# Abstract Hardy inequalities: The case $p = 1$

A Preprint

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**Abstract**

Boundedness of an abstract formulation of Hardy operators between Lebesgue spaces over general measure spaces is studied and, when the domain is $L^1$, shown to be equivalent to the existence of a Hardy inequality on the half line with general Borel measures. This is done by extending the greatest decreasing minorant construction to general measure spaces depending on a totally ordered collection of measurable sets, called an ordered core. A functional description of the greatest decreasing minorant is given, and for a large class of ordered cores, a pointwise description is provided. As an application, characterizations of Hardy inequalities for metric measure spaces are given, we note that the metric measure space is not required to admit a polar decomposition.

## 1 Introduction: Abstract Hardy inequalities

Given three Borel measures on $[0,\infty)$, simple necessary and sufficient conditions for which the inequality

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]} f\,d\lambda\right)^q d\nu(x)\right)^{1/q} \leq C\Big(\int_{[0,\infty)} f^p\,d\eta\Big)^{1/p} \tag{1}$$

holds for all positive measurable functions have been given by several authors. Letting $p = q > 1$, $\lambda$ and $\eta$ as the Lebesgue measure and $d\nu = 1/x\,d\lambda$ yields the classical Hardy inequality proved in the 1925 paper [4], which holds with best constant $p/(p-1)$. Muckenhoupt, in [8], showed that letting $\nu$ and $\eta$ be absolutely continuous with respect to the Lebesgue measure, the inequality holds if and only if a one-parameter supremum is finite. Bradley, in [3], extended the result for indices $1 < p \leq q < \infty$. Maz'ya, in [7] and Sinnamon, in [17], showed that for $0 < q < p$ and $1 < p < \infty$, the characterization is given by the finiteness of a single integral. In the case $p > 1$, simple characterizations for inequality (1) can be found in [14].

Extensions have been made in several directions; results for more general measures, higher dimensions, restrictions on the domain are available, see [6].

The case $p = 1$ must be treated differently. In [5, Theorem 3.1] the following characterization is shown:

**Theorem 1.1.** *If $0 < q < 1 = p$, then the inequality (1) holds if and only if*

$$\left(\int_{[0,\infty)}\left(\int_{[0,x]}\frac{1}{\underline{w}}\,d\nu\right)^{\frac{q}{1-q}} d\nu(x)\right)^{1/q} < \infty, \tag{2}$$

*with $\underline{w}(x) = \text{ess inf}_\lambda\{w(t) : t \in [0,x]\}$, where $d\eta = d\lambda^\perp + w d\lambda$ and $\lambda^\perp \perp \lambda$.*

In this paper we are concerned with a large class of Hardy inequalities introduced in [13], which require the following definition.

**Definition 1.2.** *Let $(U, \Sigma, \mu)$ and $(Y, \mathcal{T}, \tau)$ be two $\sigma-$finite measure spaces, a map $B : Y \to \Sigma$ is called a* ***core map*** *provided it satisfies:*

1. *(Total order) The range of $B$ is totally ordered by inclusion.*
2. *(Measurability) For each $E \in \Sigma$ the map $y \mapsto \mu(E \cap B(y))$ is $\mathcal{T}-$measurable.*
3. *($\sigma$-boundedness) There is a countable subset $Y_0 \subseteq Y$ such that $\bigcup_{y \in Y} B(y) = \bigcup_{y \in Y_0} B(y)$.*
4. *(Finite measure) For all $y \in Y$, $\mu(B(y)) < \infty$.*

Given a core map, an inequality of the form

$$\left(\int\limits_Y \left(\int\limits_{B(y)} f\, d\mu\right)^q d\tau(y)\right)^{1/q} \leq C\Big(\int\limits_U f^p\, d\eta\Big)^{1/p}, \tag{3}$$

for all positive measurable functions $f$ is called an *Abstract Hardy inequality*. Notice that setting $Y = U = [0, \infty)$ and $B(y) = [0, y]$ recovers inequality (1). In the case that $\mu = \eta$, [13, Theorem 2.4] shows that the best constant $C$ in (3) is the same as the best constant in the inequality

$$\left(\int_0^\infty \left(\int_0^{b(x)} f(t)\, dt\right)^q dx\right)^{1/q} \leq C\Big(\int\limits_0^\infty f(t)^p\, dt\Big)^{1/p}, \quad \text{for all } f \in L^+,$$

for an appropriate non-increasing function $b : (0, \infty) \to [0, \infty]$. For $p > 1$, any abstract Hardy inequality (3) can be reduced to the case where $\eta$ and $\mu$ coincide (see [13, Theorem 5.1]), however the reduction is not available for the case $p = 1$, as the formula involves a power of the form $\frac{1}{p-1}$. Our main result is the following extension of Theorem 1.1 to the abstract setting.

**Theorem.** For $\sigma-$finite measure spaces $(Y, \mathcal{T}, \tau)$, $(U, \Sigma, \mu)$, $(U, \Sigma, \nu)$ and a core map $B : Y \to \Sigma$, let $\eta = \eta_a + \eta_s$, where $d\eta_a = u d\mu$ and $\eta_s \perp \mu$. Then the best constant $C$ in the inequality

$$\left(\int\limits_Y \left(\int\limits_{B(y)} f\, d\mu\right)^q d\tau(y)\right)^{1/q} \leq C\int\limits_U f\, d\eta,$$

satisfies

$$C \approx \left(\int\limits_Y \left(\int\limits_{\mu(B(z)) \leq \mu(B(y))} R\left(\frac{1}{\underline{u}}\right) \circ \mu \circ B(y)\, d\tau(y)\right)^{\frac{q}{1-q}} d\tau(z)\right)^{\frac{1-q}{q}}, \text{ for } q \in (0, 1),$$

and

$$C = \sup_{s \in U}\left(\frac{1}{\underline{u}}(s)\right)\tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q}, \text{ for } q \in [1, \infty).$$

The map $R$ is introduced in Section 2.1 and the proof of the main Theorem will be provided in Chapter 3.

Our approach is to show that, for $p = 1$, an abstract Hardy inequality is equivalent to a Hardy inequality with measures and give necessary and sufficient conditions for such an inequality to hold.

In Section 2 we introduce the tools necessary to state our main result. The key construction is the greatest core decreasing minorant of a function, which extends the construction $\underline{w}$ of Theorem 1.1 to general measure spaces. This construction allows us to reduce inequality (3) to a suitable inequality of the form (1). This is done in Section 3. In Section 4 we give explicit examples of the greatest core decreasing minorant and apply the main result in Section 3 for Hardy inequalities in metric measure spaces. We leave Section 5 for a proof of a functional description of the least core decreasing minorant, which is the key step in proving our main result.

We finish this introduction by setting up notation and some basic results. For a $\sigma-$finite measure space $(U, \Sigma, \mu)$ and a set $\mathcal{A} \subseteq \Sigma$ we denote the $\sigma-$ring generated by $\mathcal{A}$ by $\sigma(\mathcal{A})$. By $L(\mathcal{A})$ we mean the collection of all (equivalence classes of) $[-\infty, \infty]-$valued $\sigma(\mathcal{A})-$measurable functions on $U$. The collection of non-negative functions in $L(\mathcal{A})$ is written as $L^+(\mathcal{A})$. We reserve the notation $L^0_\mu$ for the collection of $\Sigma-$measurable functions and $L^+_\mu$ for the non-negative ones.

We write $0 \leq \alpha_n \uparrow \alpha$ to indicate the limit of a non-decreasing sequence in $[0, \infty]$ and use $\alpha_n \downarrow \alpha$ when the limit is non-increasing. In the case of sets, we write $A_n \uparrow A$ or $A_n \downarrow A$ if their characteristic functions converge increasingly or decreasingly almost everywhere. We adopt the convention that expressions that evaluate to $0/0$ will be taken to be zero. For $p \in (0, \infty]$ the expression $L^p_\mu$ denotes the usual Lebesgue space of $\mu-$measurable functions. For two positive constants $C$ and $D$ we write $C \approx D$ if $d_1 D \leq C \leq d_2 D$ for positive numbers $d_1, d_2$.

For a function $f \in L(\Sigma)$, its distribution function, $\mu_f$ is given by $\mu_f(\alpha) = \mu\left(\{s \in U : |f(s)| > \alpha\}\right)$. Following [1], if $\mu_f = \tau_g$ then for any $p \in (0, \infty)$ we have $\int\limits_U |f|^p \, d\mu = \int\limits_Y |g|^p \, d\tau$.

We consider a *metric measure space* to be the triple $(\mathbb{X}, d, \mu)$ where $d$ is a distance function and $\mu$ is a Borel measure with respect to the topology induced by the metric $d$ and for every $a \in \mathbb{X}$ and $r > 0$, the closed ball of radius $r$ centered at $a$ has finite measure.

## 2 Ordered cores

In this section, we set up our tools and notation to work with monotone functions in general measure spaces without an order relation on the elements. First, we recall some key definitions in [12, Definition 1.1]:

**Definition 2.1.** *Let $(U, \Sigma, \mu)$ be a $\sigma-$finite measure space. A family of sets $\mathcal{A} \subseteq \Sigma$ is a full $\sigma$-bounded ordered core provided:*

1. The family $\mathcal{A}$ is totally ordered by inclusion.
2. Every set $E \in \mathcal{A}$ has finite $\mu-$measure.
3. The space $U$ can be realized as the union $U = \bigcup_{E \in A_0} E$ for some countable subfamily $A_0$ of $\mathcal{A}$.

We will also need following related concepts

- For a full ordered core $\mathcal{A}$ the relation $\leq_\mathcal{A}$ on $U$ is defined by $u \leq_\mathcal{A} v$ if for all $A \in \mathcal{A}$, $v \in A$ implies $u \in A$. When there is no ambiguity on the core, we omit the subscript $\mathcal{A}$. We will write $u <_\mathcal{A} v$ whenever $u \leq_\mathcal{A} v$ holds but $v \leq_\mathcal{A} u$ fails.
- For a full ordered core $\mathcal{A}$ there exists an extension $\mathcal{M}$ that does not modify the order relation and is closed under arbitrary unions and intersections, provided the result has finite measure and $\sigma(\mathcal{A}) = \sigma(\mathcal{M})$ (see [12, Lemma 4.1]). We will refer to this extension as the *maximal core* induced by $\mathcal{A}$.
- For a maximal core $\mathcal{M}$ and $E \in \sigma(\mathcal{A})$, then $E \in \mathcal{M}$ is equivalent to: For all $u, v \in U$, if $v \in E$ and $u \leq_\mathcal{A} v$, then $u \in E$. (see [12, Lemma 4.1 (c)])
- A function $f : U \to [0, \infty]$ is called *core-decreasing* relative to $\mathcal{A}$ if it is $\sigma(\mathcal{A})$-measurable and if for all $u, v \in U$, $u \leq_\mathcal{A} v$ implies $f(u) \geq f(v)$. The collection of core-decreasing functions is denoted by $L^\downarrow(\mathcal{A})$.

We define the collection of (equivalence classes of) functions

$$L^1_{\text{loc}_\mathcal{A},\mu} = \left\{ f \in L(\Sigma) : \int_A |f| \, d\mu < \infty \text{ for all } A \in \mathcal{A} \right\}.$$

Let $\mathcal{B}$ be the Borel $\sigma-$algebra on $[0, \infty)$, then by virtue of [12, Theorem 6.4], for every ordered core $\mathcal{A}$ there exists a Borel measure $\lambda$ induced by the core $\mathcal{A}$ and linear *transition maps* $R : L^1_{\text{loc}_\mathcal{A},\mu} \to L^1_{\text{loc},\lambda}$ and $Q : L^1_{\text{loc},\lambda} \to L^1_{\text{loc}_\mathcal{A},\mu}$ satisfying:

1. If $\varphi \in L^+(\mathcal{B}) \cup L^1_{\text{loc},\lambda}$, then $RQ\varphi = \varphi$ up to a set of $\lambda-$measure zero.
2. If $f \in L^+(\mathcal{A}) \cup \left(L^1_{\text{loc}_\mathcal{A},\mu} \cap L(\mathcal{A})\right)$, then $QRf = f$ up to a set of $\mu-$measure zero.
3. If $f \in L^+(\Sigma)$, $\varphi \in L^+(\mathcal{B})$ and $A \in \mathcal{A}$ then
$$\int\limits_A fQ(\varphi) \, d\mu = \int\limits_{[0,\mu(A)]} R(f)\varphi \, d\lambda \quad \text{and} \quad \int\limits_U fQ(\varphi) \, d\mu = \int\limits_{[0,\infty)} R(f)\varphi \, d\lambda.$$
4. If $f, g \in L^+_\mu \cap L^1_{\text{loc}_\mathcal{A},\mu} \cap L(\mathcal{A})$, then $R(fg) = R(f)R(g)$.

5. If $f, g \in L^+_\mu \cap L^1_{\text{loc}_{\mathcal{A}},\mu} \cap L(\mathcal{A})$ satisfy $\int_A f\,d\mu = \int_A g\,d\mu$ for all $A \in \mathcal{A}$, then $f = g$ up to a set of zero $\mu-$measure.

Notice that condition (v) follows from (ii) and the fact that the equality $\int\limits_{[0,x]} Rf\,d\lambda = \int\limits_{[0,x]} Rg\,d\lambda$ holding for all $x > 0$ forces that the functions $Rf$ and $Rg$ to be equal $\lambda-$almost everywhere.

We introduce our main technical tool, which extends the greatest non-increasing minorant (see [16, Section 2]).

**Definition 2.2.** *For a $\Sigma-$measurable function $g$, we call $h \in L^\downarrow(\mathcal{A})$ a greatest core decreasing minorant of $g$ if $0 \le h \le |g|$ $\mu-$a.e and for any $w \in L^\downarrow(\mathcal{A})$ satisfying $0 \le w \le |g|$, then $w \le h$ $\mu-$a.e.*

Note that a greatest core decreasing minorant is unique almost everywhere, provided it exists. The next lemma shows that such a greatest core decreasing minorant always exists.

**Lemma 2.3.** *Every $\Sigma-$measurable function $g$ admits a greatest core-decreasing minorant denoted $\underline{g}$, which is unique $\mu-$almost everywhere.*

**Proof:** Suppose that $|g| \le C < \infty$ and let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $A_n \uparrow U$. Set

$$\alpha_n = \sup\left\{ \int_{A_n} h\,d\mu : h \in L^\downarrow(\mathcal{A}) \text{ and } h \le |g| \right\}.$$

The collection defining the supremum is not empty as $h = 0$ is a core-decreasing function, moreover, the supremum is finite since $\int\limits_{A_n} h\,d\mu \le C\mu(A_n) < \infty$.

Let $h_n = 0$ if $\alpha_n = 0$, otherwise there exists $h_n \in L^\downarrow(\mathcal{A})$ such that $h_n \le |g|$ and $\alpha_n - 1/n < \int\limits_{A_n} h_n\,d\mu$. Since the pointwise maximum of core decreasing functions is core decreasing, we may assume that $\{h_n\}$ is an increasing sequence. Let $h = \sup_n h_n$, which is clearly a core decreasing minorant of $g$.

To show that $h$ is the greatest core decreasing minorant of $g$, let $w$ be another core decreasing minorant, then so is $\max\{h, w\}$, thus

$$\infty > \int\limits_{A_n} h\,d\mu \ge \int\limits_{A_n} h_n\,d\mu > \alpha_n - 1/n \ge \int\limits_{A_n} \max\{w, h\}\,d\mu - 1/n.$$

Then $1/n \ge \int\limits_{A_n} (\max\{w, h\} - h)\;d\mu \ge 0$. Let $n \to \infty$ to get $\max\{w, h\} = h$ almost everywhere. This completes the proof in the case that $g$ is bounded.

For the unbounded case, define $g_m = \min\{m, |g|\}$ and let $\underline{g_m}$ be its greatest core decreasing minorant which exists since $g_m$ is bounded. Since $\underline{g_{m-1}} \le \min\{m-1, |g|\} \le \min\{m, |g|\} = g_m$, then $\underline{g_{m-1}} \le \underline{g_m}$. Therefore $\{\underline{g_m}\}_{m\in\mathbb{N}}$ is an increasing sequence.

Let $h = \sup_{m\in\mathbb{N}} \underline{g_m}$. Since each $\underline{g_m}$ is bounded above by $|g|$, then $h \le |g|$, thus $h$ is a core decreasing minorant of $|g|$. If $w$ is another core decreasing minorant of $|g|$, then $\min(m, w)$ is a core decreasing minorant of $|g_m|$, thus $\min(m, w) \le \underline{g_m}$. Let $m \to \infty$ to get $w \le h$ and complete the proof.

□

The next theorem gives a functional description of the greatest core decreasing minorant; it extends the corresponding statement in [16, Theorem 2.1] to a very large class of functions. The proof follows a different argument than its real line counterpart and is left for Section 5.

**Theorem 2.4.** *Let $u$ and $f$ be non-negative measurable functions, finite $\mu-$almost everywhere, such that $\int_U uf\,d\mu < \infty$ and $f \in L^1_{loc_{\mathcal{A}},\mu}$. Then*

$$\int\limits_U f\underline{u}\,d\mu = \inf\left\{ \int\limits_U gu\,d\mu : \int\limits_A g\,d\mu \ge \int\limits_A f\,d\mu \text{ for all } A \in \mathcal{A} \right\}.$$

As the necessary and sufficient conditions for the existence of a finite constant $C$ in the abstract Hardy inequality (3) depend on the computation of this greatest core decreasing minorant, the next result gives an explicit pointwise formula of this minorant, in the case that the ordered core satisfies a mild condition. It is worth mentioning that for the ordered core constructed in [12, Example 5.4], the following formula does not hold. Hence, some condition on the core must be required.

**Theorem 2.5.** *Let $(U, \Sigma, \mu)$ be a measure space with a full $\sigma-$bounded ordered core $\mathcal{A}$ such that arbitrary unions and intersections in $\mathcal{A}$ are $\sigma(\mathcal{A})-$measurable. Then for any $\Sigma-$measurable function $g$ the formula*

$$\underline{g}(s) = \textit{ess inf}_\mu \left\{ |g(v)| : v \leq_{\mathcal{A}} s \right\}$$

*holds.*

**Proof:** Let $h(s) = \text{ess inf}_\mu \left\{ |g(v)| : v \leq_{\mathcal{A}} s \right\}$. Since the order relation $\mathcal{A}$ is unchanged if we replace $\mathcal{A}$ by its maximal core, we may assume that $\mathcal{A}$ is maximal and that arbitrary unions and intersections of core sets are in the core, provided the result has finite $\mu-$measure. It follows from the definition of the order relation that

$$\{t \in U : t <_A s\} = \bigcup \{A \in \mathcal{A} : s \in A\} \text{ and } \{t \in U : t \leq_A s\} = \bigcap \{A \in \mathcal{A} : s \notin A\}.$$

By hypothesis, all of these sets are $\sigma(\mathcal{A})-$measurable for all $s \in U$. Define $[s] = \{t \in U : t \leq_A s \text{ and } s \leq_A t\}$, which is the difference of the sets above, so it is $\sigma(\mathcal{A})-$measurable as well.

To show that $\underline{g}$ is a $\sigma(\mathcal{A})$-measurable function: Let $\alpha \in \mathbb{R}$ and define $O = h^{-1}(\alpha, \infty)$, we proceed to show that $O$ is $\sigma(\mathcal{A})-$measurable.

Clearly $O \subseteq \bigcup_{x \in O} \{t \in U : t \leq_A x\}$. Conversely, if $x \in O$ and $y \leq_A x$, then

$$h(y) = \text{ess inf}_\mu \{t \in U : t \leq_A y\} \geq \text{ess inf}_\mu \{t \in U : t \leq_A x\} = h(x) > \alpha,$$

hence $\{t \in U : t <_A x\} \subseteq O$, this proves that $O = \bigcup_{x \in O} \{t \in U : t <_A x\}$, which by hypothesis, is a $\sigma(\mathcal{A})-$measurable set. As $\alpha$ was arbitrary, then $h$ is $\sigma(\mathcal{A})-$measurable.

Since $h$ satisfies $y \leq_A x$ implies $h(y) \geq h(x)$ and is $\sigma(\mathcal{A})-$measuable, then it only remains to show that $h$ is a minorant of $|g|$ and that it is optimal.

We show the inequality $h(z) \leq |g(z)|$ by cases, depending on the measure of the set $[z]$. If $z \in U$ satisfies $\mu([z]) > 0$, notice that if $z' \in [z]$ then $h(z') = h(z)$. Hence, by definition of essential infimum we have that $\mu\left(\{z' \in [z] : |g(z')| < h(z)\}\right) = 0$. Therefore $h \leq |g|$ on $[z]$ up to a set of $\mu-$measure zero. Since $\lambda$ is a $\sigma-$finite measure, the collection of sets $U_D = \{[z] : \mu([z]) > 0\}$ must be countable. Hence, we have $h \leq |g|$ on its union up to a set of $\mu-$measure zero.

We must show the same inequality holds for the set $U_0 = \{z \in U : \mu([z]) = 0\}$. For this purpose: Fix $\epsilon > 0, n, m \in \mathbb{N}$, $\{A_n\} \in \mathcal{A}$ satisfy $U \subseteq \cup_n A_n$ and define

$$S_{m,n} = \{z \in U_0 \cap A_m : h(z) - |g(z)| > \epsilon \text{ and } n\epsilon \leq |g(z)| < (n+1)\epsilon\}.$$

By the previous estimate, we have that

$$\mu\left(\{z \in U : |g(z)| < h(z)\} \setminus \cup_{m,n} S_{m,n}\right) = 0.$$

Since $U_D \in \sigma(\mathcal{A})$, is obtained by countably many unions of set differences of core sets, then $R\chi_{U_D}$ is a characteristic function by [12, Proposition 6.2(i)]. Since $U = U_0 \cup U_D$, we have that $R\chi_{U_0}$ is also a characteristic function, and $[0, \infty)$ is a disjoint union of some Borel sets $L_0, L_D$ such that $\chi_{L_0} = R\chi_{U_0}$ and $\chi_{L_D} = R\chi_{U_D}$.

We claim that any $t \in [0, \infty)$ satisfying $\lambda(\{t\}) > 0$ must be contained in $L_D$. To see this, let $E_1, E_2$ satisfy $\mu(E_1) = \lambda(0, t)$ and $\mu(E_2) = \lambda(0, t]$, Observe that any $A \in \mathcal{A}$ must satisfy $\mu(A) \leq \mu(E_1)$ or $\mu(E_2) \leq \mu(A)$. Define

$$M = \cup\{A \in \mathcal{A} : \mu(A) < \mu(E_2)\} \quad \text{and} \quad N = \cap\{A \in \mathcal{A} : \mu(E_1) < A\}.$$

By hypothesis $M, N \in \mathcal{A}$, by the choice of $E_1, E_2$ we must have that $\mu(A) < \mu(E_2)$ implies $\mu(A) \leq \mu(E_1)$ and the monotone convergence theorem shows that $\mu(M) = \mu(E_1)$. Similarly, the dominated convergence theorem shows that $\mu(N) = \mu(E_2)$. Let $z \in M \setminus N$, then $\mu([z]) = \lambda(t) > 0$, so $M \setminus N$ is contained in $U_D$. An application of $R$ yields $t \in L_D$.

Since the support of $R\chi_{S_{m,n}}$ is contained in $L_C$, then there are no atoms, thus the function

$$\varphi(y) = \int_{[y,\infty]} R\chi_{S_{m,n}} \, d\lambda$$

is continuous. Moreover, $\varphi(0) = \mu(S_{m,n})$ and $\lim_{y \to \infty} \varphi(y) = 0$.

Suppose that $\mu(S_{m,n}) > 0$ seeking a contradiction. Pick $r_1, r_2 > 0$ such that $\varphi(r_1) = \frac{\mu(S_{m,n})}{3}$, $\varphi(r_2) = \frac{\mu(S_{m,n})}{2}$ and let $E \in \mathcal{A}$ satisfy $r_1 \leq \mu(E) \leq r_2$. Then $\mu(S_{m,n} \cap E) > 0$ and $\mu(S_{m,n} \setminus E) > 0$. Let $z \in S_{m,n} \setminus E$, then any $t \in E$ satisfies $t \leq_{\mathcal{A}} z$, thus

$$h(z) = \text{ess inf}_\mu \{|g(t)| : t \leq_{\mathcal{A}} z\} \leq \text{ess inf}_\mu \{|g(t)| : t \in E\} \leq \text{ess inf}_\mu \{|g(t)| : t \in E \cap S_{m,n}\} \leq \epsilon(n+1).$$

But since $z \in S_{m,n}$, we have $h(z) > \epsilon + |g(t)| > \epsilon + n\epsilon = (n+1)\epsilon$, which is a contradiction, therefore $\mu(S_{m,n}) = 0$ for all $m, n \in \mathbb{N}$. This shows that $h(z) \leq |g(z)|$ almost everywhere.

We have shown that $h$ is a core-decreasing minorant of $g$, thus $h \leq \underline{g}$. To show the converse, let $z \in U$, and note that if $t \leq_{\mathcal{A}} z$, then $\underline{g}(z) \leq \underline{g}(t) \leq |g(t)|$, therefore taking essential infimum yields $\underline{g}(z) \leq h(z)$ completing the proof.

□

As a consequence of this result we have the following examples where the ordered core satisfies that any arbitrary union or intersection of core sets can be reduced to a countable one, therefore it is measurable. These examples show that the terms appearing in formula (1.1) and [9, Theorem 3.1] are a particular case of the greatest core decreasing majorant.

**Example 2.6.** *Let $U = [0, \infty)$, $\mathcal{A} = \{\emptyset\} \cup \{[0, x] : x > 0\}$ and $\mu$ be a Borel measure, then*

$$\underline{g}(x) = \textit{ess inf}_{[0,x]} |g(t)| \,.$$

**Example 2.7.** *Let $U = \mathbb{X}$ be a metric measure space with distance function $d$, $a \in \mathbb{X}$ be any element, $\mu$ be any Borel measure and the core*

$$\mathcal{A} = \{\emptyset\} \cup \{B_{a,r} : r > 0\}$$

*where $B_{a,r} = \{x \in \mathbb{X} : d(a, x) \leq r\}$. Then*

$$\underline{g}(x) = \textit{ess inf}_\mu \left\{|g(t)| : t \in B_{a,|x|_a}\right\},$$

*where $|x|_a = d(a, x)$.*

## 3 Abstract Hardy inequalities with p=1

Our approach to finding necessary and sufficient conditions on the measures for inequality (3) is to find an equivalent inequality involving only two measures and a weight function, then to use Theorem 2.4 to replace the weight function with a core decreasing function. Finally we find an equivalent Hardy inequality on the half line.

**Proposition 3.1.** *Let $\eta$ and $\mu$ be $\sigma$-finite measures over $(U, \Sigma)$ and let $\tau$ be a $\sigma$-finite measure over $(Y, \tau)$. Suppose $B : Y \to \Sigma$ is a core map and $p = 1$. Then there exists a positive $\Sigma-$measurable function $u$ such that the best constant in inequality (3) is the same as the best constant in the inequality*

$$\left(\int\limits_Y \left(\int\limits_{B(y)} f \, d\mu\right)^q d\tau(y)\right)^{1/q} \leq C \int\limits_U f u \, d\mu, \ \forall f \in L^+_\mu \tag{4}$$

**Proof:** An application of the Lesbesgue decomposition theorem shows that $\mu = \mu_1 + \mu_2$, with $\mu_2 << \eta$ and $\mu_1 \perp \eta$. Also $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $\mu_2(U_1) = 0 = \eta(U_2)$. The Radon-Nikodym theorem provides a $\Sigma-$measurable non-negative function $h$ such that $d\mu_2 = h \, d\eta$. If $E = \{s \in U : h(s) = 0\}$ we can define the function $g = h\chi_{(U \setminus E)}$ and the sets $V_1 = U_1 \setminus E$ and $V_2 = U_2 \cup E$ to get a decomposition $d\mu = g \, d\eta + d\mu_1$ supported on $V_1$ and $V_2$ respectively, moreover $g$ is never zero on $V_1$. Thus the inequality (3) becomes

$$\left(\int\limits_Y \left(\int\limits_{B(y)} fg \, d\eta + \int\limits_{B(y)} f \, d\mu_1\right)^q d\tau(y)\right)^{\frac{1}{q}} \leq C \int_U f \, d\eta, \ \forall f \in L^+_\mu.$$

Fix $z \in Y$ and set $f = \chi_{(B(z) \cap V_2)}$, then if $C$ is finite, we have

$$\left(\int\limits_Y \left(\mu_1 \left(B(y) \cap B(z)\right)\right)^q d\tau(y)\right)^{\frac{1}{q}} = \left(\int\limits_Y \left(\int\limits_{B(y) \cap B(z)} d\mu_1\right)^q d\tau(y)\right)^{\frac{1}{q}} \leq C\eta \left(B(z) \cap V_2\right) = 0.$$

Therefore $\mu_1 \left(B(y) \cap B(z)\right) = 0$ for almost every $y$, since this holds for all $z \in Y$, then letting $B(y) \uparrow U$ we get $\mu_1 (U) = 0$.

Hence the inequality becomes

$$\left(\int\limits_Y \left(\int\limits_{B(y)} fg\,d\eta\right)^q d\tau(y)\right)^{\frac{1}{q}} \le C\int_U f\,d\eta,\ \forall f\in L^+_\mu.$$

Since $g$ is non-zero $\eta-$almost everywhere, then we can define $u=\frac{1}{g}$, so $d\eta = u\,d\mu$ and yields inequality (4). This shows that if the best constant in the inequality (3) is finite, then it is also the best constant in the inequality (4). For the remaining case, notice that we can decompose $d\eta = ud\mu + d\eta_2$ for some measure $\eta_2$ satisfying $\eta\perp\eta_2$. Therefore

$$\sup_{f\in L^+_\mu}\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{\frac{1}{q}}}{\int_U f\,d\eta} \le \sup_{f\in L^+_\mu}\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{\frac{1}{q}}}{\int_U fu\,d\mu},$$

thus if the best constant in inequality (3) is infinite, then it is also the best constant in the inequality (4) and completes the proof.

□

Now we replace the weight function $u$ with its greatest core decreasing minorant.

**Proposition 3.2.** *Given a $\sigma$-finite measure $\mu$ over $(U,\Sigma)$, a $\sigma$-finite measure $\tau$ over $(Y,\tau)$, and a core map $B: Y\to\Sigma$, the best constant in inequality (4) is the same as the best constant in the inequality*

$$\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \le C\int\limits_U f\underline{u}\,d\mu, \tag{5}$$

*where $\underline{u}$ is the greatest core-decreasing minorant of $u$ with respect to the ordered core $\mathcal{A}=\{\emptyset\}\cup\{B(y): y\in Y\}$.*

**Proof:** Our goal is to show that

$$\sup_{f\ge 0}\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q}}{\int\limits_U fu\,d\mu} = \sup_{f\ge 0}\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q}}{\int\limits_U f\underline{u}\,d\mu}.$$

Since $\underline{u}\le u$, then '$\le$' is clear, hence we focus on the converse. If the left hand side of the inequality above is infinite, then the result is trivial. Thus, we focus only on functions for which $\int_U fu\,d\mu$ is finite and $\int_E f\,d\mu$ is also finite for any core set $E\in\mathcal{A}$, so Theorem 2.4 applies and we get

$$\begin{aligned}
\sup_{f\ge 0}\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q}}{\int\limits_U f\underline{u}\,d\mu} &= \sup_{f\ge 0}\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q}}{\inf\left\{\int\limits_U gu\,d\mu : f\preccurlyeq g\right\}} \\
&= \sup_{f\ge 0}\sup\left\{\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q}}{\int\limits_U gu\,d\mu} : f\preccurlyeq g\right\} \\
&\le \sup_{f\ge 0}\sup\left\{\frac{\left(\int\limits_Y \left(\int\limits_{B(y)} g\,d\mu\right)^q d\tau(y)\right)^{1/q}}{\int\limits_U gu\,d\mu} : f\preccurlyeq g\right\}.
\end{aligned}$$

Where the symbol $f \preccurlyeq g$ means that $\int_E f\, d\mu \leq \int_E g\, d\mu$ for every $E \in \mathcal{A}$.

The right hand side is bounded above by $\sup_{f\geq 0} \frac{\left(\int\limits_Y \left(\int\limits_{B(y)} f\, d\mu\right)^q d\tau(y)\right)^{1/q}}{\int\limits_U fu\, d\mu}$, this completes the proof.

□

We now reduce the problem to a Hardy inequality with measures over the half line.

**Lemma 3.3.** *Given $B, \tau, \mu$ as in the previous propositions, then there exist Borel measures $\nu, \lambda$ on $[0,\infty)$ and a non-increasing function $w$ finite $\lambda-$almost everywhere, such that the best constant in inequality (5) is the best constant in*

$$\left(\int\limits_{[0,\infty)} \left(\int\limits_{[0,x]} f\, d\lambda\right)^q d\nu(x)\right)^{1/q} \leq C \int\limits_{[0,\infty)} fw\, d\lambda,\ \forall f \in L^+_\lambda \tag{6}$$

**Proof:** Since $B$ is a core map, then the function $\varphi : Y \to [0,\infty)$ defined by $\varphi(y) = \mu(B(y))$ is measurable. Let $\nu$ be the push-forward Borel measure associated to $\varphi$, that is

$$\nu(E) = \tau\big(\varphi^{-1}(E)\big), \forall E \text{ Borel}.$$

Let $\lambda$ be the Borel measure associated to the ordered core $\mathcal{A}$ with enriched core $\mathcal{M}$, and $R, Q$ the transition operators.

Fix a positive $\Sigma-$measurable function $f$ integrable over every core set $A \in \mathcal{A}$ and define the functions

$$Hf(x) = \int\limits_{[0,x]} R(f)\, d\lambda, \quad \text{and} \quad Tf(y) = \int\limits_{B(y)} f\, d\lambda.$$

We will show that $Hf$ and $Tf$ are equimeasurable with respect to the measures $\nu$ and $\tau$ by computing their distribution functions. First notice that for all $y \in Y$ we have

$$Hf \circ \varphi(y) = Hf\Big(\mu\big(B(y)\big)\Big) = \int\limits_{[0,\mu(B(y))]} R(f)\, d\lambda = \int\limits_{B(y)} f\, d\lambda = Tf(y).$$

Fix $\alpha > 0$ and define the sets

$$E_\alpha = \{x \in [0,\infty) : Hf(x) > \alpha\} \text{ and } F_\alpha = \{y \in Y : T(y) > \alpha\}.$$

Let

$$\gamma = \sup\left\{x \in [0,\infty) : \int\limits_{[0,x]} Rf\, d\lambda \leq \alpha\right\}.$$

Notice that by the monotone convergence theorem $Hf(\gamma) \leq \alpha$. We claim that $E_\alpha = (\gamma,\infty)$ and that $F_\alpha = \varphi^{-1}(E_\alpha)$.

Let $x \in E_\alpha$, then since $Hf$ is increasing, we must have that $x > \gamma$, thus $E_\alpha \subseteq (\gamma,\infty)$. Conversely, let $x > \gamma$, then $Hf(x) > \alpha$, thus $x \in E_\alpha$, this shows the first equation.

For the second equation, notice that

$$F_\alpha = \{y \in Y : T(y) > \alpha\} = \{y \in Y : H_f \circ \varphi(y) > \alpha\}.$$

So if $y \in F_\alpha$, then $\varphi(y) \in E_\alpha$, this shows $F_\alpha \subseteq \varphi^{-1}(E_\alpha)$. Conversely, if $y \in \varphi^{-1}(E_\alpha)$, then $T(y) > \alpha$, hence $y \in F_\alpha$.

Computation of the distribution functions yields

$$\nu(E_\alpha) = \tau\big(\varphi^{-1}(E_\alpha)\big) = \tau(F_\alpha).$$

Therefore $Hf$ and $Tf$ are equimeasurable, hence

$$\begin{aligned}\left(\int\limits_{[0,\infty)} \left(\int\limits_{[0,x]} R(f)\, d\lambda\right)^q d\nu\right)^{\frac{1}{q}} &= \left(\int\limits_{[0,\infty)} \Big(Hf\Big)^q d\nu\right)^{\frac{1}{q}} = \left(\int\limits_Y \Big(Tf\Big)^q d\tau\right)^{\frac{1}{q}} \\ &= \left(\int\limits_Y \left(\int\limits_{B(y)} f\, d\mu\right)^q d\tau\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\underline{u}$ is core-decreasing, then we have

$$\int\limits_U f\underline{u}\,d\mu = \int\limits_{[0,\infty)} Rf R\underline{u}\,d\lambda.$$

Therefore if inequality (5) holds, so does

$$\left(\int\limits_{[0,\infty)}\left(\int\limits_{[0,x]} Rf\,d\lambda\right)^q d\nu(x)\right)^{\frac{1}{q}} \leq C\int\limits_{[0,\infty)} RfR\underline{u}\,d\lambda,\ \forall Rf\in L^+_\lambda.$$

Note that $R\underline{u}$ must be finite almost everywhere, otherwise the original measures are not $\sigma-$finite. The result follows from letting $w = R\underline{u}$ and noting that $R$ is surjective.

□

We are ready to prove the main result.

**Theorem 3.4.** *For $\sigma-$finite measure spaces $(Y,\mathcal{T},\tau),(U,\Sigma,\mu),(U,\Sigma,\nu)$ and a core map $B: Y\to\Sigma$. Let $\eta=\eta_a+\eta_s$, where $d\eta_a = ud\mu$ and $\eta_s\perp\mu$. Then the best constant $C$ in the inequality*

$$\left(\int\limits_Y\left(\int\limits_{B(y)} f\,d\mu\right)^q d\tau(y)\right)^{1/q} \leq C\int\limits_U f\,d\eta, \tag{7}$$

*satisfies*

$$C\approx\left[\int\limits_Y\left(\int\limits_{\mu(B(z))\leq\mu(B(y))} R\left(\frac{1}{\underline{u}}\right)\circ\mu\circ B(y)\,d\tau(y)\right)^{\frac{q}{1-q}} d\tau(z)\right]^{\frac{1-q}{q}},\ \textit{for}\ q\in(0,1),$$

*and*

$$C=\sup_{s\in U}\left(\frac{1}{\underline{u}}(s)\right)\tau\left(\{y\in Y: s\in B(y)\}\right)^{1/q},\ \textit{for}\ q\in[1,\infty).$$

*Where the least core decreasing majorant is taken with respect to the core $\mathcal{A}=\{\emptyset\}\cup\{B(y): y\in Y\}$ and $R$ is the linear map mentioned in Section 2.*

**Proof:** Suppose that $q\in(0,1)$, then by Lemma 3.3 and Theorem 1.1 (Theorem 3.1 of [5]) the best constant is equivalent to

$$\left(\int\limits_{[0,\infty)}\left(\int\limits_{[0,x]}\frac{1}{\underline{w}}\,d\nu\right)^{\frac{q}{1-q}} d\nu(x)\right)^{1/q},$$

where $w=R(\underline{u})$ and $\nu$ is the push-forward measure (see [2]) for the map $\varphi(y)=\mu\circ B(y)$. Notice that $\underline{w}=w$, and it follows from Definition 2.2 (iv) that $\frac{1}{R(\underline{u})}=R\left(\frac{1}{\underline{u}}\right)$, then

$$\int\limits_{[0,x]}\frac{1}{\underline{w}}\,d\nu = \int_{[0,\infty)} R\left(\frac{1}{\underline{u}}\right)\chi_{[0,x]}\,d\nu = \int_Y R\left(\frac{1}{\underline{u}}\right)\circ\varphi(y)\chi_{[0,x]}\circ\varphi(y)\,d\tau(y)$$
$$= \int\limits_{\varphi(y)\leq x} R\left(\frac{1}{\underline{u}}\right)\circ\varphi(y)\,d\tau(y).$$

Thus

$$\int\limits_{[0,\infty)}\left(\int\limits_{[0,x]}\frac{1}{\underline{w}}\,d\nu\right)^{\frac{q}{1-q}} d\nu(x) = \int_Y\left(\int\limits_{\varphi(y)\leq\varphi(z)} R\left(\frac{1}{\underline{u}}\right)\circ\varphi(y)\,d\tau(y)\right)^{\frac{q}{1-q}} d\tau(z)$$

and completes the proof for the case $q\in(0,1)$.

The case $q\in[1,\infty)$ follows directly from duality and we include it for the sake of completeness.

By Proposition 3.1 the best constant in inequality (7) is the norm of the integral operator
$Kf(y) = \int_U k(y,s)f(s)\,d\theta(s)$ acting from $L^1_\theta \to L^q_\tau$ where $d\theta = \underline{u}d\mu$ and $k(y,s) = \frac{1}{\underline{u}(s)}\chi_{B(y)}(s)$. By duality, it is the best constant in the inequality

$$\left\| \int\limits_Y k(y,\cdot)h(y)\,d\tau(y) \right\|_{L^\infty_\theta} \leq C \left( \int_Y h^{q'}\,d\tau \right)^{\frac{1}{q'}}, \forall h \in L^+_\tau.$$

Define $\psi_s(y) = 1$ if $s \in B(y)$ and $\psi_s(y) = 0$ otherwise. Divide both sides of the equation by $\|h\|_{L^{q'}_\tau}$ to get

$$\sup \left\{ \frac{1}{\underline{u}(s)} \int\limits_Y \psi_s(y) \frac{h(y)}{\|h\|_{L^{q'}_\tau}}\,d\tau(y) : s \in U \right\} \leq C.$$

Taking supremum over non-zero positive functions $h$ yields

$$\sup_{s\in U} \frac{1}{\underline{u}(s)} \|\psi_s\|_{L^q_\tau} \leq C,$$

which is the same as

$$C \geq \sup_{s\in U} \left( \frac{1}{\underline{u}}(s) \right) \tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q}.$$

For the reverse inequality, an application of Minkowski's integral inequality yields

$$\begin{aligned} \left( \int\limits_Y \left( \int_U k(s,y)f(s)\,d\theta(s) \right)^q d\tau(y) \right)^{1/q} &\leq \left( \int\limits_U \left( \int_Y \psi_s(y)\,d\tau(y) \right)^{1/q} \frac{f(s)}{\underline{u}(s)}\,d\theta(s) \right) \\ &\leq \sup_{s\in U} \left( \frac{1}{\underline{u}}(s) \right) \tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q} \int_U f(s)\,d\theta(s) \end{aligned}$$

hence $C \leq \sup_{s\in U} \left( \frac{1}{\underline{u}}(s) \right) \tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q}$ and proves the statement for $q \in (1,\infty)$.

□

## 4 Applications to metric measure spaces

In this section we show that the framework of abstract Hardy inequalities can be used to give different proofs to [10, Theorem 2.1 Condition $\mathcal{D}_1$], [11, Theorem 2.1] and [9, Theorem 3.1]. These theorems give necessary and sufficient conditions for Hardy inequalities to hold in metric measure spaces; they cover three cases depending on the indices $p$ and $q$, provided the existence of a locally integrable function $\lambda \in L^1_{\text{loc}}$ such that for all $f \in L^1(\mathbb{X})$ the following *polar decomposition* at $a \in \mathbb{X}$ holds:

$$\int_{\mathbb{X}} f d\mu = \int_0^\infty \int_{\Sigma_r} f(r,\omega)\lambda(r,\omega)\,d\omega_r dr,$$

for a family of measures $d\omega_r$, where $\Sigma_r = \{x \in \mathbb{X} : d(x,a) = r\}$.

Our new proofs show that the polar decomposition hypothesis is not required so the results hold in all metric measure spaces.

We also give the corresponding results regarding the conjugate Hardy inequality discussed in [10, Theorem 2.2 Condition $\mathcal{D}_1^*$] and [9, Theorem 3.2].

We begin with the case $p > 1$, extending [10, Theorem 2.1 Condition $\mathcal{D}_1$], [11, Theorem 2.1] to all metric measure spaces.

**Theorem 4.1.** *Let $\mu$ be a $\sigma-$finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$ and let $p \in (1,\infty)$, $q \in (0,\infty)$ and $\omega, v$ be measurable functions, positive $\mu-$almost everywhere satisfying $\omega \in L^1_\mu(\mathbb{X}\setminus\{a\})$, $v^{1-p'} \in L^1_{Loc}(\mathbb{X})$. Then the Hardy inequality*

$$\left( \int\limits_{\mathbb{X}} \left( \int\limits_{B_{a,|x|_a}} f(y)\,d\mu(y) \right)^q \omega(x)\,d\mu(x) \right)^{\frac{1}{q}} \leq C \left( \int\limits_{\mathbb{X}} f(x)^p v(x)\,d\mu(x) \right)^{\frac{1}{p}}, \ \forall f \in L^+_\mu$$

*holds if and only if* $p \le q$ *and:*

$$\sup_{x\neq a}\left\{\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} \omega\, d\mu\right)^{\frac{1}{q}}\left(\int\limits_{B_{a,|x|_a}} v^{1-p'}\, d\mu\right)^{\frac{1}{p'}}\right\}<\infty,$$

$0<q<1<p$ *and*

$$\int\limits_{\mathbb{X}}\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} \omega\, d\mu\right)^{\frac{r}{p}}\left(\int\limits_{B_{a,|x|_a}} v^{1-p'}\, d\mu\right)^{\frac{r}{p'}} u(s)\, d\mu(s)<\infty,$$

$1<q<p$ *and*

$$\int\limits_{\mathbb{X}}\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} \omega\, d\mu\right)^{\frac{r}{q}}\left(\int\limits_{B_{a,|x|_a}} v^{1-p'}\, d\mu\right)^{\frac{r}{q'}} v^{1-p'}(s)\, d\mu(s)<\infty,$$

*where* $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$.

**Proof:** The Lebesgue decomposition theorem applied to the measures $d\mu$ and $vd\mu$ provides measures $\mu_1, \mu_2$ such that $\mu=\mu_1+\mu_2$, $\mu_1 \perp vd\mu$ and $\mu_2 << v\, d\mu$. Since $v>0$ $\mu-$almost everywhere we can take $\mu_1$ to be zero and write

$$d\mu=\frac{1}{v}d(v\, d\mu).$$

Let $d\tau=v^{-p'}d(v\, d\mu)$ and set $Y=\{s\in\mathbb{X}:\int_{B_{a,|s|_a}} v^{1-p'}\, d\mu<\infty\}$. Define the map $B:\mathbb{X}\to\Sigma$ by

$$B(y)=B_{a,|y|_a}.$$

The image of $B$ is an ordered core with respect to the measure $\tau$.

By hypothesis $Y=\mathbb{X}$ and since $\mu_1=0$, [13, Theorem 5.1] provides the equivalent abstract Hardy inequality

$$\left(\int\limits_{\mathbb{X}}\left(\int\limits_{B_{a,|y|_a}} f\, d\tau\right)^{q}\omega(y)\, d\mu(y)\right)^{\frac{1}{q}}\le C\left(\int\limits_{\mathbb{X}} f^p\, d\tau\right)^{\frac{1}{p}},\ \forall f\in L^{+}_{\mu}.$$

By definition of $\tau$ this is equivalent to

$$\left(\int\limits_{\mathbb{X}}\left(\int\limits_{B_{a,|y|_a}} fv^{1-p'}\, d\mu\right)^{q}\omega(y)\, d\mu(y)\right)^{\frac{1}{q}}\le C\left(\int\limits_{\mathbb{X}} f^p v^{1-p'}\, d\mu\right)^{\frac{1}{p}},\ \forall f\in L^{+}_{\mu}. \tag{8}$$

Let $\lambda$ be the measure on $[0,\infty)$ induced by the core, so that for every $M$ in the core

$$\int\limits_{[0,x]} Rf\, d\lambda=\int\limits_{M} fv^{1-p'}\, d\mu,\ \text{where } x=\int\limits_{M} v^{1-p'}\, d\mu.$$

We claim that inequality (8) is equivalent to the Hardy inequality

$$\left(\int\limits_{[0,\infty)}\left(\int\limits_{[0,y]} g\, d\lambda\right)^{q} R\left(\frac{\omega}{v^{1-p'}}\right)\, d\lambda(y)\right)^{\frac{1}{q}}\le C\left(\int\limits_{[0,\infty)} g^p d\lambda\right)^{\frac{1}{p}},\quad \forall g\in L^{+}_{\lambda}. \tag{9}$$

By [13, Theorem 2.4], it suffices to show that the normal form parameters of inequalities (8) and (9) coincide. Hence, it suffices to show that the maps

$$b_1(s)=\int\limits_{B_{a,|s|_a}} v^{1-p'}\, d\mu\quad \text{and}\quad b_2(x)=\lambda([0,x])$$

have the same distribution functions with respect to the measures $\omega\, d\mu$ and $R\left(\frac{\omega}{v^{1-p'}}\right)\, d\lambda$ respectively.

Fix $t > 0$ and consider the sets $E_1 = b_1^{-1}(t, \infty)$ and $E_2 = b_2^{-1}(t, \infty)$, we give a characterization for these sets. Define the set $W$ as follows

$$W = \bigcup \left\{ B_{a,|s|} : \int\limits_{B_{a,|s|_a}} v^{1-p'}\, d\mu \leq t \right\}.$$

If $z \in E_1$, then $b_1(z) > t$, thus $z \notin W$, conversely if $z \in W$ then $b_1(z) \leq t$, therefore $z \notin E_1$. Hence $W^c = E_1$. Since $W$ is a union of closed balls centered at $a$, then there exists a sequence $s_n$ such that $B(a, s_n) \uparrow W$. Let $t_n = \int\limits_{B(a,s_n)} v^{1-p'}\, d\mu$.

Let $\widetilde{t}$ be defined as

$$\widetilde{t} = \sup \left\{ z \leq t : z = \int\limits_{B_{a,|s|_a}} v^{1-p'}\, d\mu \text{ for some } s \in \mathbb{X} \right\},$$

hence $\widetilde{t} = \lambda[0, t]$.

Therefore,

$$\begin{aligned}
\int\limits_{E_1^c} \omega\, d\mu &= \sup_{n\in\mathbb{N}} \int\limits_{B(a,s_n)} \omega\, d\mu \text{ by the Monotone convergence theorem} \\
&= \sup_{n\in\mathbb{N}} \int\limits_{[0,t_n]} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda \text{ by the action of } R \\
&= \int\limits_{[0,t]} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda \text{ by Monotone convergence theorem} \\
&= \int\limits_{E_2^c} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda.
\end{aligned}$$

Since by hypothesis $\int\limits_{E_1^c} \omega\, d\mu < \infty$, then we have that

$$\int\limits_{b_1^{-1}(t,\infty)} \omega\, d\mu = \int\limits_{b_2^{-1}(t,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda.$$

It follows that the distribution functions coincide and proves that the Hardy inequalities (8) and (9) have the same normal form parameter, therefore they are equivalent.

For all the index cases, we can apply [15, Theorem 7.1] to get that in the case $1 < p \leq q < \infty$, the inequality (9) holds if and only if

$$\sup_x \left( \int\limits_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(t) \right)^{\frac{1}{q}} \left( \int\limits_{[0,x]} d\lambda \right)^{\frac{1}{p'}} < \infty$$

which is equivalent to

$$\sup_{s\neq a} \left( \int\limits_{\mathbb{X}\setminus B_{a,|s|_a}} \omega\, d\mu \right)^{\frac{1}{q}} \left( \int\limits_{B_{a,|s|_a}} v^{1-p'}\, d\mu \right)^{\frac{1}{p'}} < \infty$$

And in the case $0 < q < 1 < p < \infty$, the inequality (9) holds if and only if

$$\int\limits_{[0,\infty)} \left( \int\limits_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda \right)^{\frac{r}{p}} \left( \int\limits_{[0,x]} d\lambda \right)^{\frac{r}{p'}} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(x) < \infty$$

which is equivalent to
$$\int\limits_{\mathbb{X}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|s|_a}} \omega \, d\mu \Bigg)^{\frac{r}{p}} \Bigg( \int\limits_{B_{a,|s|_a}} v^{1-p'} \, d\mu \Bigg)^{\frac{r}{p'}} \omega(s) \, d\mu(s) < \infty.$$
In the case $1 < q < p$ we have that inequality (9) holds if and only if
$$\int\limits_{[0,\infty)} \Bigg( \int\limits_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda \Bigg)^{\frac{r}{q}} \Bigg( \int\limits_{[0,x]} d\lambda \Bigg)^{\frac{r}{q'}} d\lambda(x) < \infty$$
which is equivalent to
$$\int\limits_{\mathbb{X}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|s|_a}} \omega \, d\mu \Bigg)^{\frac{r}{q}} \Bigg( \int\limits_{B_{a,|s|_a}} v^{1-p'} \, d\mu \Bigg)^{\frac{r}{q'}} v^{1-p'} \, d\mu(s) < \infty$$
completing the proof.

□

We also have a corresponding result to the conjugate Hardy inequality, extending [10, Theorem 2.1 Condition $\mathcal{D}_1$]

**Theorem 4.2.** *Let $\mu$ be a $\sigma-$finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$ and let $p \in (1,\infty)$, $q \in (0,\infty)$ and $\omega, v$ be measurable functions, positive $\mu-$almost everywhere satisfying $v^{1-p'} \in L^1_\mu(\mathbb{X}\setminus\{a\})$, $\omega \in L^1_{Loc}(\mathbb{X})$. Then the Hardy inequality*
$$\Bigg( \int\limits_{\mathbb{X}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} f(y) \, d\mu(y) \Bigg)^{q} \omega(x) \, d\mu(x) \Bigg)^{\frac{1}{q}} \le C \Bigg( \int\limits_{\mathbb{X}} f(x)^p v(x) \, d\mu(x) \Bigg)^{\frac{1}{p}}, \ \forall f \in L^+_\mu$$
*holds if and only if $p \le q$ and:*
$$\sup_{x \ne a} \left\{ \Bigg( \int\limits_{B_{a,|x|_a}} \omega \, d\mu \Bigg)^{\frac{1}{q}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} v^{1-p'} \, d\mu \Bigg)^{\frac{1}{p'}} \right\} < \infty,$$
*$0 < q < 1 < p$ and*
$$\int\limits_{\mathbb{X}} \Bigg( \int\limits_{B_{a,|x|_a}} \omega \, d\mu \Bigg)^{\frac{r}{p}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} v^{1-p'} \, d\mu \Bigg)^{\frac{r}{p'}} u(s) \, d\mu(s) < \infty,$$
*$1 < q < p$ and*
$$\int\limits_{\mathbb{X}} \Bigg( \int\limits_{B_{a,|x|_a}} \omega \, d\mu \Bigg)^{\frac{r}{q}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} v^{1-p'} \, d\mu \Bigg)^{\frac{r}{q'}} v^{1-p'}(s) \, d\mu(s) < \infty,$$
*where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

**Proof:** We only sketch the proof as most details follow the same argument as Theorem 4.1.

Let $d\tau = v^{1-p'} \, d\mu$. Observe that the hypothesis on $v$ guarantees that, for each $y \in \mathbb{X}$,the sets $\mathbb{X}\setminus B_{a,|y|_a}$ have finite $\tau$ measure. Thus the map $B(y) = \mathbb{X}\setminus B_{a,|y|_a}$ is a core map.

Then the Lebesgue decomposition Theorem and [13, Theorem 5.1] provides the equivalent abstract Hardy inequality
$$\Bigg( \int\limits_{\mathbb{X}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|y|_a}} f \, d\tau \Bigg)^{q} \omega(y) \, d\mu(y) \Bigg)^{\frac{1}{q}} \le C \Bigg( \int\limits_{\mathbb{X}} f^p \, d\tau \Bigg)^{\frac{1}{p}}, \ \forall f \in L^+_\mu.$$
By definition of $\tau$ this is equivalent to
$$\Bigg( \int\limits_{\mathbb{X}} \Bigg( \int\limits_{\mathbb{X}\setminus B_{a,|y|_a}} f v^{1-p'} \, d\mu \Bigg)^{q} \omega(y) \, d\mu(y) \Bigg)^{\frac{1}{q}} \le C \Bigg( \int\limits_{\mathbb{X}} f^p v^{1-p'} \, d\mu \Bigg)^{\frac{1}{p}}, \ \forall f \in L^+_\mu. \tag{10}$$

Let $\lambda$ be the measure on $[0,\infty)$ induced by the core, so that for every $y \in Y$:

$$\int\limits_{[0,x]} Rf\, d\lambda = \int\limits_{\mathbb{X}\setminus B_{a,|y|_a}} f v^{1-p'}\, d\mu, \text{ where } x = \int\limits_{\mathbb{X}\setminus B_{a,|y|_a}} v^{1-p'}\, d\mu.$$

The maps

$$b_1(s) = \int\limits_{\mathbb{X}\setminus B_{a,|s|_a}} v^{1-p'}\, d\mu \quad \text{and} \quad b_2(x) = \lambda([0,x])$$

have the same distribution functions with respect to the measures $\omega\, d\mu$ and $R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda$ respectively. Then, by [13, Theorem 2.4], we get that the inequality (10) is equivalent to the Hardy inequality

$$\left(\int\limits_{[0,\infty)} \left(\int\limits_{[0,y]} g\, d\lambda\right)^q R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(y)\right)^{\frac{1}{q}} \le C\left(\int\limits_{[0,\infty)} g^p d\lambda\right)^{\frac{1}{p}}, \quad \forall g \in L^+_\lambda. \tag{11}$$

For all the index cases, we can apply [15, Theorem 7.1] to get that in the case $1 < p \le q < \infty$, the inequality (11) holds if and only if

$$\sup_x \left(\int\limits_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(t)\right)^{\frac{1}{q}} \left(\int\limits_{[0,x]} d\lambda\right)^{\frac{1}{p'}} < \infty$$

which is equivalent to

$$\sup_{s\neq a} \left(\int\limits_{B_{a,|s|_a}} \omega\, d\mu\right)^{\frac{1}{q}} \left(\int\limits_{\mathbb{X}\setminus B_{a,|s|_a}} v^{1-p'}\, d\mu\right)^{\frac{1}{p'}} < \infty$$

And in the case $0 < q < 1 < p < \infty$, the inequality (11) holds if and only if

$$\int\limits_{[0,\infty)} \left(\int\limits_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda\right)^{\frac{r}{p}} \left(\int\limits_{[0,x]} d\lambda\right)^{\frac{r}{p'}} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda(x) < \infty$$

which is equivalent to

$$\int\limits_{\mathbb{X}} \left(\int\limits_{B_{a,|s|_a}} \omega\, d\mu\right)^{\frac{r}{p}} \left(\int\limits_{\mathbb{X}\setminus B_{a,|s|_a}} v^{1-p'}\, d\mu\right)^{\frac{r}{p'}} \omega(s)\, d\mu(s) < \infty.$$

In the case $1 < q < p$ we have that inequality (11) holds if and only if

$$\int\limits_{[0,\infty)} \left(\int\limits_{[x,\infty)} R\left(\frac{\omega}{v^{1-p'}}\right) d\lambda\right)^{\frac{r}{q}} \left(\int\limits_{[0,x]} d\lambda\right)^{\frac{r}{q'}} d\lambda(x) < \infty$$

which is equivalent to

$$\int\limits_{\mathbb{X}} \left(\int\limits_{B_{a,|s|_a}} \omega\, d\mu\right)^{\frac{r}{q}} \left(\int\limits_{\mathbb{X}\setminus B_{a,|s|_a}} v^{1-p'}\, d\mu\right)^{\frac{r}{q'}} v^{1-p'}\, d\mu(s) < \infty$$

completing the proof.

□

For the case $p = 1$, our main result implies the following characterization

**Corollary 4.3.** *Let $\mu$ be a $\sigma-$finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$, let $q \in (0,\infty)$ and $\omega, v$ be measurable functions, positive $\mu-$almost everywhere satisfying $\omega \in L^1_\mu(\mathbb{X}\setminus\{a\})$, $v^{1-p'} \in L^1_{Loc}(\mathbb{X})$. Then the best constant in the Hardy inequality*

$$\left(\int\limits_{\mathbb{X}} \left(\int\limits_{B_{a,|x|_a}} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \le C \int\limits_{\mathbb{X}} f(x) v(x)\, d\mu(x),\ \forall f \in L^+_\mu$$

*satisfies*

$$C \approx \left( \int\limits_{\mathbb{X}} \left( \int\limits_{z \le_{\mathcal{A}} x} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu(x) \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}, \text{ for } q \in (0,1),$$

*and*

$$C = \sup_{x \in X} \left( \frac{1}{\underline{v}}(x) \right) \left( \int\limits_{x \le_{\mathcal{A}} t} \omega(t)\, d\mu(t) \right)^{1/q}, \text{ for } q \in [1,\infty).$$

*Where* $\underline{v}(x) = \text{ess inf}_\mu\{v(t) : t \in B_{a,|x|_a}\}$, $x \le_{\mathcal{A}} t$ *means* $B_{a,|x|_a} \subseteq B(a,|t|)$ *and* $B_{a,|x|_a} = \{z \in \mathbb{X} : \text{dist}(a,z) \le \text{dist}(a,x)\}$.

**Proof:** Let $\mathcal{A} = \{\emptyset\} \cup \{B_{a,|x|_a}\}_{x\in\mathbb{X}}$, it is the full ordered core induced by the core map $x \to B_{a,|x|_a}$. Let $d\tau = \omega d\mu$, $d\eta = v d\mu$ and $\lambda$ be the measure on $[0,\infty)$ induced by the ordered core.

Consider the function $\varphi : \mathbb{X} \to [0,\infty)$ defined by $\varphi(x) = \mu\big(B_{a,|x|_a}\big)$ and let $\nu$ be the pushforward measure. Then, if $y = \varphi(x)$ we have

$$\nu\big([0,y]\big) = \mu\big(\varphi^{-1}([0,y])\big) = \int\limits_{\varphi(t)\le y} d\mu(t) = \int\limits_{B_{a,|x|_a}} d\mu = \lambda\big([0,\varphi(x)]\big) = \lambda\big([0,y]\big).$$

It follows that the Borel measures $\nu$ and $\lambda$ coincide and are finite over $[0,y]$ for all $y > 0$, therefore $\lambda$ is the pushforward measure of $\varphi$.

We now show that $R\Big(\frac{1}{\underline{v}}\Big) = \frac{1}{\underline{v}} \circ \varphi$ up to a set of $\mu-$measure zero.

Indeed

$$\begin{aligned}
\int\limits_{B_{a,|x|_a}} \frac{1}{\underline{v}}\, d\mu &= \int\limits_{\varphi(t)\le\varphi(x)} \frac{1}{\underline{v}}(t)\, d\mu(t) = \int\limits_{[0,\varphi(x)]} R\Big(\frac{1}{\underline{v}}\Big)(t)\, d\lambda(t) = \int\limits_{[0,\infty)} R\Big(\frac{1}{\underline{v}}\Big)(t)\chi_{[0,\varphi(x)]}(t)\, d\lambda(t) \\
&= \int\limits_{\mathbb{X}} R\Big(\frac{1}{\underline{v}}\Big) \circ \varphi(t) \chi_{[0,\varphi(x)]}\varphi(t)\, d\mu(t) = \int\limits_{\varphi(t)\le\varphi(x)} R\Big(\frac{1}{\underline{v}}\Big) \circ \varphi(t)\, d\mu(t) \\
&= \int\limits_{B_{a,|x|_a}} R\Big(\frac{1}{\underline{v}}\Big) \circ \varphi(t)\, d\mu(t).
\end{aligned}$$

Since the equality holds for all core sets, then $R\Big(\frac{1}{\underline{v}}\Big) = \frac{1}{\underline{v}} \circ \varphi$ almost everywhere.

Then for $q \in (0,1)$, Theorem 3.4 yields

$$\begin{aligned}
C &\approx \left( \int\limits_{\mathbb{X}} \left( \int\limits_{\varphi(z)\le\varphi(x)} R\left(\frac{1}{\underline{v}}\right) \circ \varphi(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}} \\
&\approx \left( \int\limits_{\mathbb{X}} \left( \int\limits_{z\le_{\mathcal{A}} x} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}.
\end{aligned}$$

The statement for $q \in [1,\infty)$ follows directly from Theorem 3.4. The description of $\underline{v}$ follows from Example 2.2 and completes the proof.

□

Our result regarding the conjugate Hardy inequality to Corollary 4.1 needs a small adjustment. Since for a metric measure space $\mathbb{X}$, the sets $(\mathbb{X} \setminus B_{a,|x|_a})$ may have infinite measure, then the collection $\{\mathbb{X} \setminus B_{a,|x|_a}\}_{x\in\mathbb{X}}$ may fail condition (ii) in Definition 2.1. This obstruction is addressed in the following lemma.

**Lemma 4.4.** *Let $\mu$ be a $\sigma-$finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$. Let $\{\mathbb{X}_n\}$ be a sequence of sets with finite $\mu$-measure such that $a \in \mathbb{X}_n \uparrow \mathbb{X}$, $q \in (0, \infty)$ and $\omega, v$ be measurable functions, positive $\mu-$almost everywhere satisfying $v^{1-p'} \in L^1_\mu(\mathbb{X} \setminus \{a\})$, $\omega \in L^1_{Loc}(\mathbb{X})$.*

*For each $n \in \mathbb{N}$. Let $C_n$ be the best constant in the inequality*

$$\left(\int\limits_{\mathbb{X}_n}\left(\int\limits_{\mathbb{X}_n\setminus B_{a,|x|_a}} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \leq C_n \int\limits_{\mathbb{X}_n} f(x)v(x)\, d\mu(x),\ \forall f \in L^+_\mu \tag{12}$$

*and $C$ be the best constant in the inequality*

$$\left(\int\limits_{\mathbb{X}}\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \leq C \int\limits_{\mathbb{X}} f(x)v(x)\, d\mu(x),\ \forall f \in L^+_\mu, \tag{13}$$

*where $B_{a,|x|_a} = \{z \in \mathbb{X} : dist(a, z) \leq dist(a, x)\}$.*

*Then,*

$$C = \sup_{n\in\mathbb{N}} C_n.$$

**Proof:**

Fix $f \in L^+_\mu$, then an application of inequality (13) yields

$$\begin{aligned}\left(\int\limits_{\mathbb{X}_n}\left(\int\limits_{\mathbb{X}_n\setminus B_{a,|x|_a}} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} &\leq \left(\int\limits_{\mathbb{X}}\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} f(y)\chi_{\mathbb{X}_n}(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \\ &\leq C\int\limits_{\mathbb{X}} f(y)\chi_{\mathbb{X}_n}(y)\, v(y) d\mu(y) = C\int\limits_{\mathbb{X}_n} f(y)\, v(y) d\mu(y).\end{aligned}$$

Division by $\int\limits_{\mathbb{X}_n} f(y)\, v(y) d\mu(y)$ and taking supremum over $f$ yields $C_n \leq C$. Thus $\sup_{n\in\mathbb{N}} C_n \leq C$.

Conversely, the monotone convergence theorem together with equation (12) yields

$$\begin{aligned}\left(\int\limits_{\mathbb{X}}\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} &= \sup_n \left(\int\limits_{\mathbb{X}_n}\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} f(y)\chi_{\mathbb{X}_n}(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \\ &= \sup_n \left(\int\limits_{\mathbb{X}_n}\left(\int\limits_{\mathbb{X}_n\setminus B_{a,|x|_a}} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \\ &\leq \sup_n C_n \int\limits_{\mathbb{X}_n} f(x)v(x)\, d\mu(x) \leq \big(\sup_n C_n\big) \int\limits_{\mathbb{X}} f(x)v(x)\, d\mu(x).\end{aligned}$$

Division by $\int\limits_{\mathbb{X}} f(x)v(x)\, d\mu(x)$ and taking supremum over $f$ yields $C \leq \sup_{n\in\mathbb{N}} C_n$ and completes the proof.

□

We are ready to state our result for the conjugate Hardy inequality with $p = 1$.

**Corollary 4.5.** *Let $\mu$ be a $\sigma-$finite measure on a metric measure space $\mathbb{X}$. Fix $a \in \mathbb{X}$, let $q \in (0, \infty)$ and $\omega, v$ be measurable functions, positive $\mu-$almost everywhere satisfying $v^{1-p'} \in L^1_\mu(\mathbb{X} \setminus \{a\})$, $\omega \in L^1_{Loc}(\mathbb{X})$. Then the best constant in the Hardy inequality*

$$\left(\int\limits_{\mathbb{X}}\left(\int\limits_{\mathbb{X}\setminus B_{a,|x|_a}} f(y)\, d\mu(y)\right)^q \omega(x)\, d\mu(x)\right)^{\frac{1}{q}} \leq C \int\limits_{\mathbb{X}} f(x)v(x)\, d\mu(x),\ \forall f \in L^+_\mu$$

*satisfies*

$$C \approx \left( \int\limits_{\mathbb{X}} \left( \int\limits_{x \leq_{\mathcal{A}} z} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu(x) \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}, \text{ for } q \in (0,1),$$

*and*

$$C = \sup_{x \in X} \left( \frac{1}{\underline{v}}(x) \right) \left( \int\limits_{t \leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q}, \text{ for } q \in [1,\infty).$$

*Where* $\underline{v}(x) = \text{ess inf}_\mu \{v(t) : t \notin B_{a,|x|_a}\}$*,* $x \leq_{\mathcal{A}} t$ *means* $B_{a,|x|_a} \subseteq B_{a,|t|}$ *and* $B_{a,|x|_a} = \{z \in \mathbb{X} : \text{dist}(a,z) \leq \text{dist}(a,x)\}$*.*

**Proof:** For each $n \in \mathbb{N}^+$ define $\mathbb{X}_n = \{x \in \mathbb{X} : \text{dist}(a,x) \leq n\}$. Let $C_n$ be the best constant in the inequality (12). Let $\mathcal{A}_n = \{\emptyset\} \cup \{\mathbb{X}_n B_{a,|x|_a}\}_{x \in \mathbb{X}}$, it is the full ordered core over $\mathbb{X}_n$ induced by the core map $x \to (\mathbb{X}_n \setminus B_{a,|x|_a})$. Let $d\tau = \omega d\mu$, $d\eta = v d\mu$ and $\lambda_n$ be the measure on $[0,\infty)$ induced by the ordered core. Notice that $\lambda_n$ is supported on the compact interval $[0, \mu(\mathbb{X}_n)]$.

Consider the function $\varphi_n : \mathbb{X} \to [0,\infty)$ defined by $\varphi_n(x) = \mu\big(\mathbb{X}_n \setminus B_{a,|x|_a}\big)$ and let $\nu_n$ be the pushforward measure.

Then, if $y = \varphi_n(x)$ we have

$$\nu_n\big([0,y]\big) = \mu\big(\varphi_n^{-1}([0,y])\big) = \int\limits_{\varphi_n(t) \leq y} d\mu(t) = \int\limits_{\mathbb{X}_n \setminus B_{a,|x|_a}} d\mu = \lambda\big([0,\varphi(x)]\big) = \lambda\big([0,y]\big).$$

It follows that the Borel measures $\nu_n$ and $\lambda_n$ coincide and are finite over $[0,y]$ for all $y > 0$, therefore $\lambda_n$ is the pushforward measure of $\varphi_n$.

We now show that $R_n\left(\frac{1}{\underline{v_n}}\right) = \frac{1}{\underline{v_n}} \circ \varphi$ up to a set of $\mu-$measure zero, here $R_n$ is the transition map between $\mu$ and $\lambda_n$ and $\underline{v_n}$ is the greatest core decreasing minorant of $v$ relative to the core $\mathcal{A}_n$.

Indeed

$$\begin{aligned}
\int\limits_{\mathbb{X}_n \setminus B_{a,|x|_a}} \frac{1}{\underline{v_n}}\, d\mu &= \int\limits_{\varphi_n(t) \leq \varphi_n(x)} \frac{1}{\underline{v_n}}(t)\, d\mu(t) = \int\limits_{[0,\varphi_n(x)]} R_n\left(\frac{1}{\underline{v_n}}\right)(t)\, d\lambda(t) \\
&= \int\limits_{[0,\infty)} R_n\left(\frac{1}{\underline{v_n}}\right)(t)\chi_{[0,\varphi_n(x)]}(t)\, d\lambda(t) \\
&= \int\limits_{\mathbb{X}} R_n\left(\frac{1}{\underline{v_n}}\right) \circ \varphi_n(t)\chi_{[0,\varphi(x)]} \circ \varphi_n(t)\, d\mu(t) = \int\limits_{\varphi_n(t) \leq \varphi_n(x)} R_n\left(\frac{1}{\underline{v_n}}\right) \circ \varphi_n(t)\, d\mu(t) \\
&= \int\limits_{\mathbb{X}_n \setminus B_{a,|x|_a}} R_n\left(\frac{1}{\underline{v_n}}\right) \circ \varphi_n(t)\, d\mu(t).
\end{aligned}$$

Since the equality holds for all core sets, then $R_n\left(\frac{1}{\underline{v_n}}\right) = \frac{1}{\underline{v_n}} \circ \varphi_n$ almost everywhere.

Then for $q \in (0,1)$, Theorem 3.4 yields

$$\begin{aligned}
C_n &\approx \left( \int\limits_{\mathbb{X}_n} \left( \int\limits_{\varphi_n(z) \leq \varphi_n(x)} R_n\left(\frac{1}{\underline{v_n}}\right) \circ \varphi_n(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}} \\
&\approx \left( \int\limits_{\mathbb{X}_n} \left( \int\limits_{x \leq_{\mathcal{A}} z} \frac{1}{\underline{v_n}}(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}.
\end{aligned}$$

Notice that

$$\frac{1}{\underline{v_n}}(x) = \frac{1}{\text{ess inf}_\mu \left\{v(t) : t \in \mathbb{X}_n \setminus B_{a,|x|_a}\right\}} = \text{ess sup}_\mu \left\{ \frac{1}{v(t)} : t \in \mathbb{X}_n \setminus B_{a,|x|_a} \right\},$$

therefore
$$\sup_n \frac{1}{\underline{v_n}}(x) = \text{ess sup}_\mu \left\{ \frac{1}{v(t)} : t \in \mathbb{X} \setminus B_{a,|x|_a} \right\} = \frac{1}{\underline{v}}(x).$$

An application of Lemma 4.4 and the monotone convergence theorem yields
$$C \approx \sup_{n\in\mathbb{N}} \left( \int\limits_{\mathbb{X}_n} \left( \int\limits_{x\leq_{\mathcal{A}} z} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}$$
$$= \left( \int\limits_{\mathbb{X}} \left( \int\limits_{x\leq_{\mathcal{A}} z} \frac{1}{\underline{v}}(x)\omega(x)\, d\mu \right)^{\frac{q}{1-q}} \omega(z)\, d\mu(z) \right)^{\frac{1-q}{q}}.$$

For $q \geq 1$ we get
$$C_n = \sup_{x\in\mathbb{X}_n} \left( \frac{1}{\underline{v_n}}(x) \right) \left( \int\limits_{t\leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q} = \sup_{x\in\mathbb{X}} \left( \frac{1}{\underline{v_n}}(x) \right) \left( \int\limits_{t\leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q} \chi_{\mathbb{X}_n}(x).$$

By Lemma 4.4 we get
$$C = \sup_n C_n = \sup_n \sup_{x\in\mathbb{X}} \left( \frac{1}{\underline{v_n}}(x) \right) \left( \int\limits_{t\leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q} \chi_{\mathbb{X}_n}(x)$$
$$= \sup_{x\in\mathbb{X}} \left( \frac{1}{\underline{v}}(x) \right) \left( \int\limits_{t\leq_{\mathcal{A}} x} \omega(t)\, d\mu(t) \right)^{1/q}.$$

This completes the proof.

□

## 5 Proof of Theorem 2.4

Before proving the functional description of the greatest core decreasing majorant, we need some preparation. The use of infimum instead of supremum makes the use of approximating simple functions difficult. We need a technical lemma first, which will be the key in the 'pushing mass' technique needed to prove Theorem 2.4.

**Lemma 5.1.** *Let $u$ be a non-negative measurable function, $a > 0$ and $A = \{s \in U : \underline{u}(s) \geq a\}$ such that $0 < \mu(A)$. Then, for all $\delta > 0$ and $B \in \mathcal{M}$ such that $\mu(A) < \mu(B)$, the set*
$$\{s \in B \setminus A : \underline{u}(s) + \delta > u(s)\}$$
*has positive $\mu-$measure.*

**Proof:** Since $\underline{u}$ is core-decreasing, then, up to a set of $\mu-$measure zero, if $s \in A$ and $t \leq_{\mathcal{A}} s$ then $t \in A$. Therefore $A$ coincides with a set in $\mathcal{M}$ up to measure zero. Suppose that the statement does not hold, then there exist some $\delta > 0$ and $B \in \mathcal{M}$ such that $\mu(A) < \mu(B)$ and $\underline{u}(s) + \delta \leq u(s)$ for $\mu$-almost all $s \in B \setminus A$.

Let $b = \text{ess inf}_{B\setminus A}\underline{u}(s)$, since $\underline{u}$ is core-decreasing, then $a > b$, equality does not hold, otherwise $\mu(B \setminus A) = 0$. Without loss of generality, we may assume that $\delta < a - b$, pick $n$ big enough, such that $\frac{a-b}{n} < \delta$ and define the function
$$h = \underline{u}\chi_{(U\setminus(B\setminus A))} + \sum_{k=1}^{n} \left( b + k\frac{a-b}{n} \right) \chi_{(E_{k-1}\setminus E_k)},$$
where $E_k = \{s \in U : \underline{u}(s) \geq b + k\frac{a-b}{n}\}$. Notice that $h$ is core-decreasing by construction and $h \geq \underline{u}$ but $h(s) - \underline{u}(s) < \delta$, hence $h$ is also a minorant of $u$, by maximality we get $h = \underline{u}$. Since $\mu(B \setminus A) > 0$, there exists some $k$ such that $\mu(E_{k-1} \setminus E_k) > 0$, and notice that $k \neq n$, now define
$$h_2 = \underline{u}\chi_{(U\setminus(E_{k-1}\setminus E_k)} + \left( b + (k+1)\frac{a-b}{n} \right) \chi_{(E_{k-1}\setminus E_k)}.$$

By the same argument as before, $h_2$ is a core-decreasing minorant of $u$, but $h_2$ is strictly greater than $\underline{u}$, a contradiction.
$\square$

We now 'push the mass to the left' of $f$ to an appropriate function $g$ to achieve the desired infimum.

**Lemma 5.2.** *Let $u$ and $f$ be non-negative measurable finite $\mu-$almost everywhere functions, such that $\int_U uf\, d\mu < \infty$ and $\int_M f\, d\mu < \infty$ for all $M \in \mathcal{M}$. Then, for any $\epsilon > 0$, there exists a measurable non-negative function $g$ such that $\int_E g\, d\mu \geq \int_E f\, d\mu$ for any $E \in \mathcal{A}$ and*

$$\int_U gu\, d\mu - \epsilon < \int_U f\underline{u}\, d\mu.$$

**Proof:** Fix $\epsilon > 0$. By hypothesis, there exists $M \in \mathcal{M}$ such that

$$\int\limits_{U\setminus M} fu\, d\mu < \frac{\epsilon}{4}.$$

Let $a = \text{ess inf}_M \underline{u}(s)$, if we set $N = \{s \in U : \underline{u}(s) \geq a\}$, then, up to a set of measure zero, $M \subseteq N$ and the previous inequality still holds for $N$, therefore, without loss of generality we may assume that $M = \{s \in U : \underline{u}(s) \geq a\}$ for some $a \geq 0$.

Let $E_n = \{s \in U : \underline{u}(s) \geq a + n\frac{\epsilon}{8\int_M f\, d\mu}\}$ and

$$J = \{n \in \mathbb{N} : \mu\left((E_{n-1} \setminus E_n)\right) > 0\}.$$

Notice that $M = \cup_{j\in J}(E_{j-1} \setminus E_j)$ up to a set of $\mu-$measure zero.

Fix $j \in J$, let $\alpha_j = \inf\{\mu(E) : E \in \mathcal{M}$ and $\mu(E) > \mu(E_j)\}$. If $\alpha_j > \mu(E_j)$, let $C_j$ be an element in $\mathcal{M}$ such that $\mu(C_j) = \alpha_j$. If $\alpha_j = \mu(E_j)$, then pick $C_j \in \mathcal{M}$ such that $\mu(E_j) < \mu(C_j)$ and

$$\int\limits_{C_j\setminus E_j} fu\, d\mu < \frac{\epsilon}{2^{j+2}}.$$

In either case, by Lemma 5.1, the set

$$H_j = \{s \in C_j \setminus E_j : \underline{u}(s) + \frac{\epsilon}{8\int_M f\, d\mu} > u(s)\}$$

has positive measure.

If $\alpha_j > \mu(E_j)$, define the function

$$g_j = \left(\int\limits_{E_{j-1}\setminus E_j} f\, d\mu\right)\frac{\chi_{(H_j\cap(C_j\setminus E_j))}}{\mu\left(H_j \cap (C_j \setminus E_j)\right)}.$$

This function is supported on $E_{j-1} \setminus E_j$ and for any $E \in \mathcal{A}$ we have $\int_E g_j \, d\mu = 0$ if $\mu(E) \leq \mu(E_j)$ or $\int_E g_j \, d\mu = \int\limits_{E_{j-1}\setminus E_j} f \, d\mu$ otherwise. Moreover,

$$\begin{aligned}
\int_U u g_j \, d\mu - \int\limits_{E_{j-1}\setminus E_j} f \underline{u} \, d\mu &= \Bigg( \int\limits_{E_{j-1}\setminus E_j} f \, d\mu \Bigg) \frac{\int\limits_{H_j \cap (C_j \setminus E_j)} u \, d\mu}{\mu\left(H_j \cap (C_j \setminus E_j)\right)} - \int\limits_{E_{j-1}\setminus E_j} f \underline{u} \, d\mu \\
&< \Bigg( \int\limits_{E_{j-1}\setminus E_j} f \, d\mu \Bigg) \frac{\int\limits_{H_j \cap (C_j \setminus E_j)} \left( \underline{u} + \frac{\epsilon}{8 \int_M f \, d\mu} \right) d\mu}{\mu\left(H_j \cap (C_j \setminus E_j)\right)} - \int\limits_{E_{j-1}\setminus E_j} f \underline{u} \, d\mu \\
&= \frac{\epsilon}{8 \int_M f \, d\mu} \int\limits_{E_{j-1}\setminus E_j} f \, d\mu \\
&+ \Bigg( \int\limits_{E_{j-1}\setminus E_j} f \, d\mu \Bigg) \frac{\int\limits_{H_j \cap (C_j \setminus E_j)} \underline{u} \, d\mu}{\mu\left(H_j \cap (C_j \setminus E_j)\right)} - \int\limits_{E_{j-1}\setminus E_j} f \underline{u} \, d\mu \\
&\leq \frac{\epsilon}{8 \int_M f \, d\mu} \int\limits_{E_{j-1}\setminus E_j} f \, d\mu + \left( a + j \frac{\epsilon}{8 \int_M f \, d\mu} \right) \int\limits_{E_{j-1}\setminus E_j} f \, d\mu \\
&- \left( a + (j-1) \frac{\epsilon}{8 \int_M f \, d\mu} \right) \int\limits_{E_{j-1}\setminus E_j} f \, d\mu \\
&= \frac{\epsilon}{4 \int_M f \, d\mu} \int\limits_{E_{j-1}\setminus E_j} f \, d\mu.
\end{aligned}$$

In the case that $\alpha_j = \mu(E_j)$, define $g_j$ as follows

$$g_j = \Bigg( \int\limits_{E_{j-1}\setminus C_j} f \, d\mu \Bigg) \frac{\chi_{(H_j \cap (C_j \setminus E_j))}}{\mu\left(H_j \cap (C_j \setminus E_j)\right)} + f \chi_{(C_j \setminus E_j)}.$$

This function satisfies $\int_E g_j \, d\mu = 0$ if $\mu(E) = \mu(E_j)$, also $\int_E g_j \, d\mu \geq \int\limits_{E \setminus E_j} f \, d\mu$ if $\mu(E_j) < \mu(E) < \mu(C_j)$ and $\int_E g_j \, d\mu = \int\limits_{E_{j-1}\setminus E_j} f \, d\mu$ if $\mu(E) \geq \mu(C_j)$. A similar computation as before, shows that

$$\int_U u g_j \, d\mu - \int\limits_{E_{j-1}\setminus E_j} f \underline{u} \, d\mu < \frac{\epsilon}{4 \int_M f \, d\mu} \int\limits_{E_{j-1}\setminus E_j} f \, d\mu + \frac{\epsilon}{2^{j+2}}.$$

Now define the function

$$g = f \chi_{(U \setminus M)} + \sum_{j \in J} g_j \chi_{(E_{j-1} \setminus E_j)}.$$

Let $E \in \mathcal{M}$, if $\mu(E) \geq \mu(M)$, by construction we have

$$\int_E g \, d\mu = \int\limits_{E \setminus M} f \, d\mu + \sum_{j \in J} \int\limits_{E_{j-1}\setminus E_j} f \, d\mu = \int\limits_{E \setminus M} f \, d\mu + \int_M f \, d\mu = \int_E f \, d\mu.$$

If $\mu(E) < \mu(M)$, then there exists $k \in J$ such that $\mu(E_{k-1}) < \mu(E) \leq \mu(E_k)$ and

$$\int_E g \, d\mu = \sum_{j \in J, j < k} \int\limits_{E_{j-1}\setminus E_j} f \, d\mu + \int_{E \setminus E_{k-1}} g_k \, d\mu \geq \int\limits_{E_{k-1}} f \, d\mu + \int_{E \setminus E_{k-1}} f \, d\mu = \int_E f \, d\mu.$$

Therefore $\int_E g\,d\mu \geq \int_E f\,d\mu$ for all $E \in \mathcal{A}$.

Finally,

$$\begin{aligned}
\int_U gu\,d\mu - \int_U f\underline{u}\,d\mu &= \int_M gu\,d\mu - \int_M f\underline{u}\,d\mu + \int\limits_{U\setminus M} gu\,d\mu - \int\limits_{U\setminus M} f\underline{u}\,d\mu \\
&= \int_M gu\,d\mu - \int_M f\underline{u}\,d\mu + \int\limits_{U\setminus M} fu\,d\mu - \int\limits_{U\setminus M} f\underline{u}\,d\mu \\
&< \int_M gu\,d\mu - \int_M f\underline{u}\,d\mu + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \sum_{j\in J}\left(\int\limits_{E_{j-1}\setminus E_j} g_j u\,d\mu - \int\limits_{E_{j-1}\setminus E_j} f\underline{u}\,d\mu\right) \\
&< \frac{\epsilon}{2} + \sum_{j\in J}\left(\frac{\epsilon}{4\int_M f\,d\mu}\int\limits_{E_{j-1}\setminus E_j} f\,d\mu + \frac{\epsilon}{2^{j+2}}\right) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{4\int_M f\,d\mu}\sum_{j\in J}\int\limits_{E_{j-1}\setminus E_j} f\,d\mu + \sum_{j\in J}\frac{\epsilon}{2^{j+2}} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{4\int_M f\,d\mu}\int\limits_M f\,d\mu + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

This completes the proof.

□

With this we finish the functional description of the greatest core decreasing minorant

**Proof of Theorem 2.4:** If $g$ satisfies $\int_E g\,d\mu \geq \int_E f\,d\mu$ for all $E \in \mathcal{A}$ then

$$\begin{aligned}
\int_U gu\,d\mu &\geq \int_U g\underline{u}\,d\mu \text{ since } u \geq \underline{u} \\
&\geq \int_U f\underline{u}\,d\mu \text{ since } \underline{u} \text{ is core-decreasing.}
\end{aligned}$$

Infimum over all $g$ yields the inequality $\int_U f\underline{u}\,d\mu \leq \inf\left\{\int_U gu\,d\mu : \int_E g\,d\mu \geq \int_E f\,d\mu \text{ for all } E \in \mathcal{A}\right\}$. Equality follows from Lemma 5.2.

□

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