THE FOUNDATIONAL DEBATE

COMPLEXITY AND CONSTRUCTIVITY IN MATHEMATICS AND PHYSICS

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KLUWER ACADEMIC PUBLISHERS
DORDRECHT / BOSTON / LONDON

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COMPUTABILITY IN QUANTUM MECHANICS[1]

In this paper, the issues of computability and constructivity in the mathematics of physics are discussed. The sorts of questions to be addressed are those which might be expressed, roughly, as: Are the mathematical foundations of our current theories unavoidably non-constructive: or, Are the laws of physics computable?

These questions are relevant to the foundational debate with respect to the issue of constructivism. Our current physical theories are formulated using the powerful, non-constructive techniques of classical analysis. The constructivist will want to re-formulate such theories constructively, while retaining their empirical content. It is not immediately clear whether this possible. There are, of course, varieties of constructivism, but the core of the interesting varieties seems to be an algorithmic attitude – a concern with computing the quantities of whose existence classical analysis assures us. If this is what lies at the core of constructivism, then what is non-computable is non-constructive, and an irremovable breakdown of computability in a physical theory would preclude a constructivist re-formulation of the theory. If this were to happen, it would render unattractive any constructivism which held, not merely that constructive methods are preferable, where available, to non-constructive mathematics, but that non- constructive mathematics is devoid of cognitive significance.

The question also touches upon the issue of artificial intelligence. In discussions about whether a machine is possible which perfectly mimics the behavior of an intelligent being, one often encounters, with various degrees of explicitness, arguments such as the following: the human brain is just another physical system, and so can be simulated by a computer, if only we understand its working well enough. A lucid discussion of such arguments was presented by Hao Wang at a meeting of the Kurt Gödel Society in 1989 (Wang 1990) and again in an article published in 1993. As Wang points out, there are two critical assumptions in such arguments. One is the assumption of physicalism, that the activities of the brain, or some other physical object, suffice to account for intelligent behavior; the other is the assumption Wang calls "algorithmism of the physical"– that the actions of physical objects can be captured algorithmically. The assumption of physicalism has received a great deal of attention. The assumption of algorithmism for the physical has often gone undiscussed or even left implicit in the formulations of such arguments. A notable exception is Roger Penrose, who seeks to ground non-computability in human behavior on non-computability in the fundamental laws of physics (Penrose 1988, 1989, 1994).

So, is it true that the laws of physics are, at bottom, computable?

In any attempt to examine this question, we must examine our physical theories. It is, after all, theories, not things, which are formulated mathematically, and to which mathematical notions such as computability and constructivity apply. Physical theories, typically, represent the state of a system by a point in some metric space, such as the phase space of classical dynamics or the Hilbert space used in quantum mechanics. Along with such representations go certain real-valued functions which give the values of physical quantities in a state represented by a point in the space in question– for example, the function which gives the kinetic energy of a classical particle as a function of its location in phase space. In quantum mechanics there is no comparable function giving the kinetic energy of a particle whose state is represented by a vector in Hilbert space; in its place is a function mapping vectors in Hilbert space and Borel subsets of the real line onto numbers in the unit interval, which are interpreted as the probability that a measurement of the energy of the particle will yield a result in a given Borel subset of the real line.

If the theory is deterministic, there will also be a function mapping initial states onto states at later times. This time- evolution function, together with the functions specifying the values of physical quantities, yield testable predictions, as, presumably, some among the physical quantities specified by the state of the system will be measurable.

In actual practice, it can be very difficult to achieve usable approximations to the predictions of a theory even when it is known that the desired quantities are computable in principle. The working scientist needs, not merely an algorithm, but a feasible one, and often considerable ingenuity goes into the construction and implementation of such algorithms. We will not consider such matters here, however, concentrating instead on the existence or non-existence of algorithms for computing the functions appearing in the theories. It will be convenient to entertain the fiction of an ideal computor who can produce the value of any computable function in negligible time.

Of the functions appearing in the theory, two sort of questions may be asked:

- Are the functions computable (in the appropriate sense)?
- Do the functions preserve computability, that is, do they map computable states onto computable states or numbers?

The answers to the two questions need not be identical, although, on any reasonable explication of 'computable function', an affirmative answer to the first will entail an affirmative answer to the second; a computable function ought to map computable points onto computable points. An affirmative answer to the second does not imply an affirmative answer to the first; a function might map computable points onto computable points without doing so in a uniformly algorithmic way.

Some of the predictions of a theory concern the value of measurable quantities. Others may concern the long-range behavior of the system, such as whether the system will ever leave a certain volume of phase space. We should, therefore, distinguish between predictions which are testable by experiments taking place in a pre-defined, bounded region of space-time, and predictions which are not. Only the former should be considered measurable predictions. This is significant because, if the dynamics of the theory permit the construction of a physical instantiation of a Turing machine, the corresponding halting problem will arise, so that certain features of the long-range behavior of the system will not be an effectively computable function of its initial state. Turing machines have been constructed (conceptually), not only out of electronic components, but of colliding billiard balls and also quantum systems.[2] Such non-computability is routine, and hardly counts as an instance of the physics outstripping effective mathematics, as, in such systems, the state of the system at any given time is an effectively computable function of the initial data.

In practice, increasing the accuracy of a measurement often requires considerable effort and ingenuity, and not infrequently the development of entirely new techniques. Just as we will ignore practical limitations on computation, we will also ignore practical limitations on experimental precision, and imagine, as a companion to our ideal computor, an ideal experimenter, who can measure any measurable quantity to any desired degree of precision.

If a physical theory invokes non-computable functions, then its mathematical formulation is non-constructive. If nevertheless every measurable prediction of the theory from computable data is a computable number, the constructivist might still attempt to capture such predictions in a constructivized re-formulation of the theory. In such a case, showing that the predictions are non-computable functions of the input data makes clear what obstacles lie in the path of such an attempt. If, however, there are measurable predictions which are not computable numbers, then an ideal measurer can outdo an ideal computor, and one must either accept non-constructivity in the formulation of the theory or hold that the predictions of the theory are wrong.

Thanks to Church and Turing, we have a satisfactory explication of the notion of a computable function of the natural numbers. We can use this to define computable real numbers and computable sequences of real numbers.

In order to extend the notion of computability to functions of a real variable, imagine a computer program which computes a function $F(x)$ as follows. The program operates with rational approximations to both the arguments x and values $F(x)$. The initial input to the program consists of a number k, indicating that an output is required which approximates the value of $F(x)$ to within 2^{-k}. The program then responds with a request for an rational approximation to x within a certain degree, which it specifies. As the computation proceeds, it may request further approximations to x. After a finite amount of time, the program must respond with the desired rational approximation to $F(x)$. The class of functions computed by such programs is identical to a class of functions defined by A. Grzegorczyk (1955) and is, therefore, known as the class of *Grzegorczyk-computable* functions. Note that nothing has been said about how the rational approximations to x which are fed into the program are obtained. Nothing in the way the program works requires the value of x for which the function $F(x)$ is computed to be a computable number. As far as the program is concerned, the source of inputs could be replaced by a "magic box" which generates rational approximations to some non-computable number. We thus obtain, by this scheme, a function F which is defined for all values of the argument, not merely the computable ones.

A program for computing a function F must respond with a value of $F(x)$ after receiving only a finite approximation to the argument x, and so "knows" only that x lies within a certain small interval. In so responding, it is, in effect, asserting that, for all values of x within that interval, $F(x)$ differs from its output by an amount less than the required degree of precision. This means that a computable function is always continuous. Grzegorczyk (1957) showed that a function $F : \mathbb{R} \to \mathbb{R}$ is Grzegorczyk-computable if, and only if:[3]

1. For any computable sequence $\{x_n\}$, $\{F(x_n)\}$ is a computable sequence.
2. F is effectively uniformly continuous with respect to rational segments. That is, there is a recursive function $g(n,m,k)$ such that for all $n,m,k \in \mathbb{N}$ and $x,y \in [Q(n),Q(m)]$, $|x-y| < 2^{-g(n,m,k)}$ implies $|F(x)-F(y)| < 2^{-k}$.

That all computable functions of a real variable are continuous may seem counter-intuitive. It might seem obvious, for example, that such a simple function as the step function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

ought to be classed as computable.[4]

If, however, to be computable means that there is a uniform, effective method of finding values from arguments, then it ought to be the case that a computable function map computable sequences into computable sequences. It is not the first clause of Grzegorczyk's characterization of computability which strikes some as implausible, but rather the second clause asserting effective uniform continuity. Yet the step function does not even satisfy the first clause; it maps some computable sequences onto non-computable sequences. To see this, suppose an effective coding of Turing machines has been given. Define a computable double sequence of rationals $\{q_{kn}\}$ as follows: if Turing machine T_k does not halt, on input k, in n steps or fewer, $q_{kn} = 0$; if T_k does halt on input k in n steps or fewer, let w be the number of steps taken by the machine before halting, and take $q_{kn} = -2^{-w}$. Then $\{q_{kn}\}$ converges to a limit x_k which is equal to 0 if T_k does not halt on input k, and is less than 0 if T_k halts on input k. Moreover, $|q_{kn} - x_k| < 2^{-n}$ for all k,n. $\{x_k\}$ is, therefore, a recursive sequence of real numbers, but

$$H(x_k) = \begin{cases} 0, & = & \text{if } T_k \text{ halts on input } k \\ 1, & = & \text{if not.} \end{cases}$$

The halting problem is known to be recursively insolvable. Thus, if a computable function must map computable sequences onto computable sequences, the step function is not a computable function.

S. Mazur (1963: 100) has proven that any function which maps computable sequences onto computable sequences is continuous in the computable reals. If one accepts that a computable function should map computable sequences onto computable sequences, this theorem eliminates simple examples, such as the step function, which have jump discontinuities, as candidates for computable functions.

It is true that there are functions which map every computable sequence onto a computable sequence which, nevertheless, are unbounded, hence discontinuous, as they approach some non-computable numbers.[5] Such functions, however, are rather bizarre and less appealing as counter-examples to the claim that all computable functions of a real variable are continuous.

The thesis that computable functions must be continuous has a long history among constructivists. It was, to the best of the author's knowledge, first enunciated by Borel in 1912, in an insightful discussion of the notion of effective calculability which presages many of the later results of recursive function theory. It owes much of its currency among constructivists to Brouwer, who devoted a great deal of space to attempts to prove that an effectively calculable function (Brouwer acknowledged no other kind) must be continuous everywhere if it is defined everywhere (see, e.g., Brouwer 1927). In honor of these two, and in analogy to the Church-Turing thesis, I call the thesis that an effectively calculable function of a real variable must be a continuous function the *Borel-Brouwer thesis*.

Though there is, as yet, no defense of the identification of effectively calculability with Grzegorczyk-computability comparable to Turing's *tour de force* in defense of the Church-Turing thesis, it is hoped that the most obvious objections to this identification, based on the supposed implausibility of the thesis that no discontinuous functions are effectively calculable, have been forestalled by the discussion above.

The solutions to differential equations do not always depend continuously on the data. The now-classic example of this was made famous by Jacques Hadamard ([1922] 1952: 33-38). The example concerns the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The solution $u(x,y)$ of this equation is uniquely determined by specifying the value and rate of change of u along the line y-axis:

$$\begin{aligned} u(0,y) &= u_0(y) \\ \frac{\partial u}{\partial x}(0,y) &= u_1(y) \end{aligned}$$

Let

$$\begin{aligned} u_0(y) &= 0 \\ u_1(y) &= \frac{\sin(ny)}{n} \end{aligned}$$

where n is some number. The corresponding solution is given by

$$u(x,y) = \frac{1}{2n^2}(e^{nx} - e^{-nx})\sin(ny),$$

so

$$u(1,y) = \frac{1}{2n^2}(e^n - e^{-n})\sin(ny),$$

Suppose, now, that all we know is that $u_0(y)$ and $u_1(y)$ differ from zero by an amount less than some small positive number ε. Since e^n/n^2 increases without limit as n increases, we can make $u(1,y)$ as large as we want while maintaining $u_0(y) = 0$ and $|u_1(y)| < \varepsilon$ for all y. Chaos theorists study systems whose behavior depends continuously but sensitively on the initial conditions. This is worse– no degree of approximation to the data u_0, u_1 allows us to determine $u(1,y)$ to any degree of approximation. Nothing short of perfect information about u_0 and u_1 suffices.

About this sort of situation, Hadamard remarked,

> Strictly, mathematically speaking, we have seen (this is Holmgren's theorem) that one set of Cauchy's data u_0, u_1 corresponds (at most) to one solution of
>
> $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$
>
> so that, if these quantities u_0, u_1 were 'known', u would be determined without any possible ambiguity.
>
> But, in any concrete application, 'known', of course, signifies "known with a certain approximation," all kind of errors being possible, provided their magnitude remains smaller than a certain quantity; and, on the other hand, we have seen that the mere replacing of the value zero for u_1 by the (however small) value (15) changes the solution not by very small but by very great quantities. Everything takes place, physically speaking, as if the knowledge of Cauchy's data would *not* determine the unknown function. (Hadamard [1922] 1952: 38).

Courant and Hilbert ([1937] 1962: 227) took the condition that the solution of a problem depends continuously on the data as one of three conditions which must be satisfied for the problem to be considered "well-posed" (the other two being that a solution exist and that the solution be unique), and Hadamard, in subsequent works, followed suit (Hadamard 1964: 19-21). The point is not that there are no physical systems which exhibit discontinuous dependence on initial conditions. Rather, solutions to problems which are not well-posed are of no use in concrete applications.

Suppose a physical theory predicts the value of a physical quantity, given certain conditions. If this prediction is to be tested by a measurement of the quantity in question, the experimenter must be able to produce the desired conditions or at least ascertain that they hold. This will be done, not exactly, but with some degree of error. This error does not invalidate the experiment so long as the value of the quantity to be measured changes by only a small amount when the experimental parameters vary slightly. I adopt terminology due to Laszlo Tisza (1963: 159), and call the principle that mathematical solutions of problems arising in physics must be insensitive to small changes in the data, in order to be of use for making quantitative predictions, the "principle of regularity." This principle is an important insight into the relationship between our mathematical models and the world that they are meant to represent. It should not be taken as an a priori prohibition against the appearance of discontinuous functions in physical theories; it merely serves to distinguish the predictions of the theory which can be tested by measurement from those which cannot. Even if *natura facit saltum*, the quantities about which we can make reliable quantitative predictions will be found in the regions where Nature refrains from leaping. We may, indeed, be led by theoretical considerations to predict that some physical quantity varies discontinuously as a function of the other, and this prediction can be compared with experiment. In such a case, however, it is the qualitative behavior of the system that is predicted; no attempt is made to make precise predictions of the value of the discontinuously varying quantity in the immediate neighborhood of the discontinuity. Even if discontinuous functions appear in the theory, the result of a calculation, in order to be of use to an experimenter, must be insensitive to small changes in the data.

Given what has been said about computability and continuity, it should not be surprising that the solution $u(x,y)$ of the Laplace equation is not always a Grzegorczyk-computable function whenever u_0 and u_1 are. Perhaps the best-known example of a differential equation with non-computable solutions for computable initial data is the three-dimensional wave equation (Pour-El and Richards 1981):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

The wave equation also exhibits discontinuous dependence of the solution on initial conditions, and the construction exploits this fact.

Pour-El and Richards (1983) have proven that, under certain mild side conditions, a linear partial differential equation preserves computability– that is, yields computable solutions for computable initial data– if and only if the solution depends continuously on the data. Thus, in the space of solutions of differential equations, we have an analog of Mazur's result that a function which maps computable sequences of reals onto computable sequences are continuous in the computable reals. Corresponding to an initial-value problem is a family of operators mapping initial data onto solutions at a later time t. If the equation is a linear one, the time-evolution operators will be linear operators. We have the strong result that linear operators which map computable functions onto computable functions are effectively uniformly continuous in the data, and linear operators which are discontinuous map some computable function onto a non-computable function.

These counter-examples violate the principle of regularity, and so are disqualified as experimentally verifiable predictions of the theory. When a differential equation satisfies the principle of regularity, then, as the Pour-El and Richards classification theorem shows, computability is preserved.

The Pour-El and Richards classification theorem applies, in its full generality, to any Banach space endowed with a computability structure satisfying the axioms:[3]

Axiom 1 (Linear Forms). Let $\{x_n\}$ and $\{y_n\}$ be computable sequences in X, let $\{\alpha_{nk}\}$ and $\{\beta_{nk}\}$ be computable double sequences of real or complex numbers, and let $d(n)$ be a recursive function. Then the sequence

$$s_n = \sum_{k=0}^{d(n)} (\alpha_{nk} x_k + \beta_{nk} y_k)$$

is a computable sequence.

Axiom 2 (Limits). Let $\{x_{nk}\}$ be a computable double sequence in X such that $\{x_{nk}\}$ converges to $\{x_n\}$ as $k \to \infty$, effectively in k and n. Then $\{x_n\}$ is a computable sequence in X.

Axiom 3 (Norms). If $\{x_n\}$ is a computable sequence in X, then the norms $\{||x_n||\}$ form a computable sequence of real numbers.

Proposition 1a). (Pour-El and Richards) . Let A be a bounded, effectively determined linear operator on a Banach space X with a computability structure satisfying Axioms 1-3. Then A maps every computable sequence in X onto a computable sequence in X.

b). Let A be a closed, unbounded linear operator on a Banach space X with a computability structure satisfying Axioms 1-3, which maps some computable basis sequence e_k onto a computable sequence $\{Ae_k\}$. Then there exists a computable u in the domain of A such that Au is not computable.

If we recall that a linear operator is continuous (moreover, effectively, uniformly so) everywhere on its domain if it is bounded, and discontinuous everywhere if it is unbounded, we see that the classification theorem is a close cousin to other results concerning the continuity of computable functions.

It is easy to extend the notion of computability to a separable Hilbert space. One chooses an orthonormal basis which one wishes to take as a computable sequence. A vector is computable if and only if its coefficients of expansion in terms of this basis are a computable sequence, and the expansion converges effectively; computable sequences are defined analogously. The notion of computability so defined satisfies the axioms for a computability structure on a Banach space, and the Pour-El and Richards classification theorem applies. A closed linear operator which acts effectively on a computable basis sequence preserves computability (*i.e.* maps computable vectors onto computable vectors) if and only if it is bounded. This fact can be used to show that quantum-mechanical time-evolution according to an effectively determined Hamiltonian is computable.

Lemma 1c. If T is an effectively determined (bounded or unbounded) self-adjoint operator, and f is a bounded, Grzegorczyk-computable function, then $f(T)$ is a computable operator.

Corollary 1d. If H is an effectively determined Hamiltonian, the time-evolution operators

$$U(t) = e^{-\frac{iHt}{\hbar}}$$

depend effectively on t, $\hbar$.

The Pour-El and Richards classification theorem dictates that, whenever T is an effectively determined unbounded, closed operator, there will be some computable state ψ such that $T\psi$ is a non-computable state. Many of the interesting operators in quantum mechanics – such as position, momentum, and energy – are unbounded and closed. However, the vector $H\psi$, where H is, for example, the Hamiltonian operator, has no immediate physical significance; it is quantities defined in terms of these vectors, such as expectation values, that are of concern to the physicist. And these expectation values are computable, so long as ψ is in the domain of the operator:

Proposition 2. If T is an effectively determined self-adjoint operator and ψ is a computable vector in the domain of T, then the expectation value $(\psi, T\psi)/||\psi||^2$ is a computable real number.

One of the central results of the theory of Hilbert spaces is the Spectral Theorem. This theorem is the means by which the probabilistic predictions of quantum mechanics are generated, as it ensures that, for any self-adjoint linear operator A (which represents an "observable") and any vector ψ, there is a measure μ_ψ^A on the Borel subsets of the real line which takes on values in [0,1]. For any Borel set $\Omega \subseteq \mathbb{R}$, $\mu_\psi^A(\Omega)$ is interpreted as the probability that a measurement of the physical quantity corresponding to A on a system in state ψ will yield a result in Ω.

We now ask: is the probability $p(x) = \mu_\psi^A((-\infty, x))$ always a Grzegorczyk-computable function of x whenever ψ and A are, respectively, a computable vector and computable operator? The answer is, trivially: no, as $p(x)$ may depend discontinuously on x, at the eigenvalues of A.[6] *E.g.*, take A to be σ_z, the spin operator in the z-direction for a spin-$^1/_2$ particle, and take ψ to be $|z;+>$ where $\sigma_z\,|z;+> = {}^1/_2\,|z;+>$ Then we have

$$p(x) = \begin{cases} 0, \text{ if } x < {}^1/_2 \\ 1, \text{if } x \geq {}^1/_2 \end{cases}$$

This function is not Grzegorczyk-computable, because of the discontinuity at 0. It does, however, take on computable values at computable points. This need not be the case.

Proposition 3. There exists a computable, bounded, self-adjoint operator A, and a computable vector ψ, such that $\mu_\psi^A(I)$ is a non-computable number, where I is the open unit interval (0, 1).[7]

The Spectral Theorem, in the form invoked by physicists doing quantum mechanics, is not a constructively valid theorem.

Is this an example of a non-computable, testable prediction from computable data? No, because, in the example given, 0 is an eigenvalue of the operator A, and the probability assigned to the singleton set 0 is non-zero. In fact, the spectrum of the operator in question is entirely contained in [0, 1), so that $\mu_\psi^A([0,1)) = 1$. The principle of regularity is violated; the slightest change in one of the endpoints of the interval of measurement produces a large change in the probability predicted. This is not an accidental feature of the construction.

Proposition 4. If A is a computable self-adjoint operator, and $[a,b]$ is an interval with computable endpoints containing no eigenvalues of the operator A, then the function

$$\phi(x) = \mu_{\psi}^{A}(x)$$

is a Grzegorczyk-computable function on $[a,b]$.

Each of the cases discussed above in which non-computability has been found to arise out of computability violates the principle of regularity. Furthermore, it can be shown, in each case, that when the principle of regularity is *not* violated, then the desired quantities can be calculated effectively from the data. There do not seem to be any regular predictions of quantum mechanics which are not computable, given computable data.

If one reviews other results in computable analysis in which non-computability arises out of computability, one finds the same situation: in each case, non-computability arises out of some sort of discontinuity. One is tempted to generalize, and conjecture that, in analysis, non-computability arises, in a natural way, out of computability only as a result of some sort of discontinuity. If so, then perhaps no realistic physical theory will produce non-computable predictions which satisfy the principle of regularity. Two notes of caution must be sounded in connection with this conjecture, however. First, many of the known results are concerned with linear problems, and, for linear operators, there is no middle ground between discontinuity everywhere and uniform effective continuity. A linear operator which is continuous at any point on its domain is uniformly effectively continuous. It is conceivable that there are non-linear partial differential equations whose solutions depend continuously on the data, but not effectively so, and such equations might map computable data onto non-computable solutions. Second, the evidence we have concerns *known* results, and perhaps the examples of non-computability which arise from discontinuity are simply the easiest to find. The Borel-Brouwer thesis seems to be used a heuristic guide in the pursuit of non-computability in analysis; this tends to lead to examples which violate regularity.

The deterministic, linear dynamics of quantum mechanics leads, if it is taken to apply to all systems for all times, to seemingly unreal situations such as superpositions of dead and alive cats, and not to actual, definite events. This is the so-called "measurement problem" (though the problem exists for situations which are not ordinarily construed as measurements). In response to the problem, it has been proposed that the dynamics be modified in a non-linear way which avoids such objectionable superpositions. Penrose (1989) has proposed that the solution of the measurement problem lies in a hypothetical "correct quantum gravity" (CQG) which will not only account for the reduction of superpositions, but will account for our allegedly non-computable behavior by being a non-algorithmic theory. It has been conjectured, above, that non-computable predictions which satisfy the principle of regularity are to be found, if at all, in non-linear problems. Only non-linear operators allow for some middle ground between uniform effective continuity and discontinuity everywhere. If the laws of physics, at some fundamental level, are non-

computable, then a non-linear emendation of quantum mechanics is a plausible location for this non-computability to occur. Computability in non-linear problems has been little investigated, chiefly because non-linear problems are much more difficult to handle than linear ones. Pour-El and Richards close their book with the remark, "Non-linear analysis is a vast area, and its connections with recursion theory, at the time of this writing, remain largely untouched" (1989: 194). In 1994, they still remain largely untouched, but an interesting avenue for further investigations.

APPENDIX: DEFINITIONS

I. a) A sequence $\{x_n\}$ of rationals or reals converges as $n \to \infty$ to a limit x if and only if for every $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that, for all $n > k, x - x_n < 2^{-m}$.

b) A sequence $\{x_n\}$ of rationals or reals converges *effectively* as $n \to \infty$ to a limit x if and only if there is a recursive function $d : \mathbb{N} \to \mathbb{N}$ such that, for all $n > d(m), x - x_n < 2^{-m}$.

c) A double sequence $\{x_{nk}\}$ of rationals or reals converges as $k \to \infty$ to the sequence $\{x_n\}$, effectively in k and n, if and only if there is a recursive function $d : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for all $k > d(n,m), x_n - x_{nk} < 2^{-m}$.

Suppose that an effective coding of the rationals by natural numbers is given, such as, e.g.

$$Q(n) = \frac{\pi_1(\pi_1(n)) - \pi_2(\pi_1(n))}{\pi_2(n) + 1}$$

where π_1 and π_2 are the left and right "unpacking" functions for some effective pairing function. *E.g.* $\tau(n,m) = {}^1/_2(n+m)(n+m+1) + m$, and $\tau(\pi_1(n), \pi_2(n)) = n$ for all $n \in \mathbb{N}$.

II. a) A sequence $\{q_n\}$ of rational numbers is a *computable sequence of rationals* if and only if there is a recursive function $d : \mathbb{N} \to \mathbb{N}$ such that $q_n = Q(d(n))$ for all $n \in \mathbb{N}$.

b) A double sequence $\{q_{nk}\}$ of rational numbers is a *computable double sequence of rationals* if and only if there is a recursive function $d : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that
$q_{nk} = Q(d(n,k))$ for all $n, k \in \mathbb{N}$
(And similarly for computable n-tuple sequences of rationals.)

III. a) A real number x is a *computable real number* if and only if there is a computable sequence of rationals which converges effectively to x.

b) A sequence $\{x_n\}$ of real numbers is a *computable sequence* if and only if there is a computable double sequence $\{q_{nk}\}$ of rational numbers which converges as $k \to \infty$ to $\{x_n\}$, effectively in k and n.

IV. A function $F : \mathbb{R} \to \mathbb{R}$ is Grzegorczyk-computable if, and only if:

i) For any computable sequence $\{x_n\}$, $\{F(x_n)\}$ is a computable sequence.

ii) F is effectively uniformly continuous with respect to rational segments. That is, there is a recursive function $g(n,m,k)$ such that for all $n,m,k \in \mathbb{N}$ and $x,y \in [Q(n),Q(m)]$, $|x-y| < 2^{-g(n,m,k)}$ implies $|F(x)-F(y)| < 2^{-k}$.

V. Let $\mathcal{H}$ be a separable Hilbert space, $\{e_k\}$ an orthonormal basis for $\mathcal{H}$.

a) A vector $u \in \mathcal{H}$ is *computable with respect to* $\{e_k\}$ iff there is a computable sequence $\{\alpha_k\}$ of complex numbers such that the partial sums

$$u_m = \sum_{k=0}^{m} \alpha_k e_k$$

converge effectively to u as $m \to \infty$.

b) A sequence $\{u_n\}$ in $\mathcal{H}$ is *computable with respect to* $\{e_k\}$ iff there is a computable double sequence $\{\alpha_{nk}\}$ of complex numbers such that the double sequence

$$u_{nm} = \sum_{k=0}^{m} \alpha_{nk} e_k$$

converges to u_n as $m \to \infty$, effectively in n and m.

VI. a) A linear operator A on $\mathcal{H}$ is *computable with respect to* $\{e_k\}$ iff:

i) $\{Ae_k\}$ is a computable sequence.

ii) A is bounded.

b) A sequence $\{A_n\}$ of linear operators on $\mathcal{H}$ is a *computable sequence with respect to* $\{e_k\}$ iff, for every sequence $\{y_m\}$ which is computable with respect to $\{e_k\}$, $A_n y_m$ is a computable double sequence with respect to $\{e_k\}$.

VII. Let $\mathcal{H}$ be a Hilbert space, $\{e_k\}$ a basis for $\mathcal{H}$. A closed linear operator T on $\mathcal{H}$ is *effectively determined with respect to the basis* $\{e_k\}$ iff:

i) $\{Te_k\}$ is a computable sequence.

ii) For all $u \in Dom(T)$, there exists a sequence $\{u_k\}$ in the linear span of $\{e_k\}$ such that $u_k \to u$ and $Tu_k \to Tu$ as $k \to \infty$.

NOTES

1. Part of the research for this paper was carried out while the author held a Graduate Fellowship in the History of Science and Technology at the Dibner Institute for the History of Science and Technology, Cambridge, Massachusetts, USA.
2. For billiard-ball computers, see Fredkin and Toffoli (1982). For quantum-mechanical computers, see Feynman (1986).
3. See Appendix for definitions.
4. This point has been raised by John Earman (1986: 119).
5. See Hartley Rogers, Jr. (1967: 371) for the construction.
6. As physicists do not always distinguish between the eigenvalues of an operator and other points in the spectrum of the operator, it is worth stressing that there is, for self-adjoint operators on a Hilbert space, a distinction between two classes of points in the spectrum: the point spectrum (eigenvalues), and the continuous spectrum.
7. See Pour-El and Richards (1989: 133-142) for the construction. The operator in question is that constructed in the proof of the Eigenvector Theorem and the vector is e_0 of the same proof.

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