Monotonicity in Kernel operators and abstract Hardy inequalities

The 50 70 80 Conference in Mathematics

Alejandro Santacruz Hidalgo
Joint work with Gord Sinnamon

University of Western Ontario

August 19, 2024

Introduction: Kernel operators

Let $(U, \Sigma, \mu), (Y, \mathcal{T}, \tau)$ be σ-finite measure spaces and $k : Y \times U \to [0, \infty)$ be $\tau \otimes \mu$ measurable.
Consider the operator $Tf(y) = \int_U k(y,t) f(t)\, d\mu(t)$.

Introduction: Kernel operators

Let $(U, \Sigma, \mu), (Y, \mathcal{T}, \tau)$ be σ-finite measure spaces and $k : Y \times U \to [0, \infty)$ be $\tau \otimes \mu$ measurable.
Consider the operator $Tf(y) = \int_U k(y,t) f(t)\, d\mu(t)$.
Definition: We say that k is a *consonant kernel* if the collection $\big\{\{u \in U : k(y,u) \geq t\} : t > 0, y \in Y\big\}$ is totally ordered by inclusion and every set has finite μ-measure.

Introduction: Kernel operators

Let $(U, \Sigma, \mu), (Y, \mathcal{T}, \tau)$ be σ-finite measure spaces and $k : Y \times U \to [0, \infty)$ be $\tau \otimes \mu$ measurable.
Consider the operator $Tf(y) = \int_U k(y,t) f(t)\, d\mu(t)$.
Definition: We say that k is a *consonant kernel* if the collection $\big\{\{u \in U : k(y,u) \geq t\} : t > 0, y \in Y\big\}$ is totally ordered by inclusion and every set has finite μ-measure.
Examples: 1. $Y = U = [0, \infty)$, Lebesgue measures and $k(y,t) = \chi_{[0,y]}(t)$. The operator becomes $Tf(y) = \int_0^y f(t)\, dt$ (Hardy operator).

Introduction: Kernel operators

Let $(U, \Sigma, \mu), (Y, \mathcal{T}, \tau)$ be σ-finite measure spaces and $k : Y \times U \to [0, \infty)$ be $\tau \otimes \mu$ measurable.

Consider the operator $Tf(y) = \int_U k(y,t) f(t)\, d\mu(t)$.

Definition: We say that k is a *consonant kernel* if the collection $\big\{\{u \in U : k(y,u) \geq t\} : t > 0, y \in Y\big\}$ is totally ordered by inclusion and every set has finite μ-measure.

Examples: 1. $Y = U = [0, \infty)$, Lebesgue measures and $k(y,t) = \chi_{[0,y]}(t)$. The operator becomes $Tf(y) = \int_0^y f(t)\, dt$ (Hardy operator).

2. $Y = U = \mathbb{N}^+$, counting measures and $k(y,t) = \frac{1}{y}\chi_{\{1,\ldots,y\}}(t)$. The operator becomes $Tf(n) = \frac{1}{n}\sum_{j=1}^{n} f(n)$ (Discrete Cesaro operator).

Introduction: Kernel operators

Let $(U, \Sigma, \mu), (Y, \mathcal{T}, \tau)$ be σ-finite measure spaces and $k : Y \times U \to [0, \infty)$ be $\tau \otimes \mu$ measurable.
Consider the operator $Tf(y) = \int_U k(y,t) f(t)\, d\mu(t)$.
Definition: We say that k is a *consonant kernel* if the collection $\left\{\{u \in U : k(y,u) \geq t\} : t > 0, y \in Y\right\}$ is totally ordered by inclusion and every set has finite μ-measure.
Examples: 1. $Y = U = [0, \infty)$, Lebesgue measures and $k(y,t) = \chi_{[0,y]}(t)$. The operator becomes $Tf(y) = \int_0^y f(t)\, dt$ (Hardy operator).
2. $Y = U = \mathbb{N}^+$, counting measures and $k(y,t) = \frac{1}{y}\chi_{\{1,\ldots,y\}}(t)$. The operator becomes $Tf(n) = \frac{1}{n}\sum_{j=1}^n f(n)$ (Discrete Cesaro operator).
3. Let $B : Y \to \Sigma$ be a mapping with totally ordered range, with $\mu(B(y)) < \infty$. Let the kernel be $k(y,t) = \chi_{B(y)}(t)$. The operator becomes $Tf(y) = \int_{B(y)} f\, d\mu$. (**Abstract** Hardy operator) [Sinnamon 2022].

Introduction: Kernel operators

Let $(U, \Sigma, \mu), (Y, \mathcal{T}, \tau)$ be σ-finite measure spaces and $k : Y \times U \to [0, \infty)$ be $\tau \otimes \mu$ measurable.
Consider the operator $Tf(y) = \int_U k(y,t) f(t)\, d\mu(t)$.
Definition: We say that k is a *consonant kernel* if the collection $\big\{\{u \in U : k(y,u) \geq t\} : t > 0, y \in Y\big\}$ is totally ordered by inclusion and every set has finite μ-measure.
Examples: 1. $Y = U = [0, \infty)$, Lebesgue measures and $k(y,t) = \chi_{[0,y]}(t)$. The operator becomes $Tf(y) = \int_0^y f(t)\, dt$ (Hardy operator).
2. $Y = U = \mathbb{N}^+$, counting measures and $k(y,t) = \frac{1}{y}\chi_{\{1,\ldots,y\}}(t)$. The operator becomes $Tf(n) = \frac{1}{n}\sum_{j=1}^n f(n)$ (Discrete Cesaro operator).
3. Let $B : Y \to \Sigma$ be a mapping with totally ordered range, with $\mu(B(y)) < \infty$. Let the kernel be $k(y,t) = \chi_{B(y)}(t)$. The operator becomes $Tf(y) = \int_{B(y)} f\, d\mu$. (**Abstract** Hardy operator) [Sinnamon 2022].
Goal: Study the boundedness of these operators acting on quasi-Banach function spaces.

Introduction: Kernel operators

Let $(U, \Sigma, \mu), (Y, \mathcal{T}, \tau)$ be σ-finite measure spaces and $k : Y \times U \to [0, \infty)$ be $\tau \otimes \mu$ measurable.
Consider the operator $Tf(y) = \int_U k(y,t) f(t)\, d\mu(t)$.
Definition: We say that k is a *consonant kernel* if the collection $\big\{\{u \in U : k(y,u) \geq t\} : t > 0, y \in Y\big\}$ is totally ordered by inclusion and every set has finite μ-measure.
Examples: 1. $Y = U = [0, \infty)$, Lebesgue measures and $k(y,t) = \chi_{[0,y]}(t)$. The operator becomes $Tf(y) = \int_0^y f(t)\, dt$ (Hardy operator).
2. $Y = U = \mathbb{N}^+$, counting measures and $k(y,t) = \frac{1}{y}\chi_{\{1,\ldots,y\}}(t)$. The operator becomes $Tf(n) = \frac{1}{n}\sum_{j=1}^n f(n)$ (Discrete Cesaro operator).
3. Let $B : Y \to \Sigma$ be a mapping with totally ordered range, with $\mu(B(y)) < \infty$. Let the kernel be $k(y,t) = \chi_{B(y)}(t)$. The operator becomes $Tf(y) = \int_{B(y)} f\, d\mu$. (**Abstract** Hardy operator) [Sinnamon 2022].
Goal: Study the boundedness of these operators acting on quasi-Banach function spaces. today we will focus on the abstract Hardy inequality
$\left(\int_Y \left(\int_{B(y)} f\, d\mu \right)^q d\tau \right)^{1/q} \leq C \left(\int_U f^p\, d\mu \right)^{1/p}$ find conditions so C exists with $p = 1$.

Main tool: Generalized Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Main tool: Generalized Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying

Main tool: Generalized Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$

Main tool: Generalized Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$ and $\cup\mathcal{A} = \cup_n E_n$ for some $\{E_n\} \in \mathcal{A}$.

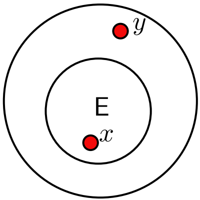
Main tool: Generalized Monotone functions

Let (U, Σ, μ) be a $\sigma-$finite measure space. **We assume no order**.

Definition: An ordered core is a totally ordered subset $\mathcal{A}$ of Σ (If $A, B \in \mathcal{A}$ then $A \subseteq B$ or $B \subseteq A$), containing $\emptyset$ and satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$ and $\cup\mathcal{A} = \cup_n E_n$ for some $\{E_n\} \in \mathcal{A}$.

Definition: A positive $\sigma(\mathcal{A})$-measurable function is core-decreasing if

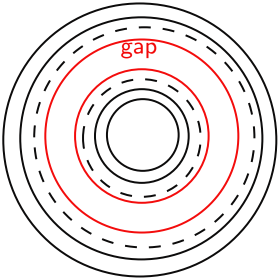
1.



if $x \in E$ and $y \notin E$, $f(x) \geq f(y)$

$\forall E \in \mathcal{A}$

2. f is constant in each gap



Examples

Every core contains $\emptyset$.

1. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, x] : x > 0\}$. Note $\sigma(\mathcal{A})$ is the Borel σ-algebra. f is core decreasing if for each $E = [0, t]$, $x \in [0, t]$ and $y \notin [0, t]$ then $f(y) \leq f(x)$.

Examples

Every core contains $\emptyset$.

1. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, x] : x > 0\}$. Note $\sigma(\mathcal{A})$ is the Borel σ-algebra. f is core decreasing if for each $E = [0, t]$, $x \in [0, t]$ and $y \notin [0, t]$ then $f(y) \leq f(x)$.

2. Let $U = \mathbb{R}^d$, μ satisfying $\mu(B[0; r]) < \infty$ for each $r > 0$. The core $\mathcal{A} = \{B[0; r] : r > 0\}$. Now $\sigma(\mathcal{A})$ is strictly smaller than the Borel σ-algebra. f is core decreasing if it is radially decreasing

Examples

Every core contains $\emptyset$.

1. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, x] : x > 0\}$. Note $\sigma(\mathcal{A})$ is the Borel σ-algebra. f is core decreasing if for each $E = [0, t]$, $x \in [0, t]$ and $y \notin [0, t]$ then $f(y) \leq f(x)$.

2. Let $U = \mathbb{R}^d$, μ satisfying $\mu(B[0; r]) < \infty$ for each $r > 0$. The core $\mathcal{A} = \{B[0; r] : r > 0\}$. Now $\sigma(\mathcal{A})$ is strictly smaller than the Borel σ-algebra. f is core decreasing if it is radially decreasing

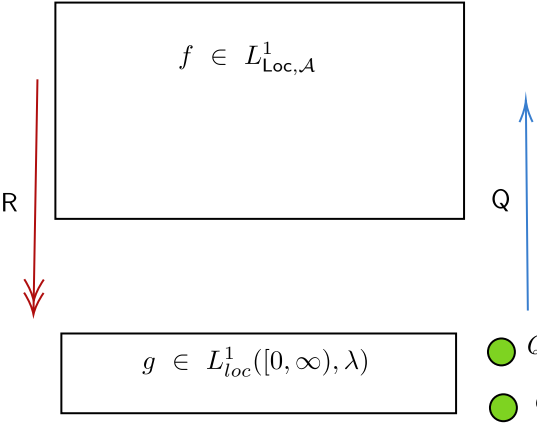
3. Let $k : Y \times U \to [0, \infty)$ be a *consonant kernel*, then the level sets $\{k^{-1}(y, \cdot)(t, \infty)\}_{y \in Y}$ form an ordered core and all the functions $t \to k(y, t)$ are core decreasing.

Induced measure on the half line

Consider the vector space.

$$L^1_{\mathsf{Loc},\mathcal{A}} = \left\{ f \text{ measurable} : \int_A |f| \, d\mu < \infty \text{ for all } A \in \mathcal{A} \right\}$$

Every ordered core induces a Borel measure λ on $[0,\infty)$ and maps R, Q:



Properties:

- $RQg = g$
- $QRf \neq f$ in general
- Q and R preserve monotonicity.
- $QRf = f$ if f is core-decreasing

Application: Abstract Hardy inequality, The case $p = 1$

Given $B : Y \to \Sigma$, characterize the best C such that

$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$ holds for all $f > 0$.

Application: Abstract Hardy inequality, The case $p = 1$

Given $B : Y \to \Sigma$, characterize the best C such that

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta \text{ holds for all } f > 0.$$

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \int_U f u \, d\mu$.

Application: Abstract Hardy inequality, The case $p = 1$

Given $B : Y \to \Sigma$, characterize the best C such that

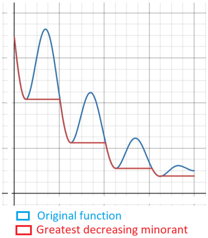
$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta \text{ holds for all } f > 0.$$

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \int_U f u \, d\mu$.

Here we would like to use the maps R, Q to go to the half line but $R(fu) \neq R(f)R(u)$.

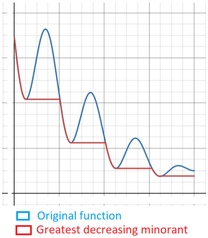
Tool: Greatest core decreasing minorant

Definition (pointwise): For $u \geq 0$, the function $\underline{u}$ is the greatest core decreasing minorant if $\underline{u}(s) \leq u(s)$ for almost all $s \in U$ and if $h(s) \leq u(s)$ and h is core decreasing, then $h \leq \underline{u}$.



Tool: Greatest core decreasing minorant

Definition (pointwise): For $u \geq 0$, the function $\underline{u}$ is the greatest core decreasing minorant if $\underline{u}(s) \leq u(s)$ for almost all $s \in U$ and if $h(s) \leq u(s)$ and h is core decreasing, then $h \leq \underline{u}$.



Theorem [S.-2024]

If $u \geq 0$,

$$\int_U \underline{u} g \, d\mu = \inf \left\{ \int_U |u| \, h \, d\mu : h \in L^+(\Sigma) \text{ and } \int_E h \, d\mu \geq \int_E g \, d\mu \text{ for all } E \in \mathcal{A} \right\}.$$

Application: Transferring monotonicity in weighted norm inequalities

Given an consonant kernel $k : Y \times U \to [0, \infty)$ and its associated operator $Tf(y) = \int_U k(y,t)\, d\mu(t)$.

Theorem [S.-Sinnamon 2024]

Let X be a quasi-Banach function space and $w > 0$, then the smallest constant C, infinite or finite, for which the inequality

$$\|Tf\|_X \leq C \int_U f w \, d\mu \quad f \in L^+_\mu$$

is unchanged when w is replaced by $\underline{w}$:

$$\|Tf\|_X \leq C \int_U f \underline{w} \, d\mu, \quad f \in L^+_\mu$$

Application: Abstract Hardy inequality, The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$

holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the

equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \int_U f u \, d\mu$.

Application: Abstract Hardy inequality, The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$ holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \int_U f u \, d\mu$.

Step 2: Replace u by $\underline{u}$, $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \underline{u} \, d\mu$.

Application: Abstract Hardy inequality, The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta$ holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \int_U f u \, d\mu$.

Step 2: Replace u by $\underline{u}$, $\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \underline{u} \, d\mu$.

Now $R(f\underline{u}) = R(f)R(\underline{u})$.

Application: Abstract Hardy inequality, The case $p = 1$

Characterize the best C such that $\left(\int_Y \left(\int_{B(y)} f\, d\mu\right)^q d\tau(y)\right)^{1/q} \leq C \int_U f\, d\eta$ holds for all $f > 0$.

Step 1: Use the Lebesgue decomposition and Radon-Nikodym to have the equivalent inequality $\left(\int_Y \left(\int_{B(y)} f\, d\mu\right)^q d\tau\right)^{1/q} \leq C \int_U f u\, d\mu$.

Step 2: Replace u by $\underline{u}$, $\left(\int_Y \left(\int_{B(y)} f\, d\mu\right)^q d\tau(y)\right)^{1/q} \leq C \int_U f \underline{u}\, d\mu$.

Now $R(f\underline{u}) = R(f)R(\underline{u})$.

Step 3: Use the maps R, Q to find an equivalent Hardy inequality on the half line and use the available results [Johansson, Stepanov, Ushakova 2008].

Main result, case $p = 1$.

Theorem [S.2024]

For $(Y, \mathcal{T}, \tau), (U, \Sigma, \mu), (U, \Sigma, \nu)$ σ-finite and a core map $B : Y \to \Sigma$. Let $\eta = \eta_a + \eta_s$, where $d\eta_a = u d\mu$ and $\eta_s \perp \mu$. Then the best constant C in the inequality

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta, \tag{1}$$

satisfies

$$C \approx \left[\int_Y \left(\int_{\mu(B(z)) \leq \mu(B(y))} R\left(\frac{1}{\underline{u}} \right) \circ \mu \circ B(y) \, d\tau(y) \right)^{\frac{q}{1-q}} d\tau(z) \right]^{\frac{1-q}{q}}, \text{ for } q \in (0,1),$$

and

$$C = \sup_{s \in U} \left(\frac{1}{\underline{u}}(s) \right) \tau \left(\{ y \in Y : s \in B(y) \} \right)^{1/q}, \text{ for } q \in [1, \infty).$$

Where $\underline{u}$ is the greatest core decreasing minorant.

References:

1. Santacruz Hidalgo, A. Abstract hardy inequalities: The case p=1. (submitted Arxiv) 2024

2. Santacruz Hidalgo, A., and Sinnamon, G. Core decreasing functions. Journal of Functional Analysis 287(4). 2024.