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Dynamical analysis on traveling wave of a reaction–diffusion model☆

Yanni Zeng, Xianbo Sun, Pei Yu *

Department of Applied Mathematics, Western University, London, Ontario, N6A 5B7, Canada

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ABSTRACT

In this paper, a class of nonlinear reaction–diffusion equations is studied, and Abelian integral method is applied to show the existence of a unique periodic traveling wave solution. Simulation is presented to verify the theoretical prediction.

1. Introduction

Reaction–convection–diffusion equation, a fundamental class of nonlinear Partial Differential Equations (PDE), plays an important role in many fields of science and engineering, which describes many interesting phenomena such as heat transfer, fluid dynamics, and population dynamics. The general form of reaction–convection–diffusion equations can be written as [1,2]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(D(u)\,\frac{\partial u}{\partial x}\Big) + F(u)\,\frac{\partial u}{\partial x} + R(u), \tag{1.1}$$

where the function $D(u)$ is the diffusion coefficient that measures dispersal rate, $F(u)$ can be regarded as a nonlinear convective flux function, and $R(u)$ is the reaction term. One of the important class of the reaction–convection–diffusion equations is the reaction–diffusion equation, and the generalized Fisher equation is a typical one in real applications. For example, Wang and Xiong have studied the explicit front wave solution of the generalized Fisher equation [3]:

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} + p\,u(1-u^\alpha)(u^\alpha + q), \quad D,\, p,\, \alpha \in \mathbb{R}^+,\, q \in \mathbb{R}, \tag{1.2}$$

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* Corresponding author.
E-mail address: pyu@uwo.ca (P. Yu).

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with the bounded conditions: $\lim_{x\to-\infty} u(t,x) = 0$, $\lim_{x\to+\infty} u(t,x) = 1$. This equation can be used to investigate the propagation of advantageous genes in populations.

On the other hand, a class of nonlinear second-order PDE is given by [1,2]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} + \frac{\partial}{\partial x}\left[(a_0 + a_1\, u^p)u\right] + u^{2-m}(1-u^p)(c_0 + c_1\, u^p), \quad u > 0, \tag{1.3}$$

where m, p, q, a_0, a_1, c_0 and c_1 are constants, which contains a number of well known reaction–convection–diffusion equations. For example, it is the Burgers–Huxley equation [4] when $m = 1$, $p = n$, $a_0 = 0$, $a_1 = -\frac{\alpha}{n+1}$, $c_1 = \beta$, $-\frac{c_0}{c_1} = \gamma \in [0, 1]$, $(c_0 < 0)$. In order to make (1.3) more general, the p in the third term on the right-hand side is changed to q [5], yielding

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} + \frac{\partial}{\partial x}\left[(a_0 + a_1\, u^p)u\right] + u^{2-m}(1-u^q)(c_0 + c_1\, u^q). \tag{1.4}$$

In this paper, we consider the system (1.4) with $m = 1$, $c_0 < 0$, and $p = 1$, $q = 2$, $c_1 = -1$. In addition, for convenience in analysis, letting $\gamma = -c_0 > 0$ we have (1.4) in the form of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (a_0 + 2a_1 u)\frac{\partial u}{\partial x} + u\,(u^2-1)(u^2+\gamma). \tag{1.5}$$

The aim of this work is to apply global bifurcation theory (e.g., see [6,7]) to prove the existence of periodic traveling wave solutions in (1.5). In the next section, system reduction is conducted and Poincaré bifurcation is studied. Then in Section 3, the monotonicity of the ratio of related Abelian integrals is examined and the existence of a unique periodic traveling wave solution is proved. Finally, simulation is presented in Section 4 to verify the theoretical prediction.

2. System reduction and Poincaré bifurcation

Assume that Eq. (1.4) has a traveling wave solution, given by

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{2.1}$$

where $c \in \mathbb{R}$ is the wave speed. For $\xi \in (-\infty, +\infty)$, (2.1) satisfies the boundary conditions: $\lim_{\xi\to+\infty} u(\xi) = m$, $\lim_{\xi\to-\infty} u(\xi) = n$. Substituting (2.1) into (1.5) yields

$$\frac{d^2 u}{d\xi^2} = (c - a_0 - 2a_1 u)\,\frac{du}{d\xi} - u(u^2-1)(u^2+\gamma). \tag{2.2}$$

Let $y = \frac{du}{d\xi}$. Then the system (2.2) can be rewritten as a dynamical system,

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = (c - a_0 - 2a_1\, u)y - u(u^2-1)(u^2+\gamma). \tag{2.3}$$

It is well known (e.g., see [7]) that $u(x,t)$ is a solitary wave solution of (1.5) if $m = n$, which corresponds to a homoclinic orbit of (2.3). If $m \neq n$, $u(x,t)$ represents a kink or anti-kink solution, which corresponds to a heteroclinic orbit of (2.3). Further, a periodic orbit of (2.3) represents a periodic traveling wave solution of (1.5), and a limit cycle of (2.3) represents an isolated periodic traveling wave solution of (1.5).

In order to apply bifurcation theory, assume $c - a_0$ and a_1 are small. Thus, let $c - a_0 = \varepsilon\alpha_0$, $-2a_1 = \varepsilon\alpha_1$, $(0 < \varepsilon \ll 1)$, and $\alpha_0, \alpha_1 \in \mathbb{R}$. Then, system (2.3) becomes

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = \varepsilon\,(\alpha_0 + \alpha_1 u)\,y - u\,(u^2-1)(u^2+\gamma). \tag{2.4}$$

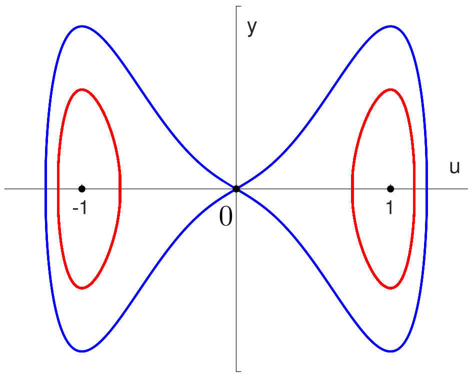

Fig. 1. Phase portrait of system (2.5) for $\gamma = \frac{1}{2}$.

Consider the unperturbed system when $\varepsilon = 0$,

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = -u(u^2-1)(u^2+\gamma), \tag{2.5}$$

which is a Hamiltonian system with the Hamiltonian function,

$$H(u,y) = \frac{1}{2}y^2 + \left[\frac{1}{6}u^6 + \frac{(\gamma-1)}{4}u^4 - \frac{1}{2}\gamma u^2\right] \triangleq \frac{1}{2}y^2 + \Psi(u). \tag{2.6}$$

It is obvious that system (2.5) has three singular points : $\mathrm{E}_1 = (0,0)$, $\mathrm{E}_2 = (1,0)$ and $\mathrm{E}_3 = (-1,0)$. Calculating the Hamiltonian values at the three singular points, we obtain

$$h_1 = H(0,0) = 0, \quad h_2 = H(1,0) = h_3 = H(-1,0) = -\frac{1}{12}(1+3\gamma). \tag{2.7}$$

It is straightforward to use the Jacobian of system (2.5) to show that E_1 is a saddle point, while E_2 and E_3 are centers.

In this paper, we consider $\gamma \in (0,1)$, for which there exist two homoclinic orbits, both of them are connected to the saddle point E_1, and the centers E_2 and E_3 are surrounded by the families of periodic orbits, defined by

$$\Gamma_h: \quad H(u,y) = h, \quad h \in \Big(-\frac{1}{12}(1+3\gamma),\, 0\Big). \tag{2.8}$$

The corresponding phase portrait of system (2.5) for $\gamma = \frac{1}{2}$ is shown in Fig. 1.

Now, suppose there exists one closed orbit around E_2 that starts from a point $A(h)$ on the positive u-axis and first intersects with the positive u-axis at a point $B(h)$. Then, the displacement function of system (2.4) can be written as [8]

$$d(h,\varepsilon) = \int_{\widehat{AB}} dH = \varepsilon\big(I(h,\delta) + O(\varepsilon)\big), \tag{2.9}$$

where $\delta = (\alpha_0, \alpha_1)$, and

$$I(h,\delta) = \oint_{\Gamma_h} (\alpha_0 + \alpha_1 u)\, y du = \alpha_0 \oint_{\Gamma_h} y\, du + \alpha_1 \oint_{\Gamma_h} uy\, du \triangleq \alpha_0 I_0(h) + \alpha I_1(h). \tag{2.10}$$

The Abelian integral method plays an important role in the study of bifurcation of limit cycles. By the Poincaré bifurcation theory [8], the number of zeros of $I(h,\delta)$ corresponds to the number of limit cycles of system (2.4). The monotonicity of the ratio $\frac{I_1}{I_0}$ implies that $I(h,\delta)$ has at most one zero, leading to the existence of one limit cycle, and so one periodic traveling wave solution of system (1.5).

3. Existence of a unique periodic traveling wave

In this section, we will show that the Abelian integral ratio $\frac{I_1(h)}{I_0(h)}$ is monotonic with respect to h, and further prove the existence of a unique periodic traveling wave in system (1.5). We need the following Lemma to prove our main result.

Lemma 3.1 ([6]). *Assume that the Hamiltonian function H(x, y) can be written as $y^2 + \Psi(x)$, satisfying*

$$\Psi'(x)(x-a) > 0 \quad \textit{for} \;\; x \in (\alpha, A),$$

then $U'(h) > 0$ (or $U'(h) < 0$) in (h_1, h_2) implies $P'(h) > 0$ (or $P'(h) < 0$) in (h_1, h_2). Here,

$$U(h) \triangleq \mu(h) + \nu(h), \quad P(h) = \frac{\oint_{\Gamma_h} uydu}{\oint_{\Gamma_h} ydu},$$

where $\mu(h)$ and $\nu(h)$ are the inverse functions of the corresponding mapping, which will be given later in the proof of Theorem 3.1.

Our main result is given in the following theorem.

Theorem 3.1. *For system (2.4), the ratio $\frac{\oint_{\Gamma_h} uydu}{\oint_{\Gamma_h} ydu}$ is monotonic for $h \in \left(-\frac{1}{12}(1+3\gamma), 0\right)$, implying that $I(h, \delta)$ has at most one zero, and the unique zero exists.*

Proof. Due to symmetry of the system (2.5), we only need to prove the case when the family of closed orbits surrounds the center $E_2 = (1, 0)$. Let Γ_h, which is defined in (2.6), represent a continuous family of closed orbits surrounding E_2 and bounded by a homoclinic loop connecting the saddle point $E_1 = (0, 0)$. Then, for $\gamma \in (0, 1)$, it is easy to show that $\Psi(u)$ is analytic in the interval $(0, A)$, where

$$A = \frac{1}{2}\sqrt{3\,(1-\gamma) + \sqrt{3(\gamma+3)(3\gamma+1)}},$$

satisfying $\Psi(0) = \Psi(A)$. Further, we have

$$\Psi'(u)(u-1) = u(u^2-1)(u^2+\gamma)(u-1) = u(u+1)(u-1)^2(u^2+\gamma) > 0$$

for $u \in (0, A)$ and $\gamma \in (0, 1)$. We define two maps:

$$\Psi: \; (0,1) \; \to \; \left(-\frac{1}{12}(1+3\gamma),\, 0\right) \quad \text{and} \quad \Psi: \; (1, A) \; \to \; \left(-\frac{1}{12}(1+3\gamma), 0\right), \tag{3.1}$$

which are strictly monotonic and have inverse functions $\mu(h)$ and $\nu(h)$, respectively, satisfying $0 < \mu(h) < 1 < \nu(h) < A$. It is easy to see that

$$\Psi(\mu(h)) = \Psi(\nu(h)) = h, \quad h \in \left(-\frac{1}{12}(1+3\gamma),\, 0\right).$$

Thus, by Lemma 3.1, it is suffice to prove that $U(h) = \mu(h) + \nu(h)$ is monotonic for $h \in \left(-\frac{1}{12}(1+3\gamma),\, 0\right)$. Let

$$s(h) = \frac{U(h)}{2} = \frac{v(h)+\mu(h)}{2}, \quad \tilde{s}(h) = \frac{v(h)-\mu(h)}{2}, \quad h \in \left(-\frac{1}{12}(1+3\gamma), 0\right).$$

Since $\Psi(\mu(h)) = \Psi(\nu(h)) = h$, we obtain

$$\int_1^{\mu(h)} u\,(u^2-1)(u^2+\gamma)du = \int_1^{v(h)} u\,(u^2-1)(u^2+\gamma)du.$$

Thus, by the change of variable (LHS: $u \to 1-u$, RHS: $u \to u+1$), we have

$$\int_0^{1-\mu(h)} u(u-1)(u-2)\left[(u-1)^2+\gamma\right] du = \int_0^{v(h)-1} u(u+1)(u+2)\left[(u+1)^2+\gamma\right] du.$$

Noticing that the integrand on RHS is bigger than that on the LHS, we have $1-\mu(h) > v(h)-1$, and so $0 < U(h) < 2$, leading to $0 < s(h) < 1$. Further, for any $h \in \left(-\frac{1}{12}\left(1+3\alpha^2\right), 0\right)$, define

$$g(t) = \Psi(s(h)+t) - \Psi(s(h)-t), \quad t \in (0, \tilde{s}(h)). \tag{3.2}$$

It can be shown that $t=0$ and $t=\pm\tilde{s}(h)$ are roots of $g(t)$. With the function Ψ, a direct computation yields that

$$g(t) = 2s(h)t\Big[t^4 + \Big(\gamma + \frac{10s^2}{3} - 1\Big)t^2 + (s^2+\gamma)(s^2-1)\Big]. \tag{3.3}$$

The term in the square bracket of the above equation has two different real roots for t^2 because $\left(s^2+\gamma\right)\left(s^2-1\right) < 0$. Thus, since $g(t)$ can have maximal three real roots, $g(t)$ has no real roots for $t \in (0, \tilde{s}(h))$, and further one can show that $g(t) < 0$ for $t \in (0, \tilde{s}(h))$.

Next, we use contradiction argument to prove that $U(h)$ is monotonic for $h \in \left(-\frac{1}{12}(1+3\gamma), 0\right)$. Suppose otherwise that there exist h_1 and h_2 such that $-\frac{1}{12}(1+3\gamma) < h_1 < h_2 < 0$ and $U(h_1) = U(h_2)$, which implies that $s(h_1) = s(h_2)$ and $\tilde{s}(h_1) < \tilde{s}(h_2)$.

Setting $h = h_2$ in $g(t)$ gives

$$g(t) = \Psi(s(h_2)+t) - \Psi(s(h_2)-t) < 0, \quad t \in (0, \tilde{s}(h_2)).$$

Noticing $\tilde{s}(h_1) < \tilde{s}(h_2)$, and then letting $t = \tilde{s}(h_1)$ and $h = h_2$ in $g(t)$ we obtain

$$\begin{aligned} g(t) &= \Psi(s(h_2)+\tilde{s}(h_1)) - \Psi(s(h_2)-\tilde{s}(h_1)) \\ &= \Psi(s(h_1)+\tilde{s}(h_1)) - \Psi(s(h_1)-\tilde{s}(h_1)) = \Psi(v(h_1)) - \Psi(\mu(h_1)) = h_1 - h_1 = 0, \end{aligned}$$

which contradicts with that $g(t) < 0$. So $U(h)$ is monotonic for $h \in \left(-\frac{1}{12}(1+3\gamma), 0\right)$.

Moreover, note that Γ_h tends to the center E_2 as $h \to -\frac{1}{12}(1+3\gamma)$, while to the homoclinic loop when $h \to 0$. Let D be the region enclosed by Γ_h. Then I_0 can be rewritten as $I_0 = \oint_{\Gamma_h} y du = \iint_D du\, dy > 0$. Now, we rewrite $I(h)$ as

$$I(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) = I_0(h)\Big(\alpha_0 + \alpha_1 \frac{I_1(h)}{I_0(h)}\Big) = \alpha_1 I_0(h)\Big(\frac{\alpha_0}{\alpha_1} + \frac{I_1(h)}{I_0(h)}\Big). \tag{3.4}$$

For α_0, $\alpha_1 \in \mathbb{R}$, there exists an h^* which satisfies $\frac{\alpha_0}{\alpha_1} = -\frac{I_1(h^*)}{I_0(h^*)}$, indicating that there exists a unique periodic solution, corresponding to a unique periodic traveling wave solution for system (1.5). $\square$

4. Simulation

In this section, to illustrate the theoretical results obtained in the previous sections, we give simulation for the dynamical system (2.4). Taking $\varepsilon = 0.01$, $\gamma = \frac{1}{4} \in (0,1)$ and choosing an initial point $(u,y) = (\frac{3}{4}, 0)$, we get $H(\frac{3}{4}, 0) = -\frac{1017}{16384}$. By a direct computation, the corresponding Abelian integral ratio equals $\frac{I_1(h)}{I_0(h)} = 0.9659$, which in turn gives $\frac{\alpha_0}{\alpha_1} = -0.9659$. Thus, we may set $\alpha_0 = -0.9659$ and $\alpha_1 = 1$. The simulated phase portrait and time history for system (2.4) are shown in Fig. 2(a) and (b), respectively. These figures show a very good agreement between the simulation and the analytical prediction.

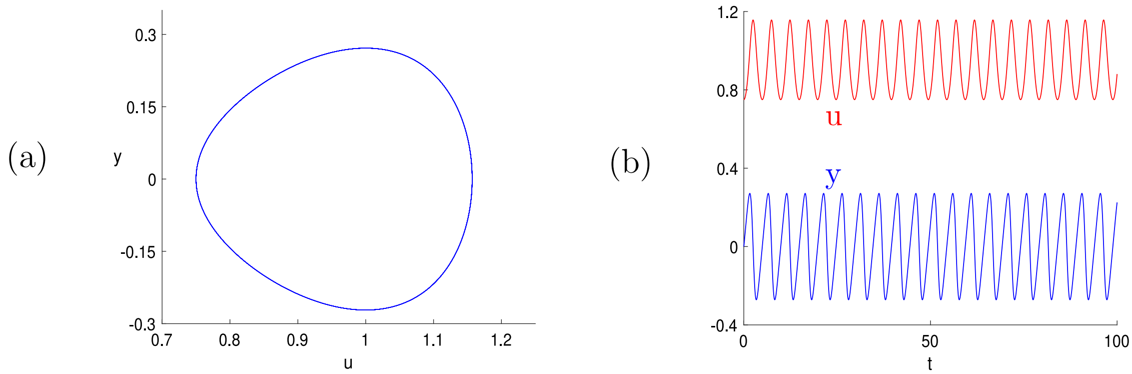

Fig. 2. Simulated periodic solution of system (2.4) for $\gamma = 0.25$, $\alpha_0 = -0.9659$, $\alpha_1 = 1$, $\varepsilon = 0.001$: (a) phase portrait; and (b) time history.

CRediT authorship contribution statement

Yanni Zeng: Conceptualization, Data curation, Formal analysis, Investigation, Writing - original draft. **Xianbo Sun:** Methodology, Visualization, Validation. **Pei Yu:** Project administration, Supervision, Validation, Writing - review & editing.

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