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Kochen–Specker ϵ-obstruction for position and momentum using a single degree of freedom

Wayne C. Myrvold [1]

Department of Philosophy, University of Western Ontario, London, ON, N6A 3K7, Canada

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Abstract

The Bell–Kochen–Specker theorem shows that, in any Hilbert space of dimension of at least 3, it is impossible to assign noncontextual definite values to all observables in such a way that the quantum-mechanical predictions are reproduced. This leaves open the issue of what subsets of observables may be assigned definite values. Clifton has shown that, for a system of at least two continuous degrees of freedom, it is not possible to assign simultaneous noncontextual values to two coordinates and their conjugate momenta. In this Letter, it is shown that, for a system of a single continuous degree of freedom, it is not possible to assign noncontextual values to the coordinate and its conjugate momenta that satisfy a continuity assumption herein called the 'ϵ-Product Rule'. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The issue of whether the uncertainty relations between position and momentum entail that these two quantities cannot simultaneously be 'elements of reality' has long played a central role in discussions concerning the interpretation of quantum theory. As is well-known, the deterministic alternative to quantum mechanics formulated by Bohm [1] attributes definite values of position to particles at the price of contextualizing the momentum value—the result of a measurement of momentum is not determined by the state of the measured system alone but depends crucially on the details of the experimental arrangement. The Bell–Kochen–Specker theorem [2] shows that, for a Hilbert space of dimension three or greater, it is not possible to attribute simultaneous noncontextual values to all physical quantities. This leaves open the question of which subsets of observables *can* be assigned noncontextual values. It seems eminently plausible that an argument analogous to the Bell–Kochen–Specker argument can show that, as a consequence of the canonical commutation relations, at most one of a pair of canonically conjugate observables can have a noncontextual value. Proof of such an assertion has been surprisingly long in coming. Clifton [3], building on the

E-mail address: wmyrvold@uwo.ca (W.C. Myrvold).

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work of Peres and Mermin [4], exhibited a Kochen–Specker obstruction utilizing two degrees of freedom. This shows that it is not possible for two independent coordinates q_1, q_2, and their conjugate momenta p_1, p_2 to all have noncontextual values. In this Letter this conclusion is extended to a single degree of freedom; of any pair of canonical conjugates q, p, at most one can have a noncontextual value. The Peres–Mermin and Clifton constructions, and the one in this Letter, do not replace the original Bell–Kochen proof, as they all require more dimensions than three—what they do is to provide more information about which subsets of observables can and cannot be ascribed noncontextual values while reproducing the quantum-mechanical predictions.

The obstruction constructed here differs from previous obstructions in one important respect, in that the usual assumptions about measured values will be supplemented with a continuity assumption. Kochen–Specker obstructions usually assume the Product Rule: if observables associated with commuting operators $\hat{A}, \hat{B}$ have definite values $v[\hat{A}], v[\hat{B}]$, then the product operator $\hat{A}\hat{B}$ has associated with it the definite value $v[\hat{A}\hat{B}] = v[\hat{A}]v[\hat{B}]$. We will extend a version of this to noncommuting operators that correspond to observables that are *approximately* co-measurable, in a sense that will be explained in Section 3.

2. The Peres/Mermin and Clifton obstructions

The following is a generalization of the simple Kochen–Specker obstruction constructed by Mermin on the basis of work by Peres [4]. Take operators $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$, such that none of them have zero as an eigenvalue, satisfying the commutation relations,

$$[\hat{A}_1, \hat{A}_2] = [\hat{A}_1, \hat{B}_2] = [\hat{B}_1, \hat{A}_2] = [\hat{B}_1, \hat{B}_2] = 0, \quad (1)$$

and the anticommutation relations

$$\hat{A}_i\hat{B}_i = -\hat{B}_i\hat{A}_i, \quad i = 1, 2. \quad (2)$$

(The specific example used by Mermin, for a pair of spin-$\frac{1}{2}$ particles, is $\hat{A}_i = \hat{\sigma}_{ix}, \hat{B}_i = \hat{\sigma}_{iy}$.) Suppose that associated with $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$ are nonzero definite values $v[\hat{A}_1], v[\hat{A}_2], v[\hat{B}_1], v[\hat{B}_2]$, respectively. As a consequence of the commutation relations (1) and the Product Rule, we have

$$v[\hat{A}_1\hat{A}_2] = v[\hat{A}_1]v[\hat{A}_2],$$
$$v[\hat{B}_1\hat{B}_2] = v[\hat{B}_1]v[\hat{B}_2]. \quad (3)$$

Moreover, $\hat{A}_1\hat{A}_2$ commutes with $\hat{B}_1\hat{B}_2$, and so

$$v[\hat{A}_1\hat{A}_2\hat{B}_1\hat{B}_2] = v[\hat{A}_1\hat{A}_2]v[\hat{B}_1\hat{B}_2] = v[\hat{A}_1]v[\hat{A}_2]v[\hat{B}_1]v[\hat{B}_2]. \quad (4)$$

As a further consequence of the commutation relations and the Product Rule, we also have

$$v[\hat{A}_1\hat{B}_2] = v[\hat{A}_1]v[\hat{B}_2],$$
$$v[\hat{A}_2\hat{B}_1] = v[\hat{A}_2]v[\hat{B}_1]. \quad (5)$$

Since $\hat{A}_1\hat{B}_2$ commutes with $\hat{A}_2\hat{B}_1$,

$$v[\hat{A}_1\hat{B}_2\hat{A}_2\hat{B}_1] = v[\hat{A}_1\hat{B}_2]v[\hat{A}_2\hat{B}_1] = v[\hat{A}_1]v[\hat{B}_2]v[\hat{A}_2]v[\hat{B}_1]. \quad (6)$$

Comparison of (4) and (6) yields

$$v[\hat{A}_1\hat{A}_2\hat{B}_1\hat{B}_2] = v[\hat{A}_1\hat{B}_2\hat{A}_2\hat{B}_1]. \quad (7)$$

However, we also have

$$\hat{A}_1\hat{B}_2\hat{A}_2\hat{B}_1 = -\hat{A}_1\hat{A}_2\hat{B}_2\hat{B}_1 = -\hat{A}_1\hat{A}_2\hat{B}_1\hat{B}_2. \quad (8)$$

Consequently,

$$v[\hat{A}_1\hat{A}_2\hat{B}_1\hat{B}_2] = -v[\hat{A}_1\hat{B}_2\hat{A}_2\hat{B}_1], \quad (9)$$

and hence

$$v[\hat{A}_1]v[\hat{B}_2]v[\hat{A}_2]v[\hat{B}_1] = -v[\hat{A}_1]v[\hat{B}_2]v[\hat{A}_2]v[\hat{B}_1], \quad (10)$$

contradicting the assumption that $v[\hat{A}_1], v[\hat{A}_2], v[\hat{B}_1], v[\hat{B}_2]$ are all nonzero.

Clifton [3] used the Weyl form of the canonical commutation relations,

$$[e^{-ia\hat{q}_i/\hbar}, e^{-ib\hat{q}_j/\hbar}] = [e^{-ia\hat{p}_i/\hbar}, e^{-ib\hat{p}_j/\hbar}] = [e^{-ia\hat{q}_i/\hbar}, e^{-ib\hat{p}_j/\hbar}] = 0, \quad i \neq j, \quad (11)$$
$$e^{-ia\hat{q}_i/\hbar}e^{-ib\hat{p}_i/\hbar} = e^{-iab/\hbar}e^{-ib\hat{p}_i/\hbar}e^{-ia\hat{q}_i/\hbar}, \quad (12)$$

to construct non-Hermitian unitary operators satisfying (1) and (2). Clifton's obstruction is, therefore, not obtained directly for real-valued observables but via a detour through the complex plane; this is supplemented by an argument that an obstruction in the complex plane entails one on the real line. It is not difficult to exhibit an obstruction directly in terms of Hermitian operators and real-valued observables. Let a_1, b_1, a_2, b_2 be real numbers such that

$$a_1 b_1 = (2m+1)\pi\hbar, \qquad a_2 b_2 = (2n+1)\pi\hbar, \tag{13}$$

for some integers m, n. Let $\hat{q}_1$, $\hat{q}_2$ be the operators corresponding to two independent coordinates, and let $\hat{p}_1$, $\hat{p}_2$ be the conjugate momenta operators. Define the operators

$$\hat{A}_i = \cos\bigl(a_i \hat{q}_i/\hbar\bigr), \qquad \hat{B}_i = \cos\bigl(b_i \hat{p}_i/\hbar\bigr). \tag{14}$$

It follows from the Weyl commutation relations (11), (12) that these operators satisfy (1) and (2), and hence we have the desired obstruction.

3. The continuity assumption

To exhibit a Kochen–Specker obstruction using only a single coordinate q and its conjugate momentum, we will construct operators satisfying the anti-commutation relations (2). However, we will not be able to satisfy at the same time the commutation relations (1). In place of the Product Rule, we will use a generalization, appropriate to observables that are approximately co-measurable, that we will call the ϵ-Product Rule.

Any hidden-variable theory will specify a set of observables to which definite values are to be ascribed (this set may depend on the quantum state), and a probability distribution over these definite values (which must, of course, depend on the quantum state). Kochen and Specker assumed that the set of observables to which definite values are ascribed form a partial algebra under the relation of comeasurability, and that the mapping ascribing definite values in $\mathbb{R}$ to these observables is a homomorphism. That is, if A and B are comeasurable observables possessing definite values $v(A)$ and $v(B)$, respectively, then to the observables $\mu A + \lambda B$ and AB are ascribed the definite values $\mu v(A) + \lambda v(B)$, and $v(A)v(B)$. The relation of comeasurability of observables is identified with commutation of the corresponding operators on physical grounds. For any two commuting operators, $\hat{A}$, $\hat{B}$, there is an operator $\hat{C}$ and functions f, g such that $\hat{A} = f(\hat{C})$ and $\hat{B} = g(\hat{C})$. One way of measuring the observables corresponding to $\hat{A}$ and $\hat{B}$ is to measure the observable corresponding to $\hat{C}$ and then to apply the functions f and g to the result. The physical justification for regarding this as a measurement of the observables A and B lies in the fact that, at least in ideal circumstances, a subsequent 'direct' measurement of A or B will yield the same result. Furthermore, for any state, an ideal measurement of an observable that commutes with $\hat{A}$ leaves unaltered the predicted statistical distribution of $\hat{A}$-measurements.

These are facts about the quantum-mechanical statistical predictions that any hidden-variables theory is bound to respect if it is to reproduce these statistical predictions, and, to the extent that these facts have been empirically verified, they must be respected by any empirically adequate hidden-variable theory.

In practice, an experimenter has only a finite degree of control over what observable is being measured, and the ideal case of exact reproducibility of results is only approximated. Empirical reasons for requiring hidden-variable theories to satisfy the usual Product Rule, therefore, have force also for observables that are *approximately* comeasurable, in the sense that a measurement of one only minimally disrupts the statistics of the other. The ϵ-Product Rule introduced below is meant to be an extension of the Product Rule to cover such cases.

We will need a measure of the degree of disruption of the statistical distribution of one observable by a measurement of another. Let $\hat{A}$ be a compact operator, and let $\{\hat{P}_i^A\}$ be its spectral projections. Suppose that a system in initial state $\hat{\rho}$ is subjected to one of two procedures—either $\hat{B}$ alone is measured, or $\hat{B}$ is measured subsequent to a prior ideal measurement of $\hat{A}$. In the former case, the expectation value of the result of the $\hat{B}$-measurement is

$$\langle \hat{B} \rangle_{\hat{\rho}} = \mathrm{Tr}\bigl(\hat{\rho}\hat{B}\bigr). \tag{15}$$

If, however, the $\hat{A}$-measurement is performed first, the expectation value for the result of the $\hat{B}$-measurement is

$$\langle \hat{B}' \rangle_{\hat{\rho}} = \mathrm{Tr}\left(\sum_i \hat{P}_i^A \hat{\rho} \hat{P}_i^A \hat{B}\right) = \mathrm{Tr}\left(\hat{\rho} \sum_i \hat{P}_i^A \hat{B} \hat{P}_i^A\right). \tag{16}$$

The difference between the two expectation values is

$$\langle \hat{B}' \rangle_{\hat{\rho}} - \langle \hat{B} \rangle_{\hat{\rho}} = \mathrm{Tr}\biggl(\hat{\rho}\biggl(\sum_i \hat{P}_i^A \hat{B} \hat{P}_i - B\biggr)\biggr) = -\,\mathrm{Tr}\biggl(\hat{\rho} \sum_i \bigl(I - \hat{P}_i^A\bigr) \hat{B} \hat{P}_i^A\biggr). \tag{17}$$

Define the disturbance of $\hat{B}$ by $\hat{A}$,

$$\Delta(\hat{B}; \hat{A}) = -\sum_i \bigl(I - \hat{P}_i^A\bigr) \hat{B} \hat{P}_i^A. \tag{18}$$

If $\hat{A}$ and $\hat{B}$ commute, the disturbance $\Delta(\hat{B}; \hat{A})$ is the zero operator. If $\hat{A}$ and $\hat{B}$ don't commute but $\Delta(\hat{B}; \hat{A})|\psi\rangle = 0$, a measurement of $\hat{A}$ leaves the expectation value of $\hat{B}$ unchanged but may alter the statistical distribution of these values about the expectation value. What we want to demand, in order to regard the value of $\hat{B}$ as minimally disturbed, is that the probability distribution for the outcome of a $\hat{B}$-measurement be minimally disrupted. If, in addition, a measurement of $\hat{A}$ only minimally disturbs the statistics regarding the product of $\hat{A}$ and $\hat{B}$, we ought to ascribe, at least with high probability, a definite value to the product of $\hat{A}$ and $\hat{B}$ that is close to the product of the values of $\hat{A}$ and $\hat{B}$. Of course, unless $\hat{A}$ and $\hat{B}$ commute, the product operator $\hat{A}\hat{B}$ will not be Hermitian, and hence not correspond to any observable; we will attribute a definite value instead to the symmetrized product,

$$\hat{A} \circ \hat{B} = \tfrac{1}{2}\bigl(\hat{A}\hat{B} + \hat{B}\hat{A}\bigr). \tag{19}$$

We will want to consider sequences of operators $\{\hat{A}_n\}$, $\{\hat{B}_n\}$ such that, for any state, the disturbance of the statistics for $\hat{B}_n$ can be made as small as one likes by taking n sufficiently large. The statistical distribution of the results of $\hat{B}$-measurements will be minimally altered if the expectation value of $\hat{B}^k$ is minimally altered for all k less than some sufficiently large K; that is, we can approximate any distribution by approximately recovering, for sufficiently high K, the first K moments of the distribution.

The assumption about definite values on which our obstruction will be based is the following.

Let $\{\hat{A}_n\}$, $\{\hat{B}_n\}$ be sequences of operators such that, for every natural number k and every vector $|\psi\rangle$, $\|\Delta(\hat{B}_n^k; \hat{A}_n)|\psi\rangle\|$ and $\|\Delta((\hat{A}_n \circ \hat{B}_n)^k; \hat{A}_n)|\psi\rangle\|$ converge to zero as $n \to \infty$. If $\hat{A}_n$ and $\hat{B}_n$ have definite values $v[\hat{A}_n]$ and $v[\hat{B}_n]$, then, for any state $\hat{\rho}$, and any $\epsilon, \delta > 0$, there exists N such that for all $n > N$ the probability is greater than $1 - \delta$ that the symmetrized product $\hat{A}_n \circ \hat{B}_n$ has a definite value satisfying

$$\bigl|v\bigl[\hat{A}_n \circ \hat{B}_n\bigr] - v\bigl[\hat{A}_n\bigr]v\bigl[\hat{B}_n\bigr]\bigr| < \epsilon.$$

A sequence of operators $\{\hat{\Delta}_n\}$ is said to *strongly converge* to a limit $\hat{\Delta}$ if and only if, for any vector $|\psi\rangle$, $\|(\hat{\Delta}_n - \hat{\Delta})|\psi\rangle\| \to 0$ as $n \to \infty$. Therefore, the condition in the first sentence of the above rule is the condition that the sequences of operators $\{\Delta(\hat{B}_n^k; \hat{A}_n)\}$ and $\{\Delta((\hat{A}_n \circ \hat{B}_n)^k; \hat{A}_n)\}$ strongly converge to zero.

A construction invoking this rule, which we will call the ϵ-Product Rule, will be called an ϵ-obstruction. Note that the usual Product Rule is entailed as a special case by the ϵ-Product Rule. It is also worth noting that the ϵ-Product Rule is satisfied by the simple noncontextual hidden-variable theories constructed by Bell and by Kochen and Specker [2] for a system consisting of a single spin-$\frac{1}{2}$ particle.

The ϵ-obstruction constructed in Section 4 will not be a state-independent one, as we will, in general, have to choose for different states different values of n to obtain a set of operators composing the obstruction. What will be shown is that, for any state, there is a set of operators that cannot consistently be assigned values in accord with the ϵ-Product Rule but which are required by that rule to have definite values if all functions of $\hat{q}$ and all functions of $\hat{p}$ are ascribed definite values. The conclusion is that no hidden-variables theory satisfying the ϵ-Product Rule can attribute noncontextual values to both of $\hat{q}$ and $\hat{p}$, in any state.

4. The obstruction

Suppose there were numbers a_1, a_2, b_1, b_2 such that

$$a_1 b_1 = (2m+1)\pi\hbar, \qquad a_1 b_2 = 2k\pi\hbar,$$
$$a_2 b_1 = 2l\pi\hbar, \qquad a_2 b_2 = (2n+1)\pi\hbar, \tag{20}$$

for some integers k, l, m, n. Then we could readily construct an obstruction for a single degree of freedom by defining the operators,

$$\hat{A}_i = \cos\bigl(a_i \hat{q}/\hbar\bigr), \qquad \hat{B}_i = \cos\bigl(b_i \hat{p}/\hbar\bigr). \tag{21}$$

As can easily be verified, such operators would satisfy the commutation/anticommutation relations (1), (2). There are no integers satisfying Eq. (20), however. To see this, note that this would require the product $a_1b_1a_2b_2$ to be both an even and an odd multiple of $\pi^2\hbar^2$. What we will do instead is satisfy these equations approximately. Define

$$a_{1n} = (2n+1)a, \qquad a_{2n} = (2+1/2n)a,$$
$$b_1 = \pi\hbar/a, \qquad b_{2n} = 2n\pi\hbar/a, \tag{22}$$

where a is an arbitrary constant. Then we have

$$a_{1n}b_1 = (2n+1)\pi\hbar,$$
$$a_{1n}b_{2n} = 2n(2n+1)\pi\hbar,$$
$$a_{2n}b_1 = (2+1/2n)\pi\hbar,$$
$$a_{2n}b_{2n} = (4n+1)\pi\hbar. \tag{23}$$

Define the projection operators

$$\hat{E}^+_{in} = \hat{E}\big[\cos\big(a_{in}\hat{q}/\hbar\big) \geqslant 0\big],$$
$$\hat{E}^-_{in} = \hat{E}\big[\cos\big(a_{in}\hat{q}/\hbar\big) < 0\big], \quad i = 1, 2, \tag{24}$$

where $\hat{E}[\cdot]$ denotes the spectral projection onto the specified subspace. We will also have occasion to consider translations of these operators,

$$\hat{E}^+_{in}(\epsilon) = \hat{E}\big[\cos\big(a_{in}\hat{q}/\hbar + \epsilon\hat{I}\,\big) \geqslant 0\big],$$
$$\hat{E}^-_{in}(\epsilon) = \hat{E}\big[\cos\big(a_{in}\hat{q}/\hbar + \epsilon\hat{I}\,\big) < 0\big]. \tag{25}$$

Now define the operators

$$\hat{A}_{in} = \hat{E}^+_{in} - \hat{E}^-_{in},$$
$$\hat{B}_1 = \cos\big(b_1\hat{p}/\hbar\big), \qquad \hat{B}_{2n} = \cos\big(b_{2n}\hat{p}/\hbar\big). \tag{26}$$

We will repeatedly use the relation,

$$f(\hat{q})e^{ib\hat{p}/\hbar} = e^{ib\hat{p}/\hbar} f\big(\hat{q} - b\hat{I}\,\big). \tag{27}$$

If f is a periodic function of period b,

$$\cos\big(b\hat{p}/\hbar\big) f(\hat{q}) = f(\hat{q}) \cos\big(b\hat{p}/\hbar\big), \tag{28}$$

and, if f changes sign under a translation by an amount b,

$$\cos\big(b\hat{p}/\hbar\big) f(\hat{q}) = -f(\hat{q}) \cos\big(b\hat{p}/\hbar\big). \tag{29}$$

$\hat{A}_{in}$ is a periodic function of $\hat{q}$ with period $2\pi\hbar/a_{in}$; it changes sign under a translation by an odd multiple of $\pi\hbar/a_{in}$. We therefore have the commutation relations

$$\big[\hat{A}_{1n}, \hat{A}_{2n}\big] = \big[\hat{A}_{1n}, \hat{B}_{2n}\big] = \big[\hat{B}_1, \hat{B}_{2n}\big] = 0, \tag{30}$$

and the anticommutation relations

$$\hat{A}_{1n}\hat{B}_1 = -\hat{B}_1\hat{A}_{1n}, \qquad \hat{A}_{2n}\hat{B}_{2n} = -\hat{B}_{2n}\hat{A}_{2n}. \tag{31}$$

The operators $\hat{E}^+_{2n}$, $\hat{E}^-_{2n}$, $\hat{A}_{2n}$ do not, for any n, commute with $\hat{B}_1$. What we have instead is

$$\hat{E}^\pm_{2n}\hat{B}_1 = \tfrac{1}{2}e^{ib_1\hat{p}/\hbar}\hat{E}^\pm_{2n}(-\epsilon_n) + \tfrac{1}{2}e^{-ib_1\hat{p}/\hbar}\hat{E}^\pm_{2n}(+\epsilon_n),$$
$$\hat{A}_{2n}\hat{B}_1 = \tfrac{1}{2}e^{ib_1\hat{p}/\hbar}\big(\hat{E}^+_{2n}(-\epsilon_n) - \hat{E}^-_{2n}(-\epsilon_n)\big)$$
$$+ \tfrac{1}{2}e^{-ib_1\hat{p}/\hbar}\big(\hat{E}^+_{2n}(+\epsilon_n) - \hat{E}^-_{2n}(+\epsilon_n)\big), \tag{32}$$

where $\epsilon_n = \pi/2n$.

Let us consider the disruption of the statistics for $\hat{B}_1$ by a measurement of $\hat{A}_{2n}$. We have

$$\Delta\big(\hat{B}_1; \hat{A}_{2n}\big) = -\hat{E}^-_{2n}\hat{B}_1\hat{E}^+_{2n} - \hat{E}^+_{2n}\hat{B}_1\hat{E}^-_{2n}. \tag{33}$$

The first term of this is

$$-\hat{E}^-_{2n}\hat{B}_1\hat{E}^+_{2n} = -\tfrac{1}{2}e^{ib_1\hat{p}/\hbar}\hat{E}^-_{2n}(-\epsilon_n)\hat{E}^+_{2n}$$
$$- \tfrac{1}{2}e^{-ib_1\hat{p}/\hbar}\hat{E}^-_{2n}(+\epsilon_n)\hat{E}^+_{2n}. \tag{34}$$

The operator $\hat{E}^-_{2n}(-\epsilon_n)\hat{E}^+_{2n}$ is the projection onto the part of the spectrum of $\hat{q}$ such that $\cos(a_{2n}q) \geqslant 0$ but $\cos(a_{2n}q - \epsilon_n) < 0$. For any state $\hat{\rho}$, the measure of this part of the spectrum in any interval $[-M, M]$ will be small if n is large, and will converge to zero as $n \to \infty$. Hence, for any state with compact support in $\hat{q}$, $\|\hat{E}^-_{2n}(-\epsilon_n)\hat{E}^+_{2n}\hat{\rho}\| \to 0$ as $n \to \infty$ (recall that $\hat{q}$ has no eigenvalues), and, since such states are norm-dense in the set of all states, $\|\hat{E}^-_{2n}(-\epsilon_n)\hat{E}^+_{2n}\hat{\rho}\| \to 0$ as $n \to \infty$ for any state ρ. Analogous considerations apply to the other terms in the expansion of $\Delta(\hat{B}_1, \hat{A}_{2n})$. We conclude that the $\Delta(\hat{B}_1; \hat{A}_{2n})$ strongly converges to zero as $n \to \infty$. A similar argument shows that, for any k, $\Delta(\hat{B}_1^k; \hat{A}_{2n})$ strongly converges to zero as $n \to \infty$.

We are now ready to begin the construction of our ϵ-obstruction. Suppose that associated with $\hat{A}_{1n}$, $\hat{A}_{2n}$, $\hat{B}_1$, $\hat{B}_{2n}$ are definite values $v[\hat{A}_{1n}]$, $v[\hat{A}_{2n}]$, $v[\hat{B}_1]$, $v[\hat{B}_{2n}]$. Since the spectrum of each $\hat{A}_{1n}$, $\hat{A}_{2n}$ is contained in $\{-1, 1\}$, the corresponding values will be nonzero. Moreover, we can, without loss of generality (or rather, a loss of measure zero) assume $v[\hat{B}_1]$, $v[\hat{B}_{2n}]$ to be nonzero also, since, for any state, the probability is zero that a measurement of either will yield a result *exactly* equal to zero. Choose some positive $\delta < \frac{1}{2}$.

Since $\hat{A}_{1n}$ commutes with $\hat{B}_{2n}$, the Product Rule requires that associated with the product operator

$\hat{A}_{1n}\hat{B}_{2n}$ is the definite value

$$v[\hat{A}_{1n}\hat{B}_{2n}] = v[\hat{A}_{1n}]v[\hat{B}_{2n}]. \tag{35}$$

As mentioned, $\hat{A}_{2n}$ does not commute with $\hat{B}_1$. However, as we have seen, for any state $\hat{\rho}$, the disturbance of the statistics of $\hat{B}$-measurements by a measurement of $\hat{A}_{2n}$ can be made arbitrarily small by taking n sufficiently large. Moreover, it is easy to verify that $\hat{A}_{2n}$ commutes with the symmetrized product $\hat{B}_1 \circ \hat{A}_{2n}$, since, for any $\hat{A}$, $\hat{B}$,

$$[\hat{A}, \hat{A} \circ \hat{B}] = \tfrac{1}{2}[\hat{A}^2, \hat{B}], \tag{36}$$

and the square of $\hat{A}_{2n}$ is the identity operator. Therefore, measurements of $\hat{A}_{2n}$ do not disrupt that statistics of measurements of $\hat{B}_1 \circ \hat{A}_{2n}$ and, for sufficiently large n, only minimally disrupt the statistics of $\hat{B}_1$-measurements. The ϵ-Product Rule dictates that, for any $\epsilon > 0$, for sufficiently large n there is a probability greater than $1 - \delta$ that the symmetrized product $\hat{B}_1 \circ \hat{A}_{2n}$ has a definite value satisfying

$$\left|v[\hat{B}_1 \circ \hat{A}_{2n}] - v[\hat{B}_1]v[\hat{A}_{2n}]\right| < \epsilon. \tag{37}$$

$\hat{B}_1 \circ \hat{A}_{2n}$ commutes with $\hat{B}_{2n}\hat{A}_{1n}$. Therefore, if $\hat{B}_1 \circ \hat{A}_{2n}$ has a definite value $v[\hat{B}_1 \circ \hat{A}_{2n}]$, then

$$\begin{aligned} v[(\hat{B}_1 \circ \hat{A}_{2n})\hat{B}_{2n}\hat{A}_{1n}] &= v[\hat{B}_1 \circ \hat{A}_{2n}]v[\hat{B}_{2n}\hat{A}_{1n}] \\ &= v[\hat{B}_1 \circ \hat{A}_{2n}]v[\hat{B}_{2n}]v[\hat{A}_{1n}], \end{aligned} \tag{38}$$

and hence

$$\begin{aligned} &v[(\hat{B}_1 \circ \hat{A}_{2n})\hat{B}_{2n}\hat{A}_{1n}] - v[\hat{B}_1]v[\hat{A}_{2n}]v[\hat{B}_{2n}]v[\hat{A}_{1n}] \\ &\quad = (v[\hat{B}_1 \circ \hat{A}_{2n}] - v[\hat{B}_1 \circ \hat{A}_{2n}])v[\hat{B}_{2n}]v[\hat{A}_{1n}]. \end{aligned} \tag{39}$$

We thus conclude that, for any ϵ, for sufficiently large n the probability is greater than $1 - \delta$ that $(\hat{B}_1 \circ \hat{A}_{2n})\hat{B}_{2n}\hat{A}_{1n}$ has a definite value satisfying

$$\begin{aligned} &\left|v[(\hat{B}_1 \circ \hat{A}_{2n})\hat{B}_{2n}\hat{A}_{1n}] - v[\hat{B}_1]v[\hat{A}_{2n}]v[\hat{B}_{2n}]v[\hat{A}_{1n}]\right| \\ &\quad = \left|v[\hat{B}_1 \circ \hat{A}_{2n}] - v[\hat{B}_1 \circ \hat{A}_{2n}]\right|\left|v[\hat{B}_{2n}]v[\hat{A}_{1n}]\right| \\ &\quad < \epsilon\left|v[\hat{B}_{2n}]v[\hat{A}_{1n}]\right| \leqslant \epsilon, \end{aligned} \tag{40}$$

where the last step is justified by the fact that the spectra of $\hat{B}_{2n}$ and $\hat{A}_{1n}$ are contained in $[-1, 1]$.

Since $\hat{A}_{1n}$ and $\hat{A}_{2n}$ commute, the Product Rule gives,

$$v[\hat{A}_{1n}\hat{A}_{2n}] = v[\hat{A}_{1n}]v[\hat{A}_{2n}]. \tag{41}$$

Similarly,

$$v[\hat{B}_1\hat{B}_{2n}] = v[\hat{B}_1]v[\hat{B}_{2n}]. \tag{42}$$

The operators $\hat{A}_{1n}\hat{A}_{2n}$ do not commute with $\hat{B}_1\hat{B}_{2n}$. However, an argument analogous to that used above for the case of $\hat{A}_{2n}$ and $\hat{B}_1$, shows that, for any k, $\Delta((\hat{B}_1\hat{B}_{2n})^k; \hat{A}_{1n}\hat{A}_{2n})$ converges strongly to zero as $n \to \infty$ (the details of this argument present no novel features and will not be rehearsed here). Moreover, $\hat{A}_{1n}\hat{A}_{2n}$ commutes with $(\hat{B}_1\hat{B}_{2n}) \circ (\hat{A}_{1n}\hat{A}_{2n})$. Therefore, for any $\epsilon > 0$, for sufficiently large n the probability is greater than $1 - \delta$ that $(\hat{B}_1\hat{B}_{2n}) \circ (\hat{A}_{1n}\hat{A}_{2n})$ has a definite value satisfying

$$\begin{aligned} &\left|v[(\hat{B}_1\hat{B}_{2n}) \circ (\hat{A}_{1n}\hat{A}_{2n})]\right. \\ &\quad \left.- v[\hat{B}_1\hat{B}_{2n}]v[\hat{A}_{1n}\hat{A}_{2n}]\right| < \epsilon, \end{aligned} \tag{43}$$

and hence

$$\begin{aligned} &\left|v[(\hat{B}_1\hat{B}_{2n}) \circ (\hat{A}_{1n}\hat{A}_{2n})]\right. \\ &\quad \left.- v[\hat{B}_1]v[\hat{B}_{2n}]v[\hat{A}_{1n}]v[\hat{A}_{2n}]\right| < \epsilon. \end{aligned} \tag{44}$$

Because of the commutation/anticommutation relations (30), (31),

$$(\hat{B}_1\hat{B}_{2n}) \circ (\hat{A}_{1n}\hat{A}_{2n}) = -(\hat{B}_1 \circ \hat{A}_{2n})\hat{B}_{2n}\hat{A}_{1n}, \tag{45}$$

and therefore,

$$v[(\hat{B}_1\hat{B}_{2n}) \circ (\hat{A}_{1n}\hat{A}_{2n})] = -v[(\hat{B}_1 \circ \hat{A}_{2n})\hat{B}_{2n}\hat{A}_{1n}]. \tag{46}$$

We therefore have the conclusion that for any ϵ, for sufficiently large n the probability is greater than $1 - \delta$ that each of (40) and (44) hold. Each choice of n will yield a different set of definite values $\{v[\hat{B}_1], v[\hat{B}_{2n}], v[\hat{A}_{1n}], v[\hat{A}_{2n}]\}$. The probability distributions for these values must mirror the quantum-mechanical predictions for the outcomes of measurements of these quantities; hence, by taking ϵ sufficiently small, it is possible to make the probability arbitrarily close to unity that

$$3\epsilon < \left|v[\hat{B}_1]v[\hat{B}_{2n}]v[\hat{A}_{1n}]v[\hat{A}_{2n}]\right|. \tag{47}$$

Let us assume that we have chosen some ϵ, n such that this is the case. For sufficiently large n, the probability is greater than $1 - \delta$ that (40) holds, and probability greater than $1 - \delta$ that (44) holds. Since δ was chosen to be less than $\frac{1}{2}$, this entails that there is a nonzero probability that both (40) and (44) hold; in fact, since

δ can be chosen arbitrarily small, this probability can be made arbitrarily close to unity. Assume, therefore, that we have ϵ, n such that (40), (44), (46), and (47) all hold. Let

$$x = v\big[(\hat{B}_1\hat{B}_{2n}) \circ (\hat{A}_{1n}\hat{A}_{2n})\big],$$
$$y = v\big[(\hat{B}_1 \circ \hat{A}_{2n})\hat{B}_{2n}\hat{A}_{1n}\big],$$
$$z = v\big[\hat{B}_1\big]v\big[\hat{B}_{2n}\big]v\big[\hat{A}_{1n}\big]v\big[\hat{A}_{2n}\big]. \tag{48}$$

Eqs. (40) and (44) require that x and y both be a distance less than ϵ from z, and hence a distance less than 2ϵ from each other. But ϵ was chosen to be smaller than $|z|/3$, and so x and y must both have absolute value greater than 2ϵ. By (46), $x = -y$, and so their distance from each other must be greater than 4ϵ, contradicting our previous conclusion that the distance between them is less than 2ϵ. Therefore, (40), (44), (46), and (47) cannot simultaneously be satisfied.

5. Comment

On the basis of an analogy with the case of a single spin-$\frac{1}{2}$ particle, Clifton [3] conjectured that no Kochen–Specker obstruction for position and momentum using only one degree of freedom was possible. Since Clifton's conjecture concerns obstructions as usually conceived, and not ϵ-obstructions, the ϵ-obstruction in this Letter does not refute this conjecture, which remains undecided. The ϵ-obstruction in this Letter does, however, reveal a disanalogy with the spin-$\frac{1}{2}$ case; since there is, in fact, a noncontextual hidden-variables theory for single spin-$\frac{1}{2}$ particles that satisfies the ϵ-Product Rule [2], there can be no ϵ-obstruction for such a case.

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