Generalized monotone functions in measure spaces

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Monotone functions on $[0, \infty)$ and Hardy inequalities

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Classic Hardy's inequality (1925)

For $p > 1$, $\left(\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\,dt \right|^p \,dx \right)^{1/p} \leq C_p \left(\int_0^\infty |f(x)|^p \,dx \right)^{1/p}$

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Conjugate: $\left(\int_0^\infty \left| \frac{1}{x} \int_x^\infty f(t)\, dt \right|^p \, dx \right)^{1/p} \leq C_p \left(\int_0^\infty |f(x)|^p \, dx \right)^{1/p}$

Differential ($f \in W_0^1$): $\left(\int_0^\infty \left| \frac{f(x)}{x} \right|^p \, dx \right)^{1/p} \leq C_p \left(\int_0^\infty |f'(x)|^p \, dx \right)^{1/p}$

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Questions/Goals: 1. Can we define monotone functions on a general measure space (S, Σ, μ) (no a priori order among the elements)?

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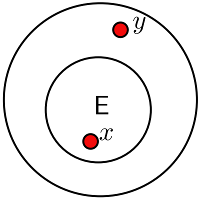
Definition: A positive measurable function is core-decreasing if

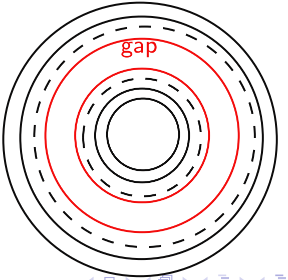
1.



if $x \in E$ and $y \notin E$, $f(x) \geq f(y)$

$\forall E \in \mathcal{A}$

2. f is constant in each gap



Examples

Every core contains $\emptyset$.

1. Let $U = [0, \infty)$, μ the Lebesgue measure. The core $\mathcal{A} = \{[0, x] : x > 0\}$. Note $\sigma(\mathcal{A})$ is the Borel σ-algebra. f is core decreasing if for each $E = [0, t]$, $x \in [0, t]$ and $y \notin [0, t]$ then $f(y) \leq f(x)$.

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3. Let $U = \mathbb{R}^d$, μ satisfying $\mu(B[0; r]) < \infty$ for each $r > 0$. The core $\mathcal{A} = \{B[0; r] : r > 0\}$. Again $\sigma(\mathcal{A})$ is strictly smaller than the Borel σ-algebra. f is core decreasing if it is radially decreasing

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4. Let (U, Σ, μ) be σ-finite and fix $\varphi : U \to \mathbb{C}$ satisfying for all $r > 0$, $\mu\big(\{y : |\varphi(y)| > r\}\big) < \infty$. The core $\mathcal{A} = \{\{y : |\varphi(y)| > r\} : r > 0\}$. f is core decreasing if $f(x) \leq f(y) \iff |\varphi(x)| \leq |\varphi(y)|$.

Maps between ordered cores

Definition: A map $r : \mathcal{A} \to \Sigma$ is a core morphism if: 1. $\tau(r(\emptyset)) = 0$.
2. There exists $c > 0$: $\tau(r(B) \setminus r(A)) \leq c\mu(B \setminus A)$ for all $A, B \in \mathcal{A}$.
3. If $A \subseteq B$ then $r(A) \subseteq r(B)$ for all $A, B \in \mathcal{A}$.

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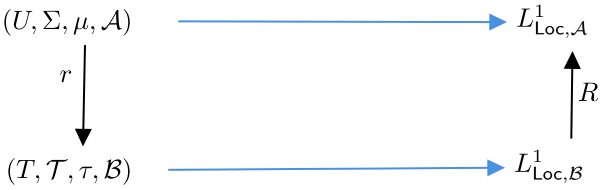
Theorem [Sinnamon, S.- 2024]

Given a core morphism $r : \mathcal{A} \to \Sigma$, there exists a linear map $R : L^1_{\text{Loc},\mathcal{B}} \to L^1_{\text{Loc},\mathcal{A}}$ such that

$$\int_A Rf\, d\mu = \int_{r(A)} f\, d\tau, \quad \forall A \in \mathcal{A}.$$



Induced measure on the half line

Every ordered core induces a Borel measure λ on $[0, \infty)$ and maps R, Q:

$$f \in L^1_{\mathrm{Loc},\mathcal{A}}$$

R ↓ Q ↑

$$g \in L^1_{loc}([0,\infty), \lambda)$$

Properties:

- $RQg = g$
- $QRf \neq f$ in general
- Q and R preserve monotonicity.
- $QRf = f$ if f is core-decreasing

An application to Hardy inequalities

More general Hardy inequalities

Recall the inequality $\left(\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\,dt \right|^p \,dx \right)^{1/p} \leq C_p \left(\int_0^\infty |f(x)|^p \,dx \right)^{1/p}$

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Extensions to measures: Find necessary and sufficient conditions (in terms of the measures and indices) such that there exists C such that

$$\left(\int_0^\infty \left| \int_{[0,x]} f(t)\, d\mu(t) \right|^q \, d\tau(x) \right)^{1/q} \leq C_{p,q,\mu,\tau,\eta} \left(\int_0^\infty |f(x)|^p \, d\eta(x) \right)^{1/p}, \forall f \geq 0.$$

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More Hardy operators:

(Sequences): $Hf(n) = \int_{[0,n]} f\,d\# = \sum_{k=0}^n f(n)$.

Higher dimensions): $Hf(x) = \int_{B(|x|)} f(s)\,ds$, where $B(|x|) = \{y \in \mathbb{R}^n : |y| \leq |x|\}$.

Previous work on metric measure spaces

M. Ruzhansky, D. Verma (2019) Metric measure spaces

Let X be a metric measure space that must admit a polar decomposition Let $1 < p \leq q < \infty$, $a \in X$ and weights $u, v > 0$, there is $C > 0$ such that

$$\left(\int_X \left(\int_{B(a,|x|_a)} f(s)\, d(s) \right)^q u(x)\, dx \right)^{\frac{1}{q}} \leq C \left(\int_X f(x)^p v(x)\, dx \right)^{\frac{1}{p}}$$

holds for all $f \in L^+$ if and only if ... conditions u, v.

Question: Is the polar decomposition hypothesis needed?

Abstract Hardy inequalities

Fix $p \geq 1$ and $q > 0$. Let $(Y, \mathcal{T}, \tau)$ and (U, Σ, μ) be σ-finite measure spaces. A map $B : Y \to \Sigma$ such that the range is an ordered core. Does there exist a constant C such that

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau \right)^{1/q} \leq C \left(\int_U f^p \, d\mu \right)^{1/p}$$

holds for all $f \in L_\mu^+$?

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holds for all $f \in L^+_\mu$?

[Sinnamon 2022]

There exists a nonincreasing function b such that the best constant for which the above inequality holds is the same as the best constant for

$$\left(\int_0^\infty \left(\int_0^{b(y)} g(s) \, ds \right)^q dy \right)^{1/q} \leq C \left(\int_0^\infty g^p(s) \, ds \right)^{1/p}, \quad \forall g \in L^+.$$

Abstract Hardy inequalities with $p = 1$.

Theorem [S.2024]

For $(Y, \mathcal{T}, \tau), (U, \Sigma, \mu), (U, \Sigma, \nu)$ σ-finite and a core map $B : Y \to \Sigma$. Let $\eta = \eta_a + \eta_s$, where $d\eta_a = u d\mu$ and $\eta_s \perp \mu$. Then the best constant C in the inequality

$$\left(\int_Y \left(\int_{B(y)} f \, d\mu \right)^q d\tau(y) \right)^{1/q} \leq C \int_U f \, d\eta, \tag{1}$$

satisfies

$$C \approx \left[\int_Y \left(\int_{\mu(B(z)) \leq \mu(B(y))} R\left(\frac{1}{\underline{u}}\right) \circ \mu \circ B(y) \, d\tau(y) \right)^{\frac{q}{1-q}} d\tau(z) \right]^{\frac{1-q}{q}}, \text{ for } q \in (0,1),$$

and

$$C = \sup_{s \in U} \left(\frac{1}{\underline{u}}(s) \right) \tau\left(\{y \in Y : s \in B(y)\}\right)^{1/q}, \text{ for } q \in [1, \infty).$$

Where $\underline{u}$ is the greatest core decreasing minorant.

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Thank you!