Monotonicity in ordered measure spaces

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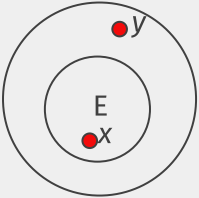
Definition: A positive measurable function is core-decreasing if

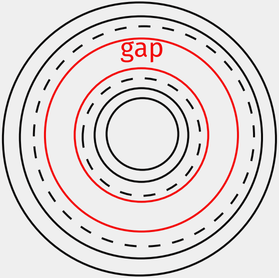
1.



if $x \in E$ and $y \notin E$, $f(x) \geq f(y)$

$\forall E \in \mathcal{A}$

2. f is constant in each gap



Constructions with monotone functions

Version in $(0, \infty)$ with Lebesgue measure.

Least decreasing majorant	Level function
$\widetilde{f}(x) = \sup_{y \geq x} f(y)$	$f^o(x) = \frac{d}{dx}\left(\text{least concave maj} \int_0^x f\right)$

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$\int_0^\infty \widetilde{f} g =$ $\sup\{\int_0^\infty fh : \int_0^x h \leq \int_0^x g\}$ g and h are positive	$\int_0^\infty f^o g =$ $\sup\{\int_0^\infty fh : \int_0^x h \leq \int_0^x g\}$ g and h are decreasing

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Version in (S, Σ, λ)

Least C.D majorant	Level function
$\widetilde{f}$	$f^{o_{\mathcal{A}}}$

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Version in (S,Σ,λ)

Least C.D majorant	**Level function**
$\widetilde{f}$	$f^{o_\mathcal{A}}$
$\int_S \widetilde{f} g\, d\lambda =$ $\sup\{\int_S fh : \int_E h \leq \int_E g, \forall E \in \mathcal{A}\}$ g and h are positive	$\int_S f^{o_\mathcal{A}} g\, d\lambda =$ $\sup\{\int_S fh : \int_E h \leq \int_E g, \forall E \in \mathcal{A}\}$ g and h are core-decreasing

Down spaces

For $p \in [1, \infty]$, the 'Down space' $(L^p_\lambda)^\downarrow$ is defined by the norm

$$\|f\|_{(L^p_\lambda)^\downarrow} = \sup\left\{ \int_S |f|\, g\, d\lambda : \|g\|_{L^{p'}_\lambda} \leq 1 \text{ and core-decreasing} \right\}$$

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We have: $\widetilde{L^p_\lambda} \subseteq L^p_\lambda \subseteq (L^p_\lambda)^\downarrow$.
Duality:

$$\left((L^p_\lambda)^\downarrow\right)' = \widetilde{L^{p'}_\lambda}.$$

Transferring monotonicity

For the operator $f \mapsto Kf(y) = \int_S k(y,s)f(s)\,d\lambda(s)$, where $k(y,\bullet)$ is core-decreasing. Let X be a Banach function space.

Theorem

The best constant C in

$$\int_S fu\,d\lambda \leq C\|Kf\|_X,\ \forall f \text{ is non-negative}$$

doesn't change when u is replaced by $\widetilde{u}$.

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Possible applications

- **Boundedness of Abstract Hardy operator $L^1 \to L^q$, $q \in (0,1)$:** Consider $B : Y \to \Sigma$ a map whose image is an ordered core. The kernel $k(y,s) = \chi_{B(y)}(s)$ is core-decreasing in s. Characterization of C for the inequality

$$\left(\int_Y \left(\int_{B(y)} f(s)\, d\lambda(s) \right)^q d\mu(y) \right)^{1/q} \le C \int_S f w \, d\lambda$$

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- **Interpolation of $(L^1_\lambda, (L^\infty_\lambda)^\downarrow)$ and $(\widetilde{L^1_\lambda}, L^\infty_\lambda)$**