Monotonicity in measure spaces and Hardy inequalities

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Weighted Hardy inequalities

Fix $p \geq 1$ and $q > 0$. For which measurable weight functions u, v does there exist a constant C such that

$$\left(\int_0^\infty \left(\int_0^y f(s)v(s)\, d(s) \right)^q u(y)\, dy \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f(s)^p v(s)\, ds \right)^{\frac{1}{p}}$$

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M. Ruzhansky, D. Verma (2019) Metric measure spaces

Let X be a metric measure space that must admit a polar decomposition Let $1 < p \leq q < \infty$, $a \in X$ and weights $u, v > 0$, there is $C > 0$ such that

$$\left(\int_X \left(\int_{B(a,|x|_a)} f(s)\, d(s) \right)^q u(y)\, dy \right)^{\frac{1}{q}} \leq C \left(\int_X f(s)^p v(s)\, ds \right)^{\frac{1}{p}}$$

holds for all $f \in L^+$ if and only if ... conditions u, v.

Ordered cores

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Definition: An Ordered core is a totally ordered subset $\mathcal{A}$ of Σ, satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$ and $S = \cup_n E_n$ for some $\{E_n\} \in \mathcal{A}$.

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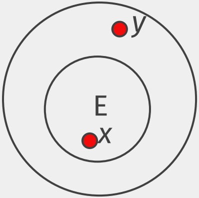
Definition: A positive measurable function is core-decreasing if

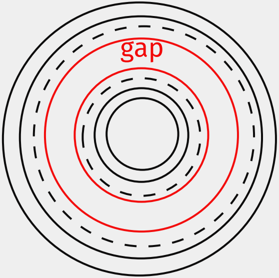
1\.



if $x \in E$ and $y \notin E$, $f(x) \geq f(y)$

$\forall E \in \mathcal{A}$

2\. f is constant in each gap



Relation with $[0, \infty)$ and the level function

Every ordered core induces a Borel measure λ on $[0, \infty)$ and maps

$$\underbrace{L^1_\mu + (L^\infty_\mu)^\downarrow}_{\text{Over } S} \underset{Q}{\overset{R}{\rightleftarrows}} \underbrace{L^1_\lambda + (L^\infty_\lambda)^\downarrow}_{\text{Over } [0,\infty)}$$

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With properties

- $\|Rh\|_{L^1_\lambda} \leq \|h\|_{L^1_\mu}$, $\|Rh\|_{(L^\infty_\lambda)^\downarrow} \leq \|h\|_{(L^\infty_\mu)^\downarrow}$, same for Q.

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- $RQ = \mathrm{Id}$ but $QRh = h$ may fail, the functions are indistinguishable in terms of the core elements but not equal.
- Core-decreasing functions on S and decreasing functions on $[0,\infty)$ are in bijective correspondence via R, Q.

Result in Hardy inequalities

Theorem: Here S a metric measure space, $p \in (1, \infty), q > 0$

If $a \in S$ and $\mu\big(B(a,s)\big) < \infty$ for all $s \in S$, where $B(a,s) = \{t \in S : \mathrm{d}(a,t) \le d(a,s)\}$. Then, the Hardy inequality

$$\left(\int_S \left(\int_{B(a,s)} f(y)\, d\mu(y) \right)^q u(x)\, d\mu(x) \right)^{\frac{1}{q}} \le C \left(\int_S f^p v(x)\, d\mu(x) \right)^{\frac{1}{p}}$$

holds for some $C > 0$ if and only if

$$\sup_{s \neq a} \left\{ \left(\int_{S \setminus B(a,s)} u\, d\mu \right)^{\frac{1}{q}} \left(\int_{B(a,s)} v^{1-p'}\, d\mu \right)^{\frac{1}{p'}} \right\} < \infty, \text{ for } p \le q \text{ and}$$

$$\int_S \left(\int_{S \setminus B(a,s)} u\, d\mu \right)^{\frac{r}{p}} \left(\int_{B(a,s)} v^{1-p'}\, d\mu \right)^{\frac{r}{p'}} u(s)\, d\mu(s) < \infty, \text{ for } q < p,$$

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For an L^p space over (S, μ) define the norms

$$\|f\|_{(L^p)^\downarrow} = \sup\left\{ \int_S |f|\, g\, d\mu : \|g\|_{L^{p'}} \leq 1 \text{ and } g \text{ is core-decreasing} \right\}.$$

Examples: $\left(L^1_\mu\right)^\downarrow = L^1_\mu$ but $L^\infty_\mu \subsetneq \left(L^\infty_\mu\right)^\downarrow$.

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Questions

1. What is the relation with the down spaces over $[0,\infty)$?
2. Can we give a better description of the 'down'-norm?
3. What is the interpolation structure of the spaces $\left(L^1_\mu, (L^\infty_\mu)^\downarrow\right)$?. Is this a Calderón Couple?

Recall the induced Borel measure λ on $[0, \infty)$ and the maps

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Theorem

For any f, there exists a non-negative core decreasing function f^o such that $\|f\|_{(L^p_\mu)^\downarrow} = \|f^o\|_{L^p_\mu}$. We call f^o the **level function** of f.

Level function

Over $[0, \infty)$ for $f \in L^1_\lambda + (L^\infty_\lambda)^\downarrow$ there is a decreasing (usual sense) function f^o satisfying

$$\int_{[0,x]} f \, d\mu \leq \int_{[0,x]} f^o \, d\mu, \forall x > 0$$

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$$\int_{[0,x]} f^o\, d\mu \leq \int_{[0,x]} g\, d\mu, \forall x > 0$$

We use the operators R and Q to lift the level function **Definition:** For $f \in L^1_\mu +$ we define the level function f^o relative to the core $\mathcal{A}$ by $f^o = Q\Big(\big(R(f)\big)^o\Big)$.

Recall the $K-$functional for a compatible couple (X_1, X_2) is
$K(x, t, X_1, X_2) = \inf \{\|x_1\|_{X_1} + t\|x_2\|_{X_2} : x = x_1 + x_2\}.$

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- $K(f, t, L^1, L^\infty) = \int_0^t f^*$.

And we have a complete description of the interpolation spaces, in terms of the rearrangement invariant spaces due to Calderón

Theorem (Calderón)

The interpolation spaces of L^1, L^∞ are exactly the rearrangement invariant spaces.

For the couple $\left(L^1_\mu, (L^\infty_\mu)^\downarrow\right)$:

Theorem (K-functional)

$K(f, t, L^1_\mu, (L^\infty_\mu)^\downarrow) = \int_0^t (f^o)^*$

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Theorem (K-functional)

$K(f, t, L^1_\mu, (L^\infty_\mu)^\downarrow) = \int_0^t (f^o)^*$

Theorem (Exact Calderón couples)

If $f, g \in L^1_\mu + \left(L^\infty_\mu\right)^\downarrow$ and $\int_0^t (f^o)^* \leq \int_0^t (g^o)^*$ for all $t > 0$ then there exists an admissible contraction $T : L^1_\mu + \left(L^\infty_\mu\right)^\downarrow$ such that $Tg = f$.

An analogous result holds for the dual couple.

Corollary

Z is an interpolation space of $(L^1_\mu, (L^\infty_\mu)^\downarrow)$ if and only if $Z = X^\downarrow$ for some rearrangement invariant space X.

Calderón couples, conclusion

We have a complete description of the interpolation spaces for the couples $(L^1_\mu, (L^\infty_\mu)^\downarrow)$ and $(\widetilde{L^1_\mu}, L^\infty_\mu)$, in terms of the rearrangement invariant spaces