Down spaces over a measure space with an ordered core

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Applications: Study of weighted norm inequalities, describe the dual of Lorentz spaces and study of Cesaro spaces given by the norm $\|f\|_{C(X)} = \|\frac{1}{x} \int_0^x |f(t)| \; dt\|_X$ Whenever $f(x) \mapsto \frac{1}{x} \int_0^x f(t) \, dt$ is bounded in X then $C(X) \approx X^{\downarrow}$.

Recall the $K-$functional for a compatible couple (X_1, X_2) is
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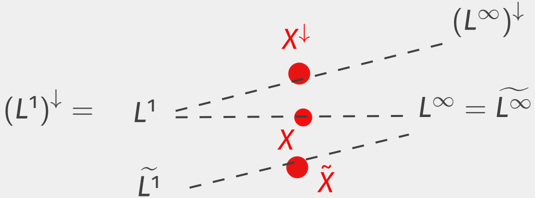
And we have a complete description of the interpolation spaces, in terms of the rearrangement invariant spaces



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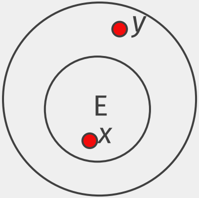
Definition: A positive measurable function is core-decreasing if

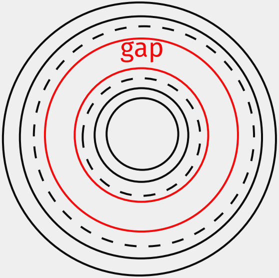
1.



if $x \in E$ and $y \notin E$, $f(x) \geq f(y)$

$\forall E \in \mathcal{A}$

2. f is constant in each gap



Let X be a rearrangement invariant Banach Function space over (S, μ) then

$$\|f\|_{X^{\downarrow}} = \sup \left\{ \int_S |f|\, g \, d\mu : \|g\|_{X'} \leq 1 \text{ and } g \text{ is core-decreasing} \right\}.$$

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Questions

1. What is the relation with the down spaces over $[0, \infty)$?
2. Do we have a level function 'f^o'?
3. What is the dual (associate space) of $X^{\downarrow}$?
4. Is $\left(L^1_\mu, (L^\infty_\mu)^{\downarrow}\right)$ a Calderón Couple?

Relation with $[0,\infty)$ and the level function

Every ordered core induces a Borel measure λ on $[0,\infty)$ and maps

$$\underbrace{L^1_\mu + (L^\infty_\mu)^\downarrow}_{\text{Over } S} \underset{Q}{\overset{R}{\rightleftarrows}} \underbrace{L^1_\lambda + (L^\infty_\lambda)^\downarrow}_{\text{Over } [0,\infty)}$$

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- $RQ = \mathrm{Id}$ but $QRh = h$ may fail, the functions are indistinguishable in terms of the core elements but not equal.
- Core-decreasing functions on S and decreasing functions on $[0,\infty)$ are in bijective correspondence via R, Q.

We use the operators R and Q to lift the level function

Definition: For $f \in L^1_\mu + (L^\infty_\lambda)^\downarrow$ we define the level function f^o relative to the core $\mathcal{A}$ by $f^o = Q\Big(\big(R(f)\big)^o\Big)$.

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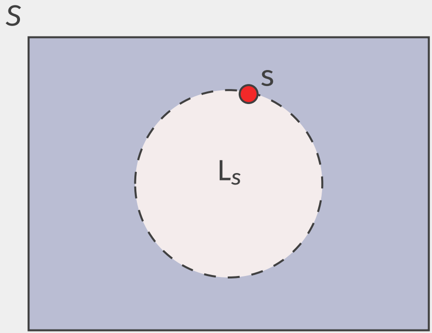
Theorem

Let X be a rearrangement invariant Banach function space over (S, μ) and $\mathcal{A}$ be an ordered core then

$$\|f\|_{X^\downarrow} = \|f^o\|_X.$$

We define the least core decreasing majorant of f as follows:

$$\widetilde{f}(s) = \sup_{t \notin L_s} |f(t)| .$$



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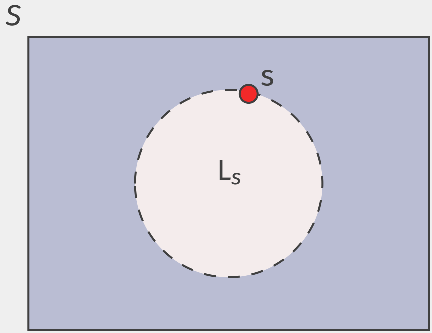

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Theorem (Duality)

Let $\|f\|_{\widetilde{X}} = \|\widetilde{f}\|_X$, then $\left(X^{\downarrow}\right)' = \widetilde{X}'$.

For the couples $\left(L^1_\mu, (L^\infty_\mu)^\downarrow\right)$ and its dual $\left(\widetilde{L^1_\mu}, L^\infty_\mu\right)$:

Theorem (K-functional)

$K(f, t, L^1_\mu, (L^\infty_\mu)^\downarrow) = \int_0^t (f^{o_\mathcal{A}})^*$ and $K(f, t, \widetilde{L^1_\mu}, L^\infty_\mu) = \int_0^t (\widetilde{f})^*$

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Theorem (Exact Calderón couples)

If $f, g \in L^1_\mu + \left(L^\infty_\mu\right)^\downarrow$ and $\int_0^t (f^{o_\mathcal{A}})^* \leq \int_0^t (g^{o_\mathcal{A}})^*$ for all $t > 0$ then there exists an admissible contraction $T : L^1_\mu + \left(L^\infty_\mu\right)^\downarrow$ such that $Tg = f$.

An analogous result holds for the dual couple.

Corollary

Z is an interpolation space of $(L^1_\mu, (L^\infty_\mu)^\downarrow)$ if and only if $Z = X^\downarrow$ for some rearrangement invariant space X.

We have a complete description of the interpolation spaces for the couples $(L^1_\mu, (L^\infty_\mu)^\downarrow)$ and $(\widetilde{L^1_\mu}, L^\infty_\mu)$, in terms of the rearrangement invariant spaces