolian vibration, needs further studies.

5 Concluding Remarks

An analytic methodology has been developed to find closed-form solutions of the natural frequencies and vibration mode shapes of an elastic cable under both distributed and concentrated loads. This method leads to solving for an equation of one unknown variable only, and therefore, avoids computing large dimensional matrix form of equations by using some other approaches such as finite element method. It has been shown that there always exist two sets of independent mode shapes associated with two sets of independent frequencies. This approach is particularly useful in dealing with high-frequency motions. However, for high-frequency modes, in order to obtain more accurate results, the local flexural boundary layers or the
longitudinal inertia should not be neglected. That is, Eq. (3) needs to be modified such that $h(t)$ is also function of $s$, $h(s,t)$. It is also possible to extend this method to consider multiple spans of cables as well as inclined cables under both distributed and concentrated loads.

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References