SUPPLEMENTARY MATERIALS: SPATIOTEMPORAL PATTERNS IN A LENGYEL–EPSTEIN MODEL NEAR A TURING–HOPF SINGULAR POINT*

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SM1. The expressions of u_1^2 and $u_1 v_{1_\tau}$.

$$\begin{aligned} u_1^2 = {} & |p_{T1}|^2 \sum_{j=1}^3 |W_j|^2 + |p_{H1}|^2 |W_H|^2 + 2p_{H1}p_{T1}W_H e^{i\omega_*\tau_* T_0} \sum_{j=1}^3 W_j e^{i\mathbf{k}_j\cdot\mathbf{r}} \\ & + 2\bar{p}_{H1}p_{T1}\bar{W}_H e^{-i\omega_*\tau_* T_0} \sum_{j=1}^3 W_j e^{i\mathbf{k}_j\cdot\mathbf{r}} + p_{T1}^2 \sum_{j=1}^3 W_j^2 e^{2i\mathbf{k}_j\cdot\mathbf{r}} + p_{H1}^2 W_H^2 e^{2i\omega_*\tau_* T_0} \\ & + 2p_{T1}^2 \sum_{\substack{m<n \\ m,n\in\{1,2,3\}}} W_m W_n e^{i(k_m+k_n)\cdot\mathbf{r}} + 2|p_{T1}|^2 \sum_{\substack{m\in n \\ mn\in(1,2,3)}} W_m \bar{W}_n e^{i(k_m-k_n)\cdot\mathbf{r}} + c.c. \end{aligned}$$

$$\begin{aligned} u_1 v_{1\tau} = {} & p_{T1}\bar{p}_{T2} \sum_{j=1}^3 |W_j|^2 + p_{H1}\bar{p}_{H2} e^{i\omega_*\tau_*} |W_H|^2 + p_{H1}p_{T2}W_H e^{i\omega_*\tau_* T_0} \sum_{j=1}^3 W_j e^{i\mathbf{k}_j\cdot\mathbf{r}} \\ & + p_{H2}p_{T1}W_H e^{i\omega_*\tau_*(T_0-1)} \sum_{j=1}^3 W_j e^{i\mathbf{k}_j\cdot\mathbf{r}} + \bar{p}_{H1}p_{T2}\bar{W}_H e^{-i\omega_*\tau_* T_0} \sum_{j=1}^3 W_j e^{i\mathbf{k}_j\cdot\mathbf{r}} \\ & + \bar{p}_{H2}p_{T1}\bar{W}_H e^{-i\omega_*\tau_*(T_0-1)} \sum_{j=1}^3 W_j e^{i\mathbf{k}_j\cdot\mathbf{r}} + p_{T1}p_{T2} \sum_{j=1}^3 W_j^2 e^{2i\mathbf{k}_j\cdot\mathbf{r}} \\ & + p_{H1}p_{H2} e^{i\omega_*\tau_*(2T_0-1)} W_H^2 + 2p_{T1}p_{T2} \sum_{\substack{m<n \\ m,n\{1,2,3\}}} W_m W_n e^{i(k_m+k_n)\cdot\mathbf{r}} \\ & + (p_{T1}\bar{p}_{T2} + p_{T2}\bar{p}_{T1}) \sum_{\substack{m<n \\ m,n\in\{1,2,3\}}} W_m \bar{W}_n e^{i(k_m-k_n)\cdot\mathbf{r}} + c.c. \end{aligned}$$

SM2. The expressions of q_1-q_6 and q_8 in equation (3.12).

$$q_{11} = \frac{b_* b_{22} A_1 - b_{12} A_2}{b_*(a_{21} b_{12} - a_{11} b_{22})}, \quad q_{12} = -\frac{A_1 + a_{11} q_{11}}{b_{12}},$$

$$q_{21} = \frac{b_{12} B_2 - b_* b_{22} B_1}{b_*(a_{11} b_{22} - a_{21} b_{12})}, \quad q_{22} = -\frac{B_1 + a_{11} q_{21}}{b_{12}},$$

$$q_{31} = \frac{\left(i\omega_* + d_2 k_T^2 - b_* b_{22} e^{-i\omega_*\tau_*}\right) C_1 + b_{12} e^{-i\omega_*\tau_*} C_2}{\left(i\omega_* + d_2 k_T^2 - b_* b_{22} e^{-i\omega_*\tau_*}\right)\left(i\omega_* + d_1 k_T^2 - a_{11}\right) - b_* a_{21} b_{12} e^{-i\omega_*\tau_*}}, \quad q_{32} = \frac{\left(i\omega_* + d_1 k_T^2 - a_{11}\right) q_{31} - C_1}{b_{12} e^{-i\omega_*\tau_*}},$$

$$q_{41} = \frac{\left(i\omega_* - d_2 k_T^2 + b_* b_{22} e^{i\omega_*\tau_*}\right) D_1 - b_{12} e^{i\omega_*\tau_*} D_2}{\left(d_1 k_T^2 - i\omega_* - a_{11}\right)\left(i\omega_* - d_2 k_T^2 + b_* b_{22} e^{i\omega_*\tau_*}\right) + b_* a_{21} b_{12} e^{i\omega_*\tau_*}}, \quad q_{42} = \frac{-D_2 - b_* a_{21} q_{41}}{i\omega_* - d_2 k_T^2 + b_* b_{22} e^{i\omega_*\tau_*}},$$

$$q_{51} = \frac{\left(4 d_2 k_T^2 - b_* b_{22}\right) E_1 + b_{12} E_2}{\left(4 d_1 k_T^2 - a_{11}\right)\left(4 d_2 k_T^2 - b_* b_{22}\right) - b_* a_{21} b_{12}}, \quad q_{52} = \frac{E_2 + b_* a_{21} q_{51}}{4 d_2 k_T^2 - b_* b_{22}},$$

$$q_{61} = \frac{\left(2i\omega_* - b_* b_{22} e^{-2i\omega_*\tau_*}\right) F_1 + \left(a_{11} + b_{12} e^{-2i\omega_*\tau_*}\right) F_2}{2i\omega_*(2i\omega_* - b_* b_{22} e^{-2i\omega_*\tau_*}) - b_* a_{21}(a_{11} + b_{12} e^{-2i\omega_*\tau_*})}, \quad q_{62} = \frac{F_2 + b_* a_{21} q_{61}}{2i\omega_* - b_* b_{22} e^{-2i\omega_*\tau_*}},$$

$$q_{81} = \frac{\left(3 d_2 k_T^2 - b_* b_{22}\right) H_1 + b_{12} H_2}{\left(3 d_1 k_T^2 - a_{11}\right)\left(3 d_2 k_T^2 - b_* b_{22}\right) - b_* a_{21} b_{12}}, \quad q_{82} = \frac{H_2 + b_* a_{21} q_{81}}{3 d_2 k_T^2 - b_* b_{22}},$$

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with $A_1 = \delta_{2000} \left|p_{T1}\right|^2 + \delta_{1001} p_{T1} \bar{p}_{T2}$,

$A_2 = \left(b_* + \varepsilon^2 \mu_1\right) \left(\beta_{2000} \left|p_{T1}\right|^2 + \beta_{1001} p_{T1} \bar{p}_{T2}\right)$,

$B_1 = \delta_{2000} \left|p_{H1}\right|^2 + \delta_{1001} p_{H1} \bar{p}_{H2} e^{i\omega_* \tau_*}$,

$B_2 = \left(b_* + \varepsilon^2 \mu_1\right) \left(\beta_{2000} \left|p_{H1}\right|^2 + \beta_{1001} p_{H1} \bar{p}_{H2} e^{i\omega_* \tau_*}\right)$,

$C_1 = 2\delta_{2000} p_{H1} p_{T1} + \delta_{1001} \left(p_{H1} p_{T2} + p_{T1} p_{H2} e^{-i\omega_* \tau_*}\right)$,

$C_2 = \left(b_* + \varepsilon^2 \mu_1\right) \left(2\beta_{2000} p_{H1} p_{T1} + \beta_{1001} \left(p_{H1} p_{T2} + p_{T1} p_{H2} e^{-i\omega_* \tau_*}\right)\right)$,

$D_1 = 2\delta_{2000} p_{T1} \bar{p}_{H1} + \delta_{1001} \left(\bar{p}_{H1} p_{T2} + \bar{p}_{H2} p_{T1} e^{i\omega_* \tau_*}\right)$,

$D_2 = \left(b_* + \varepsilon^2 \mu_1\right) \left(2\beta_{2000} p_{T1} \bar{p}_{H1} + \beta_{1001} \left(\bar{p}_{H1} p_{T2} + \bar{p}_{H2} p_{T1} e^{i\omega_* \tau_*}\right)\right)$,

$E_1 = \delta_{2000} p_{T1}^2 + \delta_{1001} p_{T1} p_{T2}$,

$E_2 = \left(b_* + \varepsilon^2 \mu_1\right) \left(\beta_{2000} p_{T1}^2 + \beta_{1001} p_{T1} p_{T2}\right)$,

$F_1 = \delta_{2000} p_{H1}^2 + \delta_{1001} p_{H1} p_{H2} e^{-i\omega_* \tau_*}$,

$F_2 = \left(b_* + \varepsilon^2 \mu_1\right) \left(\beta_{2000} p_{H1}^2 + \beta_{1001} p_{H1} p_{H2} e^{-i\omega_* \tau_*}\right)$,

$G_1 = 2p_{T1} \left(\delta_{2000} p_{T1} + \delta_{1001} p_{T2}\right)$,

$G_2 = 2p_{T1} \left(b_* + \varepsilon^2 \mu_1\right) \left(\beta_{2000} p_{T1} + \beta_{1001} p_{T2}\right)$,

$H_1 = 2\delta_{2000} \left|p_{T1}\right|^2 + \delta_{1001} \left(p_{T1} \bar{p}_{T2} + p_{T2} \bar{p}_{T1}\right)$,

$H_2 = \left(b_* + \varepsilon^2 \mu_1\right) \left(2\beta_{2000} \left|p_{T1}\right|^2 + \beta_{1001} \left(p_{T1} \bar{p}_{T2} + p_2 \bar{p}_{T1}\right)\right)$.

SM3. The expressions of $\mathbf{u}_i = \left(u_i^{(1)}, u_i^{(2)}\right)$, $i = 1, 2, 3, 4$ in equation (3.13).

$$
\begin{cases}
u_1^{(1)} = \mathcal{A}_1 \frac{\partial W_H}{\partial T_2} + \mathcal{A}_2 W_H + \mathcal{A}_3 W_H |W_H|^2 + \mathcal{A}_4 \left(|W_1|^2 + |W_2|^2 + |W_3^2|\right) W_H, \\
u_1^{(2)} = \mathcal{B}_1 \frac{\partial W_H}{\partial T_2} + \mathcal{B}_2 W_H + \mathcal{B}_3 W_H |W_H|^2 + \mathcal{B}_4 \left(|W_1|^2 + |W_2|^2 + |W_3^2|\right) W_H, \\
u_2^{(1)} = \mathcal{C}_1 \frac{\partial W_1}{\partial T_2} + \mathcal{C}_2 W_1 + \mathcal{C}_3 \left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \mathcal{C}_4 |W_1|^2 W_1 + \mathcal{C}_5 (|W_2|^2 + |W_3|^2) W_1 + \mathcal{C}_6 |W_H|^2 W_1, \\
u_2^{(2)} = \mathcal{D}_1 \frac{\partial W_1}{\partial T_2} + \mathcal{D}_2 W_1 + \mathcal{D}_3 \left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \mathcal{D}_4 |W_1|^2 W_1 + \mathcal{D}_5 (|W_2|^2 + |W_3|^2) W_1 + \mathcal{D}_6 |W_H|^2 W_1, \\
u_3^{(1)} = \mathcal{C}_1 \frac{\partial W_2}{\partial T_2} + \mathcal{C}_2 W_2 + \mathcal{C}_3 \left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \mathcal{C}_4 |W_2|^2 W_2 + \mathcal{C}_5 (|W_1|^2 + |W_3|^2) W_2 + \mathcal{C}_6 |W_H|^2 W_2, \\
u_3^{(2)} = \mathcal{D}_1 \frac{\partial W_2}{\partial T_2} + \mathcal{D}_2 W_2 + \mathcal{D}_3 \left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \mathcal{D}_4 |W_2|^2 W_2 + \mathcal{D}_5 (|W_1|^2 + |W_3|^2) W_2 + \mathcal{D}_6 |W_H|^2 W_2, \\
u_4^{(1)} = \mathcal{C}_1 \frac{\partial W_3}{\partial T_2} + \mathcal{C}_2 W_3 + \mathcal{C}_3 \left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \mathcal{C}_4 |W_3|^2 W_3 + \mathcal{C}_5 (|W_1|^2 + |W_2|^2) W_3 + \mathcal{C}_6 |W_H|^2 W_3, \\
u_4^{(2)} = \mathcal{D}_1 \frac{\partial W_3}{\partial T_2} + \mathcal{D}_2 W_3 + \mathcal{D}_3 \left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \mathcal{D}_4 |W_3|^2 W_3 + \mathcal{D}_5 (|W_1|^2 + |W_2|^2) W_3 + \mathcal{D}_6 |W_H|^2 W_3,
\end{cases}
$$

where $\mathcal{A}_1 = -p_{H1} - \tau_* b_{12} p_{H2} e^{-i\omega_* \tau_*}$,

$\mathcal{A}_2 = \mu_2 \left(a_{11} p_{H1} + b_{12} p_{H2} e^{-i\omega_* \tau_*}\right)$,

$$
\begin{aligned}
\mathcal{A}_3 = \tau_* & \left(2\delta_{2000} \left(\bar{p}_{H1} q_{61} + 2p_{H1} \mathbf{Re}\left\{q_{21}\right\}\right) + \delta_{1001} \left(2 \left(p_{H1} \mathbf{Re}\left\{q_{22}\right\}\right.\right.\right. \\
& \left.\left. + p_{H2} \mathbf{Re}\left\{q_{21}\right\} e^{-i\omega_* \tau_*}\right) + \bar{p}_{H1} q_{62} e^{-2i\omega_* \tau_*} + \bar{p}_{H2} q_{61} e^{i\omega_* \tau_*}\right) \\
& \left. + 3\delta_{3000} p_{H1} \left|p_{H1}\right|^2 + \delta_{2001} \left(2p_{H2} \left|p_{H1}\right|^2 e^{-i\omega_* \tau_*} + \bar{p}_{H2} p_{H1}^2 e^{i\omega_* \tau_*}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_4 = \tau_* & \left(2\delta_{2000} \left(p_{T1} \bar{q}_{41} + \bar{p}_{T1} q_{31} + 2p_{H1} \mathbf{Re}\left\{q_{11}\right\}\right) + \delta_{1001} \left(p_{T2} \bar{q}_{41} + \bar{p}_{T2} q_{31}\right.\right. \\
& \left. + \left(p_{T1} \bar{q}_{42} + \bar{p}_{T1} q_{32}\right) e^{-i\omega_* \tau_*} + 2 \left(p_{H1} \mathbf{Re}\left\{q_{12}\right\} + p_{H2} \mathbf{Re}\left\{q_{11}\right\} e^{-i\omega_* \tau_*}\right)\right) \\
& \left. + 6\delta_{3000} p_{H1} \left|p_{T1}\right|^2 + 2\delta_{2001} \left(p_{H1} \left(p_{T1} \bar{p}_{T2} + p_{T2} \bar{p}_{T1}\right) + p_{H2} \left|p_{T1}\right|^2 e^{-i\omega_* \tau_*}\right)\right),
\end{aligned}
$$

$$\mathcal{B}_1 = -p_{H2} - \tau_* b_* b_{22} p_{H2} e^{-i\omega_*\tau_*},$$

$$\mathcal{B}_2 = \tau_*\mu_1\left(a_{21}p_{H1} + b_{22}p_{H2}e^{-i\omega_*\tau_*}\right) + \mu_2\left(b_*a_{21}p_{H1} + b_*b_{22}p_{H2}e^{-i\omega_*\tau_*}\right),$$

$$\begin{aligned}\mathcal{B}_3 = &\left(b_* + \varepsilon^2\mu_1\right)\tau_*\left(2\beta_{2000}\left(\bar{p}_{H1}q_{61} + 2p_{H1}\mathbf{Re}\left\{q_{21}\right\}\right) + \beta_{1001}\left(2\left(p_{H1}\mathbf{Re}\left\{q_{22}\right\}\right.\right.\right.\\ &\left.\left.+p_{H2}\mathbf{Re}\left\{q_{21}\right\}e^{-i\omega_*\tau_*}\right) + \bar{p}_{H1}q_{62}e^{-2i\omega_*\tau_*} + \bar{p}_{H2}q_{61}e^{i\omega_*\tau_*}\right)\\ &\left.+3\beta_{3000}p_{H1}\left|p_{H1}\right|^2 + \beta_{2001}\left(2p_{H2}\left|p_{H1}\right|^2e^{-i\omega_*\tau_*} + \bar{p}_{H2}p_{H1}^2e^{i\omega_*\tau_*}\right)\right),\end{aligned}$$

$$\begin{aligned}\mathcal{B}_4 = &\left(b_*+\varepsilon^2\mu_1\right)\tau_*\left(2\beta_{2000}(p_{T1}\bar{q}_{41}+\bar{p}_{T1}q_{31}+2p_{H1}\mathbf{Re}\{q_{11}\})+\beta_{1001}(p_{T2}\bar{q}_{41}+\bar{p}_{T2}q_{31}\right.\\ &\left.+\left(p_{T1}\bar{q}_{42} + \bar{p}_{T1}q_{32}\right)e^{-i\omega_*\tau_*} + 2\left(p_{H1}\mathbf{Re}\left\{q_{12}\right\} + p_{H2}\mathbf{Re}\left\{q_{11}\right\}e^{-i\omega_*\tau_*}\right)\right)\\ &\left.+6\beta_{3000}p_{H1}\left|p_{T1}\right|^2 + 2\beta_{2001}\left(p_{H1}\left(p_{T1}\bar{p}_{T2} + p_{T2}\bar{p}_{T1}\right) + p_{H2}\left|p_{T1}\right|^2e^{-i\omega_*\tau_*}\right)\right),\end{aligned}$$

$$\mathcal{C}_1 = -p_{T1} - \tau_* b_{12} p_{T2},$$

$$\mathcal{C}_2 = \mu_2\left(\left(-d_1k_T^2 + a_{11}\right)p_{T1} + b_{12}p_{T2}\right),$$

$$\mathcal{C}_3 = 2\tau_*\left(\delta_{2000}\bar{p_{T1}}^2 + \delta_{1001}\bar{p_{T1}}\bar{p_{T2}}\right),$$

$$\begin{aligned}\mathcal{C}_4 = &\tau_*\left(2\delta_{2000}\left(2p_{T1}\mathbf{Re}\left\{q_{11}\right\} + \bar{p}_{T1}q_{51}\right)\right.\\ &+\delta_{1001}\left(2\left(p_{T1}\mathbf{Re}\left\{q_{12}\right\} + p_{T2}\mathbf{Re}\left\{q_{11}\right\}\right) + \bar{p}_{T1}q_{52} + \bar{p}_{T2}q_{51}\right)\\ &\left.+3\delta_{3000}p_{T1}\left|p_{T1}\right|^2 + \delta_{2001}\left(p_{T1}^2\bar{p}_{T2} + 2\left|p_{T1}\right|^2p_{T2}\right)\right),\end{aligned}$$

$$\begin{aligned}\mathcal{C}_5 = &\tau_*\left(2\delta_{2000}\left(p_{T1}q_{81} + 2p_{T1}\mathbf{Re}\left\{q_{11}\right\}\right)\right.\\ &+\delta_{1001}\left(p_{T1}q_{82} + p_{T2}q_{81} + 2\left(p_{T1}\mathbf{Re}\left\{q_{12}\right\} + p_{T2}\mathbf{Re}\left\{q_{11}\right\}\right)\right)\\ &\left.+ 6\delta_{3000}p_{T1}\left|p_{T1}\right|^2 + 2\delta_{2001}\left(p_{T1}^2\bar{p}_{T2} + 2\left|p_{T1}\right|^2p_{T2}\right)\right),\end{aligned}$$

$$\begin{aligned}\mathcal{C}_6 = &\tau_*\left(2\delta_{2000}\left(p_{H1}q_{41}+\bar{p}_{H1}q_{31}+2p_{T1}\mathbf{Re}\left\{q_{21}\right\}\right)+\delta_{1001}\left(\left(\bar{p}_{H1}q_{32}+p_{H2}q_{41}\right)e^{-i\omega_*\tau_*}\right.\right.\\ &\left.+\left(p_{H1}q_{42} + \bar{p}_{H2}q_{31}\right)e^{i\omega_*\tau_*} + 2\left(p_{T1}\mathbf{Re}\left\{q_{22}\right\} + p_{T2}\mathbf{Re}\left\{q_{21}\right\}\right)\right),\\ &\left.+6\delta_{3000}p_{T1}\left|p_{H1}\right|^2 + 2\delta_{2001}\left(\left|p_{H1}\right|^2p_{T2} + 2p_{T1}\mathbf{Re}\left\{p_{H1}\bar{p}_{H2}e^{i\omega_*\tau_*}\right\}\right)\right),\end{aligned}$$

$$\mathcal{D}_1 = -p_{T2} - \tau_* b_* b_{22} p_{T2},$$

$$\mathcal{D}_2 = \tau_*\mu_1\left(a_{21}p_{T1} + b_{22}p_{T2}\right) + \mu_2\left(b_*a_{21}p_{T1} + \left(-d_2k_T^2 + b_*b_{22}\right)p_{T2}\right),$$

$$\mathcal{D}_3 = 2\left(b_* + \varepsilon^2\mu_1\right)\tau_*\left(\beta_{2000}\bar{p_{T1}}^2 + \beta_{1001}\bar{p_{T1}}\bar{p_{T2}}\right),$$

$$\begin{aligned}\mathcal{D}_4 = &\left(b_* + \varepsilon^2\mu_1\right)\tau_*\left(2\beta_{2000}\left(2p_{T1}\mathbf{Re}\left\{q_{11}\right\} + \bar{p}_{T1}q_{51}\right)\right.\\ &+\beta_{1001}\left(2\left(p_{T1}\mathbf{Re}\left\{q_{12}\right\} + p_{T2}\mathbf{Re}\left\{q_{11}\right\}\right) + \bar{p}_{T1}q_{52} + \bar{p}_{T2}q_{51}\right)\\ &\left.+ 3\beta_{3000}p_{T1}\left|p_{T1}\right|^2 + \beta_{2001}\left(p_{T1}^2\bar{p}_{T2} + 2\left|p_{T1}\right|^2p_{T2}\right)\right),\end{aligned}$$

$$\begin{aligned}\mathcal{D}_5 = &\left(b_* + \varepsilon^2\mu_1\right)\tau_*\left(2\beta_{2000}\left(p_{T1}q_{81} + 2p_{T1}\mathbf{Re}\left\{q_{11}\right\}\right)\right.\\ &+\beta_{1001}\left(p_{T1}q_{82} + p_{T2}q_{81} + 2\left(p_{T1}\mathbf{Re}\left\{q_{12}\right\} + p_{T2}\mathbf{Re}\left\{q_{11}\right\}\right)\right)\\ &\left.+ 6\beta_{3000}p_{T1}\left|p_{T1}\right|^2 + 2\beta_{2001}\left(p_{T1}^2\bar{p}_{T2} + 2\left|p_{T1}\right|^2p_{T2}\right)\right),\end{aligned}$$

$$\begin{aligned}\mathcal{D}_6 = &\left(b_* + \varepsilon^2\mu_1\right)\tau_*\left(2\beta_{2000}\left(p_{H1}q_{41} + \bar{p}_{H1}q_{31} + 2p_{T1}\mathbf{Re}\left\{q_{21}\right\}\right) + \beta_{1001}\left(\left(\bar{p}_{H1}q_{32}\right.\right.\right.\\ &\left.\left.+p_{H2}q_{41}\right)e^{-i\omega_*\tau_*}+\left(p_{H1}q_{42}+\bar{p}_{H2}q_{31}\right)e^{i\omega_*\tau_*}+2\left(p_{T1}\mathbf{Re}\left\{q_{22}\right\}+p_{T2}\mathbf{Re}\left\{q_{21}\right\}\right)\right)\\ &\left.+6\beta_{3000}p_{T1}\left|p_{H1}\right|^2 + 2\beta_{2001}\left(\left|p_{H1}\right|^2p_{T2} + 2p_{T1}\mathbf{Re}\left\{p_{H1}\bar{p}_{H2}e^{i\omega_*\tau_*}\right\}\right)\right).\end{aligned}$$

SM4. The elements ($\mathbf{p}_i^*$, $i = 1, 2, 3, 4$) in the null space of the homogeneous adjoint problem corresponding to equation (3.13) and the equations that amplitudes W_H and W_j, $j = 1, 2, 3$ meet.

$\mathbf{p}_i^*$, $i = 1, 2, 3, 4$ are listed as follows:

$$\mathbf{p}_1^* = \mathrm{col}\left(\frac{b_* a_{21} b_{12} e^{i\omega_* \tau_*}}{b_* a_{21} b_{12} e^{i\omega_* \tau_*} + \left(i\omega_* - a_{11}\right)^2}, \frac{b_{12}\left(-i\omega_* - a_{11}\right) e^{i\omega_* \tau_*}}{b_* a_{21} b_{12} e^{i\omega_* \tau_*} + \left(-i\omega_* - a_{11}\right)^2}\right) \triangleq \begin{pmatrix} p_{11}^* \\ p_{12}^* \end{pmatrix},$$

$$\mathbf{p}_2^* = \mathrm{col}\left(\frac{b_* a_{21} b_{12}}{\left(d_1 k_T^2 - a_{11}\right)^2 + b_* a_{21} b_{12}}, \frac{b_{12}\left(d_1 k_T^2 - a_{11}\right)}{\left(d_1 k_T^2 - a_{11}\right)^2 + b_* a_{21} b_{12}}\right) \triangleq \begin{pmatrix} p_{21}^* \\ p_{22}^* \end{pmatrix},$$

$$\mathbf{p}_3^* = \mathbf{p}_2^* \triangleq \mathrm{col}\left(p_{31}^*, p_{32}^*\right), \quad \mathbf{p}_4^* = \mathbf{p}_2^* \triangleq \mathrm{col}\left(p_{41}^*, p_{42}^*\right).$$

The equations satisfied by amplitude W_H and W_j, $j = 1, 2, 3, 4$:

$$\begin{cases} \frac{\partial W_H}{\partial T_2} = \widetilde{\mu} W_H + \widetilde{\varphi}\left|W_H\right|^2 W_H + \widetilde{\psi}\left(\left|W_1\right|^2 + \left|W_2\right|^2 + \left|W_3\right|^2\right) W_H, \\ \frac{\partial W_1}{\partial T_2} = \widetilde{\kappa} W_1 + \widetilde{\chi}\left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \widetilde{\eta}\left|W_1\right|^2 W_1 + \widetilde{\zeta}\left(\left|W_2\right|^2 + \left|W_3\right|^2\right) W_1 + \widetilde{\xi}\left|W_H\right|^2 W_1, \\ \frac{\partial W_2}{\partial T_2} = \widetilde{\kappa} W_2 + \widetilde{\chi}\left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \widetilde{\eta}\left|W_2\right|^2 W_2 + \widetilde{\zeta}\left(\left|W_1\right|^2 + \left|W_3\right|^2\right) W_2 + \widetilde{\xi}\left|W_H\right|^2 W_2, \\ \frac{\partial W_3}{\partial T_2} = \widetilde{\kappa} W_3 + \widetilde{\chi}\left(\bar{W}_2 \bar{V}_3 + \bar{W}_3 \bar{V}_2\right) + \widetilde{\eta}\left|W_3\right|^2 W_3 + \widetilde{\zeta}\left(\left|W_1\right|^2 + \left|W_2\right|^2\right) W_3 + \widetilde{\xi}\left|W_H\right|^2 W_3, \end{cases}$$

where $\widetilde{\mu} = -\frac{\left(\bar{p}_{11}^* \mathcal{A}_2 + \bar{p}_{12}^* \mathcal{B}_2\right)}{\left(\bar{p}_{11}^* \mathcal{A}_1 + \bar{p}_{12}^* \mathcal{B}_1\right)}$, $\widetilde{\varphi} = -\frac{\left(\bar{p}_{11}^* \mathcal{A}_3 + \bar{p}_{12}^* \mathcal{B}_3\right)}{\left(\bar{p}_{11}^* \mathcal{A}_1 + \bar{p}_{12}^* \mathcal{B}_1\right)}$, $\widetilde{\psi} = -\frac{\left(\bar{p}_{11}^* \mathcal{A}_4 + \bar{p}_{12}^* \mathcal{B}_4\right)}{\left(\bar{p}_{11}^* \mathcal{A}_1 + \bar{p}_{12}^* \mathcal{B}_1\right)}$, $\widetilde{\kappa} = -\frac{\left(\bar{p}_{21}^* \mathcal{C}_2 + \bar{p}_{22}^* \mathcal{D}_2\right)}{\left(\bar{p}_{21}^* \mathcal{C}_1 + \bar{p}_{22}^* \mathcal{D}_1\right)}$, $\widetilde{\chi} = -\frac{\left(\bar{p}_{21}^* \mathcal{C}_3 + \bar{p}_{22}^* \mathcal{D}_3\right)}{\left(\bar{p}_{21}^* \mathcal{C}_1 + \bar{p}_{22}^* \mathcal{D}_1\right)}$, $\widetilde{\eta} = -\frac{\left(\bar{p}_{21}^* \mathcal{C}_4 + \bar{p}_{22}^* \mathcal{D}_4\right)}{\left(\bar{p}_{21}^* \mathcal{C}_1 + \bar{p}_{22}^* \mathcal{D}_1\right)}$, $\widetilde{\zeta} = -\frac{\left(\bar{p}_{21}^* \mathcal{C}_5 + \bar{p}_{22}^* \mathcal{D}_5\right)}{\left(\bar{p}_{21}^* \mathcal{C}_1 + \bar{p}_{22}^* \mathcal{D}_1\right)}$, $\widetilde{\xi} = -\frac{\left(\bar{p}_{21}^* \mathcal{C}_6 + \bar{p}_{22}^* \mathcal{D}_6\right)}{\left(\bar{p}_{21}^* \mathcal{C}_1 + \bar{p}_{22}^* \mathcal{D}_1\right)}$.

SM5. Amplitude equations in complex coordinate system and its coefficient expressions.

$$\begin{cases} \frac{\partial H^v}{\partial t} = \mu H^v + \varphi\left|H^v\right|^2 H^v + \psi\left(\left|T_1^v\right|^2 + \left|T_2^v\right|^2 + \left|T_3^v\right|^2\right) H^v, \\ \frac{\partial T_1^v}{\partial t} = \kappa T_1^v + \chi \bar{T}_2^v \bar{T}_3^v + \eta\left|T_1^v\right|^2 T_1^v + \zeta\left(\left|T_2^v\right|^2 + \left|T_3^v\right|^2\right) T_1^v + \xi\left|H^v\right|^2 T_1^v, \\ \frac{\partial T_2^v}{\partial t} = \kappa T_2^v + \chi \bar{T}_2^v \bar{T}_3^v + \eta\left|T_2^v\right|^2 T_2^v + \zeta\left(\left|T_1^v\right|^2 + \left|T_3^v\right|^2\right) T_2^v + \xi\left|H^v\right|^2 T_2^v, \\ \frac{\partial T_3^v}{\partial t} = \kappa T_3^v + \chi \bar{T}_2^v \bar{T}_3 + \eta\left|T_3^v\right|^2 T_3^v + \zeta\left(\left|T_1^v\right|^2 + \left|T_2^v\right|^2\right) T_3^v + \xi\left|H^v\right|^2 T_3^v, \end{cases}$$

where $\mu = \varepsilon^2 \widetilde{\mu}$, $\varphi = \frac{\widetilde{\varphi}}{p_{H2}^2}$, $\psi = \frac{\widetilde{\psi}}{p_{T2}^2}$, $\kappa = \varepsilon^2 \widetilde{\kappa}$, $\chi = \frac{\widetilde{\chi}}{p_{T2}}$, $\eta = \frac{\widetilde{\eta}}{p_{T2}^2}$, $\zeta = \frac{\widetilde{\zeta}}{p_{T2}^2}$, $\xi = \frac{\widetilde{\xi}}{p_{H2}^2}$.

SM6. Amplitude equations in real coordinate system.

$$\begin{cases} \frac{\partial \Theta}{\partial t} = -\beta \frac{z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2}{z_1 z_2 z_3} \sin\Theta, \\ \frac{d\rho}{dt} = \mathbf{Re}\{\mu\}\rho + \mathbf{Re}\{\varphi\}\rho^3 + \mathbf{Re}\{\psi\}\left(z_1^2 + z_2^2 + z_3^2\right)\rho, \\ \frac{\partial z_1}{\partial t} = \kappa z_1 + \chi z_2 z_3 \cos\Theta + \eta z_1^3 + \zeta\left(z_2^2 + z_3^2\right) z_1 + \mathbf{Re}\{\xi\}\rho^2 z_1, \\ \frac{\partial z_2}{\partial t} = \kappa z_2 + \chi z_1 z_3 \cos\Theta + \eta z_2^3 + \zeta\left(z_1^2 + z_3^2\right) z_2 + \mathbf{Re}\{\xi\}\rho^2 z_2, \\ \frac{\partial z_3}{\partial t} = \kappa z_3 + \chi z_1 z_2 \cos\Theta + \eta z_3^3 + \zeta\left(z_1^2 + z_2^2\right) z_3 + \mathbf{Re}\{\xi\}\rho^2 z_3, \\ \frac{dC}{dt} = \mathbf{Im}\{\mu\} + \mathbf{Im}\{\varphi\}\rho^2 + \mathbf{Im}\{\psi\}\left(z_1^2 + z_2^2 + z_3^2\right), \end{cases}$$

with $\Theta = \Theta_1 + \Theta_2 + \Theta_3$.