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Bifurcation of Multiple Limit Cycles in an Epidemic Model on Adaptive Networks

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Very recently, Zhang *et al.* considered an epidemic model on adaptive networks [Zhang *et al.*, 2019], in which Hopf bifurcation, homoclinic bifurcation and Bogdanov–Takens bifurcation are studied. Degenerate Hopf bifurcation is investigated via simulation and a numerical example is given to show the existence of two limit cycles. However, whether the codimension of the Hopf bifurcation is two is still open. In this paper, we will rigorously prove that the codimension of the Hopf bifurcation is two. That is, the maximal two limit cycles can bifurcate from the Hopf critical point. Moreover, the conditions for the existence of two limit cycles are derived.

Keywords: Epidemic model; adaptive network; Hopf bifurcation; limit cycle; normal form.

1. Introduction

In recent years, complex social network structure has been introduced into epidemic modeling to address that the assumption of random and homogeneous mixing population may be too idealized, for example, see [Keeling & Eames, 2005; Kuperman & Abramson, 2001; Newman, 2002, 2003; Strogatz, 2001]. In particular, to better understand the effects of complex network topologies on disease transmission, Pastor-Satorras and Vespignani [2001a, 2001b] established the first SIS model, using a heterogeneous mean-field approach, to study the epidemic spreading. Very recently, using a five-dimensional network-based pairwise epidemic model proposed by Keeling and Eames [2005], Zhang *et al.* [2019] obtained a simplified three-dimensional system by taking the average degree of the network and assuming that there are no self-loops and multiple connections between nodes in a rewiring process. This simple three-dimensional model is described by

$$\begin{aligned} \frac{di}{d\tilde{t}} &= -\gamma i + \frac{\tau\bar{k}}{2}P_{\mathrm{SI}}, \\ \frac{dP_{\mathrm{SI}}}{d\tilde{t}} &= \frac{\tau\bar{k}}{2(1-i)}P_{\mathrm{SI}}(2P_{\mathrm{SS}} - P_{\mathrm{SI}}) \\ &\quad - (\tau + 3\gamma + w)P_{\mathrm{SI}} - 2\gamma P_{\mathrm{SS}} + 2\gamma, \\ \frac{P_{\mathrm{SS}}}{d\tilde{t}} &= P_{\mathrm{SI}}\left(\gamma + w - \tau\bar{k}\frac{P_{\mathrm{SS}}}{1-i}\right), \end{aligned} \tag{1}$$

where i represents the population of infectious individuals, P_{SS} denotes the population counting local spatial connections between the susceptible individuals, while P_{SI} denotes the population counting local spatial connections between the susceptible

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and infectious individuals. τ is the rate per unit of time for infectious individuals to infect their susceptible neighbors, and γ is the recovery rate of the infectious individuals. w is the rate with which a susceptible individual can disconnect the link with an infectious individual, and $\bar{k}$ denotes the average degree of the network.

To further simplify the model, let

$$\begin{aligned} &i = 1 - \frac{1}{x}, \quad P_{\mathrm{SI}} = y, \quad P_{\mathrm{SS}} = z, \quad t = \gamma\tilde{t}, \\ &A = \frac{\tau}{\gamma} + 2, \quad B = \frac{\tau\bar{k}}{2\gamma}, \quad E = \frac{w}{\gamma} + 1. \end{aligned} \tag{2}$$

Then we obtain

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + Bxy), \\ \frac{dy}{dt} &= 2(1 - z) - (A + E)y + Bxy(2z - y), \\ \frac{dz}{dt} &= y(E - 2Bxz). \end{aligned} \tag{3}$$

Since $\bar{k} > 1$, we have $A < 2(1+B)$. For convenience, define the parameter space as

$$\Gamma = \{(A, B, E) \,|\, 2 < A < 2(1 + B), \; B > 0, E > 1\}. \tag{4}$$

It is easy to verify that

$$\Omega = \{(x, y, z) \in R^3_+ \,|\, x \geq 1, y \geq 0, \; z \geq 0, 0 \leq y + z \leq 1\} \tag{5}$$

is a positive invariant set of system (3). Therefore, we consider the equilibria of system (3) in Ω below.

System (1) has two equilibrium solutions: disease-free equilibrium E_0 and disease equilibrium E_1, given by

$$\begin{aligned} &\mathrm{E}_0: \quad (x_0, y_0, z_0) = (1, 0, 1), \\ &\mathrm{E}_1: \quad (x_1, y_1, z_1) = \left(x_1, \frac{x_1 - 1}{Bx_1}, \frac{E}{2Bx_1}\right), \quad x_1 \geq 1, \end{aligned} \tag{6}$$

where x_1 is determined from the quadratic polynomial equation:

$$F_1(x_1) = x_1^2 + (A - 2 - 2B)x_1 - A + E + 1, \tag{7}$$

given in the form of

$$\begin{aligned} x_{1\pm} &= \frac{1}{2}(2 + 2B - A \pm \sqrt{\Delta}), \\ \Delta &= (2 + 2B - A)^2 + 4(A - E - 1) \geq 0. \end{aligned} \tag{8}$$

We further define

$$\mathrm{E}_{1\pm} : (x_{1\pm}, y_{1\pm}, z_{1\pm}) = \left(x_{1\pm}, \frac{x_{1\pm} - 1}{Bx_{1\pm}}, \frac{E}{2Bx_{1\pm}}\right). \tag{9}$$

Note that E_0 is a boundary equilibrium and exists for all real parameter values. For the disease equilibrium E_1, it is easy to see that E_1 does not exist if $\Delta < 0$, and two equilibria E_{1+} and E_{1-} exist if $\Delta > 0$, and the two equilibria coincide at $\Delta = 0$. The stability of these equilibria will be discussed in the next section.

2. Stability and Bifurcation of Equilibrium Solutions

The stability of the equilibria is determined by the Jacobian,

$$J = \begin{bmatrix} 2x(By - 1) + 1 & Bx^2 & 0 \\ By(2z - y) & -A - E + 2Bx(z - y) & 2(Bxy - 1) \\ -2Byz & E - 2Bxz & -2Bxy \end{bmatrix}. \tag{10}$$

Evaluating the J on the disease-free equilibrium E_0 yields the characteristic polynomial,

$$P_0(\lambda) = (\lambda + 1)\{\lambda^2 + [A + (E - 2B)]\lambda + 2(E - 2B)\}, \tag{11}$$

indicating that the equilibrium E_0 is stable (unstable) for $E > 2B$ ($E < 2B$). Define the reproduction number as

$$R_0 = \frac{2B}{E}. \tag{12}$$

Then, E_0 is stable (unstable) for $R_0 < 1$ ($R_0 > 1$), as expected.

The characteristic polynomial for the disease equilibrium E_1 can be obtained as

$$P_1(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{13}$$

where

$$\begin{aligned} a_1 &= A + 4x_1 - 3 > 0, \quad \text{for } x_1 \geq 1, \\ a_2 &= A(2x_1 - 1) + (x_1 - 1)(5x_1 - E - 1), \\ a_3 &= 2(x_1 - 1)(x_1^2 + A - E - 1). \end{aligned} \tag{14}$$

The equilibrium E_1 is stable if

$$a_1 > 0, \quad a_3 > 0, \quad \Delta_2 = a_1a_2 - a_3 > 0. \tag{15}$$

In order to show the bifurcation property of system (1), we may choose E as the bifurcation parameter, and treat A and B as control parameters. Thus, with the function $F_1(x_1) = 0$, we can classify the bifurcation diagram into three categories:

(a) $0 < B \leq \frac{1}{2}$;
(b) $B > \frac{1}{2}$, $\max\{2, 2B\} < A < 2(1 + B)$;
(c) $B > 1$, $2 < A < 2B$.

The bifurcation diagrams corresponding to the three cases are shown in Figs. 1(a)–1(c), respectively, where the vertex of the parabola $F_1(x_2) = 0$ is defined as

$$\begin{aligned} S : (x, E) &= (\overline{x}, \overline{E}) \\ &= \left(1 + B - \frac{A}{2}, 2B + \left(B - \frac{A}{2}\right)^2\right) \end{aligned} \tag{16}$$

and the intersection point of the parabola and the line $x = 1$ is given by

$$P : (x, E) = (1, 2B). \tag{17}$$

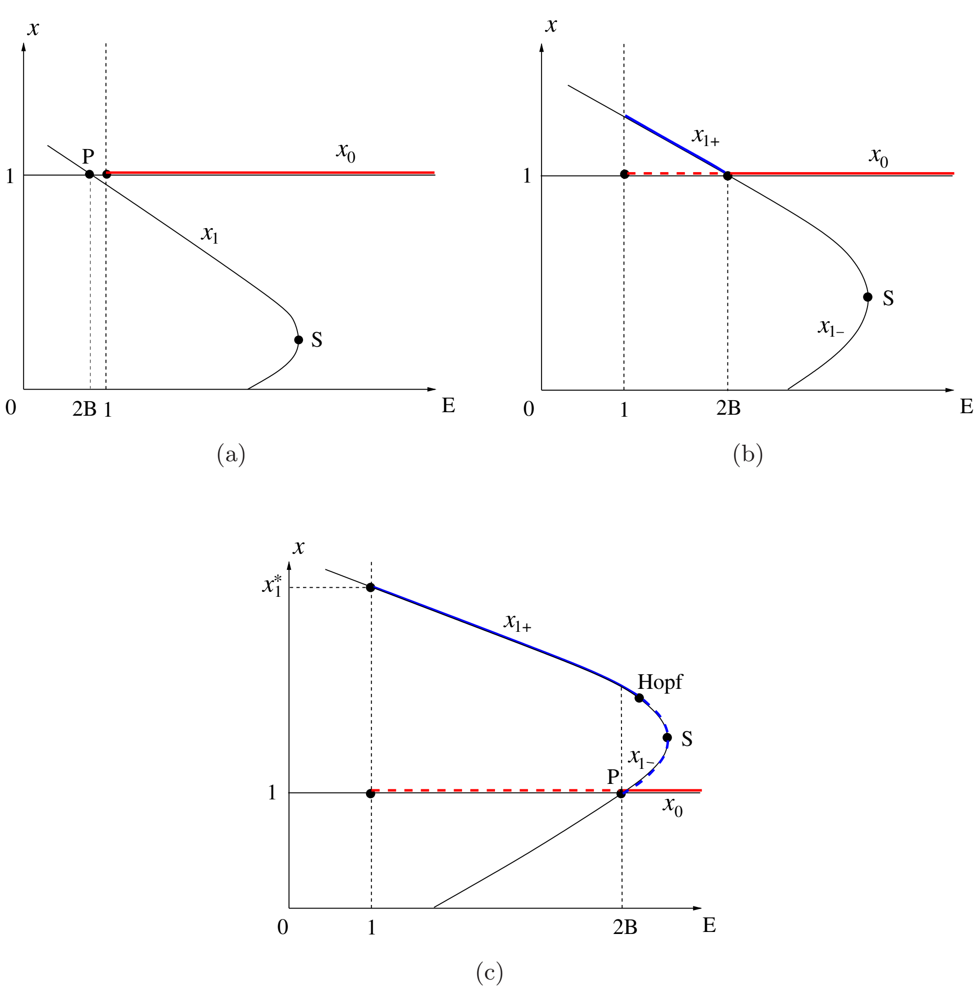


Fig. 1. Bifurcation diagrams for system (1) when (a) $B \leq \frac{1}{2}$, (b) $B > \frac{1}{2}$, $\max\{2, 2B\} < A < 2(1 + B)$ and (c) $B > 1$, $2 < A < 2B$, with stable and unstable equilibrium solutions shown in red and blue colors, respectively.

The x-coordinate of the intersection point of the parabola and the line $E = 1$ is denoted by

$$x_1^* = \frac{1}{2}[2+2B-A+\sqrt{(2+2B-A)^2+4(A-2)}]. \tag{18}$$

In addition, define

$$C_s = (A-2B)^2 + 4(A-4B). \tag{19}$$

We have the following theorem on the stability and bifurcation of the equilibria.

Theorem 1. *For stability of the equilibria* E_0 *and* E_1 *of system (3), there are three cases.*

(a) *When* $B \in (0, \frac{1}{2}]$, E_0 *is globally asymptotically stable while* E_1 *does not exist.*
(b) *When* $B > \frac{1}{2}$ *and* $\max\{2, 2B\} < A < 2(1+B)$, E_0 *is globally asymptotically stable for* $E > 2B$, *for which* E_1 *does not exist; and* E_0 *is unstable for* $E < 2B$ *for which* E_{1+} *is globally asymptotically stable.*
(c) *When* $B > 1$ *and* $2 < A < 2B$, E_0 *is stable (unstable) for* $E > 2B$ $(E < 2B)$. E_{1-} *is unstable, and* E_{1+} *is stable for* $E \in (1, \overline{E})$ *if* $C_s < 0$. *Hopf bifurcation occurs from* E_{1+} *at the critical point* $E_H \in (2B, \overline{E})$ *if* $C_s > 0$.

Proof. First note that the positions of the points P and S classify the bifurcation diagrams into three categories.

(a) If $0 < B \le \frac{1}{2}$, then the point P is located on the left side of the line $E = 1$ [see Fig. 1(a)], and thus only the equilibrium E_0 exists for $E > 1$. It is globally asymptotically stable since there is no interior equilibria.

(b) If $B > \frac{1}{2}$ and $\max\{2, 2B\} < A < 2(1+B)$, then the point P is on the right side of the line $E = 1$, and the point S is below the line $x = 1$ [see Fig. 1(b)]. This is a typical *forward* bifurcation, and it is easy to see from (6) that $E = 2B$ defines a transcritical bifurcation point at which E_0 and E_1 exchange their stability, though E_1 is biologically meaningless for $x_1 < 1$. E_0 is asymptotically stable for $E > 2B$, and is actually globally asymptotically stable since it is a unique equilibrium that exists under the condition. It is unstable when $E < 2B$ for which E_1 emerges asymptotically stable, and its proof is given as follows. The condition $\max\{2, 2B\} < A < 2(1+B)$ implies $A > 2B > E$. Thus, using (13) and (14), and the conditions $x_1 \ge 1$ and $A > 2$, we obtain

$$\begin{aligned} a_3 &= 2(x_1-1)(x_1^2-1+A-E) > 0,\\ a_2 &= (x_1-1)(A-E+5x_1) + (A-1)x_1 + 1 > 0,\\ \Delta_2 &= (A+4x_1-3)[(A-1)x_1+1]\\ &\quad + (x_1-1)^2[3(A-E)+12x_1-2] + (x_1-1)\\ &\quad \times [x_1(A-E+5x_1) + (A-2)(A-E)] > 0, \end{aligned} \tag{20}$$

where x_1 is actually x_{1+}, implying that E_{1+} is asymptotically stable for $1 < E < 2B$, and therefore for this case no Hopf bifurcation can occur from E_{1+}.

(c) If $B > 1$ and $2 < A < 2B$, then the vertex S is above the line $x = 1$, as depicted in Fig. 1(c), showing a typical *backward* bifurcation. Again, like in case (b), the disease-free equilibrium E_0 is asymptotically stable (unstable) for $E > 2B$ $(E < 2B)$. But now E_0 is not globally asymptotically stable since there exist bistable states. The disease equilibrium E_{1-}, for which $1 \le x \le \overline{x}$, is unstable due to $a_3 < 0$. To show this, let $a_3 = 2(x_{1-}-1)F_2$, where $F_2 = x_{1-}^2 + A - E - 1$. Then, F_2 and a_3 have the same sign. A direct calculation shows that

$$\begin{aligned} F_2 &= x_{1-}^2 + A - E - 1\\ &= \frac{1}{2}[\Delta - (2+2B-A)\sqrt{\Delta}]\\ &= -\frac{1}{2}\sqrt{\Delta}[2+2B-A-\sqrt{\Delta}]\\ &= -x_{1-}\sqrt{\Delta}\\ &< 0. \end{aligned}$$

For the disease equilibrium E_{1+}, similarly we can prove that $a_3 = 2x_{1+}(x_{1+}-1)\sqrt{\Delta} > 0$ for $1 < E < \overline{E}$ and $a_3 = 0$ at the vertex S where $\Delta = 0$. Note that x_{1+} runs from the point S to the point $(x_1, E) = (x_1^*, 1)$. At the point $(x_1^*, 1)$, (18) gives $x_1^* > 1 + \sqrt{3}$, and thus we have

$$\begin{aligned} a_2 &= (x_1^*-1)(A+5x_1^*) + (A-2)x_1^* + 2 > 0,\\ \Delta_2 &= x_1^*(18x_1^{*2} - 41x_1^* + 33) + A(13x_1^{*2} - 19x_1^* + 7)\\ &\quad + A^2(2x_1^*-1) - 10 > 0. \end{aligned}$$

This, by the solution continuity on the parameters, implies that E_{1+} is stable at least for $E \in (1, \tilde{E}_1)$ with some $\tilde{E}_1 > 1$. If under certain condition $\tilde{E}$ can extend to $\overline{E}$ (the vertex S), then the whole branch E_{1+} is stable, and no bifurcation occurs from E_{1+}. Let us find the condition. Note that at the vertex S, we have

$$a_3 = 0, \quad a_2 = -\frac{1}{8}(2 + 2B - A)C_s,$$

$$\Delta_2 = (1 + 4B - A)a_2.$$

Therefore, if $C_s < 0$, namely if

$$\max\{2, 2(B - 1 - \sqrt{1 + 2B})\} < A < 2(B - 1 + \sqrt{1 + 2B}), \quad B > 1, \tag{21}$$

which yields $a_2 > 0$ and $\Delta_2 > 0$, then E_{1+} is stable at least for $E \in (\tilde{E}_2, \overline{E})$ with some $\tilde{E}_2 < \overline{E}$. In fact, we can prove that when $C_s < 0$, the whole branch E_{1+} is asymptotically stable, that is, $\tilde{E}_2 = 1$. To prove this, we solve E from $F_1 = 0$ to obtain

$$E = 2B + \left(B - \frac{A}{2}\right)^2 - \left(x_1 - 1 - B + \frac{A}{2}\right)^2.$$

Then, using (20) we obtain

$$\begin{aligned} a_2 &= x_1[x_1^2 + (A - 2B + 2)x_1 + 2B - 3] \\ &\equiv x_1 M_1, \\ \Delta_2 &= x_1[4x_1^3 + (5A - 8B + 1)x_1^2 \\ &\quad + (A^2 - 2AB - 3A + 18B - 10)x_1 \\ &\quad + 2AB - A - 10B + 5] \\ &\equiv x_1 M_2. \end{aligned}$$

Thus, we only need to show that $M_1 > 0$ and $M_2 > 0$ for $x_1 > \overline{x} = 1 + B - \frac{A}{2}$. A direct calculation yields that

$$\begin{aligned} M_1(\overline{x}) &= -\frac{1}{4}C_s > 0, \\ \frac{dM_1}{dx_1} &= 2x_1 + A - 2B + 2 \\ &> 4 > 0, \quad \text{for } x_1 > \overline{x}, \\ M_2(\overline{x}) &= -\frac{1}{4}(4B + 1 - A)C_s > 0, \\ \frac{dM_2}{dx_1} &= A^2 - 2AB + 10Ax_1 - 16Bx_1 \\ &\quad + 12x_1^2 - 3A + 18B + 2x_1 - 10, \\ \frac{dM_2(\overline{x})}{dx_1} &= -C_s + 12B - 2A + 4 \\ &> 0, \quad \text{for } x_1 > \overline{x}, \\ \frac{d^2 M_2}{dx_1^2} &= 10A - 16B + 24x_1 + 2 \\ &> 26 + 8B - 2A > 0, \quad \text{for } x_1 > \overline{x}, \end{aligned}$$

which clearly indicates that $M_1 > 0$ and $M_2 > 0$ for $x_1 > \overline{x}$.

Next, consider the condition $C_s > 0$, i.e. if

$$\begin{cases} A > 2(B - 1 + \sqrt{1 + 2B}), \quad \text{when } B > 1, \\ 2 < A < 2(B - 1 - \sqrt{1 + 2B}), \\ \qquad\qquad \text{when } B > 3 + \sqrt{6}, \end{cases} \tag{22}$$

then Hopf bifurcation occurs from E_{1+}. Note that since $\Delta_2 = a_1 a_2 - a_3$, and $a_1 > 0$ and $a_3 > 0$ for $E \in (1, \overline{E})$, Δ_2 becomes zero before a_2 does, and so the only possible bifurcation from E_{1+} is Hopf bifurcation, arising from the critical point E_H, which is determined from the condition $\Delta_2 = 0$, leading to the following equation:

$$\begin{aligned} F_3(E_H) &= 8E_H^2 - (3A^2 - 14AB + 16B^2 - 27A \\ &\quad + 122B + 34)E_H - 108AB + 60A^2B \\ &\quad - 164AB^2 + 76B - 2A + 11A^2 + 284B^2 \\ &\quad - 7A^3 + 144B^3 + (7A^2 - 46AB + 3AE \\ &\quad + 72B^2 - 8BE - 9A + 70B - 17E + 4) \\ &\quad \times \sqrt{(2B - A)^2 + 4(2B - E)} \\ &= 0, \quad \text{under the condition (22).} \end{aligned} \tag{23}$$

It will be shown in Theorem 2 that the Hopf critical point, denoted by E_H satisfies $E_H \in (2B, \overline{E})$.

This completes the proof of Theorem 1. ■

3. Bifurcation of Multiple Limit Cycles

Next, we want to ask what is the maximal number of limit cycles which can bifurcate in system (3) from the Hopf critical point E_H when the condition (22) holds. To achieve this, we may use the normal form associated with the Hopf bifurcation to find the focus values which can be used to determine the number of limit cycles. A sufficient condition for a dynamical system to have multiple limit cycles bifurcating from a Hopf critical point is given in the following lemma (the detailed proof can be found, for example, in [Han & Yu, 2012]). Without loss of generality, suppose that the amplitude equation of the normal form of a general n-dimensional dynamical system $\dot{x} = f(x, \mu)$ with a Hopf bifurcation is given by

$$\frac{dr}{d\tau} = r[v_0 + v_1 r^2 + \cdots + v_{k-1} r^{2k-2} + v_k r^{2k} + O(r^{2k+2})], \tag{24}$$

where v_i is the ith-order focus value, expressed in terms of the system parameters. If we can find the conditions on k parameters, say, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, such that $v_0 = v_1 = \cdots = v_{k-1} = 0$, but $v_k \neq 0$, at the critical point defined by $\mu_c = (\mu_{1c}, \mu_{2c}, \ldots, \mu_{kc})$ and

$$\text{rank}\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(\mu_1, \mu_2, \ldots, \mu_k)}\right]_{\mu=\mu_c} = k,$$

then k small-amplitude limit cycles can bifurcate from the critical point near the equilibrium by performing appropriate perturbations on μ.

We have the following result.

Theorem 2. *For system (3), when the condition (22) holds, Hopf bifurcation occurs from the equilibrium* E_{1+} *at the critical point* $\mathrm{E}_\mathrm{H} \in (2B, \overline{E})$. *Moreover, the codimension of the Hopf bifurcation is two and so maximal two small-amplitude limit cycles can bifurcate from the Hopf critical point near the equilibrium* E_{1+}. *In fact, there exists an infinite number of solutions* (x_1, A) *such that for each* $A > 2$, *there is a unique solution* x_1 *satisfying* $x_1 > 3 + \sqrt{A+4}$ *for the first Lyapunov constant to vanish but the second Lyapunov constant is positive.*

Proof. To prove this theorem, we use the method of normal forms to compute the focus values. To achieve this, we use the parameter E to solve the equilibrium equation $F_1(x_1) = 0$, and B to solve the Hopf critical condition $\Delta_2 = 0$ to obtain

$$\begin{aligned} B &= \frac{(A+4x_1)(x_1^2-1) + (A^2-4)x_1 + 3Ax_1(x_1-1) + (Ax_1-2)x_1 + x_1^2 + 5}{2(x_1-1)(4x_1+A-5)} > 0, \\ E &= 2B + \left(B - \frac{A}{2}\right)^2 - \left(x_1 - 1 - B + \frac{A}{2}\right)^2 \\ &= \frac{(2x_1+A-1)[A(x_1-1) + (A-2)x_1 + (3x_1-2)^2 + 1]}{(x_1-1)(4x_1+A-5)} = E_H > 0, \end{aligned} \tag{25}$$

for $A > 2$ and $x_1 \geq 1$. Moreover, for the above solutions, by simple calculations we can show that $2 < A < 2B$ and $2B - 1 < E - 1 < 2B - 1 + (B - \frac{A}{2})^2 = \overline{E} - 1$ because

$$\begin{aligned} A < 2B &\Leftrightarrow A^2 + (Ax_1-2)x_1 + 4x_1(x_1^2-1) + 6A(x_1-1) + (x_1-2)^2 + 1 > 0, \\ E > 2B &\Leftrightarrow A^2 + (A-2)x_1 + 7A(x_1-1) + \frac{1}{14}(14x_1-11)^2 + \frac{19}{14} > 0, \\ 2B - 1 &= \frac{x_1[A(A+5x_1-4) + (x_1-1)(4x_1+1)]}{(x_1-1)(4x_1+A-5)} > 0, \\ \overline{E} - E &= \frac{(A+4x_1-3)^2(x_1^2-6x_1+5-A)^2}{4(x_1-1)^2(4x_1+A-5)^2} > 0, \end{aligned}$$

for $A > 2$ and $x_1 \geq 1$. Therefore, any solutions (x_1, A) satisfying $A > 2$ and $x_1 \geq 1$ guarantee $B > 0$ and $E > 1$. Also it is seen from the above conditions that a necessary condition for Hopf bifurcation to occur is that the value of E must be taken from the interval $(2B, \overline{E})$, yielding bistable phenomenon.

At the Hopf critical point E_H, which is now determined from (25) in terms of A and x_1, the system (3) has one negative real eigenvalue and a purely imaginary pair:

$$\begin{aligned} \lambda_1 &= -(A + 4x_1 - 3), \\ \lambda_{2,3} &= \pm i\omega_c, \\ \omega_c &= \sqrt{\frac{2x_1[(x_1-1)(x_1-5) - A]}{A + 4x_1 - 5}}. \end{aligned} \tag{26}$$

The requirement $\omega_c^2 > 0$ for a Hopf bifurcation gives an additional constraint on the parameters, i.e.

$$(x_1 - 1)(x_1 - 5) - A > 0 \;\;\Rightarrow\;\; x_1 > 3 + \sqrt{4 + A}, \tag{27}$$

which implies that when $1 + B - \frac{A}{2} < x_1 \le 3 + \sqrt{4 + A}$, there is no Hopf bifurcation arising from E_{1+}. This condition is equivalent to $C_s < 0$ [see Eq. (21)], as expected.

Now, introducing the affine transformation,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{bmatrix} \dfrac{-T_a}{(2x_1 + A - 1)T_b} & \dfrac{-T_a\omega_c}{2(x_1 - 1)(2x_1 + A - 1)T_b} & \dfrac{T_a}{2(x_1 - 1)T_b} \\ \dfrac{-2T_c}{(2x_1 + A - 1)T_b} & \dfrac{\omega_c(1 - 2x_1)(4x_1 + A - 5)}{(2x_1 + A - 1)T_b} & \dfrac{(4 - 4x_1 - A)(4x_1 + A - 5)}{T_b} \\ 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where

$$\begin{aligned} T_a &= x_1^2[Ax_1(A + x_1) + 3Ax_1(x_1 - 1) \\ &\quad + 4x_1(x_1^2 - 1) + (x_1^2 - 10x_1 + 5 - A)] \\ &> 0, \quad (x_1 > 3 + \sqrt{4 + A}), \\ T_b &= 2Ax_1 + 8x_1(x_1 - 1) + (x_1^2 - 6x_1 + 5 - A) \\ &> 0, \quad (x_1 > 3 + \sqrt{4 + A}), \\ T_c &= -x_1^3 + 2Ax_1 + 10x_1^2 - A - 14x_1 + 5, \end{aligned} \tag{28}$$

into (3) with a time rescaling $\xi = \omega_c t$, we obtain the following dynamical system,

$$\begin{aligned} \frac{du}{d\xi} &= v + \sum_{i+j+k=2}^{5} a_{ijk}u^i v^j w^k, \\ \frac{dv}{d\xi} &= -u + \sum_{i+j+k=2}^{5} b_{ijk}u^i v^j w^k, \\ \frac{dw}{d\xi} &= \frac{\lambda_1}{\omega_c}w + \sum_{i+j+k=2}^{5} c_{ijk}u^i v^j w^k, \end{aligned} \tag{29}$$

whose linear part is in Jordan canonical form, where a_{ijk}, b_{ijk} and c_{ijk} are coefficients, given in terms of x_1 and A. We now apply the Maple program developed in [Yu, 1998] or in [Tian & Yu, 2013, 2014] to system (29) to obtain the focus values:

$$\begin{aligned} v_1 &= \frac{T_a^2\omega_c x_1^3}{16x_1(x_1 - 1)^4(A + 4x_1 - 3)(4x_1 + A - 5)(2x_1 + A - 1)^2(x_1^2 - 6x_1 + 5 - A)^2 T_b T_d T_e} G_1, \\ v_2 &= \frac{T_a^4\omega_c x_1^3}{1152x_1^3(x_1 - 1)^8(A + 4x_1 - 3)^3(4x_1 + A - 5)^3(2x_1 + A - 1)^4(x_1^2 - 6x_1 + 5 - A)^4 T_b^2 T_d^3 T_e^3 T_f} G_2, \end{aligned} \tag{30}$$

where T_a and T_b are given in (28), and T_d, T_e, T_f and G_1 are given by

$$\begin{aligned} T_d &= 2(x_1^2 - 6x_1 + 5 - A)(24A + 104 + 36x_1) + T_g, \\ T_e &= 2(x_1^2 - 6x_1 + 5 - A)(24A + 104 + 33x_1) + T_g, \\ T_f &= 2(x_1^2 - 6x_1 + 5 - A)(24A + 104 + 41x_1) + T_g, \\ T_g &= A(A^2 + 37A + 7) + 12A(A + 22)x_1 + 1084x_1 - 1085 \end{aligned} \tag{31}$$

and

$$\begin{aligned} G_1 = &-8640x_1^{12} - 48(501A - 5399)x_1^{11} - 32(708A^2 - 19901A + 71455)x_1^{10} - 8(1339A^3 - 73369A^2 \\ &+ 644045A - 1251047)x_1^9 - 8(357A^4 - 35800A^3 + 552337A^2 - 2570424A + 3223546)x_1^8 \end{aligned}$$

$$
\begin{aligned}
&- (433A^5 - 83259A^4 + 2003574A^3 - 15856826A^2 + 47669881A - 41992043)x_1^7 \\
&- 2(17A^6 - 7456A^5 + 267851A^4 - 3149846A^3 + 15967525A^2 - 34296314A + 22017263)x_1^6 \\
&- (A^7 - 1601A^6 + 86129A^5 - 1408457A^4 + 10477043A^3 - 38203363A^2 + 62066907A - 28773971)x_1^5 \\
&+ (92A^7 - 7765A^6 + 170928A^5 - 1750781A^4 + 9479036A^3 - 26783159A^2 + 33820792A \\
&- 10019063)x_1^4 + (A-5)(2A^7 - 261A^6 + 6688A^5 - 81451A^4 + 512082A^3 - 1563923A^2 \\
&+ 1835788A - 55613)x_1^3 + (A-5)^2(11A^6 - 303A^5 + 1986A^4 - 574A^3 - 20553A^2 \\
&+ 19405A + 48412)x_1^2 + (A-3)(A-5)^3(A^5 - 37A^4 + 294A^3 - 698A^2 + 641A - 1161)x_1 \\
&- (A-3)^2(A-5)^4(A^3 - 7A^2 + 7A - 9). \qquad (32)
\end{aligned}
$$

The lengthy polynomial G_2 is omitted here for brevity. It is obvious that $T_d > 0$, $T_e > 0$ and $T_f > 0$ for $x_1 > 3 + \sqrt{4+A}$. Hence, v_1 has the same sign of G_1, and v_2 has the same sign of G_2.

The two equations $v_1 = v_2 = 0$, i.e. $G_1 = G_2 = 0$, have two independent coefficients A and x_1, and thus the best result we can have is to find solutions such that $v_1 = v_2 = 0$, but $v_3 \neq 0$, possibly yielding three small-amplitude limit cycles due to Hopf bifurcation. Eliminating A from the two equations $G_1 = G_2 = 0$ yields a resultant equation $\mathrm{R}_{12}(x_1) = 0$, where

$$
\begin{aligned}
\mathrm{R}_{12} = &-112280250439315076893468241361371136x_1^{82}(2x_1-1)^5(3x_1-2)^2(9x_1-8)^2(x_1-1)^{26} \\
&\times (x_1-2)^{74}(x_1^2-x_1+1)(x_1^2-2x_1+2)^{12}(x_1^2-3x_1+3)^2(5x_1^2-10x_1+4)^2(9x_1^2-16x_1+6)^2 \\
&\times (2x_1^2-4x_1+1)^2(x_1^3-3x_1^2+5x_1-2)^2(x_1^4-3x_1^3+7x_1^2-5x_1+1)^2(63x_1^7-334x_1^6+823x_1^5 \\
&- 1135x_1^4+1362x_1^3-1344x_1^2+672x_1-108)^2(16641x_1^{17}-286896x_1^{16}+2298802x_1^{15} \\
&- 11484468x_1^{14}+40401670x_1^{13}-106977720x_1^{12}+221881186x_1^{11}-368588272x_1^{10}+495777864x_1^9 \\
&- 541599904x_1^8+478330840x_1^7-337439096x_1^6+186622432x_1^5-78817632x_1^4+24451200x_1^3 \\
&- 5231232x_1^2+686592x_1-41472)^2\mathrm{R}_{12a},
\end{aligned}
$$

in which R_{12a} is omitted here for simplicity.

The factors in the polynomial R_{12} except for R_{12a} have no real solutions for $x_1 > 3 + \sqrt{4+A} > 3 + \sqrt{6}$. The 114th-degree polynomial R_{12a} gives two real solutions for $x_1 > 3 + \sqrt{6}$: $x_1 = 31.285826\cdots$ and $x_1 = 155.401579\cdots$, but both of them do not satisfy $G_1 = G_2 = 0$. Hence, there do not exist feasible parameter values satisfying $v_1 = v_2 = 0$, and so the bifurcation of three small-amplitude limit cycles is not possible. The next best possibility is to have $v_1 = 0$, but $v_2 \neq 0$, leading to the existence of two small-amplitude limit cycles. Now, since there are two free parameters A and x_1 in the equation $v_1 = 0$, we can have infinitely many solutions for the existence of two limit cycles. However, the stability of these two limit cycles are the same for all feasible parameter values, because the sign of v_2 does not change. Otherwise, we could have solutions for three limit cycles.

To find the solutions for two limit cycles, we plot the curves $G_1 = 0$ and $G_0 = (x_1-1)(x_1-5) - A = 0$ on the x_1–A plane, as shown in Fig. 2, where the red and blue curves represent the curves G_1 and G_0, respectively. It is seen from this figure that for $G_0 > 0$, i.e. $x_1 > 3 + \sqrt{A+4}$, there always exists the solution for $G_1 = 0$ or $v_1 = 0$. More precisely, we can find the minimum value of x_1 at which a vertical tangent line touches the curve $G_1 = 0$, as shown in Fig. 2. This unique minimum value is solved from the two equations $G_1 = \frac{dG_1}{dA} = 0$ as $(x_1, A) = (16.93779566, 11.69311552)$, implying that for each value of $A > 2$, there exists a unique value of x_1 satisfying $G_1 = 0$ (i.e. $v_1 = 0$). Therefore, when $3 + \sqrt{6} \approx 5.449490 < x_1 < 16.937796$ (or for the original variable, $0.816497 < i < 0.940960$), $G_1 > 0$ and so $v_1 > 0$, implying that when $5.449490 < x_1 < 16.937796$, the Hopf bifurcation

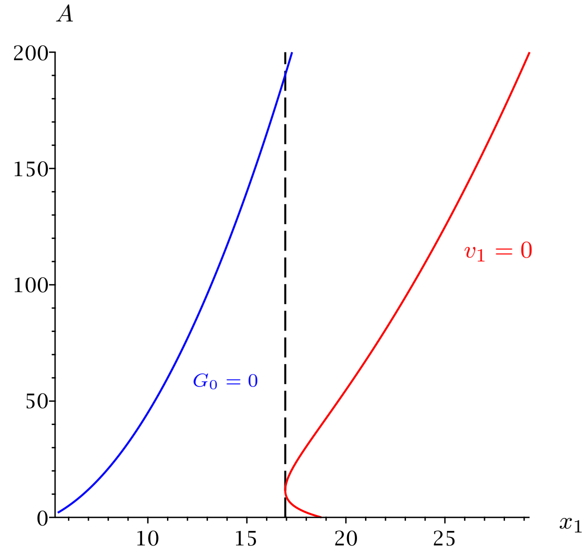


Fig. 2. Graphs of $v_1 = 0$ (i.e. $G_1 = 0$) and $G_0 = 0$, showing that $v_1 = 0$ is always satisfied with a unique solution x_1 for each value of $A > 2$.

is subcritical, and the one bifurcating limit cycle is unstable.

The necessary condition for system (3) to have two limit cycles from the Hopf critical point is $x_1 > 16.93779566\cdots$ (i.e. $0.940960 < i < 1$), for which the equation $G_1 = 0$ always has a unique solution x_1 for each value of $A > 2$. For example, choosing $A = 2.137571\cdots$ we obtain $x_1 = 18$ (or the original variable $i = 0.944444$), and then

$$B = 11.445442\cdots, \quad E = 86.697241\cdots,$$

$$\omega_c = 10.675292\cdots,$$

under which $v_1 = 0$ and $v_2 = 15.301167\cdots > 0$. Then by proper perturbations on the parameters, we can have $0 < v_0 \ll -v_1 \ll v_2$, yielding two limit cycles. Note that the inner limit cycle is stable while the outer limit cycle is unstable, both of them enclose the unstable equilibrium E_{1+}. ■

To end this section, we give an example of perturbations to generate two limit cycles. We take $x_1 = 18$ and perturb A from $A = 2.137571$ to $A = 2.137571 + 0.5 = 2.637571$, and then perturb B using (25) as $B = 11.711908 + 0.00001 = 11.711918$, and then $E = 87.790350$. For these parameter values, we obtain

$$v_0 = 0.00002031, \quad v_1 = -0.16516938,$$

$$v_2 = 172.31402801.$$

Thus, the truncated equation of the normal form is

$$0.00002031 - 0.16516938r^2 + 172.31402801r^4 = 0,$$

which gives the approximations of the two limit cycles as $r_1 \approx 0.012036$ and $r_2 \approx 0.028525$. To check if higher-order focus values affect the number of limit cycles, we compute v_3 and v_4 to obtain $v_3 = 1380.154340$ and $v_4 = 27705.622410$. This shows that higher-order focus values do not change the stability of the limit cycles, since all v_2, v_3 and v_4 have positive sign (in fact, even v_5 and v_6 also have positive signs). Adding v_3 and v_4 to the above equation we have

$$0.00002031 - 0.16516938r^2 + 172.31402801r^4 + 1380.154340r^6 + 27705.622410r^8 = 0,$$

which again yields two positive solutions: $r_1 \approx 0.012037$ and $r_2 \approx 0.028412$, which are almost exactly the same as that obtained above using only the focus values up to v_2. This means that the outer limit cycle must be unstable, and thus if the equilibrium E_{1+} is unstable (a saddle-focus), then there must exist a stable limit cycle inside the unstable limit cycle, restricted to an invariant manifold and enclosing the equilibrium. As a matter of fact, when

$$A = 2.637571, \quad B = 11.711918, \quad E = 87.790350,$$

there are three equilibria E_0, $E_{1\pm}$, and $E_{1\pm}$ as given by

$$E_{1-}: \quad (4.786266, 0.303266, 0.783054),$$

$$E_{1+}: \quad (18.0, 0.080640, 0.208217).$$

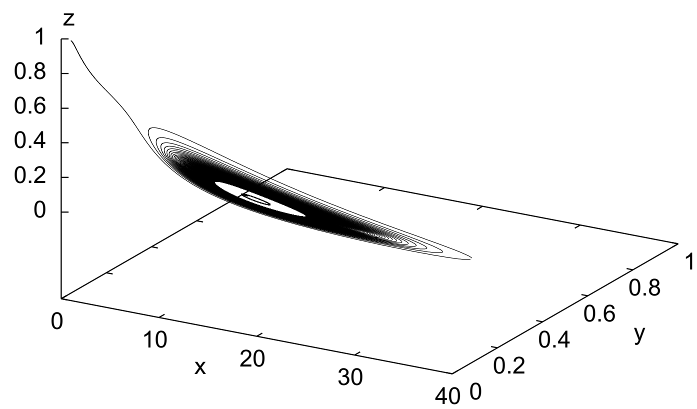
The corresponding eigenvalues for the three equilibria are

$$\text{For } E_0: \quad -1, -1.979767, -65.024317;$$

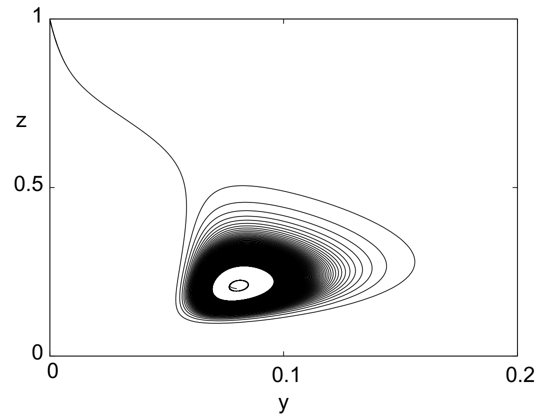
$$\text{For } E_{1-}: \quad -26.511331, -1.880034, 9.608732;$$

$$\text{For } E_{1+}: \quad -71.637652, 0.000041 \pm 10.624728.$$

Therefore, E_0 is a stable node, E_{1-} is a saddle, and E_{1+} is a saddle-focus. The simulation is shown in Fig. 3. Note that for this case, bistable states exist, including the stable disease-free equilibrium E_0 and the stable limit cycle (the smaller one). Therefore, depending upon initial conditions, a trajectory may converge to the equilibrium E_0 or to the stable limit cycle. Actually, when we choose the initial point as $(x, y, z) = (18, 0.08, 0.20)$, which is very close to the unstable equilibrium E_{1+}, the



(a)



(b)

Fig. 3. Simulated trajectories of system (1) for $A = 2.637571$, $B = 11.711918$ and $E = 87.790350$: (a) the phase portrait in the 3D x–y–z space and (b) projected on the y–z plane, with two initial points: (i) $(x, y, z) = (18.0, 0.08, 0.20)$, converging to the stable limit cycle; and (ii) $(x, y, z) = (18.0, 0.10, 0.22)$, converging to the equilibrium $E_0 : (1, 0, 1)$.

trajectory converges to the stable limit cycle. When we choose the initial point a little bit away from E_{1+} as $(x, y, z) = (18, 0.10, 0.22)$, the trajectory converges to the stable equilibrium E_0. This implies that the second initial point, though it is still very close to the equilibrium E_{1+}, is outside the unstable limit cycle, indicating the existence of the unstable limit cycle. However, we cannot obtain the exact unstable limit cycle (the outer one) from simulation. The simulated phase portrait is given in Fig. 3.

4. Conclusion

In this paper, we have carried out a detailed bifurcation analysis for an epidemic model on adaptive networks. Particular attention is focused on the bifurcation of multiple limit cycles arising from degenerate Hopf bifurcation. We have rigorously shown that the codimention of the Hopf bifurcation and maximal two small-amplitude limit cycles can occur near the Hopf critical point. Numerical simulation is presented to show the two limit cycles in the three-dimensional dynamical system on an invariant manifold.

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