Hardy operators and monotonicity in general spaces

Spring School in Analysis 2023 - Function Spaces and Applications XII

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Weighted Hardy inequalities

Fix $p \geq 1$ and $q > 0$. For which measurable weight functions u, v does there exist a constant C such that

$$\left(\int_0^\infty \left(\int_0^x f(s)v(s)\, d(s) \right)^q u(x)\, dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f(x)^p v(x)\, ds \right)^{\frac{1}{p}}$$

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M. Ruzhansky, D. Verma (2019) Metric measure spaces

Let X be a metric measure space that must admit a polar decomposition Let $1 < p \leq q < \infty$, $a \in X$ and weights $u, v > 0$, there is $C > 0$ such that

$$\left(\int_X \left(\int_{B(a,|x|_a)} f(s)\, d(s) \right)^q u(x)\, dx \right)^{\frac{1}{q}} \leq C \left(\int_X f(x)^p v(x)\, dx \right)^{\frac{1}{p}}$$

holds for all $f \in L^+$ if and only if ... conditions u, v.

Ordered cores

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Definition: An Ordered core is a totally ordered subset $\mathcal{A}$ of Σ, satisfying $\mu(E) < \infty$ for all $E \in \mathcal{A}$ and $S = \cup_n E_n$ for some $\{E_n\} \in \mathcal{A}$.

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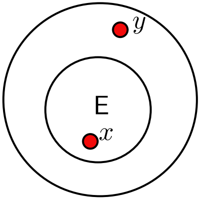
Definition: A positive measurable function is core-decreasing if

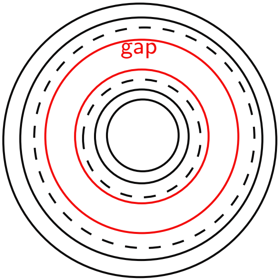
1.



if $x \in E$ and $y \notin E$, $f(x) \geq f(y)$

$\forall E \in \mathcal{A}$

2. f is constant in each gap



Induced measure on $[0, \infty)$

Every ordered core induces a Borel measure λ on $[0, \infty)$ and maps R, Q:

$$\begin{tikzcd} f \in L^1_{loc}(S, \mu) \arrow[d, shift right, "R"'] \\ g \in L^1_{loc}([0\infty), \lambda) \arrow[u, shift right, "Q"'] \end{tikzcd}$$

Properties:

- $RQg = g$
- $QRf \neq f$ in general
- Q and R preserve monotonicity.
- $QRf = f$ if f is core-decreasing

Result in Hardy inequalities

Theorem: Here S a metric measure space, $p \in (1,\infty), q > 0$

If $a \in S$ and $\mu\big(B(a,s)\big) < \infty$ for all $s \in S$, where $B(a,s) = \{t \in S : \mathsf{d}(a,t) \leq d(a,s)\}$. Then, the Hardy inequality

$$\left(\int_S \left(\int_{B(a,s)} f(y)\, d\mu(y)\right)^q u(s)\, d\mu(s)\right)^{\frac{1}{q}} \leq C \left(\int_S f^p v\, d\mu\right)^{\frac{1}{p}} \text{ holds for some } C > 0$$

if and only if $\sup_{s \neq a} \left\{ \left(\int_{S \setminus B(a,s)} u\, d\mu\right)^{\frac{1}{q}} \left(\int_{B(a,s)} v^{1-p'}\, d\mu\right)^{\frac{1}{p'}} \right\} < \infty$, for $p \leq q$

and $\int_S \left(\int_{S \setminus B(a,s)} u\, d\mu\right)^{\frac{r}{p}} \left(\int_{B(a,s)} v^{1-p'}\, d\mu\right)^{\frac{r}{p'}} u(s)\, d\mu(s) < \infty$, for $q < p$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Associated function spaces

For an L^p space over (S, μ) define the norms

$$\|f\|_{(L^p)^{\downarrow}} = \sup\left\{\int\limits_S |f|\, g\, d\mu : \|g\|_{L^{p'}} \leq 1 \text{ and } g \text{ is core-decreasing}\right\}.$$

Examples: $\left(L^1_\mu\right)^{\downarrow} = L^1_\mu$ but $L^\infty_\mu \subsetneq \left(L^\infty_\mu\right)^{\downarrow}$:

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Questions:

1. What is the relation with the down spaces over $[0, \infty)$?
2. Can we give a better description of the 'down'-norm?
3. What is the interpolation structure of the spaces $\left(L^1_\mu, \left(L^\infty_\mu\right)^{\downarrow}\right)$?. Is this a Calderón Couple?

General measures vs half-line down spaces

Recall the induced Borel measure λ on $[0,\infty)$ with the maps R, Q

$f \in L^1_\mu(S) + (L^\infty_\mu(S))^\downarrow$

R

Q

$g \in L^1_\lambda[0,\infty) + (L^\infty_\lambda[0,\infty))^\downarrow$

Properties

- $\|Rf\|_{L^1_\lambda(0,\infty]} \leq \|f\|_{L^1_\mu(S)}$
- $\|Rf\|_{(L^\infty_\lambda(0,\infty])^\downarrow} \leq \|f\|_{(L^\infty_\mu(S))^\downarrow}$

Lifting operators

- $\mathcal{T}f := Q \circ T \circ Rf$

For an operator T

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Theorem

For any f, there exists a non-negative core decreasing function f^o such that $\|f\|_{(L^p_\mu)^{\downarrow}} = \|f^o\|_{L^p_\mu}$. We call f^o the **level function** of f relative to the core $\mathcal{A}$.

Interpolation of down spaces

Recall the $K-$functional for a compatible couple (X_1, X_2) is
$K(x, t, X_1, X_2) = \inf\{\|x_1\|_{X_1} + t\|x_2\|_{X_2} : x = x_1 + x_2\}$.

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Theorem (K-functional)

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Theorem (Exact Calderón couples)

If $f, g \in L^1_\mu + \left(L^\infty_\mu\right)^\downarrow$ and $\int_0^t (f^o)^* \leq \int_0^t (g^o)^*$ for all $t > 0$ then there exists an admissible contraction $T : L^1_\mu + \left(L^\infty_\mu\right)^\downarrow$ such that $Tg = f$.

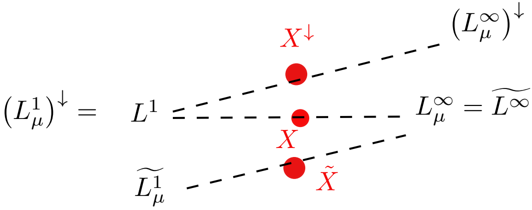
An analogous result holds for the dual couple.

Corollary

Z is an interpolation space of $(L^1_\mu, (L^\infty_\mu)^\downarrow)$ if and only if $Z = X^\downarrow$ for some rearrangement invariant space X.

Calderón couples, conclusion

We have a complete description of the interpolation spaces for the couples $(L^1_\mu, (L^\infty_\mu)^\downarrow)$ and $(\widetilde{L^1_\mu}, L^\infty_\mu)$, in terms of the rearrangement invariant spaces



Thank you!