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A HIERARCHICAL PARAMETRIC ANALYSIS ON HOPF BIFURCATION OF AN EPIDEMIC MODEL

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Abstract. A common task in studying nonlinear dynamical systems is to derive the conditions on stability and bifurcations, which becomes difficulty when the system contain multiple parameters. In particular, finding the explicit conditions under which Hopf bifurcation can occur is not easy and becomes very involved even for simple models. In this paper, an epidemic model is presented to illustrate how to use a hierarchical parametric analysis for bifurcation study, in particular to demonstrate how to choose proper parameters as bifurcation parameters, how to deal with other "control" parameters, and how to derive the conditions on stability and Hopf bifurcation, which are explicitly expressed in terms of system parameters.

1. **Introduction.** Bifurcation and limit cycle theory plays a very important role in the study of nonlinear dynamical systems, related to the well-known phenomenon of self-oscillations arising from sciences and engineering [5, 6]. Hopf and Bogdanov-Takens bifurcations are two main bifurcations generating limit cycles in real world systems. A common task of the study for such systems is to determine the conditions under which bifurcations may happen and to derive the associated normal form, which is not easy even for lower-dimensional systems. Particularly, when considering practical systems, due to physical limitations on the system parameters, determining the conditions on stability and bifurcations becomes very difficult such as determining the codimension of bifurcations. When a dynamical system contains multiple parameters, the classical method for stability and bifurcation analysis is often to put all parameters in a general formula, which makes it very difficult in applications, since such a general formula does not provide any clues for choosing parameter values in identifying different bifurcation phenomena. A more sophisticated approach is to choose appropriate parameters as bifurcation/perturbation parameters so that the unfolding in the normal form can be properly determined. The chosen bifurcation parameters usually play an important role in the bifurcation analysis, but their choice is not unique and often depends upon the physical properties. Once the bifurcation parameters are determined, then how to treat the remaining parameters, called "control parameters", is also important since some of these parameters may determine the "marginal values" in the parameter space for

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bifurcation analysis. Therefore, developing a hierarchical parametric analysis, like what was used in [7, 14], is necessary for the applications in solving stability and bifurcations arising from real world problems.

Another difficulty in solving bifurcation problems is to determine the maximal number of bifurcating limit cycles, since it is related to complex behaviours of the system. For example, suppose in a 2-dimensional dynamical system there exists one unstable limit cycle arising from Hopf bifurcation, which encloses a stable focus, then the phase portraits of the system have at least two regions on the 2-dimensional plane, which are separated by the unstable limit cycle, and each of the regions may exhibit complete different dynamical behaviours. However, due to more parameters involved in the system, there may simultaneously exist two limit cycles enclosing a stable focus, the situation becomes more complex since there are three regions separated by the two limit cycles, and each region may exhibit different dynamical behaviour. In reality, there always exist physical constraints on system parameters, which makes it much more difficult in determining the number of bifurcating limit cycles. For instance, considering the limit cycles arising from Hopf bifurcation in a nonlinear system described by ordinary differential equations in a 2-dimensional plane, we suppose that the system involves 4 real parameters. In general, the maximal number of limit cycles may be 4, the same as the number of parameters, if there are no restrictions on these parameters, for example, all parameters are assumed to be real, leading to solving multi-variate polynomial equations in real domain. However, if it is a biological system, due to physical limitation on the parameters, the maximal number of limit cycles might be 3, 2, or even only 1. In such a case, it is much more difficult to determine the maximal number of bifurcation limit cycles, that is, to determine the codimension of Hopf bifurcation. The difficulty is not only from computing the normal form (or the focus values) associated with generalized Hopf bifurcation, but also from solving the polynomial systems, since one needs to determine the sign of the polynomials with many variables (parameters).

For Bogdanov-Takens bifurcation, the analysis of codimension-2 Bogdanov-Takens bifurcation has become standard [5, 8]. However, for codimension-3 or higher-codimension Bogdanov-Takens bifurcations, the computation of the normal forms becomes much more involved, particularly in order to establish the relation between the original system and the simplified system (the normal form). Recently, an efficient computation approach, called "one-step" transformation method, has been developed [14, 15] for higher-codimensional Bogdanov-Takens bifurcations, which is based on the simplest normal form theory [4, 13]. This approach not only derives the simplest normal form with unfolding in one unified step, but also generates the nonlinear transformations between the normal form and the original system, as well as that between the new and the original bifurcation parameters. That makes it very convenient in real applications. However, this approach demands more computational effort.

In this paper, we will use a simple epidemic model to illustrate how to determine the explicit conditions on Hopf bifurcation. The epidemic model has been studied in [10] for Hopf bifurcation and codimension-2 Bogdanov-Takens bifurcation. Later, Li *et al.* [9] gave a complete analysis on the codimension-3 Bogdanov-Takens bifurcation using the method developed in [3]. In the next section, we introduce the simple epidemic model and briefly discuss the solution property of the model. In Section 3, we provide a complete stability analysis and obtain all conditions for generating Hopf bifurcation. A concluding remark is given in Section 4.

2. **System modelling and solution property.** In this paper, the simple SI-epidemic model studied in [10] will be reinvestigated and main attention is focused on stability and bifurcations, specially on Hopf bifurcation. This model is described by the following ordinary differential equations:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= A - dS - \beta(1+\varepsilon I)SI, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta(1+\varepsilon I)SI - (d+\alpha)I, \end{aligned} \tag{1}$$

where S and I represent the numbers of the susceptible and infective populations, respectively; A, d and α denote the recruitment rate of susceptibles, the nature death rate, and the sum of the recover rate and the disease-related death rate, respectively; and $\beta(1+\varepsilon I)SI$ is the incidence rate, suggested in [1, 2, 11]. All the parameters A, d, α, β and ε take real positive values.

Let $N = S + I$ be the total number of the susceptibles and infectives. Then system (1) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\tau} &= X\big[k(1+\varepsilon X)(Y-X) - (n+1)\big], \\ \frac{\mathrm{d}Y}{\mathrm{d}\tau} &= m - nY - X, \end{aligned} \tag{2}$$

where

$$X = I, \quad Y = N, \quad \mathrm{d}\tau = \alpha\,\mathrm{d}t, \quad k = \frac{\beta}{\alpha}, \quad m = \frac{A}{\alpha}, \quad n = \frac{d}{\alpha}. \tag{3}$$

It should be pointed out that solutions of most epidemic models have well-posedness property, that is, solutions remain positive if initial points are chosen positive, and are bounded. However, for the model (2), the first quadrant in the X-Y plane is not invariant. Trajectories starting from the initial points in the first quadrant may pass through the X-axis to enter the fourth quadrant and then return to the first quadrant, as shown in Figure 1. Note that since the Y-axis is invariant, any trajectories starting from the initial points in the first or fourth quadrant will stay in these two quadrants and eventually enter the first quadrant. In other words, if the domain is restricted to the region: $\{(X,Y)\,|\,X \geqslant 0\}$, then the well-posedness property on the solutions of (2) is well defined. The domain of interest for system (2) may be defined as (see Figure 1)

$$\Omega = \left\{ (X,Y) \,\middle|\, 0 \leqslant X < Y \leqslant \frac{m}{n} \right\}. \tag{4}$$

It should be noted that Ω is invariant, but it does not serve as a trapping region for the dynamical solutions in the first quadrant. However, any trajectory starting from a point in Ω will be remained in Ω. This can be simply proved as follows. The Y-axis is invariant, and more precisely, the trajectory $X = 0$ $(Y \in R)$ moves towards the equilibrium $(X,Y) = (0, \frac{m}{n})$. On the line segment $Y = \frac{m}{n}$ $(0 < X < \frac{m}{n})$, it is obvious that $\frac{\mathrm{d}Y}{\mathrm{d}\tau} < 0$ and $\frac{\mathrm{d}X}{\mathrm{d}\tau} < 0$ as long as $X > \frac{m}{n} - \frac{n(n+1)}{k(n+m\varepsilon)}$, which holds for the whole line segment if $\frac{m}{n} - \frac{n(n+1)}{k(n+m\varepsilon)} \leqslant 0$, and at least for X near the end point $(X,Y) = (\frac{m}{n}, \frac{m}{n})$ if $\frac{m}{n} - \frac{n(n+1)}{k(n+m\varepsilon)} > 0$. On the line segment $Y = X$ $(0 < X < \frac{m}{n})$, it can be shown that the trajectory passes through this line to enter Ω, since when $Y = X$, we have $\frac{\mathrm{d}X}{\mathrm{d}\tau} = -(n+1)X < 0$ and $\frac{\mathrm{d}Y}{\mathrm{d}\tau} = m - (n+1)X$, which yields $\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{(n+1)X - m}{(n+1)X} = 1 - \frac{m}{(n+1)X} < 1$ for positive m, n and X. As a matter of fact, on the whole line $Y = X$ in the first quadrat, trajectories move through the line from the right side to the left side. It is seen from Figure 1 that the nullcline, $m - nY - X = 0$

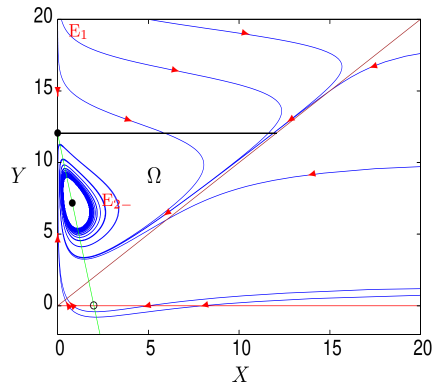

FIGURE 1. Simulation of the epidemic model (2) for $m = 2$, $n = \frac{1}{6}$, $\varepsilon = \frac{1}{2}$ and $k = \frac{13}{100}$, showing non-well-posedness solution preperty.

(in green color) on which trajectories moving in the horizontal direction, divides the X-axis into two parts. On the right part, trajectories enter the fourth quadrant from the first quadrant, while on the left part, trajectories enter the first quadrant from the fourth quadrant. Although the epidemic model is not perfect, this paper does not intend to improve the model, since the aim of this paper is to use this simple model to introduce a hierarchical parametric analysis, providing a complete stability analysis related to Hopf bifurcation.

3. **Stability and bifurcation analysis.** In this section, we will first derive the equilibrium solutions of system (2) and their stability. Although a stability analysis on the equilibria of system (2) was given in [10], the results on the bifurcation, in particular, for the conditions of Hopf bifurcations, are not completely explored. In the following, we first derive the conditions for the existence of the equilibrium solutions of system (2) and their stability. We will give a complete partition in the parameter space for the bifurcation analysis. Setting $\frac{\mathrm{d}X}{\mathrm{d}\tau} = \frac{\mathrm{d}Y}{\mathrm{d}\tau} = 0$ in system (2) yields two equilibrium solutions:

$$\begin{aligned} &\mathrm{E}_1: \quad (X_1,\, Y_1) = \Big(0,\, \frac{m}{n}\Big), \\ &\mathrm{E}_2: \quad (X_2,\, Y_2) = \big(m - nY_2,\, Y_2\big), \quad \Big(0 < Y_2 < \frac{m}{n}\Big), \end{aligned} \tag{5}$$

where E_1 is the infection-free equilibrium (boundary equilibrium) and E_2 is the infectious equilibrium (positive equilibrium), with Y_2 determined from the following quadratic polynomial,

$$F_2(Y_2, k) = 1 + k\varepsilon n(Y_2 - Y_{2\mathrm{L}})(Y_2 - Y_{2\mathrm{U}}), \tag{6}$$

where

$$Y_{2\mathrm{L}} = \frac{m}{n+1}, \quad Y_{2\mathrm{U}} = \frac{m}{n} + \frac{1}{\varepsilon n}. \tag{7}$$

Note that at $Y_2 = \frac{m}{n}$, E_2 becomes E_1, at which a transcritical bifurcation will be shown to occur between these two equilibrium solutions.

Solving $F_2 = 0$ gives the infectious equilibrium solutions:

$$Y_{2\pm} = \frac{1}{2k\varepsilon n(n+1)} \left\{ k\big[(2n+1)m\varepsilon + n + 1\big] \pm \sqrt{\Delta} \right\}, \tag{8}$$

in which

$$\Delta = k\,(\varepsilon m + n + 1)^2 \left[k - \frac{4\,\varepsilon n(n+1)^2}{(\varepsilon m + n + 1)^2}\right]. \tag{9}$$

For convenience, we define the non-zero equilibria:

$$\mathrm{E}_{2\pm}:\ \big(m - n\,Y_{2\pm},\, Y_{2\pm}\big), \tag{10}$$

the marginal values of ε:

$$\begin{aligned}
\varepsilon_1 &= \frac{n+1}{m},\\
\varepsilon_2 &= \frac{4n(n+1)^2}{m} = 4n(n+1)\varepsilon_1,\\
\varepsilon_3 &= \frac{(n+1)^2}{m(1-n)} = \Big(\frac{2}{1-n} - 1\Big)\varepsilon_1, \quad (n<1),\\
\varepsilon_4 &= \frac{(n+1)^2}{mn} = \Big(1 + \frac{1}{n}\Big)\varepsilon_1,\\
\varepsilon^* &= \frac{n+1}{2nm}\left[n+1+\sqrt{(n+1)(1-3n)}\right], \quad \Big(n \leqslant \frac{1}{3}\Big),\\
\varepsilon_* &= \frac{n+1}{2nm}\left[1+n-\sqrt{(1+n)(1-3n)}\right], \quad \Big(n \leqslant \frac{1}{3}\Big),
\end{aligned} \tag{11}$$

and the critical values of k and Y_2:

$$\begin{aligned}
k_{\mathrm{T}} &= n\,\varepsilon_1,\\
k_{\mathrm{SN}} &= \frac{\varepsilon\,\varepsilon_2}{m(\varepsilon+\varepsilon_1)^2},\\
\mathrm{R}_0 &= \frac{k}{k_{\mathrm{T}}},\\
k^* &= \frac{2n\,\varepsilon\,\varepsilon_1\varepsilon_4}{(\varepsilon+\varepsilon_1)(\varepsilon+\varepsilon_4)},\\
k_{\mathrm{H}\pm} &= \frac{\varepsilon[n\,(\varepsilon+\varepsilon_4)+\varepsilon_2] \pm n\,(\varepsilon_4-\varepsilon)\sqrt{\varepsilon(\varepsilon-\varepsilon_2)}\}}{2m(\varepsilon+\varepsilon_1)^2}, \quad (\varepsilon \geqslant \varepsilon_2),\\
Y_{2\,\mathrm{SN}} &= \frac{m(2n+1)}{2n(n+1)} + \frac{1}{2n\varepsilon} \in (Y_{2\mathrm{L}}, Y_{2\mathrm{U}}),\\
Y_{2\mathrm{T}} &= \frac{m}{n} \in (Y_{2\mathrm{L}}, Y_{2\mathrm{U}}),
\end{aligned} \tag{12}$$

where R_0 represents the basic reproduction number. Note that E_2 denotes $\mathrm{E}_{2\pm}$. It is easy to show that $k_{\mathrm{T}} > k_{\mathrm{SN}}$, where the subscripts T and SN denote the transcritical bifurcation and saddle-node bifurcation, respectively. In the following analysis, we treat k as a bifurcation parameter, and other parameters ε, m and n as control parameters. In addition, we call Type-I bistable phenomenon (or Type-I coexistence of bistable states) if two stable equilibria coexist, and Type-II bistable phenomenon (or Type-II coexistence of bistable states) if a stable equilibrium and a stable limit cycle coexist.

We have the following results for stability of the equilibria E_1 and E_2, and the conditions for Hopf bifurcation emerging from E_{2-}.

Theorem 3.1. *For the epidemic model* (2)*, the infection-free equilibrium* E_1 *is asymptotically stable for* $k < k_{\mathrm{T}}$ *(i.e.,* $\mathrm{R}_0 < 1$*) and unstable for* $k > k_{\mathrm{T}}$ *(i.e.,* $\mathrm{R}_0 > 1$*). The infectious equilibrium* E_2 *does not exist for* $k < k_{\mathrm{SN}}$*; only* E_{2-} *exists for* $k \geqslant k_{\mathrm{T}}$ *and* $\varepsilon \leqslant \varepsilon_1$*; both* $\mathrm{E}_{2\pm}$ *exist for* $k > k_{\mathrm{SN}}$ *and* $\varepsilon > \varepsilon_1$*. Moreover,* E_{2+} *is a saddle* *point when it exists. A trascritical bifurcation between* E_1 *and* E_2 *happens at the critical point* $k=k_{\mathrm{T}}$. *The model may undergo Hopf bifurcation from* E_{2-} *for certain conditions. The conditions on the stability of* E_{2-} *and Hopf bifurcation are given below.*

(1) *Suppose* $\varepsilon \leqslant \varepsilon_1$ *holds.*
  (1a) *If* $\varepsilon \leqslant \min\{\varepsilon_1, \varepsilon_2\}$, E_{2-} *is asymptotically stable for* $k > k_{\mathrm{T}}$.
  (1b) *If* $n \leqslant \frac{\sqrt{2}-1}{2}$ *and* $\varepsilon_2 < \varepsilon \leqslant \varepsilon_1$, *two Hopf bifurcations occur at* $k=k_{\mathrm{H}_-}$ *and* $k=k_{\mathrm{H}_+}$; E_{2-} *is asymptotically stable for* $k \in (k_{\mathrm{T}}, k_{\mathrm{H}_-}) \bigcup (k_{\mathrm{H}_+}, \infty)$, *and unstable for* $k \in (k_{\mathrm{H}_-}, k_{\mathrm{H}_+})$.
  No bistable phenomenon can happen.

(2) *Suppose* $\varepsilon > \frac{n+1}{m}$ *is satisfied.*
  (2a) E_{2-} *is asymptotically stable if one of the following conditions holds:*
    (i) $\varepsilon \geqslant \varepsilon_4$ *and* $k > \max\{k_{\mathrm{SN}}, k^*\}$;
    (ii) $n<1$, $\varepsilon > \max\{\varepsilon_3, \varepsilon_4\}$ *and* $k = k^*$; *and*
    (iii) $n \geqslant \frac{\sqrt{2}-1}{2}$, $\varepsilon_1 < \varepsilon < \min\{\varepsilon_2, \varepsilon_4\}$ *and* $k > k_{\mathrm{SN}}$ $(> k^*)$.
    Type-I bistable phenomenon can occur.
  (2b) E_{2-} *is unstable if* $0<n<\frac{1}{2}$, $\varepsilon_3 < \varepsilon \leqslant \varepsilon_4$ *and* $k_{\mathrm{SN}} < k \leqslant k^*$.
  (2c) (i) *If* $n<\frac{1}{2}$, $\varepsilon_3 < \varepsilon < \varepsilon_4$ *and* $k > k^*$ $(> k_{\mathrm{SN}})$, *one Hopf bifurcation occurs at* $k=k_{\mathrm{H}_+}$; E_{2-} *is asymptotically stable for* $k \in (k_{\mathrm{SN}}, k^*) \bigcup (k_{\mathrm{H}_+}, \infty)$, *and unstable for* $k \in (k^*, k_{\mathrm{H}_+})$. *Type-I bistable states coexist, and Type-II bistable states coexist if* $\frac{1}{3} \leqslant n < \frac{1}{2}$, *or if* $n<\frac{1}{3}$ *with* $\varepsilon > \varepsilon^*$, *for which* $k_{\mathrm{H}_+} < k_{\mathrm{T}}$.
    (ii) *If* $n<1$, $\varepsilon > \max\{\varepsilon_3, \varepsilon_4\}$ *and* $k_{\mathrm{SN}} < k < k^*$, *one Hopf bifurcation occurs at* $k=k_{\mathrm{H}_+}$; E_{2-} *is asymptotically stable for* $k > k_{\mathrm{H}_+}$, *and unstable for* $k \in (k_{\mathrm{SN}}, k_{\mathrm{H}_+})$. *Both Type-I and Type-II bistable states coexist, since* $k_{\mathrm{H}_+} < k_{\mathrm{T}}$.
  (2d) *Two Hopf bifurcations happen at* $k=k_{\mathrm{H}_-}$ *and* $k=k_{\mathrm{H}_+}$ *if* $n<\frac{1}{2}$, $\max\{\varepsilon_1, \varepsilon_2\} < \varepsilon < \varepsilon_3$ *and* $k \geqslant k_{\mathrm{SN}}$ $(> k^*)$; E_{2-} *is asymptotically stable for* $k \in (k_{\mathrm{SN}}, k_{\mathrm{H}_-}) \bigcup (k_{\mathrm{H}_+}, \infty)$, *and unstable for* $k \in (k_{\mathrm{H}_-}, k_{\mathrm{H}_+})$. *Type-I bistable states coexist, and Type-II bistable states coexist if* $\frac{\sqrt{5}-1}{4} \leqslant n < \frac{1}{2}$ *and* $\varepsilon_2 < \varepsilon < \varepsilon_3$, *or* $n < \frac{\sqrt{5}-1}{4}$ *and* $\varepsilon_* < \varepsilon < \varepsilon_3$ *for which* $k_{\mathrm{H}_-} < k_{\mathrm{T}}$; *and if* $\frac{\sqrt{5}-1}{4} \leqslant n < \frac{1}{3}$ *and* $\varepsilon_2 < \varepsilon < \varepsilon_*$, *or* $\frac{1}{3} \leqslant n < \frac{1}{2}$ *and* $\varepsilon_2 < \varepsilon < \varepsilon_3$ *for which* $k_{\mathrm{H}_+} < k_{\mathrm{T}}$.

Bogdanov-Takens bifurcation occurs at the critical point, defined by

$$\varepsilon = \frac{(n+1)^2}{m(1-n)}, \quad k = \frac{n(1-n)(n+1)^2}{m}, \quad (0<n<1), \tag{13}$$

associated with the equilibrium,

$$(X_2, Y_2) = \left(\frac{mn}{(n+1)^2}, \frac{m(n^2+n+1)}{n(n+1)^2}\right). \tag{14}$$

Proof. It is easy to see from (9) that $F_2=0$ does not have real solutions and so E_2 does not exist if $k<k_{\mathrm{SN}}$. E_2 exists for $k \geqslant k_{\mathrm{SN}}$. However, the part of the solution Y_2 satisfying $Y_2 > Y_{2\mathrm{T}}$ is biologically meaningless since $X_2<0$ when $Y_2 > Y_{2\mathrm{T}}$. Thus, the curve $F_2=0$ in the bifurcation diagram, projected on the k-Y_2 plane, represents the equilibrium solution E_2 (see Figures 2(a), 3(a), 4(a) and 5(a)). It is clear that $Y_2=Y_{2\mathrm{L}}$ and $Y_2=Y_{2\mathrm{U}}$ are two horizontal asymptotes of this curve, serving as the lower and upper boundaries of the solution E_2. The curve has a unique vertex at

$(k, Y_2) = (k_{\rm SN}, Y_{2\rm SN})$. Moreover, the derivative $\frac{{\rm d}k}{{\rm d}Y_2}$, given by

$$\frac{{\rm d}k}{{\rm d}Y_2} = \frac{2\,(Y_2 - Y_{2\,\rm SN})}{n\,\varepsilon(Y_2 - Y_{2\rm L})^2(Y_2 - Y_{2\rm U})^2},$$

implies that the solution Y_2, determined by a function $k = k(Y_2)$, is monotonically decreasing for $Y_2 < Y_{2\,\rm SN}$ and monotonically increasing for $Y_2 > Y_{2\,\rm SN}$, like a parabola. Hence, when $Y_{2\,\rm SN} \geqslant Y_{2\rm T}$, i.e., when $\varepsilon \leqslant \varepsilon_1$, $\rm E_2$ has one solution $\rm E_{2-}$, see Figure 2(a); while when $Y_{2\,\rm SN} < Y_{2\rm T}$, i.e., when $\varepsilon > \varepsilon_1$, $\rm E_2$ has two solutions: $\rm E_{2+}$ and $\rm E_{2-}$, and $\rm E_{2+}$ exists for $Y_{2\,\rm SN} \leqslant Y_2 \leqslant Y_{2\rm T}$, while $\rm E_{2-}$ exists for $\frac{m}{n+1} < Y_2 \leqslant Y_{2\,\rm SN}$, see Figures 3(a), 4(a) and 5(a).

To find stability of the equilibria, we use the Jacobian of system (2), given by

$$J(X, Y) = \begin{bmatrix} k[Y - 2X + \varepsilon\, X(2Y - 3X)] - (n+1) & k\, X(1 + \varepsilon X) \\ -1 & -n \end{bmatrix}. \tag{15}$$

Evaluating J at $\rm E_1$ yields the eigenvalues $\lambda_1 = -\,n$ and $\lambda_2 = (k - k_{\rm T})Y_{2\rm T}$, indicating that $\rm E_1$ is a stable focus for $k < k_{\rm T}$ (i.e., $\rm R_0 < 1$), and it becomes a saddle point for $k > k_{\rm T}$ (i.e., $\rm R_0 > 1$).

For stability of $\rm E_2$, we consider two cases: $\varepsilon \leqslant \varepsilon_1$ for which only $\rm E_{2-}$ exists, and $\varepsilon > \varepsilon_1$ for which both $\rm E_{2-}$ and $\rm E_{2+}$ exist.

We first consider the case $\varepsilon \leqslant \varepsilon_1$. It is easy to use

$$X_{2-} = \frac{2(k - k_{\rm T})}{k(\varepsilon_1 - \varepsilon) + (\varepsilon_1 + \varepsilon)\sqrt{k(k - k_{\rm SN})}}$$

to show that $k > k_{\rm T}$ for the existence of $\rm E_{2-}$. Then, calculating the Jacobian of system (2) at $\rm E_{2-}$ yields the trace and determinant as follows:

$$\begin{aligned} {\rm Tr}(J({\rm E}_{2-})) &= \frac{2\,{\rm Tr_n}}{{\rm Tr_d} + m^2(\varepsilon_1 + \varepsilon)[n(\varepsilon_1 - \varepsilon) + \varepsilon_1]\sqrt{k(k - k_{\rm SN})}}, \\ \text{where} \quad {\rm Tr_n} &= m^3(\varepsilon_1 + \varepsilon)^2 k^2 - m^2 n\varepsilon[\varepsilon + (4n+1)\varepsilon_4]\, k + \varepsilon n(n+1)^4, \\ {\rm Tr_d} &= n\big[m^2 k\,(\varepsilon_1 + \varepsilon)(\varepsilon + \varepsilon_4) - 2\varepsilon(n+1)^3\big], \\ \det(J({\rm E}_{2-})) &= \frac{2m(\varepsilon + \varepsilon_1)^2(k - k_{\rm T})(k - k_{\rm SN})}{(\varepsilon + \varepsilon_1)^2(k - k_{\rm SN}) + (\varepsilon_1^2 - \varepsilon^2)\sqrt{k(k - k_{\rm SN})}}. \end{aligned} \tag{16}$$

It is clearly seen from (16) that $\det(J({\rm E}_{2-})) > 0$ for $k > k_{\rm T}$ $(> k_{\rm SN})$, and that a transcritical bifurcation exists between $\rm E_1$ and $\rm E_{2-}$ at the critical point $k = k_{\rm T}$. Thus, the stability of $\rm E_{2-}$ is determined by the sign of ${\rm Tr}(J({\rm E}_{2-}))$. It can be shown that

$${\rm Tr_d} \geqslant {\rm Tr_d}|_{k=k_{\rm T}} > {\rm Tr_d}|_{k=k_{\rm SN}} = \frac{2\,n\,\varepsilon\,(n+1)^2[n(\varepsilon_1 + \varepsilon) + (\varepsilon_1 - \varepsilon)]}{\varepsilon_1 + \varepsilon} > 0.$$

Hence, the sign of ${\rm Tr}(J({\rm E}_{2-})$ is determined by $\rm Tr_n$ which is a quadratic polynomial in k. The discriminant of this quadratic polynomial is equal to

$$\Delta_1 = m^4 n^2 \varepsilon(\varepsilon - \varepsilon_4)^2(\varepsilon - \varepsilon_2). \tag{17}$$

Thus, when $\varepsilon \leqslant \min\{\varepsilon_1, \varepsilon_2\}$, $\Delta_1 \leqslant 0$ and so ${\rm Tr_n} > 0$, implying that the whole solution $\rm E_{2-}$ is asymptotically stable. If $\varepsilon_2 < \varepsilon \leqslant \varepsilon_1$, which requires $n \leqslant \frac{\sqrt{2}-1}{2}$, then ${\rm Tr_n} = 0$ has two positive solutions $k_{\rm H_\pm}$. Moreover, a direct computation shows that $k_{\rm H_-} \geqslant k_{\rm T}$:

$$\begin{aligned} k_{\rm H_-} \geqslant k_{\rm T} \quad &\Longleftrightarrow \quad \frac{m^2\varepsilon(n\varepsilon + \varepsilon_2) - \sqrt{\Delta_1}}{2m^3(\varepsilon_1 + \varepsilon)^2} \geqslant n\,\varepsilon_1 \\ &\Longleftrightarrow \quad \sqrt{\Delta_1} \leqslant -n(2n+1)(m\varepsilon)^2 + (n+1)^2 m\varepsilon - 2n(n+1)^3 \overset{\triangle}{=} F_3(m\varepsilon). \end{aligned}$$

Since F_3 is a quadratic polynomial in $m\varepsilon$, and moreover for $n \leqslant \frac{\sqrt{2}-1}{2}$,

$$F_3(4n(n+1)^2) = 2n(n+1)^3(2n+1)^2(1-2n-4n^2) \geqslant 0,$$
$$F_3(n+1) = (n+1)^2(1-2n-4n^2) \geqslant 0,$$

we know that $F_3 \geqslant 0$ for $4n(n+1)^2 < m\varepsilon \leqslant n+1$. Then,

$$\sqrt{\Delta_1} \leqslant F_3 \quad \Longleftrightarrow \quad \Delta_1 - F_3^2 = -4n^2(n+1)(n+1m\varepsilon)^2\, F_4(m\varepsilon) \leqslant 0,$$

where

$$F_4(m\varepsilon) = n(m\varepsilon)^2 - (n+1)^2 m\varepsilon + (n+1)^3. \tag{18}$$

It is easy to show $F_4 \geqslant 0$ because that

$$F_4(4n(n+1)^2) = (n+1)^3(4n^2+4n-1)^2 \geqslant 0,$$
$$F_4(n+1) = n(n+1)^2 > 0,$$

and the minimum point of F_4 is on the right-hand side of the end point $n+1$:

$$\frac{(n+1)^2}{2n} > n+1, \quad \text{for} \;\; n \leqslant \frac{\sqrt{2}-1}{2}.$$

Note that when $\varepsilon = \varepsilon_1$, $k_{\mathrm{H}_-} = k_{\mathrm{T}}$, and that when $\varepsilon = \varepsilon_2$, $k_{\mathrm{H}_+} = k_{\mathrm{H}_-} > k_{\mathrm{T}}$, leading to an isolated Hopf critical point.

Therefore, both k_{H_+} and k_{H_-} define Hopf critical points due to $k_{\mathrm{H}_+} > k_{\mathrm{H}_-}$.

Next, we turn to the case $\varepsilon > \varepsilon_1$. For this case, both E_{2+} and E_{2-} exist for $k > k_{\mathrm{SN}}$. Moreover, using the condition $X_{2+} > 0$ we can show that E_{2+} exists for $k_{\mathrm{SN}} < k < k_{\mathrm{T}}$ for which $Y_{2\,\mathrm{SN}} < Y_2 < Y_{2\mathrm{T}}$. Similarly, a linear analysis yields

$$\det(J(\mathrm{E}_{2+})) = \frac{-\,2m(\varepsilon_1+\varepsilon)^2(k-k_{\mathrm{SN}})(k_{\mathrm{T}}-k)}{(\varepsilon_1+\varepsilon)^2(k-k_{\mathrm{SN}}) + (\varepsilon^2-\varepsilon_1^2)\sqrt{k(k-k_{\mathrm{SN}})}} < 0, \tag{19}$$

for $k_{\mathrm{SN}} < k < k_{\mathrm{T}}$, indicating that E_{2+} is a saddle point.

For the equilibrium E_{2-}, we have

$$\det(J(\mathrm{E}_{2-})) = \frac{m(\varepsilon_1+\varepsilon)^2(k-k_{\mathrm{SN}}) + m(\varepsilon^2-\varepsilon_1^2)\sqrt{k(k-k_{\mathrm{SN}})}}{2\,\varepsilon_1\,\varepsilon} > 0, \tag{20}$$

and thus the stability of E_{2-} is determined by the sign of $\mathrm{Tr}(J(\mathrm{E}_{2-}))$. Using the Jacobian of system (2) evaluated at E_{2-}, we obtain

$$\mathrm{Tr}(J(\mathrm{E}_{2-})) = \frac{G_1+G_2}{2\,\varepsilon(n+1)^2}, \tag{21}$$

where

$$\begin{aligned} G_1 &= m^2(\varepsilon_1+\varepsilon)(\varepsilon+\varepsilon_4)\,(k-k^*), \\ G_2 &= m^2(\varepsilon_1+\varepsilon)(\varepsilon-\varepsilon_4)\sqrt{k(k-k_{\mathrm{SN}})}. \end{aligned} \tag{22}$$

To determine the sign of $\mathrm{Tr}(J(\mathrm{E}_{2-}))$, we need to discuss four cases, based on the signs of G_1 and G_2:

(a) $G_1 \geqslant 0,\ G_2 \geqslant 0$ (excluding $G_1 = G_2 = 0$), E_{2-} asymptotically stable,

(b) $G_1 \leqslant 0,\ G_2 \leqslant 0$ (excluding $G_1 = G_2 = 0$), E_{2-} unstable,

(c) $G_1 G_2 < 0$, Hopf bifurcation from E_{2-},

where $G_1 = G_2 = 0$ defines a special Hopf critical point. $G_1 = 0$ gives the unique solution $k = k^*$, and $G_2 = 0$ yields the unique solution $\varepsilon = \varepsilon_4$. Combining the two solutions easily generates the results for the item (a): E_{2-} is asymptotically stable for $\varepsilon \geqslant \varepsilon_4$ and $k > \max\{k_{\rm SN}, k^*\}$; or for $n<1$, $\varepsilon > \max\{\varepsilon_3, \varepsilon_4\}$ and $k = k^*$. Further, we can show that $k_{\rm T} > k^*$ for $\varepsilon \geqslant \varepsilon_4$ by proving that

$$k_{\rm T} - k^* = \frac{\varepsilon_1(n\varepsilon^2 - \varepsilon_1\varepsilon + n\,\varepsilon_1\varepsilon_4)}{(\varepsilon+\varepsilon_1)(\varepsilon+\varepsilon_4)} > 0.$$

This easily follows that the term in the bracket of the numerator equals $2n\varepsilon_1\varepsilon_4 > 0$ at $\varepsilon = \varepsilon_4$, and its derivative w.r.t. ε is given by $2n\varepsilon - \varepsilon_1 \geqslant (2n+1)\varepsilon_1 > 0$ for $\varepsilon \geqslant \varepsilon_4$. Therefore, Type-I bistable phenomenon can occur for the cases (2a)(i) and (ii).

For the item (b), we can similarly obtain that E_{2-} is unstable if $n < \frac{1}{2}$, $\varepsilon_3 < \varepsilon < \varepsilon_4$ and $k_{\rm SN} < k < k^*$. The case shown in Figure 3 satisfies $k_{\rm H_+} < k_{\rm T}$ and thus both Type-I and Type-II bistable phenomena exist. More precisely, if k takes values from the interval $(k_{\rm SN}, k_{\rm H_+})$, then trajectories converge either to the stable node E_1 or to a stable limit cycle, as shown in Figure 3(b). Here, the saddle-node bifurcation can generate limit cycles. If k takes values from the interval $(k_{\rm H_+}, k_{\rm T})$, then trajectories converge either to the stable node E_1 or to the stable focus E_{2-}, as depicted in Figure 3(c). For both of the two bistable phenomena (Type-I and Type-II), the separators between the two stable states are the stable and unstable manifolds generated from the saddle point E_{2+}.

For the item (c), there are two cases: (c-1) $G_1 > 0$, $G_2 < 0$, which requires the conditions $\varepsilon_1 < \varepsilon < \varepsilon_4$ and $k > \max\{k_{\rm SN}, k^*\}$; and (c-2) $G_1 < 0$, $G_2 > 0$, which needs the conditions $n < 1$, $\varepsilon > \max\{\varepsilon_3, \varepsilon_4\}$ and $k_{\rm SN} < k < k^*$. We first consider the case (c-1). Note that $\mathrm{Tr}(J(\mathrm{E}_{2-})) \gtrless 0$ is equivalent to $G_1^2 - G_2^2 \gtrless 0$, where

$$G_1^2 - G_2^2 = \frac{4\varepsilon(n+1)^2}{n}\big\{m^3(\varepsilon+\varepsilon_2)^2k^2 - m^2n\varepsilon[\varepsilon + (4n+1)\varepsilon_4]\,k + \varepsilon n(n+1)^4\big\},$$

in which the term in the script bracket is a quadratic polynomial in k, with its discriminant Δ_1 given in (17). Hence, when $\varepsilon \leqslant \varepsilon_2$, $\Delta_1 \leqslant 0$ and so E_{2-} is asymptotically stable for $k > k_{\rm SN}$ since $k^* \leqslant k_{\rm SN}$. When $\varepsilon > \varepsilon_2$, $\Delta_1 > 0$ and $\mathrm{Tr}(J(\mathrm{E}_{2-})) = 0$ has two real positive solutions, $k_{\rm H_-}$ and $k_{\rm H_+}$ ($k_{\rm H_-} < k_{\rm H_+}$). Further, a direct computation shows that $k_{\rm H_-} > k_{\rm SN}$, and

$$k_{\rm H_-} - k^* \gtrless 0 \iff -4m^3n(1-n)\varepsilon(n+1)^3(\varepsilon+\varepsilon_1)(\varepsilon-\varepsilon_3) \gtrless 0,$$

which indicates that when $\varepsilon < \varepsilon_3$, there exist two Hopf critical points satisfying $k_{\rm H_+} > k_{\rm H_-} > k_{\rm SN}$ ($> k^*$) under the condition $n < \frac{1}{2}$ and $\max\{\varepsilon_1, \varepsilon_2\} < \varepsilon < \varepsilon_3$. If $\varepsilon \geqslant \varepsilon_3$, then it can be shown that $k^* > k_{\rm SN}$, and $k_{\rm H_-} < k^* < k_{\rm H_+}$ under the condition $n < \frac{1}{2}$ and $\varepsilon_3 < \varepsilon < \varepsilon_4$, implying that there exists one Hopf critical point at $k = k_{\rm H_+}$. E_{2-} is asymptotically stable for $k > k_{\rm H_+}$ and unstable for ($k_{\rm SN} <$) $k^* < k < k_{\rm H_+}$.

Next, consider the case (c-2). For this case, $k_{\rm H_-} > k_{\rm H_+}$ and the conditions become $n < 1$, $\varepsilon > \max\{\varepsilon_3, \varepsilon_4\}$ and $k_{\rm SN} < k < k^*$. Similar to case (c-1), we can prove that $k_{\rm SN} < k_{\rm H_+} < k^* < k_{\rm H_-}$. This shows that there exists one Hopf critical point at $k = k_{\rm H_+} \in (k_{\rm SN}, k^*)$. E_{2-} is asymptotically stable for $k_{\rm H_+} < k < k^*$ and unstable for $k_{\rm SN} < k < k_{\rm H_+}$. However, for this case E_{2-} is also asymptotically stable for $k \geqslant k^*$, and so E_{2-} is also asymptotically stable for $k > k_{\rm H_+}$.

It is easy to see from the above results that Type-I bistable phenomenon always exists for the cases (2a), (2c) and (2d), while Type-II bistable states coexist if $k_{\rm H_+} < k_{\rm T}$ (for the Case (2c)), or at least if $k_{\rm H_-} < k_{\rm T}$ (for the Case (2d)). In the following, we derive the explicit conditions under which $k_{\rm H_\pm} < k_{\rm T}$.

First, consider the Case (2c)(i) for which $n < \frac{1}{2}$ and $\varepsilon_3 < \varepsilon < \varepsilon_4$. Using the formulas k_{T} and k_{H_+} given in (12), we obtain

$$\begin{aligned} k_{\mathrm{T}} - k_{\mathrm{H}_+} &= \frac{F_{5\mathrm{a}} - F_{5\mathrm{b}}}{2m(n+1+m\varepsilon)^2}, \\ \text{where} \qquad F_{5\mathrm{a}} &= n(2n+1)(m\varepsilon)^2 - (n+1)^2 m\varepsilon + 2n(n+1)^3, \\ F_{5\mathrm{b}} &= [(n+1)^2 - mn\varepsilon]\sqrt{m\varepsilon\,[m\varepsilon - 4n(n+1)^2]}. \end{aligned} \tag{23}$$

Note that $m\varepsilon - 4n(n+1)^2 > 0$ due to $\varepsilon > \varepsilon_3$, and $(n+1)^2 - mn\varepsilon > 0$ because of $\varepsilon < \varepsilon_4$. Also, note that $F_{5\mathrm{a}}$ is a quadratic polynomial in $m\varepsilon$, with the discriminant,

$$\Delta_2 = (n+1)^3(1+n-8n^2-16n^3) \begin{cases} \leqslant 0 & \text{for} \quad n \geqslant 0.317382 \quad \Longrightarrow \quad F_{5\mathrm{a}} > 0, \\ > 0 & \text{for} \quad n < 0.317382. \end{cases}$$

When $n < 0.317382$, $F_{5\mathrm{a}} = 0$ has two positive solutions:

$$\varepsilon_{\mathrm{L}} = \frac{(n+1)^2 - \sqrt{\Delta_2}}{2mn(2n+1)}, \quad \varepsilon_{\mathrm{U}} = \frac{(n+1)^2 + \sqrt{\Delta_2}}{2mn(2n+1)}. \tag{24}$$

A direct computation shows that

$$\begin{aligned} &\varepsilon_{\mathrm{L}} < \varepsilon_3 < \varepsilon_{\mathrm{U}} < \varepsilon_4 \quad \text{for} \quad n \leqslant 0.298036, \\ &\varepsilon_{\mathrm{L}} < \varepsilon_{\mathrm{U}} < \varepsilon_3 < \varepsilon_4 \quad \text{for} \quad 0.298036 < n < 0.317383. \end{aligned} \tag{25}$$

Hence, for $n \leqslant 0.298036$ and $\varepsilon_3 < \varepsilon < \varepsilon_{\mathrm{U}}$, we have $F_{5\mathrm{a}} \leqslant 0$, which yields $k_{\mathrm{T}} < k_{\mathrm{H}_+}$ and so no Type-II bistable phenomenon can occur. The remaining cases are for $F_{5\mathrm{a}} > 0$:

(A) $0.298036 < n < 0.5$ and $\varepsilon_3 < \varepsilon < \varepsilon_4$

or

(B) $n \leqslant 0.298036$ and $\varepsilon_{\mathrm{U}} < \varepsilon < \varepsilon_4$.

For these two cases, the sign of $F_{5\mathrm{a}} - F_{5\mathrm{b}}$ is the same as that of $F_{5\mathrm{a}}^2 - F_{5\mathrm{b}}^2$, which yields

$$F_{5\mathrm{a}}^2 - F_{5\mathrm{b}}^2 = 4n^2(n+1)(n+1+m\varepsilon)^2\, F_4, \tag{26}$$

where F_4 is given in (18), which is a quadratic polynomial in $m\varepsilon$, with the discriminat,

$$\Delta_3 = (n+1)^3(1-3n), \tag{27}$$

and its minimum value $\frac{(n+1)^3(3n-1)}{4n}$ is obtained at the point $\varepsilon = \frac{(n+1)^2}{2mn}$. Thus, for the Case (A), $F_{5\mathrm{a}}^2 - F_{5\mathrm{b}}^2 > 0$ for $\frac{1}{3} \leqslant n < \frac{1}{2}$, implying that $k_{\mathrm{H}_+} < k_{\mathrm{T}}$ and so Type-II bistable states coexist. When $0.298036 \leqslant n < \frac{1}{3}$, we have

$$F_4|_{\varepsilon=\varepsilon_3} = -\frac{n(n+1)^3(1-3n)}{(1-n)^2}, \quad F_4|_{\varepsilon=\varepsilon_4} = (n+1)^3, \tag{28}$$

indicating that there exists a unique solution, $\varepsilon = \varepsilon^*$ such that $F_4 = 0$. Moreover, one may use a direct computation to show that

$$\begin{aligned} &\varepsilon_{\mathrm{L}} < \varepsilon_3 < \varepsilon_{\mathrm{U}} < \varepsilon^* < \varepsilon_4 \quad \text{for} \quad n \leqslant 0.298036, \\ &\varepsilon_{\mathrm{L}} < \varepsilon_{\mathrm{U}} < \varepsilon_3 < \varepsilon^* < \varepsilon_4 \quad \text{for} \quad 0.298036 < n < 0.317383. \end{aligned} \tag{29}$$

Therefore, we obtain for the Case (A) when $0.298036 \leqslant n < \frac{1}{3}$ that $k_{\mathrm{T}} < k_{\mathrm{H}_+}$ for $\varepsilon_3 < \varepsilon < \varepsilon^*$, and $k_{\mathrm{H}_+} < k_{\mathrm{T}}$ for $\varepsilon^* < \varepsilon < \varepsilon_4$.

Now, consider the Case (B). Similarly we have

$$F_4|_{\varepsilon=\varepsilon_{\mathrm{U}}} = \frac{(n+1)^3(4n^2+2n-1)}{n + \sqrt{(n+1)(1+n-8n^2-16n^3)}} < 0 \quad \text{for} \quad n \leqslant 0.298036, \tag{30}$$

implying that F_4 has a unique solution $\varepsilon=\varepsilon^*\in(\varepsilon_{\rm U},\varepsilon_4)$ since $F_4>0$ at $\varepsilon=\varepsilon_4$. This shows that for this case, $k_{\rm T}<k_{\rm H_+}$ for $\varepsilon_{\rm U}<\varepsilon<\varepsilon^*$, and $k_{\rm H_+}<k_{\rm T}$ for $\varepsilon^*<\varepsilon<\varepsilon_4$. Summarizing the above results for the cases having $k_{\rm H_+}<k_{\rm T}$ proves the Case (2c)(i) in Theorem 3.1.

For the Case (2c)(ii) with $n<1$ and $\varepsilon>\max\{\varepsilon_3,\varepsilon_4\}$, it is easy to show that $(n+1)^2-mn\varepsilon<0$, and thus $-F_{\rm 5b}>0$. Now, we prove that $F_{\rm 5a}>0$. Since it has been shown in the above that $F_{\rm 5a}>0$ for $n\geqslant 0.317383$, we only need to prove that $F_{\rm 5a}>0$ for $n<0.317383$. It is noted from (25) that $\varepsilon>\max\{\varepsilon_3,\varepsilon_4\}>\varepsilon_{\rm U}$ which implies that $F_{\rm 5a}>0$ because $\varepsilon_{\rm U}$ is the larger root of $F_{\rm 5a}$. Hence, $k_{\rm H_+}<k_{\rm T}$ and Type-II bistable states coexists for the Case (2c)(ii).

Next, consider the Case (2d). For this case, Type-II bistable states coexist as long as at least $k_{\rm H_-}<k_{\rm T}$. If in addition, $k_{\rm H_+}<k_{\rm T}$, then Type-II bistable phenomenon appears near both the two Hopf critical points. First, it can be shown that $k_{\rm SN}<k_{\rm H_-}<k_{\rm H_+}$ under the condition $\max\{\varepsilon_1,\varepsilon_2\}<\varepsilon<\varepsilon_3$. Thus, we only need to find the condition under which $k_{\rm H_-}<k_{\rm T}$ and that for which $k_{\rm H_+}<k_{\rm T}$. Similar to (23), we have

$$k_{\rm T}-k_{\rm H_-}=\frac{F_{\rm 5a}+F_{\rm 5b}}{2m(n+1+m\varepsilon)^2},\tag{31}$$

where $F_{\rm 5a}$ and $F_{\rm 5b}$ are given in (23).

Similar to the discussion in the above for the Case (2c)(i), we divide the proof for the following two cases:

$$(\alpha)\quad n\geqslant 0.317383 \quad \text{and} \quad (\beta)\quad n<0.317383.$$

For the Case (α), we have $\varepsilon_2>\varepsilon_1$, and it is obvious that $k_{\rm H_-}<k_{\rm T}$ since both $F_{\rm 5a}$ and $F_{\rm 5b}$ are positive. Considering the sign of $k_{\rm T}-k_{\rm H_+}$, we use F_4 in (18) and Δ_3 in (27) to obtain $k_{\rm H_+}<k_{\rm T}$ for $\frac{1}{3}\leqslant n<\frac{1}{2}$ due to $\Delta_3\leqslant 0$. When $0.317383\leqslant n<\frac{1}{3}$, we have

$$F_4\big|_{\varepsilon=\varepsilon_2}=(n+1)^3(4n^2+2n-1)^2>0,$$

which, together with $F_4\big|_{\varepsilon=\varepsilon_3}<0$ (see (28)), implies that there exists a unique solution $\varepsilon=\varepsilon_*$ such that $F_4=0$. Thus, for $0.317383\leqslant n<\frac{1}{3}$, we have that $k_{\rm H_+}<k_{\rm T}$ if $\varepsilon_2<\varepsilon<\varepsilon_*$, and that $k_{\rm H_+}\geqslant k_{\rm T}$ if $\varepsilon_*\leqslant\varepsilon<\varepsilon_3$.

For the Case (β) $n<0.317383$, both $\varepsilon_{\rm L}$ and $\varepsilon_{\rm U}$ are positive. Further, it is easy to show that for this case, the following holds:

$$\varepsilon_{\rm L}<\varepsilon_1<\varepsilon_{\rm U},\quad \text{and} \quad \begin{cases} \varepsilon_{\rm L}<\varepsilon_2\leqslant\varepsilon_{\rm U} & \text{for}\ \ n\leqslant\frac{\sqrt{5}-1}{4},\\ \varepsilon_{\rm L}<\varepsilon_{\rm U}\leqslant\varepsilon_2 & \text{for}\ \ \frac{\sqrt{5}-1}{4}<n<0.317383.\end{cases}$$

The relation between ε_3 and $\varepsilon_{\rm U}$ is given in (29). More precisely, we have

$$\begin{aligned}\varepsilon_{\rm U}-\varepsilon_3 &= \frac{(1+n)[(1-n)\sqrt{(1+n)(1+n-8n^2-16n^3)}-(n+1)^2(4n-1)]}{2mn(1-n)(2n+1)}\\ &= \frac{2(1+n)^2(1-3n-4n^3)}{m(1-n)[(1-n)\sqrt{(1+n)(1+n-8n^2-16n^3)}+(n+1)^2(4n-1)]}.\end{aligned}$$

Then, a direct computation with the above formulas leads to the following results:

$$\begin{aligned} n<\tfrac{\sqrt{5}-1}{4},&\quad k_{\rm H_+}>k_{\rm T}\quad \text{for}\quad \max\{\varepsilon_1,\varepsilon_2\}<\varepsilon<\varepsilon_3,\\ \tfrac{\sqrt{5}-1}{4}\leqslant n<0.317383,&\quad k_{\rm H_+}>k_{\rm T}\quad \text{for}\quad \varepsilon_*<\varepsilon<\varepsilon_3,\\ \tfrac{\sqrt{5}-1}{4}\leqslant n<0.317383,&\quad k_{\rm H_+}<k_{\rm T}\quad \text{for}\quad \varepsilon_2<\varepsilon<\varepsilon_*.\end{aligned}$$

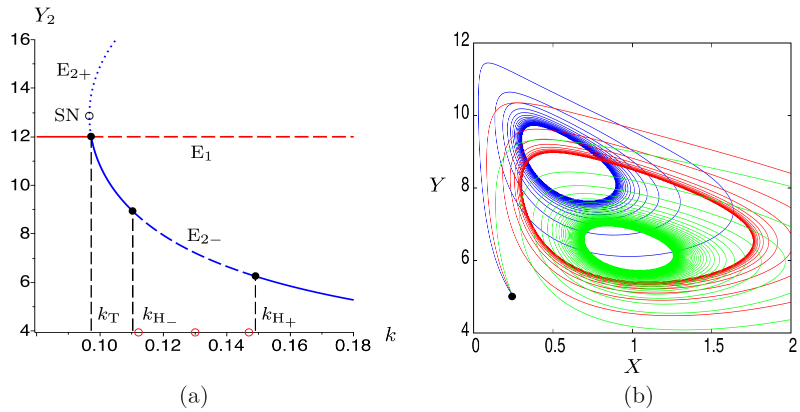

FIGURE 2. (a) bifurcation diagram for the epidemic model (2) projected on the k-Y_2 plane with $m=2$, $n=\frac{1}{6}$, $\varepsilon=\frac{1}{2}$, corresponding to Case (1b) in Theorem 3.1 having two Hopf critical points, with $k_{\rm T}=0.097222$, $k_{\rm H_-}=0.110329$ and $k_{\rm H_+}=0.149040$; and (b) simulations of three stable limit cycles, starting from the initial point $(X,Y)=(0.25,5)$, for three values of k (marked by the red circles on the k-axis): $k=0.112$ (blue color), $k=0.13$ (red color), $k=0.147$ (green color).

Similarly, for $k_{\rm H_-}$, we have that

$$\begin{aligned}
n &< \tfrac{\sqrt{5}-1}{4}, \quad k_{\rm H_-} \geqslant k_{\rm T} \quad \text{for} \quad \max\{\varepsilon_1,\varepsilon_2\}<\varepsilon<\varepsilon_*,\\
n &< \tfrac{\sqrt{5}-1}{4}, \quad k_{\rm H_-} < k_{\rm T} \quad \text{for} \quad \varepsilon_*<\varepsilon<\varepsilon_3,\\
\tfrac{\sqrt{5}-1}{4} \leqslant n &< 0.317383, \quad k_{\rm H_-} < k_{\rm T} \quad \text{for} \quad \varepsilon_2<\varepsilon<\varepsilon_3.
\end{aligned}$$

Combining the above results in the Cases (α) and (β) yields the conclusion for the coexistence of bistable states in (2d) of Theorem 3.1.

Finally, to find the critical point of Bogdanov-Takens bifurcation, we may use the formulas in (16) to solve the equations ${\rm Tr}(J({\rm E}_{2-}))=\det(J({\rm E}_{2-}))=0$ to determine the critical values of k and ε, which implies that the Bogdanov-Takens bifurcation critical point is the intersection of saddle-node and Hopf bifurcations. Thus, $k=k_{\rm SN}$ and $Y_2=\frac{n+1+m\varepsilon(2n+1)}{2n(n+1)\varepsilon}$, for which the equation ${\rm Tr}(J({\rm E}_{2-}))=0$ becomes $\frac{(n+1)^2-m(1-n)\varepsilon}{n+1+m\varepsilon}=0$, yielding the solution $\varepsilon=\frac{(n+1)^2}{m(1-n)}$, $(0<n<1)$. Then, substituting this solution into Y_2 and k we obtain the solutions given in (13) and (14).

This completes the proof for Theorem 3.1. □

Bifurcation diagrams, projected on the k-Y_2 plane, and their simulations for the examples taken from the Cases (1b), (2c)(i), (2c)(ii) and (2d) are shown in Figures 2, 3, 4 and 5, respectively. The parameter values taken for Figure 2 are $m=2$, $n=\frac{1}{6}<\frac{\sqrt{2}-1}{2}$ and $\varepsilon=\frac{1}{2}\in(\varepsilon_2,\varepsilon_1)=(\frac{49}{108},\frac{7}{12})$, satisfying the conditions for the Case (1b). It can be seen from Figure 2 that periodic solutions bifurcating near the two Hopf critical points are stable, implying that both the two Hopf bifurcations are supercritical. In fact, for any values of $k\in(k_{\rm H_-},k_{\rm H_+})$, the bifurcating limit cycles are stable. The simulated three limit cycles as depicted in Figure 2(b) take the values of k from this interval and are indeed stable. Note that the amplitudes of the limit cycles are small for the k near the two Hopf critical points and large

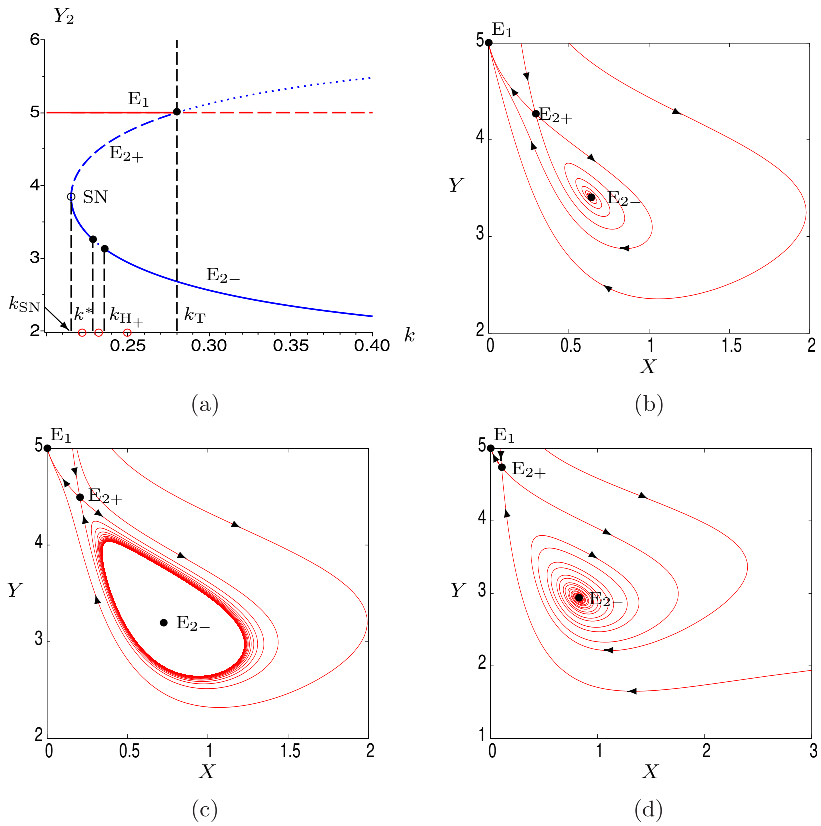

FIGURE 3. (a) bifurcation diagram for the epidemic model (2) projected on the k-Y_2 plane with $m=2$, $n=\frac{2}{5}$, $\varepsilon=2$, corresponding to the Case (2c)(i) in Theorem 3.1 having one Hopf critical point, with $k_{\mathrm{SN}}=0.215089$, $k^*=0.228381$, $k_{\mathrm{H}_+}=0.235367$ and $k_{\mathrm{T}}=0.28$; (b) simulated phase portrait with $k=0.222$, showing the bistable phenomenon with two stable equilibria E_1 and E_{2-}; (c) simulated phase portrait with $k=0.232$, showing the bistable phenomenon with the stable equilibrium E_1 and a stable limit cycle; and (d) simulated phase portrait with $k=0.25$, showing the bistable phenomenon with two stable equilibria E_1 and E_{2-}. The three values of k are marked by the circles on the k-axis in Figure 3(a).

when k is near the middle point of the two Hopf critical points. It is clear that no bistable phenomenon can happen since this is a forward bifurcation.

Figure 3 shows the results for the Case (2c)(i) by choosing the parameter values: $m=2$, $n=\frac{2}{5}\in(\frac{1}{3},\frac{1}{2})$ and $\varepsilon=2\in(\varepsilon_3,\varepsilon_4)=(\frac{49}{30},\frac{49}{20})$, which indeed yields $k_{\mathrm{SN}}<k^*<k_{\mathrm{H}_+}<k_{\mathrm{T}}$. Three values of k are used for simulations: $k=0.222\in(k_{\mathrm{SN}},k^*)$, showing Type-I bistable phenomenon in Figure 3(b) with two stable equilibria E_1 and E_{2-}; $k=0.232\in(k^*,k_{\mathrm{H}_+})$, showing Type-II bistable phenomenon in Figure 3(c) with stable equilibrium E_1 and a stable limit cycle; and $k=0.25\in(k_{\mathrm{H}_+},k_{\mathrm{T}})$, again showing Type-I bistable phenomenon in Figure 3(d) with two stable equilibria E_1 and E_{2-}.

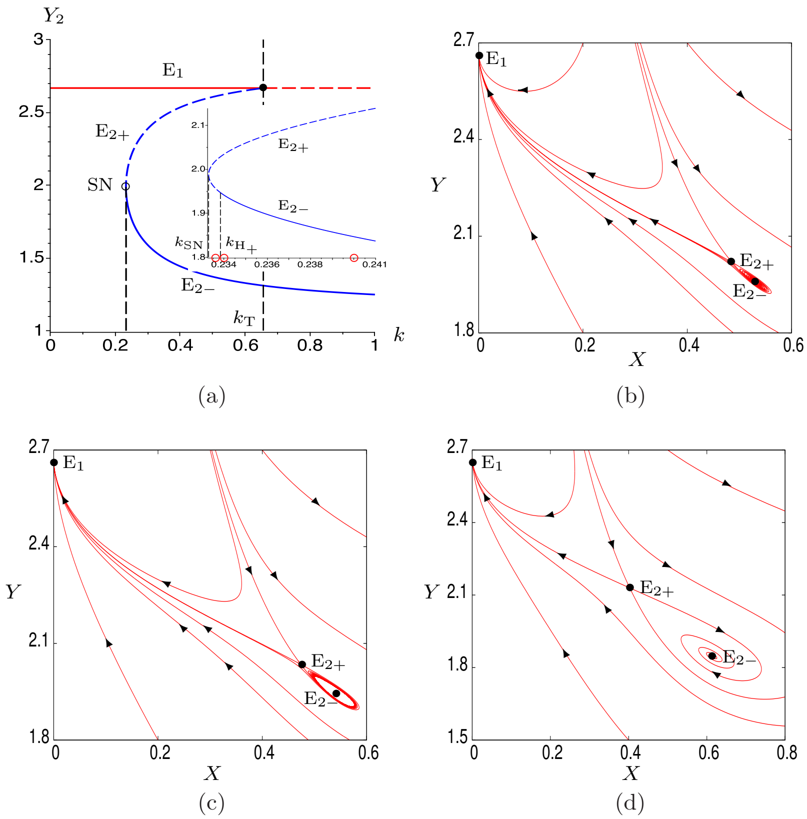

FIGURE 4. (a) bifurcation diagram for the epidemic model (2) projected on the k-Y_2 plane with $m = 2$, $n = \frac{3}{4}$, $\varepsilon = 8$, corresponding to the Case (2c)(ii) in Theorem 3.1 having one Hopf critical point, with $k_{\rm SN} = 0.233287$, $k_{\rm H_+} = 0.233834$, $k^* = 0.240547$, and $k_{\rm T} = 0.656250$; (b) simulated phase portrait with $k = 0.2336$, showing the stable node $\rm E_1$ and the unstable focus $\rm E_{2-}$; (c) simulated phase portrait with $k = 0.234$, showing the stable node $\rm E_1$ and an unstable limit cycle; and (d) simulated phase portrait with $k = 0.24$, showing the bistable phenomenon with two stable equilibria $\rm E_1$ and $\rm E_{2-}$. The three values of k are marked by the circles on the k-axis in Figure 4(a).

The results for the Case (2c)(ii) are shown in Figure 4, with the parameter values: $m = 2$, $n = \frac{3}{4} < 1$ and $\varepsilon = 8 > \max\{\varepsilon_3, \varepsilon_4\} = (\frac{98}{16}, \frac{49}{24})$, which indeed gives $k_{\rm SN} < k_{\rm H_+} < k^* < k_{\rm T}$. It can be seen from the bifurcation diagram in Figure 4(a) that the interval $(k_{\rm SN}, k_{\rm H_+})$ is very narrow, see the zoomed figure in Figure 4(a). The Hopf bifurcation shown for this example is subcritical. But it may be supercritical for choosing some parameter values. For example, taking $m = 2$, $n = \frac{5}{12}$ yields $\varepsilon_3 = \frac{289}{168}$ and $\varepsilon_4 = \frac{289}{120}$. Then, we can show that $v_1 < 0$ for $\varepsilon = 3$ and $v_1 > 0$ for $\varepsilon = 5$. Three simulations are depicted in Figures 4(b), (c) and (d), respectively. The value of k taken for Figure 4(b) is $k = 0.2336 \in (k_{\rm SN}, k_{\rm H_+})$ (see the left red circle in Figure 4(a)), showing the stable node $\rm E_1$ and an unstable focus $\rm E_{2-}$. The unstable limit cycle, as shown in Figure 4(c), is obtained by using the value of

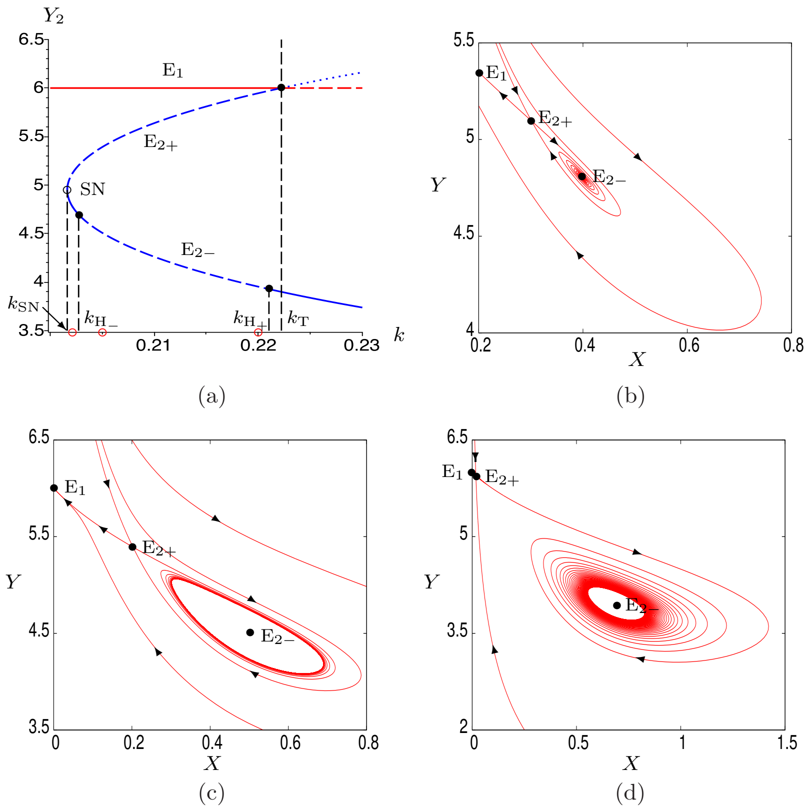

FIGURE 5. (a) bifurcation diagram for the epidemic model (2) projected on the k-Y_2 plane with $m = 2$, $n = \frac{1}{3}$, $\varepsilon = \frac{5}{4}$, corresponding to the Case (2d) in Theorem 3.1 having two Hopf critical points, with $k^* = 0.197348$, $k_{\rm SN} = 0.201638$, $k_{\rm H_-} = 0.202731$, $k_{\rm H_+} = 0.221025$ and $k_{\rm T} = 0.222222$; (b) simulated phase portrait with $k = 0.202$, showing the bistable phenomenon with two stable equilibria $\rm E_1$ and $\rm E_{2-}$; (c) simulated phase portrait with $k = 0.205$, showing the bistable phenomenon with the stable equilibrium $\rm E_1$ and a stable limit cycle; and (d) simulated phase portrait with $k = 0.22$, showing the bistable phenomenon with the stable equilibrium $\rm E_1$ and a stable limit cycle. The three values of k are marked by the circles on the k-axis in Figure 5(a).

$k > k_{\rm H_+}$ but very close to $k_{\rm H_+}$ (see the middle red circle in Figure 4(a)). If the value of k is increasing from $k_{\rm H_+}$, the limit cycle disappears, resulting in the bistable phenomenon with two stable equilibria $\rm E_1$ and $\rm E_{2-}$.

Figure 5 demonstrates the Case (2d) with two Hopf bifurcations by choosing the parameter values: $m = 2$, $n = \frac{1}{3} \in (\frac{\sqrt{5}-1}{4}, \frac{1}{2})$ and $\varepsilon = \frac{5}{4} \in (\varepsilon_2, \varepsilon_3) = (\frac{32}{27}, \frac{4}{3})$, satisfying the conditions in the Case (2d) for $k_{\rm H_-} < k_{\rm H_+} < k_{\rm T}$. This is similar to the Case (1b) but this case can have both Type-I and Type-II bistable phenomena because the chosen parameter values satisfy $k_{\rm SN} < k_{\rm H_-} < k_{\rm T} < k_{\rm T}$. Hence, two stable equilibria $\rm E_1$ and $\rm E_{2-}$ coexist for $k \in (k_{\rm SN}, k_{\rm H_-}) \bigcup (k_{\rm H_+}, k_{\rm T})$, as shown in Figure 5(b), while stable $\rm E_1$ and a stable limit cycle coexist for $k \in (k_{\rm H_-}, k_{\rm H_+})$, as shown in Figures 5(c) and 5(d).

Since the model (2) contains four parameters, more parameters may be included in the bifurcation diagram. For example, if two parameters are used to plot the bifurcation diagram, we may obtain a bifurcation diagram like the one shown in Figure 6, where ε and k are chosen as bifurcation parameters. The v_1, called the first-order focus value, is obtained under the condition of Hopf bifurcation. Thus, the intersection of the curve $v_1 = 0$ and the Hopf bifurcation curve determines the generalized Hopf bifurcation. There are many methods for computing the focus values, and a perturbation approach can be found in [12]. The difference between this bifurcation diagram in Figure 6 and others in Figure 2–5 is obvious since the diagram in Figure 6 does not show any state variables while others do, which clearly shows the relation between the bifurcation parameters and the state variables. In order to show the impact of the bifurcation parameters on dynamical behaviours of the system using the bifurcation diagram in Figure 6, the traditional method is to plot some phase portraits using a typical parameter value chosen from each of the regions in the bifurcation diagram. The phase portraits for this bifurcation diagram are omitted here for brevity.

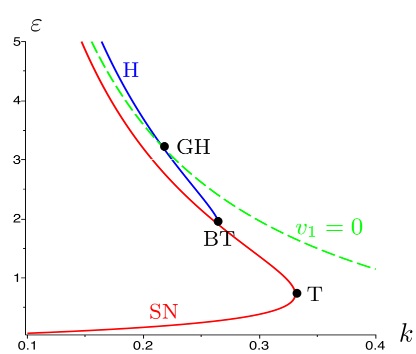

FIGURE 6. Bifurcation diagram for the epidemic model (2) on the k-ε parameter plane with $m = 2$, $n = \frac{5}{11}$, where SN, T, H, BT and GH denote the saddle-node, transcritical, Hopf, Bogdanov-Takens and generalized Hopf bifurcations, respectively.

4. **Conclusion.** In this paper, we have reinvestigated an SI-epidemic model and provided a further study on stability and bifurcation. In particular, we have introduced a hierarchical parametric analysis which enables us to give a complete study on Hopf bifurcation and derived explicit conditions for the occurrence of different Hopf bifurcations. Simulations are presented to demonstrate that the theoretical predictions match very well with the numerical results. Note that the proof of Theorem 3.1 does not rigorously show whether the Hopf bifurcation is supercritical or subcritical, since it needs the computation of the normal form (or the focus value) associated with the Hopf bifurcation (e.g., see [12]). It should be pointed out that it is impossible to compute the focus values using the solution Y_{2-} given in (8) and the Hopf critical points $k_{\mathrm{H}_\pm}$ given in (12). A different method must be developed for classifying the Hopf bifurcations or determining the stability of bifurcating limit cycles. Other future works include the study on the comdimension of Hopf bifurcation, as well as bifurcation analysis on higher-codimensional Bogdanov-Takens bifurcation, with particular attention to the computation of the simplest normal form.

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