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A NEW COMPARISON METHOD FOR STABILITY THEORY OF DIFFERENTIAL SYSTEMS WITH TIME-VARYING DELAYS

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In this paper, a new comparison method is developed by using increasing and decreasing mechanisms, which are inherent in time-delay systems, to decompose systems. Based on the new method, whose expected performance is compared with the state of the original system, some new conditions are obtained to guarantee that the original system tracks the expected values. The locally exponential convergence rate and the convergence region of the polynomial differential equations with time-varying delays are also investigated. In particular, the comparison method is used to improve the 3/2 stability theorems of differential systems with pure delays. Moreover, the comparison method is applied to identify a threshold, and to consider the disease-free equilibrium points of an HIV endemic model with stages of progress to AIDs and time-varying delay. It is shown that if the threshold is smaller than 1, the equilibrium point of the model is globally, exponentially stable. Another application of the comparison method is to investigate the global, exponential stability of neural networks, and some new theoretical results are obtained. Numerical simulations are presented to verify the theoretical results.

Keywords: Comparison method; stability; time-varying delay; pure delay; neural network; endemic model.

1. Introduction

Lyapunov function method is a classical but powerful tool for studying the stability of differential equations [Hale, 1977]. However, there are no general rules for constructing Lyapunov functions. In many engineering problems and hardware implementations, time delays or even time-varying delays are often inevitable because of internal or external uncertainties. Therefore, the stability of differential equations with time-varying delays deserves

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in-depth investigation. However, it is very difficult to choose a Lyapunov function (or Lyapunov functional) in differential equations with pure time delays since the derivative of the Lyapunov function must be negative definite. In fact, Lyapunov function method is also a comparison method; i.e. the states of a system are compared with the contour curves of the Lyapunov function. Since the contour curves are monotonously descending to an equilibrium point, the states of the system approach the equilibrium point as the contour curves go to lower levels. By comparing the expected performance (stability, exponential stability, etc.) of the system with the state of the original system, a new comparison method is developed.

In 1970, Yorke [1970] developed a 3/2-type criterion for one-dimensional (1-D) linear differential equations with a pure delay. Later, Yoneyama [1986, 1987, 1992], Hara *et al.* [1992], Muroya [2000], and Zhang and Yan [2004] improved the Yorke's 3/2-type stability theory. They extended the Yorke's theory to 1-D nonlinear differential equations with a pure delay under the Yorke condition. Recently, for a nonautonomous Lotka–Volterra competition model with distributed delays but without instantaneous negative feedbacks, Tang and Zou [2002, 2003] established some 3/2-type criteria for global attractivity of positive equilibrium points of the system. Muroya [2006], on the other hand, considered separable nonlinear delay differential systems and established conditions for global asymptotic stability of the zero solution. He improved the 3/2-type criteria for global asymptotic stability of nonautonomous Lotka–Volterra systems with delays. However, with the results in [Yorke, 1970; Yoneyama, 1986, 1987, 1992; Hara *et al.*, 1992; Muroya, 2000; Zhang & Yan, 2004; Tang & Zou, 2002, 2003; Muroya, 2006], it is very difficult to completely characterize the state of the system, and very difficult to obtain the exponential convergence rate.

Studies of epidemic models have become one of the important areas in the mathematical theory of epidemiology, mainly inspired by the works of Anderson and May [1979] and May and Anderson [1979]. Since epidemic models often contain strong nonlinearity, it is very difficult to choose constructing a Lyapunov function (or Lyapunov functional) for stability analysis. For example, the Human Immunodefiency Virus (HIV) is the source of causing the Acquired Immunodefiency Syndrome in humans (AIDS). Because of the ever-increasing numbers of reported cases of HIV infection and AIDS, much collaborative research is being conducted by mathematicians, biologists and physicians with the hope to get better insight into the transmission dynamics of HIV in order to design effective control methods (e.g. see [Hsieh & Sheu, 2001; Driessche & Watmough, 2002; Hyman & Li, 2000; Huang *et al.*, 1992; Moghadas, 2002]).

Some budding recurrent neural network models may be traced back to the nonlinear difference-differential equations in learning theory or prediction theory [Grossberg, 1967, 1968]. In particular, a general neural network, which is called the Cohen–Grossberg neural network (CGNN) and can function as stable associative memory, was developed and studied [Cohen & Grossberg, 1983]. As a special case of the Cohen–Grossberg neural network, the continuous-time Hopfield neural network [Hopfield, 1984] was proposed. Since Hopfield's fundamental work on stability of the Hopfield neural network (HNN) using an energy function, extensive studies on the quantitative analysis of various neural networks have been reported. At the same time, development of cellular neural network (CNN) [Chua & Yang, 1988] has attracted great attention due to the valuable perspective of applications. The stability of recurrent neural networks is a prerequisite for almost all neural network applications, which is primarily concerned with the existence and uniqueness of equilibrium points, and the global asymptotic stability, global exponential stability, and global robust stability of neural networks at equilibria. In recent years, stability studies on recurrent neural networks with time delays have received considerable attention (e.g. see [Zhang *et al.*, 2001; Arik, 2002a, 2002b; Chen *et al.*, 2002; Dong, 2002; Huang *et al.*, 2002; Gopalsamy & Sariyasa, 2002; Cao & Wang, 2003; Liao & Wang, 2003; Mohamad & Gopalsamy, 2003; Zhang *et al.*, 2003; Zeng *et al.*, 2004; Zeng *et al.*, 2005a; Zeng *et al.*, 2005b; Zeng *et al.*, 2005c; Zeng & Wang, 2006]). In many engineering applications and hardware implementation of neural networks, time delays or even time-varying delays in neuron signals are often inevitable, because of internal or external uncertainties. It is very important to know what can guarantee the stability of neural networks with pure multidelays.

Motivated by the above mentioned research works, one of our aims in this paper is to develop a new comparison method by using increasing and decreasing mechanisms to decompose systems. Based on the new method, whose expected performance is compared with the state of the original system, some conditions have been obtained to guarantee that the original system tracks the expected performance. The locally exponential convergence rate and the convergence region of the polynomial differential equation with time-varying delays have also been investigated. In order to demonstrate the validity and characteristics of the comparison method, stabilities on the epidemic models and neural networks with pure time delays are considered. Some sufficient conditions for the stability of these systems are derived.

The rest of this paper comprises five sections. Section 2 presents some preliminaries. Section 3 is devoted to describe the new comparison method. In Sec. 4, the comparative method is used to improve the 3/2 stability theorems of differential systems with pure delays. The conditions of Yorke are extended to the systems with multiple delays, and the 3/2 stability theorems are generalized to multi-dimensional systems with pure time-delays. In Sec. 5, with the comparative method, two applications are presented. A threshold is identified. It is shown that if the threshold is smaller than 1, the disease-free equilibrium point of an HIV endemic model with stages of progress to AIDs and time-varying delay is globally, exponentially stable. Also with the comparative method, some new theoretical results on global exponential stability of neural networks with time-varying delays and Lipschitz continuous activation functions are obtained, which depend upon only the parameters of the networks. These stability conditions improve the existing ones with constant time delays and without time delays. The new results are convenient to estimate the exponential convergence rates of neural networks. Numerical simulations are given for these two applications, showing that simulation results agree with the analytical predictions. Finally, concluding remarks are given in Sec. 6.

2. Preliminaries

Consider the following differential system with delays:

$$\dot{x}(t) = f(t, x(t), x(t-\tau(t))), \tag{1}$$

where $x(t-\tau(t)) = (x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_n(t)))^T$, $f(t, x(t), x(t-\tau(t))) \in C(\Re \times \Re^n \times \Re^n, \Re^n)$, and time delays $\tau_i(t)$ $(i = 1, 2, \ldots, n)$ are continuous functions.

For $H > 0$ and $t \geq 0$, let $C_H(t)$ be the set of continuous function $\varphi = (\varphi_1, \ldots, \varphi_n)^T : [\min_{1\leq i\leq n}\{t-\tau_i(t)\}, t] \to \Re^n$ with $\|\varphi\|_t = \sup_{s\in[\min_{1\leq i\leq n}\{t-\tau_i(t)\},t]} \{\max_{1\leq i\leq n}\{|\varphi_i(s)|\}\} \leq H$. For $t_0 \geq 0$ and $\psi \in C_H(t_0)$, denote the solution of (1) as $x(t; t_0, \psi)$, implying that $x(t; t_0, \psi)$ is continuous with respect to t and satisfies (1), and $x(s; t_0, \psi(s)) = \psi(s)$ for $s \in [\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\}, t_0]$. In the following, we also simply use $x(t)$ to denote the solution of (1). For $x, y \in \Re^n$, $x \leq y$ means that $x_i \leq y_i$ $(i = 1, 2, \ldots, n)$; and $x < y$ means that $x_i < y_i$ $(i = 1, 2, \ldots, n)$, where $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$.

Definition 1. If there exists a function $F_l \in C(\Re \times \Re^n \times \Re^n \times \Re^n \times \Re^n, \Re^n)$ such that for $x^{(1)} \leq x^{(2)}$ and $x_i^{(1)} = x_i^{(2)}$, $F_{il}(t, x^{(1)}, y, u, v) \leq F_{il}(t, x^{(2)}, y, u, v)$; for $u^{(1)} \leq u^{(2)}$, $F_l(t, x, y, u^{(1)}, v) \leq F_l(t, x, y, u^{(2)}, v)$; and for $({y^{(1)}}^T, {v^{(1)}}^T)^T \leq ({y^{(2)}}^T, {v^{(2)}}^T)^T$,

$$F_l(t, x, y^{(1)}, u, v^{(1)}) \geq F_l(t, x, y^{(2)}, u, v^{(2)}),$$

and $F_l(t, x, x, y, y) \leq f(t, x, y)$, then $f(t, x, y)$ is said to have lower mixed quasi-monotone decomposition, where F_{il} is the ith element of F_l, $i = 1, 2, \ldots, n$.

Definition 2. If there exists a function $F_r \in C(\Re \times \Re^n \times \Re^n \times \Re^n \times \Re^n, \Re^n)$ such that for $x^{(1)} \leq x^{(2)}$ and $x_i^{(1)} = x_i^{(2)}$, $F_{ir}(t, x^{(1)}, y, u, v) \leq F_{ir}(t, x^{(2)}, y, u, v)$; for $u^{(1)} \leq u^{(2)}$, $F_r(t, x, y, u^{(1)}, v) \leq F_r(t, x, y, u^{(2)}, v)$; and for $({y^{(1)}}^T, {v^{(1)}}^T)^T \leq ({y^{(2)}}^T, {v^{(2)}}^T)^T$,

$$F_r(t, x, y^{(1)}, u, v^{(1)}) \geq F_r(t, x, y^{(2)}, u, v^{(2)}),$$

and $F_r(t, x, x, y, y) \geq f(t, x, y)$, then $f(t, x, y)$ is said to have upper mixed quasi-monotone decomposition, where F_{ir} is the ith element of F_r, $i = 1, 2, \ldots, n$.

Definition 3. If $f(t, x, y)$ has, respectively, the lower and upper mixed quasi-monotone decompositions $F_l(t, x, x, y, y)$ and $F_r(t, x, x, y, y)$, and $F_l(t, x, x, y, y) = F_r(t, x, x, y, y)$, then $f(t, x, y)$ is said to have mixed quasi-monotone decomposition, and $F_l(t, x, x, y, y)$ or $F_r(t, x, x, y, y)$ is called the mixed quasi-monotone transformation of $f(t, x, y)$.

Definition 4 [Liao, 1993]. If $A = [a_{ij}]_{n\times n}$ satisfies that: (i) $a_{ij} \leq 0$ $(i \neq j,\ i, j = 1, 2, \ldots, n)$; and (ii) there exists a vector $u > 0$ such that $Au > 0$, then A is called a nonsingular M-matrix.

Remark 1 [Liao, 1993]. There are many equivalent definitions on M-matrix. Here are two examples.

E_1: If $a_{ij} \leq 0$ $(i \neq j,\ i, j = 1, 2, \ldots, n)$, and there exists positive diagonal matrix Q such that

$QA + A^TQ$ is positive definite, then A is a nonsingular M-matrix.

E_2: If $a_{ij} \le 0$ ($i \ne j$, $i, j = 1, 2, \ldots, n$), and all real eigenvalues of A are positive, then A is a nonsingular M-matrix.

Lemma 1 [Muroya, 2000]. *Assume that $a(t)$ is a continuous function on $[0, +\infty)$, and $g(t) \le t$ is an increasing function. Let $g^{-1}(t) = \sup\{s : g(s) = t\}$. Define $\alpha = \int_{g(\underline{t})}^{\underline{t}} a(s)ds$. Then for $\alpha > 1$, there exists s_1 such that $\underline{t} < s_1 < g^{-1}(\underline{t})$ and $\int_{g(s_1)}^{\underline{t}} a(s)ds = 1$; while for $\alpha \le 1$, simply let $s_1 = \underline{t}$. Then*

$$\int_{\underline{t}}^{s_1} a(\sigma)d\sigma \int_{g(s_1)}^{\underline{t}} a(s)ds + \int_{s_1}^{g^{-1}(\underline{t})} a(\sigma) \int_{g(\sigma)}^{\underline{t}} a(s)dsd\sigma \le P(\lambda), \quad (2)$$

where

$$\lambda = \sup_{t \ge 0} \int_{g(t)}^{t} a(s)ds,$$

$$P(\lambda) = \begin{cases} \dfrac{\lambda^2}{2} & \text{for } 0 \le \lambda \le 1, \\ \dfrac{\lambda - 1}{2} & \text{for } \lambda > 1. \end{cases}$$

Proof. The inequality (2) can be directly obtained by exchanging the order of integration. The detailed proof can be found in [Muroya, 2000, Lemma 2.2], and thus omitted here for brevity. ■

In the following, the definition of function P is always assumed the same as that given in Lemma 1.

3. The Comparison Method

In this section, we develop the new comparison method, which will be used in the next two sections.

Theorem 1. *If $f(t, x, y)$ has mixed quasi-monotone transformation $F(t, x, x, y, y)$, and there exist continuous functions $u(t)$ and $v(t)$ such that $\forall\, t \ge t_0$,*

$$Du(t) > F(t, u(t), v(t), u(t - \tau(t)), v(t - \tau(t))), \quad (3)$$

$$Dv(t) < F(t, v(t), u(t), v(t - \tau(t)), u(t - \tau(t))), \quad (4)$$

and for $\theta \in [\min_{1 \le i \le n}\{t_0 - \tau_i(t_0)\}, t_0]$, $v(\theta) \le y(\theta)$, $x(\theta) \le u(\theta)$, then $\forall\, t \ge t_0$, the inequalities $v(t) \le y(t)$ and $x(t) \le u(t)$ hold, where $(x^T(t), y^T(t))^T$ is the solution of the following differential equations:

$$\begin{cases} \dot{x}(t) = F(t, x(t), y(t), x(t - \tau(t)), y(t - \tau(t))), \\ \dot{y}(t) = F(t, y(t), x(t), y(t - \tau(t)), x(t - \tau(t))), \end{cases} \quad (5)$$

with the initial condition $(x^T(\theta), y^T(\theta))^T$, $\theta \in [\min_{1 \le i \le n}\{t_0 - \tau_i(t_0)\}, t_0]$, $Du(t)$ denotes one of the Dini derivatives D^+, D_+, D^- and D_-.

Proof. We use the argument of contradiction. Suppose the conclusion of Theorem 1 is not true. Then, there exist $t_1 \ge t_0$, $i \in \{1, 2, \ldots, n\}$ or $j \in \{1, 2, \ldots, n\}$ such that $\forall\, \theta \in [\min_{1 \le i \le n}\{t_0 - \tau_i(t_0)\}, t_1]$,

$$v(\theta) \le y(\theta), \quad x(\theta) \le u(\theta), \quad (6)$$

and one of following two cases must be true:

Case I. $x_i(t_1) = u_i(t_1)$ and

$$x_{it_1}^{(D)} \,\widehat{=}\, D(x_i(t) - u_i(t))|_{t=t_1} \ge 0; \quad (7)$$

Case II. $y_j(t_1) = v_j(t_1)$ and

$$y_{jt_1}^{(D)} \,\widehat{=}\, D(y_j(t) - v_j(t))|_{t=t_1} \le 0. \quad (8)$$

First consider Case I where $x_i(t_1) = u_i(t_1)$. From (3) and (5),

$$x_{it_1}^{(D)} < F_i(t_1, x(t_1), y(t_1), x(t_1 - \tau(t_1)), y(t_1 - \tau(t_1))) - F_i(t_1, u(t_1), v(t_1), u(t_1 - \tau(t_1)), v(t_1 - \tau(t_1))). \quad (9)$$

Since $F(t, x, x, y, y)$ is the mixed quasi-monotone transformation of $f(t, x, y)$, (6) and (9) imply $x_{it_1}^{(D)} < 0$, which contradicts (7). Thus Case I does not hold.

For Case II, $y_j(t_1) = v_j(t_1)$. It follows from (4) and (5) that

$$y_{jt_1}^{(D)} > F_j(t_1, y(t_1), x(t_1), y(t_1 - \tau(t_1)), x(t_1 - \tau(t_1))) - F_j(t_1, v(t_1), u(t_1), v(t_1 - \tau(t_1)), u(t_1 - \tau(t_1))). \quad (10)$$

Since $F(t, x, x, y, y)$ is the mixed quasi-monotone transformation of $f(t, x, y)$, (6) and (10) imply $y_{jt_1}^{(D)} > 0$, which contradicts (8). Hence, Case II does not hold either.

The proof is complete. ■

According to the above discussion, in order to estimate the behavior of the solution of (5), one needs to look for appropriate functions $u(t)$ and $v(t)$. In fact, for the linear systems and the weak nonlinear systems, we can always find the functions satisfying the conditions. The functions $u(t)$ and $v(t)$ can be used to estimate the behaviors of the linear systems and the weak nonlinear systems. Hence, the constructive criterion on stability of the systems can be derived. In the following, the linear systems will be carefully studied. The method finding the appropriate functions $u(t)$ and $v(t)$ will be obtained. The need stability behavior is compared with the state of the original system. Thus, the estimation of the solution of the original system can be obtained. In other words, if a stability character is required, then the corresponding stability is obtained by comparing the stability character with the state of the original system and finding appropriate conditions.

Now, consider the linear system with time delays, described by

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau(t)), \tag{11}$$

where $A(t) = [a_{ij}(t)]_{n\times n} \in C(\Re, \Re^n \times \Re^n)$, $B(t) = [b_{ij}(t)]_{n\times n} \in C(\Re, \Re^n \times \Re^n)$, $\tau(t) = (\tau_1(t), \tau_2(t), \ldots, \tau_n(t))^T$ is the time-varying delay with continuous functions $\tau_i(t)$ $(i = 1, 2, \ldots, n)$.

Let $A^+(t) = [a^+_{ij}(t)]_{n\times n}$, $A^-(t) = [a^-_{ij}(t)]_{n\times n}$, where for $i \neq j$, $a^+_{ij}(t) = \max\{0, a_{ij}(t)\}$, $a^-_{ij}(t) = \max\{0, -a_{ij}(t)\}$, and $a^+_{ii}(t) = a_{ii}(t)$, $a^-_{ii}(t) = 0$. Similarly, let $B^+(t) = [b^+_{ij}(t)]_{n\times n}$, $B^-(t) = [b^-_{ij}(t)]_{n\times n}$, where $b^+_{ij}(t) = \max\{0, b_{ij}(t)\}$, $b^-_{ij}(t) = \max\{0, -b_{ij}(t)\}$ for all $i, j \in \{1, 2, \ldots, n\}$. Then, system (11) can be rewritten as

$$\begin{aligned}\dot{x}(t) = A^+(t)x(t) - A^-(t)x(t) + B^+(t)x(t-\tau(t))\\ - B^-(t)x(t-\tau(t)).\end{aligned} \tag{12}$$

Obviously, $A^+(t)x(t) - A^-(t)x(t) + B^+(t)x(t-\tau(t)) - B^-(t)x(t-\tau(t))$ is a mixed quasi-monotone transformation of $A(t)x(t) + B(t)x(t-\tau(t))$.

As a special case of Theorem 1, we have the following result for the linear system (11) with time-varying delays.

Corollary 1. *If there exist continuous functions $u(t)$ and $v(t)$ such that $\forall\, t \geq t_0$,*

$$\begin{aligned}Du(t) > A^+(t)u(t) - A^-(t)v(t) + B^+(t)u(t-\tau(t))\\ - B^-(t)v(t-\tau(t)),\end{aligned}$$

$$\begin{aligned}Dv(t) < A^+(t)v(t) - A^-(t)u(t) + B^+(t)v(t-\tau(t))\\ - B^-(t)u(t-\tau(t)),\end{aligned}$$

and $\forall\, \theta \in [\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\}, t_0]$, $v(\theta) \leq x(\theta) \leq u(\theta)$, then $\forall\, t \geq t_0$, any solution of system (11) with the initial condition $x(\theta), \theta \in [\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\}, t_0]$, satisfies $v(t) \leq x(t) \leq u(t)$.

Proof. Since $A^+(t)x(t) - A^-(t)x(t) + B^+(t)x(t-\tau(t)) - B^-(t)x(t-\tau(t))$ is a mixed quasi-monotone transformation of $A(t)x(t) + B(t)x(t-\tau(t))$, according to Theorem 1, the conclusion of Corollary 1 is true. ■

Theorem 2. *If there exists a vector function $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ such that $u_i(t)$, $i = 1, 2, \ldots, n$ are continuous, non-negative decreasing functions, and $\forall\, t \geq t_0$,*

$$\begin{aligned}Du(t) \geq A^+(t)u(t) + A^-(t)u(t) + B^+(t)u(t-\tau(t))\\ + B^-(t)u(t-\tau(t)),\end{aligned} \tag{13}$$

and $\forall\, \theta \in [\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\}, t_0]$, $-u(\theta) \leq x(\theta) \leq u(\theta)$, then $\forall\, t > t_0$, any solution of the linear system (11) with the initial condition $x(\theta)$, $\theta \in [\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\}, t_0]$, satisfies $-u(t) \leq x(t) \leq u(t)$.

Proof. Suppose the conclusion of Theorem 2 does not hold. Then there exist t_1, t_2 and $r_\ell > 1$, $\ell \in \{1, 2, \ldots, n\}$ such that $t_0 \leq t_1 < t_2$, $|x_\ell(t_1)|/u_\ell(t_1) = 1$, $|x_\ell(t_2)|/u_\ell(t_2) = r_\ell$. Moreover, $\forall\, k \in \{1, 2, \ldots, n\}$, $\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\} \leq s \leq t_1$, $|x_k(s)|/u_k(s) \leq 1$; $\forall\, r \in (t_1, t_2)$, $|x_k(r)|/u_k(r) \leq r_\ell$; and $D[|x_\ell(t)|/u_\ell(t)]|_{t=t_2} > 0$; i.e. $D[|x_\ell(t)| - r_\ell u_\ell(t)]|_{t=t_2} > 0$. Hence, one of the following two cases must hold:

Case I. $x_\ell(t_1) = u_\ell(t_1)$, $x_\ell(t_2) = r_\ell u_\ell(t_2)$, $\forall\, k \in \{1, 2, \ldots, n\}$ and for $\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\} \leq s \leq t_1$, $|x_k(s)| \leq u_k(s)$; and $\forall\, s \in (t_1, t_2)$, $|x_k(s)| \leq r_\ell u_k(s)$ and

$$x^{(D)}_{+\ell t_2} := D[x_\ell(t) - r_\ell u_\ell(t)]|_{t=t_2} > 0; \tag{14}$$

Case II. $x_\ell(t_1) = -u_\ell(t_1)$, $x_\ell(t_2) = -r_\ell u_\ell(t_2)$, $\forall\, k \in \{1, 2, \ldots, n\}$ and for $\min_{1\leq i\leq n}\{t_0 - \tau_i(t_0)\} \leq s \leq t_1$, $|x_k(s)| \leq u_k(s)$; and $\forall\, s \in (t_1, t_2), |x_k(s)| < r_\ell u_k(s)$; and

$$x^{(D)}_{-\ell t_2} := D[x_\ell(t) + r_\ell u_\ell(t)]|_{t=t_2} < 0. \tag{15}$$

First, consider Case I. From (12) and (13), we have

$$\begin{aligned}
x^{(D)}_{+\ell t_2} &\le \sum_{k=1}^{n} a_{\ell k}(t_2)x_k(t_2) + \sum_{k=1}^{n} b_{\ell k}(t_2)x_k(t_2-\tau_k(t_2)) \\
&\quad - r_\ell\Bigg[a_{\ell\ell}(t_2)u_\ell(t_2) + \sum_{k=1,k\ne\ell}^{n}|a_{\ell k}(t_2)|u_k(t_2) \\
&\quad + \sum_{k=1}^{n}|b_{\ell k}(t_2)|u_k(t_2-\tau_k(t_2))\Bigg] \\
&\le (r_\ell - r_\ell)a_{\ell\ell}(t_2)u_\ell(t_2) \\
&\quad + \sum_{k=1,k\ne\ell}^{n}|a_{\ell k}(t_2)|(|x_k(t_2)| - r_\ell u_k(t_2)) \\
&\quad + \sum_{k=1}^{n}|b_{\ell k}(t_2)|(|x_k(t_2-\tau_k(t_2))| \\
&\quad - r_\ell u_k(t_2-\tau_k(t_2))),
\end{aligned}$$

since $\forall s \in (\min_{1\le i\le n}\{t_0 - \tau_i(t_0)\}, t_2)$, $\forall k \in \{1,2,\ldots,n\}$, $|x_k(s)| \le r_\ell u_k(s)$ and $x^{(D)}_{+\ell t_2} \le 0$. This contradicts (14), and thus Case I does not hold.

Next, for Case II, it follows from (12) and (13) that

$$\begin{aligned}
x^{(D)}_{-\ell t_2} &\ge \sum_{k=1}^{n} a_{\ell k}(t_2)x_k(t_2) + \sum_{k=1}^{n} b_{\ell k}(t_2)x_k(t_2-\tau_k(t_2)) \\
&\quad + r_\ell\Bigg[a_{\ell\ell}(t_2)u_\ell(t_2) + \sum_{k=1,k\ne\ell}^{n}|a_{\ell k}(t_2)|u_k(t_2) \\
&\quad + \sum_{k=1}^{n}|b_{\ell k}(t_2)|u_k(t_2-\tau_k(t_2))\Bigg] \\
&\ge -(r_\ell - r_\ell)a_{\ell\ell}(t_2)u_\ell(t_2) \\
&\quad + \sum_{k=1,k\ne\ell}^{n}|a_{\ell k}(t_2)|(r_\ell u_k(t_2) - |x_k(t_2)|) \\
&\quad + \sum_{k=1}^{n}|b_{\ell k}(t_2)|(r_\ell u_k(t_2-\tau_k(t_2)) \\
&\quad - |x_k(t_2-\tau_k(t_2))|),
\end{aligned}$$

since $\forall s \in (\min_{1\le i\le n}\{t_0 - \tau_i(t_0)\}, t_2]$ and $\forall k \in \{1,2,\ldots,n\}$, $|x_k(s)| \le r_\ell u_k(s)$ and $x^{(D)}_{-\ell t_2} \ge 0$. This contradicts (15), and thus Case II does not hold either.

Theorem 2 is proved. ■

Based on Theorem 2, we can construct the explicit conditions of uniform stability and globally exponential stability for the linear system (11) with time-varying delays, listed in the following three corollaries.

Corollary 2. *If* $\forall t \ge t_0$, $\forall i \in \{1,2,\ldots,n\}$,

$$a_{ii}(t) + \sum_{j=1,j\ne i}^{n}|a_{ij}(t)| + \sum_{j=1}^{n}|b_{ij}(t)| \le 0, \tag{16}$$

then the zero solution of system (11) is uniformly stable.

Proof. $\forall t \ge \min_{1\le i\le n}\{t_0 - \tau_i(t_0)\}$, choose $u(t) \equiv \|x\|_{t_0}$. Then (16) implies that (13) holds. According to Theorem 2, the conclusion of Corollary 2 is true. ■

Corollary 3. *If* $\forall t \ge t_0$, $\forall i \in \{1,2,\ldots,n\}$, $\tau_i(t) \le \tau$ *(constant) and*

$$\sup_{t\ge t_0}\left\{a_{ii}(t) + \sum_{j=1,j\ne i}^{n}|a_{ij}(t)| + \sum_{j=1}^{n}|b_{ij}(t)|\right\} < 0, \tag{17}$$

then the zero solution of system (11) is globally, exponentially stable.

Proof. (17) implies that there exists small enough constant $\theta > 0$ such that $\forall t \ge t_0, \forall i \in \{1,2,\ldots,n\}$,

$$a_{ii}(t) + \sum_{j=1,j\ne i}^{n}|a_{ij}(t)| + \sum_{j=1}^{n}|b_{ij}(t)|\,e^{\theta\tau} + \theta \le 0. \tag{18}$$

$\forall t \ge \min_{1\le i\le n}\{t_0 - \tau_i(t_0)\}$, choose $u(t) = \|x\|_{t_0}e^{-\theta(t-t_0)}$. Then (18) implies that (13) holds. According to Theorem 2, Corollary 3 holds. ■

Corollary 4. *If for* $i,j = 1,2,\ldots,n$, $\forall t \ge t_0$, $a_{ij}(t) \equiv a_{ij}$ *(constant)*, $b_{ij}(t) \equiv b_{ij}$ *(constant)*, $\tau_i(t) \le \tau$ *(constant) and* $-(A^+ + A^- + B^+ + B^-)$ *is a nonsingular* M*-matrix, then the zero solution of system (11) is globally, exponentially stable.*

Proof. Since $-(A^+ + A^- + B^+ + B^-)$ is a nonsingular M-matrix, there exist positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\forall i \in \{1,2,\ldots,n\}$,

$$-\alpha_i a_{ii} > \sum_{j=1,j\ne i}^{n}\alpha_j|a_{ij}| + \sum_{j=1}^{n}\alpha_j|b_{ij}|. \tag{19}$$

$\forall t \ge \min_{1\le i\le n}\{t_0 - \tau_i(t_0)\}$, let $y_i(t) = x_i(t)/\alpha_i$. Then system (11) can be rewritten as

$$\dot{y}_i(t) = \sum_{j=1}^{n}\frac{\alpha_j}{\alpha_i}\left[a_{ij}y_j(t) + b_{ij}y_j(t-\tau_j(t))\right]. \tag{20}$$

From (19) and Corollary 3, we know that the zero solution of (20) is globally, exponentially stable. The definition of $y_i(t)$ implies that Corollary 4 holds. ■

The method, in which the function $u(t) = e^{-\theta(t-t_0)}$ is compared with the state of the original system, is not solely used to the linear systems or the weak nonlinear systems, it can be used to study the local stability of the strong nonlinear systems too.

Next, consider a set of polynomial differential equations with time-varying delays, given by

$$\dot{x}_i(t) = \sum_{k=1}^{m}\sum_{j=1}^{n} a_{kij}(t)x_j^k(t) + \sum_{k=1}^{m}\sum_{j=1}^{n} b_{kij}(t)x_j^k(t-\tau_j(t)), \quad i = 1, 2, \ldots, n, \tag{21}$$

where the coefficient functions $a_{kij}(t)$, $b_{kij}(t)$ and the delay $\tau_i(t)$ are continuous functions.

Theorem 3. *If $\forall\, i \in \{1, 2, \ldots, n\}$, $\forall\, t \geq t_0$, $\tau_i(t) \leq \tau$ (constant) and*

$$\sup_{t\geq t_0}\left\{a_{1ii}(t) + \sum_{j=1, j\neq i}^{n} |a_{1ij}(t)| + \sum_{j=1}^{n} |b_{1ij}(t)|\right\} < 0, \tag{22}$$

then the zero solution of system (21) is locally, exponentially stable.

Proof. (22) implies that there exist constants $\theta > 0$ and $ß > 0$ such that

$$a_{1ii}(t) + \sum_{j=1,j\neq i}^{n} |a_{1ij}(t)| + \sum_{k=2}^{m}\sum_{j=1}^{n} |a_{kij}(t)|ß^{k-1} + \sum_{k=1}^{m}\sum_{j=1}^{n} |b_{kij}(t)|ß^{k-1}\, e^{\theta\tau} + \theta \leq 0. \tag{23}$$

$\forall\, t \geq t_0$, $\forall\, i \in \{1, 2, \ldots, n\}$, choose $u_i(t) = \|x\|_{t_0}\, e^{-\theta(t-t_0)}$. Then (23) implies that for $\|x\|_{t_0} \leq ß$,

$$\dot{u}_i(t) \geq a_{1ii}(t)u_i(t) + \sum_{j=1,j\neq i}^{n} |a_{1ij}(t)|u_j(t) + \sum_{k=2}^{m}\sum_{j=1}^{n} |a_{kij}(t)|u_j^k(t) + \sum_{k=1}^{m}\sum_{j=1}^{n} |b_{kij}(t)|u_j^k(t-\tau(t)). \tag{24}$$

Proceeding the proof as that of Theorem 2, the conclusion of Theorem 3 can be assured. ■

Remark 2. According to Theorem 3, the rate of locally exponential convergence of the zero solution of system (21) is at least equal to θ, and any solution of system (21) with the initial condition $\|x\|_{t_0} \leq ß$ satisfies that $\forall\, t \geq t_0$, $|x_i(t)| \leq \|x\|_{t_0}\, e^{-\theta(t-t_0)}$; i.e. the region of locally exponential convergence of the zero solution of system (21) is at least equal to $[-ß, ß]^n$. In particular, if $m = 2$ and $\sum_{j=1}^{n} |a_{2ij}(t)| + \sum_{j=1}^{n} |b_{2ij}(t)| \neq 0$, then the positive number ß, given as

$$ß < \inf_{t\geq t_0}\left\{\frac{-a_{1ii}(t) - \sum_{j=1,j\neq i}^{n} |a_{1ij}(t)| - \sum_{j=1}^{n} |b_{1ij}(t)|}{\sum_{j=1}^{n} |a_{2ij}(t)| + \sum_{j=1}^{n} |b_{2ij}(t)|}\right\},$$

satisfies (23).

4. A Generalization of the 3/2 Stability Theory for Pure Time-Delay Systems

Consider the nonlinear differential system with pure delays:

$$\dot{x}(t) = F(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_n(t))), \tag{25}$$

where $F(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_n(t)))$ is continuous on t.

In the following, we assume that there exist continuous functions c_{ij}: $[0, \infty) \to [0, \infty)$ such that $\forall\, t \geq 0$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T \in C_H(t)$, $i \in \{1, 2, \ldots, n\}$,

$$F_{il}(t, \psi_1, \psi_2, \ldots, \psi_n) \leq F_i(t, \psi_1, \psi_2, \ldots, \psi_n) \leq F_{ir}(t, \psi_1, \psi_2, \ldots, \psi_n), \tag{26}$$

where for $i \in \{1, 2, \ldots, n\}$,

$$F_{il}(t, \psi_1, \psi_2, \ldots, \psi_n) = -c_{ii}(t)N_t(\psi_i) - \sum_{j=1,j\neq i}^{n} c_{ij}(t)N_t(-\psi_j),$$

$$F_{ir}(t, \psi_1, \psi_2, \ldots, \psi_n) = c_{ii}(t)N_t(-\psi_i) + \sum_{j=1,j\neq i}^{n} c_{ij}(t)N_t(\psi_j),$$

in which $N_t(\psi_i) = \max\{0, \sup_{s\in[t-\tau_i(t),t]} \psi_i(s)\}$.

Yoneyama [1986] considered the 1-D delay differential equation:

$$\dot{x}(t) = -a(t)f(x(t - r(t))), \tag{27}$$

where $a : [0, +\infty) \to [0, \infty)$, $f : (-\infty, \infty) \to (-\infty, \infty)$, $r : [0, +\infty) \to [0, q]$, $q \geq 0$, and obtained some sufficient conditions which guarantee that (27) is uniformly stable or asymptotically stable. Burton and Haddock [1976] and the references therein have also analyzed the asymptotic behavior of (27). When $n = 1$, $F(t, x(t - \tau_1(t)), x(t - \tau_2(t)), \ldots, x(t - \tau_n(t))) = -c_{11}(t)x(t - \tau_1(t))$, Hara *et al.* [1992] has shown that if $\int_0^{+\infty} c_{11}(s)ds = +\infty$ and $\sup_{t\geq 0} \int_{t-\tau_1(t)}^{t} c_{11}(s)ds < 1$, the zero solution of (25) is uniformly, asymptotically stable. When $n = 1$, (26) is so-called Yorke condition. When $n = 1$ and $t - \tau_1(t)$ is an increasing function, Yorke [1970], Yoneyama [1987] and Muroya [2000] considered the stability of (25) and developed the 3/2 stability theorem. Under the assumption that $t - \tau_1(t)$ is an increasing function, Yoneyama [1987, 1992] improved the stability theorem of Yorke [1970]: if the Yorke's conditions: $\lambda = \sup_{t\geq 0} \int_{t-\tau_1(t)}^{t} c_{11}(s)ds < 3/2$ and $\mu = \inf_{t\geq 0} \int_{t-\tau_1(t)}^{t} c_{11}(s)ds > 0$ hold, then the zero solution of (25) is uniformly, asymptotically stable. Muroya [2000] studied the uniformly asymptotic stability of (25) in the case of $\mu = 0$, while Yoneyama [1991] and Pituk [1997] investigated n-dimensional systems.

Theorem 4. *Let $t - \tau_i(t) = g_i(t)$, $g_i^{-1}(t) = \sup\{s, g_i(s) = t\}$, $\lambda_{ij} = \sup_{t\geq t_0} \int_{g_i(t)}^{t} c_{ij}(s)ds$, and $g_{\max}^{-1}(t_0) = \max_{1\leq i\leq n}\{t_0 + g_i^{-1}(g_i^{-1}(g_i^{-1}(t_0)))\}$. If $g_i(t)$ is increasing, $g_i(t) \to \infty$ as $t \to \infty$, and there exist continuous decreasing functions $\eta_i(t, t_0) > 0$ $(i = 1, 2, \ldots, n)$ such that $\forall\, t \geq g_{\max}^{-1}(t_0)$, $\eta_i(t, t_0)$ is differentiable and*

$$\eta_i(g_{\max}^{-1}(t_0), t_0) \geq 1, \tag{28}$$

$$\begin{aligned} &P(\lambda_{ii})\eta_i(g_i(g_i(g_i(g_i(t)))), t_0) \\ &\quad + \lambda_{ii} \sum_{j=1, j\neq i}^{n} \lambda_{ij}\eta_j(g_j(g_i(g_i(g_i(t)))), t_0) \\ &\quad + 2 \sum_{j=1, j\neq i}^{n} \lambda_{ij}\eta_j(g_j(g_i(g_i(t))), t_0) \leq \eta_i(t, t_0). \end{aligned} \tag{29}$$

In addition, $\forall\, t \geq t_0$ and $k \in \{1, 2, \ldots, n\}$, the following cases hold:

(I) *when $\inf_{u\in[g_k(t),t]} \varphi_k(u) \geq \|\bar{\varphi}\| \left[\eta_k(t, t_0) - \sum_{j=1, j\neq k}^{n} \lambda_{kj}\eta_j(g_j(g_k(t)), t_0)\right]$ and $\forall\, j \in \{1, 2, \ldots, n\}$, $\sup_{u\in[g_j(t),t]} \varphi_j(u) \leq \|\bar{\varphi}\| \times \eta_j(g_j(g_k(t)), t_0)$,*

$$F_k(t, \varphi_1, \varphi_2, \ldots, \varphi_n) - \|\bar{\varphi}\|\dot{\eta}_k(t, t_0) \leq 0; \tag{30}$$

(II) *when $\sup_{u\in[g_k(t),t]} \varphi_k(u) \leq -\|\bar{\varphi}\| \left[\eta_k(t, t_0) - \sum_{j=1, j\neq k}^{n} \lambda_{kj}\eta_j(g_j(g_k(t)), t_0)\right]$ and $\forall\, j \in \{1, 2, \ldots, n\}$, $\inf_{u\in[g_j(t),t]} \varphi_j(u) \geq -\|\bar{\varphi}\| \times \eta_j(g_j(g_k(t)), t_0)$,*

$$F_k(t, \varphi_1, \varphi_2, \ldots, \varphi_n) + \|\bar{\varphi}\|\dot{\eta}_k(t, t_0) \geq 0.$$

Then $\forall\, t \geq t_0, i \in \{1, 2, \ldots, n\}$, any solution $x(t)$ of (25) with the initial condition $x(s) = \varphi(s) \in C_H(t_0)$, $\min_{1\leq i\leq n}\{g_i(t_0)\} \leq s \leq t_0$, satisfies

$$|x_i(t; t_0, \varphi)| \leq \|\bar{\varphi}\|\eta_i(t, t_0), \tag{31}$$

where $\|\bar{\varphi}\| = \|\varphi\|_{t_0} \max_{1\leq i\leq n}\{e^{\int_{t_0}^{g_{\max}^{-1}(t_0)} \sum_{j=1}^{n} c_{ij}(s)ds}\}$.

Proof. If (31) does not hold, then there exist t_1, t_2, $g_{\max}^{-1}(t_0) \leq t_1 < t_2$, $k \in \{1, 2, \ldots, n\}$ and $r_k > 1$ such that

$$\frac{|x_k(t_1)|}{(\|\bar{\varphi}\|\eta_k(t_1, t_0))} = 1, \quad \frac{|x_k(t_2)|}{(\|\bar{\varphi}\|\eta_k(t_2, t_0))} = r_k. \tag{32}$$

For any $s \in (t_1, t_2)$,

$$1 < \frac{|x_k(s)|}{\|\bar{\varphi}\|\eta_k(s, t_0)} < r_k, \tag{33}$$

$$\left[D|x_k(t)| - \frac{|x_k(t)|}{\|\bar{\varphi}\|\eta_k(t, t_0)}\,\dot{\eta}_k(t, t_0)\right]_{t=t_2} > 0; \tag{34}$$

and $\forall\, u \in [\min_{1\leq i\leq n}\{g_i(t_0)\}, t_2]$, $\forall\, j \in \{1, 2, \ldots, n\}$,

$$\frac{|x_j(u)|}{\|\bar{\varphi}\|\eta_j(u, t_0)} \leq r_k; \tag{35}$$

$\forall\, s \in [\min_{1\leq i\leq n}\{g_i(t_0)\}, t_1]$, $\forall\, j \in \{1, 2, \ldots, n\}$,

$$\frac{|x_j(u)|}{\|\bar{\varphi}\|\eta_j(u, t_0)} \leq 1. \tag{36}$$

Assume $x_k(t_1) > 0$. The proof for the case of $x_k(t_1) < 0$ is similar.

(1) If $g_k(t_2) \leq t_1$, there exists t_3 such that $g_k(t_3) \leq g_k(t_1) \leq g_k(t_2) \leq t_3 < t_1 < t_2, x_k(t_3) = 0$, and for $s \in (t_3, t_2)$, $x_k(s) > 0$. Then, from (26)

and (36), $\forall\, i \in \{1, 2, \ldots, n\}$, $\forall\, s \in [g_k(t_3), t_1]$,

$$\dot{x}_i(s) \le c_{ii}(s)N_s(-x_i) + \sum_{j=1, j\ne i}^{n} c_{ij}(s)N_s(x_j) \le \|\overline{\varphi}\| \sum_{j=1}^{n} c_{ij}(s)\eta_j(g_j(g_i(g_i(t_2))), t_0)$$

$$< \|\overline{\varphi}\| r_k \sum_{j=1}^{n} c_{ij}(s)\eta_j(g_j(g_i(g_i(t_2))), t_0).$$

Hence, for $s \in [g_k(t_3), t_3]$,

$$|x_i(s) - x_i(t_3)| = |x_i(s)| < \|\overline{\varphi}\| r_k \int_s^{t_3} \sum_{j=1}^{n} c_{ij}(r)\eta_j(g_j(g_k(g_k(t_2))), t_0)dr. \tag{37}$$

If $\overline{\lambda}_{kk} = \int_{g_k(t_3)}^{t_3} c_{kk}(r)dr \le 1$, then from (26) and (36), we obtain

$$\begin{aligned} x_k(t_2) - x_k(t_3) &\le \int_{t_3}^{t_2} [c_{kk}(s)N_s(-x_k) + \sum_{j=1, j\ne k}^{n} c_{kj}(s)N_s(x_j)]ds \\ &< \|\overline{\varphi}\| r_k(\eta_k(g_k(g_k(g_k(t_2))), t_0) \int_{t_3}^{t_2} c_{kk}(s) \int_{g_k(s)}^{t_3} c_{kk}(r)drds \\ &\quad + \int_{t_3}^{t_2} c_{kk}(s) \int_{g_k(s)}^{t_3} \sum_{j=1, j\ne k}^{n} c_{kj}(r)\eta_j(g_j(g_k(g_k(t_2))), t_0)drds) \\ &\quad + \int_{t_3}^{t_2} \sum_{j=1, j\ne k}^{n} c_{kj}(s)N_s(x_j)ds \\ &\le \|\overline{\varphi}\| r_k \Bigg(\eta_k(g_k(g_k(g_k(t_2))), t_0) \int_{t_3}^{t_2} c_{kk}(s) \int_{g_k(s)}^{t_3} c_{kk}(r)drds \\ &\quad + \int_{t_3}^{t_2} c_{kk}(s) \int_{g_k(s)}^{t_3} \sum_{j=1, j\ne k}^{n} c_{kj}(r)\eta_j(g_j(g_k(g_k(t_2))), t_0)drds \\ &\quad + \int_{t_3}^{t_2} \sum_{j=1, j\ne k}^{n} |c_{kj}(s)|\eta_j(g_j(g_k(t_2)), t_0)ds \Bigg) \\ &\le \|\bar{\varphi}\| r_k \Bigg(\eta_k(g_k(g_k(g_k(t_2))), t_0)P(\lambda_{kk}) + \lambda_{kk} \sum_{j=1, j\ne k}^{n} \lambda_{kj}\eta_j(g_j(g_k(g_k(t_2))), t_0) \\ &\quad + \sum_{j=1, j\ne k}^{n} \lambda_{kj}\eta_j(g_j(g_k(t_2)), t_0) \Bigg). \end{aligned}$$

Then by (29), $x_k(t_2) < \|\overline{\varphi}\| r_k \eta_k(t_2, t_0)$, which contradicts (32).

If $\overline{\lambda}_{kk} = \int_{g_k(t_3)}^{t_3} c_{kk}(r)dr > 1$, then there exists t_4 such that $g_k(t_3) < g_k(t_4) < t_3$ and $\int_{g_k(t_4)}^{t_3} c_{kk}(r)dr = 1$. If $s \in [t_3, t_4]$, then $g_k(s) \in [g_k(t_3), g_k(t_4)] \subset [g_k(t_3), t_1]$. From (37), $\forall\, i \in \{1, 2, \ldots, n\}$,

$$|x_i(g_k(s))| \le \|\overline{\varphi}\| r_k \sum_{j=1}^{n} c_{ij}(g_k(s))\eta_j(g_j(g_k(g_k(t_2))), t_0).$$

If $s \in [t_4, g_k^{-1}(t_3)]$, then $g_k(s) \in [g_k(t_4), t_3]$. It follows from (37) that

$$|x_k(g_k(s))| < \|\overline{\varphi}\| r_k \int_{g_k(s)}^{t_3} \sum_{j=1}^{n} c_{kj}(r)\eta_j(g_j(g_k(g_k(t_2))), t_0)dr.$$

Hence,

$$\begin{aligned}
x_k(t_2) &\le \int_{t_3}^{t_4} c_{kk}(s)N_s(-x_k)ds \\
&\quad + \int_{t_4}^{t_2} c_{kk}(s)N_s(-x_k)ds \\
&\quad + \int_{t_3}^{t_2} \sum_{j=1,j\neq k}^{n} c_{kj}(s)N_s(x_j)ds \\
&\le \int_{t_3}^{t_4} c_{kk}(s)N_s(-x_k)ds \\
&\quad + \int_{t_4}^{g_k^{-1}(t_3)} c_{kk}(s)N_s(-x_k)ds \\
&\quad + \int_{t_3}^{g_k^{-1}(t_3)} \sum_{j=1,j\neq k}^{n} c_{kj}(s)N_s(x_j)ds \\
&< \|\overline{\varphi}\|r_k\left\{ \int_{t_3}^{t_4} c_{kk}(s)\eta_k(g_k(g_k(t_2)),t_0)ds \right.\\
&\quad + \int_{t_4}^{g_k^{-1}(t_3)} c_{kk}(s)\int_{g_k(s)}^{t_3}\sum_{j=1}^{n} c_{kj}(r) \\
&\quad \times \eta_j(g_j(g_k(g_k(t_2))),t_0)dr \\
&\quad + \int_{t_3}^{g_k^{-1}(t_3)} \sum_{j=1,j\neq k}^{n} c_{kj}(s) \\
&\quad \left. \times \eta_j(g_j(g_k(t_2)),t_0)ds \right\}.
\end{aligned}$$

Using Lemma 1, we deduce that

$$\begin{aligned}
x_k(t_2) &< \|\overline{\varphi}\|r_k\left\{ P(\overline{\lambda}_{kk})\eta_k(g_k(g_k(g_k(t_2))),t_0) \right.\\
&\quad + \lambda_{kk}\sum_{j=1,j\neq k}^{n} \lambda_{kj}\eta_j(g_j(g_k(g_k(t_2))),t_0) \\
&\quad \left. + \sum_{j=1,j\neq k}^{n} \lambda_{kj}\eta_j(g_j(g_k(t_2)),t_0) \right\},
\end{aligned}$$

which, by (29), implies that $x_k(t_2) < \|\overline{\varphi}\|r_k\eta_k(t_2,t_0)$. This contradicts (32). Thus, it is impossible to have $t_3 \in [g_k(t_2),t_1]$ such that $x_k(t_3)=0$; i.e. $\forall\, s \in [g_k(t_2),t_2]$, $x_k(s)>0$.

(2) If $g_k(g_k(t_2)) \le t_1$, the proof is similar to that of part (1) for $x_k(s)>0\ \forall\, s\in[g_k(g_k(t_2)),t_2]$. Thus, $\forall\, u \in [g_k(t_2),t_2]$,

$$\begin{aligned}
x_k(t_2) - x_k(u) &= \int_u^{t_2} \dot{x}_k(s)ds \\
&\le \int_{g_k(t_2)}^{t_2} \sum_{j=1,j\neq k}^{n} c_{kj}(s)N_s(x_j)ds,
\end{aligned}$$

i.e. $\forall\, u \in [g_k(t_2),t_2]$,

$$\begin{aligned}
x_k(u) &\ge \|\overline{\varphi}\|r_k\left\{ \eta_k(t_2,t_0) - \int_{g_k(t_2)}^{t_2}\sum_{j=1,j\neq k}^{n} c_{kj}(s) \right.\\
&\quad \left. \times \eta_j(g_j(g_k(t_2)),t_0)ds \right\}.
\end{aligned}$$

Also from (35), $\forall\, u \in [g_k(t_2),t_2]$, $|x_k(u)| \le \|\overline{\varphi}\|r_k\eta_k(u,t_0)$. Let $\overline{\eta}_j(t,t_0) = r_k\eta_j(t,t_0)$. Then, from (30) we have

$$\begin{aligned}
&\dot{x}_k(t_2) - \|\overline{\varphi}\|\dot{\eta}_k(t_2,t_0) \\
&\quad \le \dot{x}_k(t_2) - \|\overline{\varphi}\|r_k\dot{\eta}_k(t_2,t_0) \le 0,
\end{aligned}$$

which contradicts (34).

(3) If $g_k(g_k(t_2)) > t_1$, then from (33) we have $\forall\, s \in [g_k(g_k(t_2)),t_2]$, $x_k(s)>0$. Thus, similar to the proof of part (2), we can show that

$$\begin{aligned}
&\dot{x}_k(t_2) - \|\overline{\varphi}\|\dot{\eta}_k(t_2,t_0) \\
&\quad \le \dot{x}_k(t_2) - \|\overline{\varphi}\|r_k\dot{\eta}_k(t_2,t_0) \le 0,
\end{aligned}$$

which contradicts (34).

The proof of Theorem 4 is complete. ■

By using Theorem 4, we can obtain the explicit conditions of uniform stability, uniformly asymptotic stability and globally exponential stability for the one-dimensional linear system (25) with pure delays, given in Corollary 5. In Corollaries 6 and 7, we present two results for the globally exponential stability of the n-dimensional linear system (38) with pure time delays.

Corollary 5. *If* $n=1$, $\lambda_{11} = \sup_{t\ge 0}\int_{g_1(t)}^{t} c_{11}(s)ds \le 3/2$, *then the zero solution of (25) is uniformly stable. If* $F(t,x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_n(t))) = -c_{11}(t)x(t-\tau_1(t))$, $\lambda_{11} < 3/2$ *and* $\int_0^{+\infty} c_{11}(s)ds = +\infty$, *then the zero solution of (25) is uniformly and asymptotically stable. Moreover, if there exist* $\beta_1 > 0$ *and* β_2 *such that* $\int_{t_0}^{t} c_{11}(s)ds \ge \beta_1(t-t_0)+\beta_2$, *then the zero solution of (25) is exponentially, asymptotically stable.*

Proof. If $\lambda \leq 3/2$, choosing $\eta(t,t_0) \equiv 1$, condition (28) holds. From Lemma 1, $P(\lambda) \leq 1$, condition (29) is satisfied. Further (26) implies that condition (30) holds.

If $F(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_n(t))) = -c_{11}(t)x(t-\tau_1(t))$, $\lambda_{11} < 3/2$, let $\theta = \max\{-1, (16\ln P(\lambda_{11}))/81\} < 0$. Further, by choosing

$$\eta(t,t_0) = e^{\theta\int_{t_0+g_1^{-1}(g_1^{-1}(g_1^{-1}(t_0)))}^{t} c_{11}(s)ds},$$

condition (28) holds. Since $\theta \geq (16\ln P(\lambda_{11}))/81$ and $\lambda_{11} < 3/2$,

$$e^{\theta\int_{g_1(g_1(g_1(g_1(t))))}^{t} c_{11}(s)ds} \geq e^{\frac{81}{16}\theta} \geq e^{\ln P(\lambda_{11})} = P(\lambda_{11}),$$

condition (29) is satisfied. For $t \geq t_0$, when $\inf_{u\in[g_1(t),t]}\varphi_1(u) \geq \|\overline{\varphi}\|\eta(t,t_0)$ and $\sup_{u\in[g_1(t),t]}\varphi_1(u) \leq \|\overline{\varphi}\|\eta(g_1(g_1(t)),t_0)$,

$$\begin{aligned} F(t,\varphi_1) &- \|\overline{\varphi}\|\dot{\eta}(t,t_0) \\ &= -c_{11}(t)\varphi_1(g_1(t)) - \|\overline{\varphi}\|\dot{\eta}(t,t_0) \\ &\leq -c_{11}(t)\|\overline{\varphi}\|\eta(t,t_0) - \|\overline{\varphi}\|\dot{\eta}(t,t_0) \\ &= -c_{11}(t)(1+\theta)\|\overline{\varphi}\|\eta(t,t_0) \leq 0, \end{aligned}$$

due to $\theta \geq -1$. This indicates that condition (30) holds.

According to Theorem 4, the conclusion of Corollary 5 is true. ■

Remark 3. Corollary 5 improves the relevant results of [Yorke, 1970; Yoneyama, 1987; Muroya, 2000]. That is, in Theorem 4, the so-called Yorke condition was extended to situation with n delays, and the 3/2 stability theorem was extended to systems with n delays.

Consider the following linear system with pure time delays:

$$\dot{x}(t) = C(t)x(t-\tau(t)), \tag{38}$$

where $C(t) = [c_{ij}(t)]_{n\times n} \in C(\Re, \Re^n \times \Re^n)$ and the delay $\tau(t) = (\tau_1(t), \ldots, \tau_n(t))^T$, $t-\tau_i(t)$ is a continuous increasing function.

Corollary 6. *Let* $\lambda_{ij} = \sup_{t\geq t_0}\int_{t-\tau_i(t)}^{t}|c_{ij}(s)|ds$. *If* $\forall\, t \geq t_0$, $\tau_i(t) \leq \tau$ *(constant) and* $\forall\, i \in \{1,2,\ldots,n\}$,

$$P(\lambda_{ii}) + \lambda_{ii}\sum_{j=1,j\neq i}^{n}\lambda_{ij} + 2\sum_{j=1,j\neq i}^{n}\lambda_{ij} < 1, \tag{39}$$

$$\sup_{t\geq t_0}\left\{c_{ii}(t)\left(1-\sum_{j=1,j\neq i}^{n}\lambda_{ij}\right) + \sum_{j=1,j\neq i}^{n}|c_{ij}(t)|\right\} < 0, \tag{40}$$

then the zero solution of system (38) is globally, exponentially stable.

Proof. Since $\tau_i(t) \leq \tau$ (constant), (39) and (40) imply that there exists a small enough constant $\bar{\theta} > 0$ such that $\forall\, t \geq t_0$, $\forall\, i \in \{1,2,\ldots,n\}$,

$$\begin{aligned} P(\lambda_{ii})\, & e^{\bar{\theta}(t-g_i(g_i(g_i(g_i(t)))))} \\ &+ \lambda_{ii}\sum_{j=1,j\neq i}^{n}\lambda_{ij}\, e^{\bar{\theta}(t-g_j(g_i(g_i(g_i(t)))))} \\ &+ 2\sum_{j=1,j\neq i}^{n}\lambda_{ij}\, e^{\bar{\theta}(t-g_j(g_i(g_i(t))))} \leq 1, \end{aligned} \tag{41}$$

and

$$\begin{aligned} c_{ii}(t)&\left[1-\sum_{j=1,j\neq i}^{n}\lambda_{ij}\, e^{\bar{\theta}(t-g_j(g_i(t)))}\right] \\ &+ \sum_{j=1,j\neq i}^{n}|c_{ij}(t)|\, e^{\bar{\theta}(t-g_j(g_i(t)))} + \bar{\theta} \leq 0. \end{aligned} \tag{42}$$

For $\forall\, t \geq t_0-\tau$, choose $\eta(t,t_0) = \|x\|_{t_0+\tau} \times e^{-\bar{\theta}(t-t_0-\tau)}$. Then (41) and (42) imply that conditions (29) and (30) hold. By Theorem 4, Corollary 6 is proved. ■

Corollary 7. *If* $\forall\, t \geq t_0$, $c_{ij}(t) \equiv c_{ij}$ *(constant)*, $\tau_i(t) \leq \tau$ *(constant), then let* $\tilde{C} = [\tilde{c}_{ij}]_{n\times n}$, *where for* $i \neq j$, $\tilde{c}_{ij} = |c_{ii}||c_{ij}|\tau + |c_{ij}|$ *and* $\tilde{c}_{ii} = c_{ii}$. *Further, if*

$$P(|c_{ii}|\tau) + 2\tau|c_{ii}| < 1, \tag{43}$$

and $-\tilde{C}$ *is a nonsingular M-matrix, then the zero solution of system (38) is globally, exponentially stable.*

Proof. Since $-\tilde{C}$ is a nonsingular M-matrix, there exist positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\forall\, i \in \{1,2,\ldots,n\}$, $c_{ii} < 0$ and

$$\alpha_i c_{ii} + \sum_{j=1,j\neq i}^{n}\alpha_j|c_{ii}||c_{ij}|\tau + \sum_{j=1,j\neq i}^{n}\alpha_j|c_{ij}| < 0. \tag{44}$$

Hence, from (43) we obtain

$$\begin{aligned} P(|c_{ii}|\tau) &+ \sum_{j=1,j\neq i}^{n}\alpha_j|c_{ii}||c_{ij}|\frac{\tau^2}{\alpha_i} + 2\sum_{j=1,j\neq i}^{n}\alpha_j|c_{ij}|\frac{\tau}{\alpha_i} \\ &\leq P(|c_{ii}|\tau) + 2\tau|c_{ii}| \\ &< 1. \end{aligned} \tag{45}$$

For any $t \geq t_0 - \tau$, let $y_i(t) = x_i(t)/\alpha_i$. Then, system (38) can be rewritten as

$$\dot{y}_i(t) = \sum_{j=1}^{n} \frac{\alpha_j}{\alpha_i} c_{ij}\, y_j(t - \tau_j(t)), \quad i = 1, 2, \ldots, n. \tag{46}$$

For the new system (46), the condition (40) holds by (44), and the condition (39) is satisfied due to (45). Thus, according to Corollary 6, the zero solution of system (46) is globally, exponentially stable. Definition of $y_i(t)$ implies that the conclusion of Corollary 7 is true. ■

From Corollaries 5–7, it is easy to find the functions $\eta_i(t, t_0)$ satisfying the conditions (29) and (30) in Theorem 4 for the linear pure time-delay systems. Hence, the stability of the linear pure time-delay systems can be analyzed by the new comparison method.

5. Applications

In this section, we apply the theory established in the precious sections to consider two problems: one is an epidemic model and the other is neural network.

5.1. *An epidemic model*

Consider the following HIV endemic model, formulated for homogeneous population with stages of progress to AIDs and time-varying delay:

$$\begin{cases} \dot{S}(t) = b - \mathcal{C}(N(t-\tau(t)))S(t-\tau(t)) \displaystyle\sum_{r=1}^{m} \beta_r \frac{I_r(t-\tau(t))}{N(t-\tau(t))} - \alpha S(t), \\ \dot{I}_1(t) = \mathcal{C}(N(t-\tau(t)))S(t-\tau(t)) \displaystyle\sum_{r=1}^{m} \beta_r \frac{I_r(t-\tau(t))}{N(t-\tau(t))} - (v_1 + \alpha) I_1(t), \\ \dot{I}_j(t) = v_{j-1} I_{j-1}(t) - (v_j + \alpha) I_j(t), \quad j = 2, 3, \ldots, m, \end{cases} \tag{47}$$

$$\dot{A}(t) = v_m I_m - (\alpha + l) A(t), \tag{48}$$

where $S(t)$, $I_j(t)$ and $A(t)$ denote the numbers of the population susceptible to the disease, of the jth stage infectivity and of HIV, respectively; the positive constants b, α and l represent the renewed rates of susceptibles, the death rates of the HIV-independent, and the death rates of HIV-related, respectively; v_j is the probability of transmission from the jth stage infectivity to the $(j+1)$th stage infectivity; β_j is the probability of transmission from the jth stage infected individual; the delay $\tau(t)$ satisfies $0 \leq \tau(t) \leq \tau$; $N(t) = S(t) + \sum_{j=1}^{m} I_j(t)$; $\mathcal{C}(N)$ satisfies $\mathcal{C}(N) > 0$ and $\dot{\mathcal{C}}(N) \geq 0$. The detailed description of the model can be found in [Hsieh & Sheu, 2001; Driessche & Watmough, 2002; Hyman & Li, 2000; Huang *et al.*, 1992; Moghadas, 2002].

Obviously, system (47) always has a disease-free equilibrium $E^* := (b/\alpha, 0, \ldots, 0)^T$. Let

$$R_0 = \mathcal{C}\left(\frac{b}{\alpha}\right) \left[\frac{\beta_1}{(v_1 + \alpha)} + \frac{\beta_2 v_1}{(v_1 + \alpha)(v_2 + \alpha)} + \frac{\beta_3 v_1 v_2}{(v_1 + \alpha)(v_2 + \alpha)(v_3 + \alpha)} + \cdots + \frac{\beta_m \prod_{j=1}^{m-1} v_j}{\prod_{j=1}^{m} (v_j + \alpha)} \right].$$

Theorem 5. *If $R_0 < 1$, then the disease-free equilibrium E^* of system (47) is globally, exponentially stable.*

Proof. Let $m_{11} = -(v_1 + \alpha) + \mathcal{C}(b/\alpha)\beta_1$. For $j = 2, 3, \ldots, m$, let $m_{1j} = \mathcal{C}(b/\alpha)\beta_j$, $m_{jj} = -(v_j + \alpha)$, $m_{j-1,j} = v_{j-1}$. For $j = 2, 3, \ldots, m$, and $i \neq j$, $i \neq j-1$, let $m_{ij} = 0$. Let matrix $\tilde{M} = [m_{ij}]_{m \times m}$, then $\tilde{M}$ is a nonsingular M-matrix. Since $\mathcal{C}(N(t-\tau(t)))/\mathcal{C}(b/\alpha) \leq 1$ and $S(t-\tau(t))/N(t-\tau(t)) \leq 1$, the last m equations of system (47) imply that

$$\begin{cases} \dot{I}_1(t) \le \mathcal{C}\left(\dfrac{b}{\alpha}\right) \displaystyle\sum_{r=1}^{m} \beta_r I_r(t-\tau(t)) - (v_1+\alpha)I_1(t), \\ \dot{I}_j(t) = v_{j-1}I_{j-1}(t) - (v_j+\alpha)I_j(t), \quad j = 2,3,\ldots,m. \end{cases}$$

Since $\tilde{M}$ is a nonsingular M-matrix, according to Corollary 4, the zero solution of the following system

$$\begin{cases} \dot{I}_1(t) = \mathcal{C}\left(\dfrac{b}{\alpha}\right) \displaystyle\sum_{r=1}^{m} \beta_r I_r(t-\tau(t)) - (v_1+\alpha)I_1(t), \\ \dot{I}_j(t) = v_{j-1}I_{j-1}(t) - (v_j+\alpha)I_j(t), \quad j = 2,3,\ldots,m \end{cases} \tag{49}$$

is globally, exponentially stable. Hence, there exist θ ($\alpha > \theta > 0$), $\Upsilon > 1$ such that the solution of (49) with initial value $I(s) = (I_1(s), I_2(s), \ldots, I_m(s))^T$, $s \in [t_0 - \tau, t_0]$ satisfies

$$I_i(t) \le \Upsilon \|I\|_{t_0} e^{-\theta(t-t_0)}, \quad t \ge t_0, \quad i = 1,2,\ldots,m,$$

where $\|I\|_{t_0} = \sup_{s\in[t_0-\tau,\,t_0]}\{\max_{1\le i\le m}\{I_i(s)\}\}$. From the first equation of system (47),

$$\dot{S}(t) \le b - \mathcal{C}\left(\frac{b}{\alpha}\right)\sum_{j=1}^{m}\beta_j I_j(t-\tau(t)) - \alpha S(t).$$

Now applying the method of constants variation to the above equation shows that

$$\begin{aligned} S(t) - \frac{b}{\alpha} &\le S(t_0)\left(e^{-\alpha(t-t_0)} - \int_{t_0}^{t} e^{-\alpha(t-r)}\mathcal{C}\left(\frac{b}{\alpha}\right)\sum_{j=1}^{m}\beta_j I_j(r-\tau)dr\right) \\ &\le S(t_0)\left(e^{-\alpha(t-t_0)} + \mathcal{C}\left(\frac{b}{\alpha}\right)\sum_{j=1}^{m}\beta_j \Upsilon\|I\|_{t_0}\frac{(e^{-\theta(t-t_0)-\tau} - e^{-\alpha(t-t_0-\tau)})}{\alpha-\theta}\right). \end{aligned}$$

Hence, the disease-free equilibrium E^* of system (47) is globally, exponentially stable. ∎

Remark 4. Similar to the proof of Theorem 5, one can use Theorem 3 to obtain a new sufficient condition for the locally exponential stability of the endemic equilibrium of system (47).

To end this subsection, we show a simulation of this example to verify the above theoretical results. Choose $m = 2$, and the parameters are given by $b = 0.1$, $\alpha = 0.2$, $v_1 = 0.2$, $\beta_1 = 0.2$, $v_2 = 0.1$, $\beta_2 = 0.3$, $l = 0.8$. Further suppose $\tau(t) = |\sin(t)|$, $\mathcal{C}(N) = N^2$, and $t_0 = 0$. Then system (47) becomes

$$\begin{cases} \dot{S}(t) = 0.1 - N(t-|\sin(t)|)S(t-|\sin(t)|)[0.2I_1(t-|\sin(t)|) + 0.3I_2(t-|\sin(t)|)] - 0.2S(t), \\ \dot{I}_1(t) = N(t-|\sin(t)|)\,S(t-|\sin(t)|)[0.2I_1(t-|\sin(t)|) + 0.3I_2(t-|\sin(t)|)] - 0.4I_1(t), \\ \dot{I}_2(t) = 0.2I_1(t) - 0.3I_2(t), \\ \dot{A}(t) = 0.1I_2(t) - A(t). \end{cases} \tag{50}$$

For the above example, it is easy to verify that

$$\begin{aligned} R_0 &= \mathcal{C}\left(\frac{b}{\alpha}\right)\left[\frac{\beta_1}{v_1+\alpha} + \frac{\beta_2 v_1}{(v_1+\alpha)(v_2+\alpha)}\right] \\ &= \frac{1}{4} < 1. \end{aligned}$$

Hence, according to Theorem 4, system (50) is globally, exponentially stable.

The time histories of simulation results for $S(t)$, $I_1(t)$, $I_2(t)$ and $A(t)$ are shown in Figs. 1(a)–1(d), respectively. Different initial points are chosen as $(S(t_0), I_1(t_0), I_2(t_0), A(t_0)) = (0.2, 0.3, 0.5, 1.0)$, $(0.4, 0.2, 0.4, 2.0)$, $(0.5, 0.1, 0.2, 0.5)$ and $(0.8, 0.35, 0.1, 0.3)$. It is clearly seen from these figures that all the trajectories converge to the equilibrium point $(1/2, 0, 0, 0)^T$ exponentially.

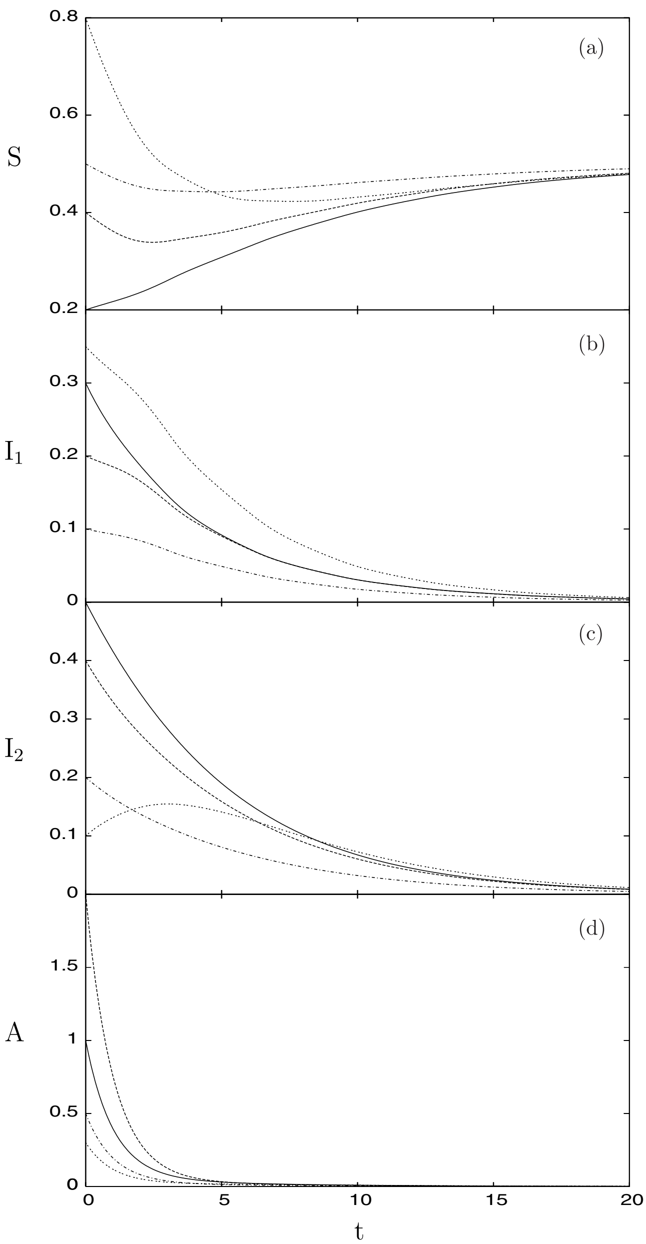

Fig. 1. The time histories of simulated results for system (50): (a) $S(t)$; (b) $I_1(t)$; (c) $I_2(t)$ and (d) $A(t)$.

5.2. *A neural network*

Consider a recurrent neural network model with time-varying delays, described by the following differential equations:

$$\frac{du_i(t)}{dt} = -c_i u_i(t-\tau_i(t)) + \sum_{j=1}^{n} a_{ij}\overline{f}_j(u_j(t-\delta_j(t))) + \sum_{j=1}^{n} b_{ij}\overline{\Upsilon}_j(u_j(t-\Delta_j(t))) + I_i, \quad i = 1,2,\ldots,n, \tag{51}$$

where a_{ij} and b_{ij} are constant connection weights; c_i is a positive constant, $\tau_i(t), \delta_i(t), \Delta_i(t)$, $(i = 1,2,\ldots,n)$ are time-varying delays, satisfying $\tau_i(t) \le \tau_i$, $\delta_i(t) \le \delta_i$, $\Delta_i(t) \le \Delta_i$; τ_i, δ_i, Δ_i are non-negative constants; I_i is an external input or bias, and $\overline{f}_j$ and $\overline{\Upsilon}_j$ are neuron activation functions, $i, j \in \{1,2,\ldots,n\}$. Let $\tau = \max\{\tau_i, i = 1,2,\ldots,n\}$, $\delta = \max\{\delta_i, i = 1,2,\ldots,n\}$, $\Delta = \max\{\Delta_i, i = 1,2,\ldots,n\}$, $\iota = \max\{\tau,\delta,\Delta\}$.

In the following, we assume that for $i = 1,2,\ldots,n$,

A_1: $\overline{f}_i$ and $\overline{\Upsilon}_i$ are bounded functions;
A_2: $\overline{f}_i$ and $\overline{\Upsilon}_i$ are Lipschitz continuous functions; i.e. there exist constants $\mu_i > 0$, $\kappa_i > 0$ such that for any $r_1, r_2, r_3, r_4 \in \Re$,

$$|\overline{f}_i(r_1) - \overline{f}_i(r_2)| \le \mu_i|r_1 - r_2| \quad \text{and}$$
$$|\overline{\Upsilon}_i(r_3) - \overline{\Upsilon}_i(r_4)| \le \kappa_i|r_3 - r_4|;$$

A_3: $t - \tau_i(t)$ is a continuous increasing function.

Obviously, the sigmoid activation function in the Hopfield neural networks [Hopfield, 1984], the linear saturation activation function in the cellular neural networks [Chua & Yang, 1988], and the radial basis function (RBF) in the RBF network all satisfy the above assumptions A_1 and A_2.

It is well known that the equilibrium points of the neural network (51) exist by the Schauder fixed point theorem and assumption A_1. Let $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ be an equilibrium point of the neural network (51), and $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T = (u_1(t) - u_1^*, u_2(t) - u_2^*, \ldots, u_n(t) - u_n^*)^T$. Then, the neural network (51) can be rewritten as

$$\dot{x}_i(t) = -c_i x_i(t-\tau_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\delta_j(t))) + \sum_{j=1}^{n} b_{ij}\Upsilon_j(x_j(t-\Delta_j(t))), \tag{52}$$

where $f_j(x_j(t)) = \overline{f}_j(x_j(t) + u_j^*) - \overline{f}_j(u_j^*)$, $\Upsilon_j(x_j(t)) = \overline{\Upsilon}_j(x_j(t) + u_j^*) - \overline{\Upsilon}_j(u_j^*)$.

Let Ç $= \mathrm{diag}\{c_i\}$, $\mathcal{A} = [\mathbf{a}_{ij}]_{n\times n}$, $\mathbf{a}_{ij} = (c_i\tau_i + 1)(|a_{ij}|\mu_j + |b_{ij}|\kappa_j)$. Consider the linear system with pure time delays:

$$\dot{z}_i(t) = -c_i z_i(t-\tau_i(t)) + \sum_{j=1}^{n} |a_{ij}|\mu_j z_j(t-\delta_j(t)) + \sum_{j=1}^{n} |b_{ij}|\kappa_j z_j(t-\Delta_j(t)). \tag{53}$$

Lemma 2. *Let $\lambda_i = \sup_{t\geq t_0}\{c_i\tau_i(t)\}$. If $\forall\, t \geq t_0$, $\forall\, i \in \{1, 2, \ldots, n\}$,*

$$P(\lambda_i) + c_i\tau_i^2\sum_{j=1}^{n}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j) + 2\tau_i\sum_{j=1}^{n}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j) < 1,$$

and

$$-c_i\left(1 - \tau_i\sum_{j=1}^{n}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j)\right) + \sum_{j=1}^{n}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j) < 0,$$

then the zero solution of system (53) is globally, exponentially stable.

Proof. The proof is similar to that of Theorem 4 and Corollary 6, and thus omitted here. ■

Theorem 6. *If $P(\lambda_i)+c_i\tau_i < 1$ and $Ç-\mathcal{A}$ is a nonsingular M-Matrix, then the neural network (51) is globally, exponentially stable.*

Proof. Form Assumption A_2 and (52) we have

$$\dot{x}_i(t) \leq -c_ix_i(t-\tau_i(t)) + \sum_{j=1}^{n}|a_{ij}|\mu_j|x_j(t-\delta_j(t))| + \sum_{j=1}^{n}|b_{ij}|\kappa_j|x_j(t-\Delta_j(t))|. \tag{54}$$

Since $Ç - \mathcal{A}$ is a nonsingular M-Matrix, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ such that

$$-\alpha_ic_i\left(1 - \tau_i\alpha_j\sum_{j=1}^{n}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j)\right) + \sum_{j=1}^{n}\alpha_j(|a_{ij}|\mu_j + |b_{ij}|\kappa_j) < 0.$$

Let $y_i(t) = x_i(t)/\alpha_i$, $i = 1, 2, \ldots, n$. Then, from (54) we obtain

$$\frac{dy_i(t)}{dt} \leq \frac{1}{\alpha_i}\left[-c_i\alpha_iy_i(t-\tau_i(t)) + \sum_{j=1}^{n}\alpha_j|a_{ij}|\mu_j|y_j(t-\delta_j(t))| + \sum_{j=1}^{n}\alpha_j|b_{ij}|\kappa_j|y_j(t-\Delta_j(t))|\right].$$

The remaining proof can follow the proof of Corollary 7 by using Lemma 2. ■

From Theorem 6, we can directly derive the following results when $\tau_i(t) \equiv 0$.

Corollary 8. *Let $|\overline{A}| = [|a_{ij}|\mu_j]_{n\times n}$, $|\overline{B}| = [|b_{ij}|\kappa_j]_{n\times n}$. If $\tau_i(t) \equiv 0$ and $Ç - |\overline{A}| - |\overline{B}|$ is a nonsingular M-matrix, then the neural network (51) is globally, exponentially stable.*

Proof. $\tau_i(t) \equiv 0$ implies $P(\lambda_i) + c_i\tau_i = 0 < 1$. Also, since $Ç - |\overline{A}| - |\overline{B}|$ is a nonsingular M-matrix, according to Theorem 6, the neural network (51) is globally, exponentially stable. ■

Corollary 9. *If $\tau_i(t) \equiv 0$ and $c_i > \sum_{j=1}^{n}|a_{ij}|\mu_j + \sum_{j=1}^{n}|b_{ij}|\kappa_j$, then the neural network (51) is globally, exponentially stable.*

Proof. Since $c_i > \sum_{j=1}^{n}|a_{ij}|\mu_j + \sum_{j=1}^{n}|b_{ij}|\kappa_j$ and $Ç - |\overline{A}| - |\overline{B}|$ is a nonsingular M-matrix, according to Corollary 6, the neural network (51) is globally, exponentially stable. ■

Remark 5. Corollaries 8 and 9 generalize and improve the corresponding results in [Zhang *et al.*, 2001; Arik, 2002a, 2002b; Chen *et al.*, 2002; Dong, 2002; Huang *et al.*, 2002; Cao & Wang, 2003; Mohamad & Gopalsamy, 2003; Zhang *et al.*, 2003].

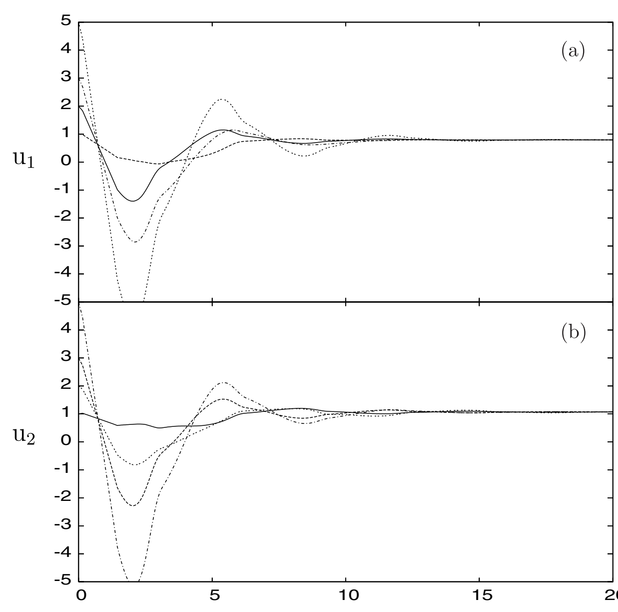

Fig. 2. The time histories of simulated results for system (57): (a) $u_1(t)$; and (b) $u_2(t)$.

Finally, consider a numerical example for the neural network system (51). We take $n = 2$, $c_1 = c_2 = 1$, $I_1 = 0.3$, $I_2 = 0.6$, $a_{11} = a_{22} = 0.1$, $a_{12} = 0.15$, $a_{21} = 0.2$, $b_{11} = b_{22} = 0.1$, $b_{12} = 0.25$, $b_{21} = 0.2$, $t_0 = 0$; $\tau_1(t) = \tau_2(t) = 0.5|\sin(t)|$, $\delta_1(t) = \delta_2(t) = |\sin(t)|$, $\Delta_1(t) = \Delta_2(t) = 0.8|\sin(t)|$, and $\forall\, u \in \Re$,

$$\overline{f}_1(u) = \overline{f}_2(u) = \frac{|u+1| - |u-1|}{2}, \tag{55}$$

$$\overline{\Upsilon}_1(u) = \overline{\Upsilon}_2(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}. \tag{56}$$

Thus, system (51) becomes

$$\begin{cases} \dot{u}_1(t) = -u_i(t - 0.5|\sin(t)|) + 0.1\overline{f}_1(u_1(t - |\sin(t)|)) + 0.15\overline{f}_2(u_2(t - |\sin(t)|)) \\ \qquad\quad + 0.1\,\overline{\Upsilon}_1(u_1(t - 0.8|\sin(t)|)) + 0.25\,\overline{\Upsilon}_2(u_2(t - 0.8|\sin(t)|)) + 0.3, \\ \dot{u}_2(t) = -u_i(t - 0.5|\sin(t)|) + 0.2\overline{f}_1(u_1(t - |\sin(t)|)) + 0.1\overline{f}_2(u_2(t - |\sin(t)|)) \\ \qquad\quad + 0.2\,\overline{\Upsilon}_1(u_1(t - 0.8|\sin(t)|)) + 0.1\,\overline{\Upsilon}_2(u_2(t - 0.8|\sin(t)|)) + 0.6. \end{cases} \tag{57}$$

Therefore, $\lambda_1 = \sup_{t \ge t_0}\{c_1\tau_1(t)\} = 1/2$, $\lambda_2 = \sup_{t \ge t_0}\{c_2\tau_2(t)\} = 1/2$. According to Lemma 1, $P(\lambda_1) = P(\lambda_2) = 1/8$. It follows from (55) and (56) that $\mu_1 = \mu_2 = \kappa_1 = \kappa_2 = 1$. Hence, for $i = 1, 2$,

$$P(\lambda_i) + c_i\tau_i^2 \sum_{j=1}^{2}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j) + 2\tau_i \sum_{j=1}^{2}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j) = \frac{1}{8} + \frac{3}{4} < 1,$$

$$-c_i\left(1 - \tau_i \sum_{j=1}^{n}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j)\right) + \sum_{j=1}^{n}(|a_{ij}|\mu_j + |b_{ij}|\kappa_j) = -1 + 0.9 < 0.$$

According to Theorem 6, (57) is globally, exponentially stable.

Simulated solutions of system (57) are depicted in Figs. 2(a) and 2(b) for $u_1(t)$ and $u_2(t)$, respectively. Again, all solutions starting from different initial points converge to an equilibrium point $(u_1, u_2) = (0.7925, 1.0694)$. The initial points are chosen as $(u_1, u_2) = (2, 1),\ (1, 3),\ (3, 5)$ and $(5, 2)$.

6. Conclusions

In this paper, we present a new comparison method (different from the Lyapunov function method) to study stability of differential systems with multiple delays. The new method is based on the comparison between the expected performance (stability, exponential stability, etc.) and the state of the original differential system. The basic idea of the method is to use the increasing and decreasing mechanisms, which are inherent in time-delay systems, to decompose the system. Based on this method, some conditions have been obtained, which guarantee that the original system tracks the expected values. The locally exponential convergence rate and the convergence region of polynomial differential equations with time-varying delays have also been studied. The results presented in this paper have improved the 3/2 stability theory for differential systems with pure delays. The comparison method has been applied to consider an HIV endemic model and a neural network, and numerical simulation results for these examples verify the theoretical predictions.

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